Minkowski compactness measure

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***Abstract*—Many compactness measures are available in the literature. In this paper we present a generalised compactness measure $C_q(S)$ which unifies previously existing definitions of compactness. The new measure is based on Minkowski distances and incorporates a parameter q which modifies the behaviour of the compactness measure. Different shapes are considered to be most compact depending on the value of q: for $q = 2$, the most compact shape in $2D$ ($3D$) is a circle (a sphere); for $q \to \infty$, the most compact shape is a square (a cube); and for $q = 1$, the most compact shape is a square (a octahedron).**

For a given shape S, measure $C_q(S)$ can be understood as a function of q and as such it is possible to calculate a spectum of $C_q(S)$ for a range of q. This produces a particular compactness signature for the shape S, which provides additional shape information.

The experiments section of this paper provides illustrative examples where measure $C_q(S)$ is applied to various shapes and describes how measure and its spectrum can be used for image processing applications.

Keywords: shape compactness, shape description, image processing, computer vision.

I. Introduction

Shape descriptors provide a quantitative representation of a particular characteristic of the shape. Afterwards these values can be used in intelligent image processing applications, for example image classification or image retrieval. Classification is performed by grouping together images which represent similar shapes, for example grouping together oranges and apples (similar because of their roundness), and separating them from pineapples and bananas. Image retrieval consists on searching a database for images which have similar shape to the given query image.

A variety of shape descriptors for $2D$ and $3D$ shapes exists in literature, such as the well known moment invariants [1], asymmetries in the distribution of roughness (ADR) [2] and the anisotropy measure for multiple component shapes [3]. Shape descriptors have been used in a range of image processing applications ranging from shape classification [4] to medical applications [5].

One of the most intuitive and useful shape characteristics is the compactness of the shape. A common way to understand compactness in $2D$ is as a ratio between the area of and perimeter of a given shape as follows:

$$C_{2D}(S) = \frac{4\pi \cdot Area(S)}{Perimeter(S)^2} \quad (1)$$

Using the definition in equation (1), a circle will be the most compact shape possible. This definition can easily be extended to $3D$ in which case the most compact possible shape is a sphere:

$$C_{3D}(S) = \frac{36\pi \cdot Volume(S)^2}{Surface(S)^3} \quad (2)$$

Other definitions of compactness exist in the literature, such as [6] which defines a cube as the most compact shape. In [7], [8] and [9] the authors give alternative definitions of compactness which make a sphere, a cube or an octahedron the most compact shapes. These definitions are derived in a very similar way and compare the distribution of mass in the shape with the distribution in the most compact shape. They only differ by using different distance metrics as their starting point – Euclidean distance in the case of the sphere and Chebyshev distance in the case of the cube and Manhattan distance in the case of the octahedron. These distance metrics are in fact particular cases of the Minkowski distance.

The Minkowski distance between D-dimensional vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is

$$\|\boldsymbol{x}-\boldsymbol{y}\|_q = (|x_1-y_1|^q+|x_2-y_2|^q+\ldots+|x_D-y_D|^q)^{\frac{1}{q}} \quad (3)$$

When $q = 2$ the standard Euclidean distance is recovered; when $q = 1$ the distance corresponds to the Manhattan distance and as $q \to \infty$ the Chebyshev distance, $\|\boldsymbol{x}-\boldsymbol{y}\|_\infty = \max(|x_1-y_1|, |x_2-y_2|, \ldots, |x_D-y_D|)$ is obtained.

In this paper we unify the compactness, cubeness and octahedroness definitions introduced in [7], [8] and [9] by defining a compactness measure based on any one of the Minkowski distances. As such, the compactness measures introduced in [7], [8] and [9] are special cases of the compactness measure introduced here. Furthermore, by using a range of values for the Minkowski distance it is possible to obtain a a spectrum of compactness measures that give a more complete characterisation of the shape.

The paper is organised as follows. In the next section we define our measure of compactness and establish its properties, after which in section III we discuss how it may be efficiently calculated. In section IV we apply the compactness measure to a number of $2D$ and $3D$ shapes. Section V includes conclusions and final remarks.

II. Generalized compactness measure

In order to construct the Minkowski compactness measure, we begin by defining the q-ball:

$$B_q(a) = \{\boldsymbol{x} \;:\; \|\boldsymbol{x}\|_q \leq a\} \tag{4}$$

Here a is the radius of the q-ball, i.e. any point on the boundary of the q-ball, is at a distance $\|\boldsymbol{x}\|_q = a$ from the origin and the boundary of $B_q(a)$ intersects the coordinate axes at a distance a from the origin [1]. The boundaries of a selection of these in are shown in Figure 1.

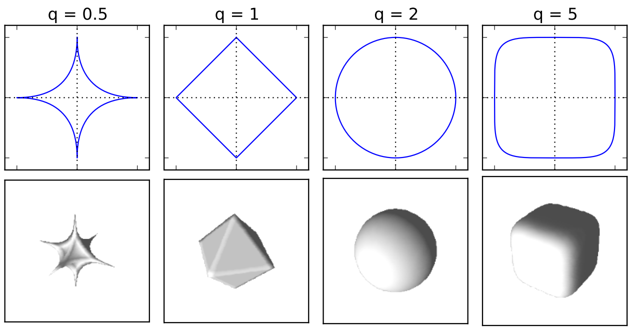


Fig. 1: Boundaries of unit q-balls $B_q(1)$ for various q in $2D$ (top row) and $3D$ (bottom row).

Notice that in $2D$ the q-ball for $q = 1$ and $q \to \infty$ is in both cases a square, only differing in rotation and scale (see Figure 1); a compactness measure derived using metrics $\|\boldsymbol{x}\|_{q=1}$ and $\|\boldsymbol{x}\|_{q\to\infty}$ would be equivalent. However this is not the case in $3D$: the q-ball for $q \to \infty$ is a cube while the q-ball for $q = 1$ is a octahedron; compactness measures derived from these $\|\boldsymbol{x}\|_{q=1}$ and $\|\boldsymbol{x}\|_{q\to\infty}$ in $3D$ would yield different results.

We consider a shape S which comprises the set of points $\boldsymbol{x}$ belonging to the shape. As usual we define the area (in $2D$) or volume (in $3D$) of the shape as:

$$\text{Vol}(S) = \int_S d\boldsymbol{x} \tag{5}$$

The existing compactness measures provide a numerical indication of the similarity between a given shape and a prototypical 'most compact' shape. To begin with our definition of compactness consider following measure for a shape S, whose centre of mass lies at the origin:

$$M_q(S) = \int_S \|\boldsymbol{x}\|_q \, d\boldsymbol{x} \tag{6}$$

The quantity $M_q(S)$ is the integrated distance, using the Minkowski distance, of all points in the shape S to the origin (its centre).

[1]Note: For simplicity and without loss of generality it is assumed throughout that the centre of mass of S lies at the origin.

The radius $a^\star$ can be selected such that the q-ball $B_q(a^\star)$ has volume equal to the volume of S:

$$\text{Vol}(B_q(a^\star)) = \int_{B_q(a^\star)} d\boldsymbol{x} = \text{Vol}(S) \tag{7}$$

Based on the above, we provide the following theorem:

Theorem 1: For any given shape S and $a^\star$ as defined by (7), then $M_q(S) \geq M_q(B_q(a^\star))$ and $M_q(S) = M_q(B_q(a^\star))$ if and only if $S = B_q(a^\star)$.

Proof 1: For convenience write $B_q^\star \equiv B_q(a^\star)$. Consider the regions $S \cap B_q^\star$, $S \setminus B_q^\star$ and $B_q^\star \setminus S$. Note that $\text{Vol}(S \setminus B_q^\star) = \text{Vol}(B_q^\star \setminus S)$ because $\text{Vol}(S) = \text{Vol}(B_q^\star)$.

Also notice that $\|\boldsymbol{x}\|_q > a^\star$ for $\boldsymbol{x} \in S \setminus B_q^\star$, while $\|\boldsymbol{x}\|_q < a^\star$ for $\boldsymbol{x} \in B_q^\star \setminus S$. Consequently:

$$\int_{B_q^\star \setminus S} \|\boldsymbol{x}\|_q \, d\boldsymbol{x} < \int_{S \setminus B_q^\star} \|\boldsymbol{x}\|_q \, d\boldsymbol{x}. \tag{8}$$

Therefore

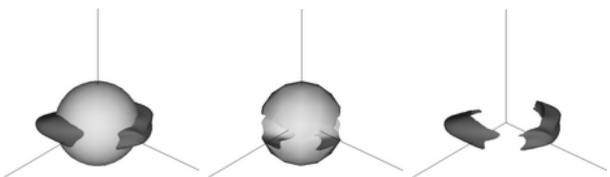
$$M_q(S) = \int_{S \cap B_q^\star} \|\boldsymbol{x}\|_q \, d\boldsymbol{x} + \int_{S \setminus B_q^\star} \|\boldsymbol{x}\|_q \, d\boldsymbol{x} \tag{9}$$

$$> \int_{S \cap B_q^\star} \|\boldsymbol{x}\|_q \, d\boldsymbol{x} + \int_{B_q^\star \setminus S} \|\boldsymbol{x}\|_q \, d\boldsymbol{x} \tag{10}$$

$$= M_q(B_q^\star) \tag{11}$$

If $S = B_q^\star$, then clearly $M_q(S) = M_q(B_q^\star)$. In the reverse direction, suppose that $M_q(S) = M_q(B_q^\star)$ and $S \neq B_q^\star$. Then equations (8) and (9) show that $M_q(S) > M_q(B_q^\star)$ and the contradiction implies that $S \neq B_q^\star$. □

Theorem 1 might be more easily understood by simply observing the shapes in Figure 2. Figure (a) which shows a given shape S (dark grey) and its corresponding $B_q(a^\star)$ for $q = 2$ (light grey); figures (b) and (c) show $B_q^\star \setminus S$ and $S \setminus B_q^\star$ respectively. It can be seen that the Euclidean distance from any point in $B_q^\star \setminus S$ to the origin is smaller than the Euclidean distance from any given point in $S \setminus B_q^\star$ to the origin.



(a) S and $B_q^\star$ (b) $B_q^\star \setminus S$ (c) $S \setminus B_q^\star$

Fig. 2: A given shape S and its equivalent $B_{q=2}(a^\star)$. With $q = 2$, it is clear that any given point in $B_q^\star \setminus S$ is closer to the origin than any given point in $S \setminus B_q^\star$.

Now we give a definition for the Minkowski compactness measure:

Definition 1: Let S be any given shape, and let $R(S, \theta)$ be the rotation of S parametrised by an angle θ (an angle in $2D$ and 2 angles in $3D$). Then, the compactness measure $C_q(S)$ is defined as:

$$C_q(S) = \min_\theta \frac{M_q(R(S, \theta))}{M_q(B_q(a^\star))} \tag{12}$$

For the instances where $q = 1$, $q = 2$ and $q = \infty$, the integrals for the measure $C_q(S)$ can be simplified and calculated in terms of shape moments [8], [7], [9]. In the generalised form this is not so straightforward, but the integrals can still be calculated numerically; this is discussed further in section III.

The compactness measure $C_q(S)$ has the following properties:

a) $C_q(S) \in [1, \infty)$, for any given shape;
b) $C_q(S)$ is invariant with respect to similarity transformations (rotation, translation and scaling);
c) $C_q(S) = 1$ if and only if S is related to B_q by a similarity transform;
d) For any given shape S_1 there is a shape S_2 such that $C_q(S_1) < C_q(S_2)$.

III. Calculation of the compactness measure

Calculation of the compactness requires finding the numerator and denominator of equation (12). Except for a few shapes and particular values of q the numerator must be found numerically. When the shapes are defined by pixels or voxels this is straightforwardly done by approximating the integral (6) by a summation over the pixels. We note that it is more accurate to rotate the coordinate system which defines $\|\boldsymbol{x}\|_q$ rather than rotating the shape itself.

The denominator of equation (12) is $M_q(B_q(a^\star))$. This can be found without numerical integration except to find $\mathrm{Vol}(S)$, which again can be approximated by summation over the pixels (pixel count). $M_q(B_q(a^\star))$ can be calculated as follows: First we note that changing variables in the integration from equation (6) shows that:

$$M_q(B_q(a)) = a^{D+1} M_q(B_q(1)) \tag{13}$$

The quantity $M_q(B_q(1))$ can be related to $\mathrm{Vol}(B_q(1))$ in the following way. In (hyper-) spherical polar coordinates $(r, \phi_1, \phi_2, \dots, \phi_{D-1})$ we have:

$$\begin{aligned}\|\boldsymbol{x}\|_q &= r\,(|\cos\phi_1|^q + |\sin\phi_1\cos\phi_2|^q + \dots \\ &\quad + |\sin\phi_1\sin\phi_2\dots\sin\phi_{D-2}\cos\phi_{D-1}|^q)^{\frac{1}{q}} \qquad (14)\\ &\equiv r b(\boldsymbol{\phi}) \qquad (15)\end{aligned}$$

The last equation shows that $b^{-1}(\boldsymbol{\phi}) = b^{-1}(\phi_1, \dots, \phi_{D-1})$ is the distance from the origin to the boundary of $B_q(1)$ in the $\boldsymbol{\phi}$ direction. In polar coordinates the volume of the $B_q(1)$ can be found by integrating over the positive orthant:

$$\mathrm{Vol}(B_q(1)) = 2^D \int_{\phi_{D-1}=0}^{\pi} \int_{\phi_{D-2}=0}^{\pi/2} \dots \int_{\phi_1=0}^{\pi/2} \int_{r=0}^{b^{-1}(\boldsymbol{\phi})} r^{D-1} J(\boldsymbol{\phi})\, dr d\boldsymbol{\phi} \tag{16}$$

where $J(\boldsymbol{\phi})$ is the angular part of the volume element. The integration with respect to r can be carried out immediately to yield:

$$\mathrm{Vol}(B_q(1)) = \frac{2^D}{D} \int_{\phi_{D-1}=0}^{\pi} \int_{\phi_{D-2}=0}^{\pi/2} \dots \int_{\phi_1=0}^{\pi/2} [b(\boldsymbol{\phi})]^{-D} J(\boldsymbol{\phi}) d\boldsymbol{\phi} \tag{17}$$

In a similar way and using the fact that $\|\boldsymbol{x}\|_q = r \cdot b(\phi)$, we may write $M_q(B_q(1))$ as:

$$\begin{aligned}M(B_q(1)) &= 2^D \int_{\phi_{D-1}=0}^{\pi} \int_{\phi_{D-2}=0}^{\pi/2} \dots \int_{\phi_1=0}^{\pi/2} \int_{r=0}^{b^{-1}(\boldsymbol{\phi})} r \cdot b(\phi) \cdot r^{D-1} J(\boldsymbol{\phi})\, dr d\boldsymbol{\phi} \qquad (18)\\ &= \frac{2^D}{D+1} \int_{\phi_{D-1}=0}^{\pi} \int_{\phi_{D-2}=0}^{\pi/2} \dots \int_{\phi_1=0}^{\pi/2} [b(\boldsymbol{\phi})]^{-D} J(\boldsymbol{\phi}) d\boldsymbol{\phi} \qquad (19)\end{aligned}$$

Consequently

$$M(B_q(1)) = \frac{D}{D+1} \mathrm{Vol}(B_q(1)) \tag{20}$$

Using this result, (13) and the fact that

$$a^\star = \left[\frac{\mathrm{Vol}(S)}{\mathrm{Vol}(B_q(1))}\right]^{1/D} \tag{21}$$

which follows from $\mathrm{Vol}(B_q(a)) = a^D\,\mathrm{Vol}(B_q(1))$, allows $M_q(B_q(a^\star))$ to be written as:

$$M_q(B_q(a^\star)) = \frac{D}{D+1} \left[\mathrm{Vol}(S)\right]^{\frac{D+1}{D}} \left[\mathrm{Vol}(B_q(1))\right]^{-\frac{1}{D}} \tag{22}$$

Finally we note that the volume of a unit q-ball in D-dimensions can be efficiently computed as [10]:

$$\mathrm{Vol}(B_q(1)) = 2^D \frac{\Gamma(\frac{1}{q}+1)^D}{\Gamma(\frac{D}{q}+1)} \tag{23}$$

A. Special cases

As it was mentioned above, the compactness, cubeness and octahedroness measures introduced in [7], [8] and [9] are indeed particular cases of the compactness measure $C_q(S)$ introduced here. In these cases, measure $C_q(S)$ indicates a similarity with a geometric shape (sphere, cube and octahedron) and therefore the denominator of (12) can be calculated from the well known formulas for volume of these geometric shapes.

Further simplification is possible for $q = 2$: given that euclidean distance is rotationally invariant, i.e. $\|\boldsymbol{x}\|_{q=2} = \|R(\boldsymbol{x}, \theta)\|_{q=2}$ for any θ, it is unnecessary to minimise for any rotation of S, so (12) can be simplified to:

$$C_{q=2}(S) = \frac{M_{q=2}(S)}{M_{q=2}(B_{q=2}(a^\star))} \tag{24}$$

Recalling that moment for a $3D$ shape centred at the origin are defined as [1]:

$$\mu_{p,q,r}(S) = \iiint\limits_{S} x^p y^q z^r \, dxdydz \tag{25}$$

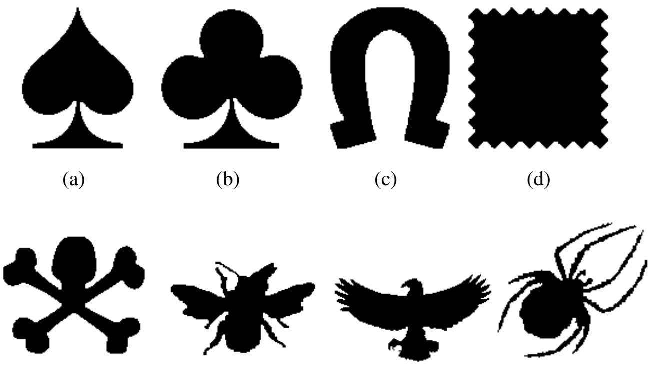
it is easy to see that $C_{q=2}(S)$ can expressed in terms of second order moments $\mu_{2,0,0}(S)$, $\mu_{0,2,0}(S)$ and $\mu_{0,0,2}(S)$. It is well known that moments can be computed efficiently [11], which implies $C_{q=2}(S)$ can also be computed efficiently.

IV. Experiments

In this section we show how the measure $C_q(S)$ behaves for different $2D$ and $3D$ shapes and provide an indication of how it can be used for image processing applications.

A. Experiments on two-dimensional shapes

Figure 3 shows a number of shapes to be analysed[2]. Table I shows the computed $C_q(S)$ values for different values of q.



(a) (b) (c) (d) (e) (f) (g) (h)

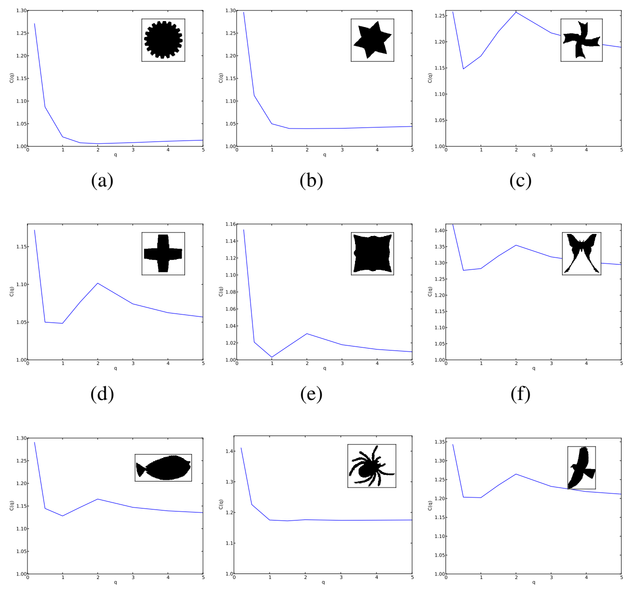
Fig. 3: A selection of shapes to be analysed.

	$q = 0.1$	$q = 0.5$	$q = 1.0$	$q = 2.0$	$q = 5.0$
(a)	1.573	1.193	1.154	1.172	1.155
(b)	1.502	1.134	1.092	1.107	1.092
(c)	1.917	1.458	1.414	1.439	1.416
(d)	1.371	1.043	1.012	1.030	1.014
(e)	1.745	1.328	1.289	1.315	1.290
(f)	1.561	1.173	1.124	1.135	1.120
(g)	1.526	1.145	1.097	1.105	1.095
(h)	1.845	1.380	1.317	1.318	1.314

TABLE I: Measure $C_q(S)$ for shapes in Figure 3 for different values of q.

As it can be seen in Figure 3, using measure $C_q(S)$ with a range of q values provides additional information which may be useful for processing these images. For example, shapes (b) and (g) have very different $C_q(S)$ scores for $q = 0.1$, but very similar scores for $q = 2.0$. Shape (c) is the least compact shape from the set, having the highest $C_q(S)$ score for any value of q, while shape (d) is the most compact shape from the set, having the lowest $C_q(S)$ score for any value of q. Notice that shape (d) reaches its minimum $C_q(S)$ value for $q = 1.0$ (1.012) and also has a very similar value for $q = 5.0$ (1.014), this is to be expected as the shape is almost a square, and both $C_{q=1.0}(S)$ and $C_{q\to\infty}(S)$ reach their minimum for a square.

For any given shape S, the measure $C_q(S)$ can be seen as a function of q. As such, a spectrum of $C_q(S)$ can be calculated. The spectrum $C_q(S)$ provides characteristic information of the shape, which may allow to differentiate between shapes.



(a) (b) (c) (d) (e) (f) (g) (h) (i)

Fig. 4: A selection of shapes and their compactness spectra.

As mentioned above, a spectrum of measure $C_q(S)$ can be calculated for a given shape. Figure 4 shows some figures and their compactness spectra in range $q \in (0, 5)$.

We point out that the minimum value of $C_q(S)$ is 1, which is achieved when S is the Minkowski ball for that q. Thus in Figure (b) the shape is similar to a circle and achieves a minimum C_q for $q = 2$. Likewise the shape in Figure (e) is approximately square and has small C_q values when $q \approx 1$ and for large q (recall that in two dimensions the Minkowski balls for $q = 1$ and $q \to \infty$ differ only by a rotation and isotropic scaling).

It is expected that similar shapes will have similar spectra. Thus the spectra of the ‘cross-like’ shapes in Figures (c), (d) are similar, although their magnitudes are different. In addition, the spectra reveal the similarity of these two shape to the ‘butterfly’ (Figure (f)) and the ‘bird’ (Figure (i)).

It is advantageous to have a spectrum of $C_q(S)$ instead of a single value of $C_q(S)$ because two shapes may have the same $C_q(S)$ value for a given value of q, but they may be distinguished by the rest of their spectra. For instance in Figure 4, shapes (b) and (d) have almost identical $C_q(S) \approx 1.05$ for $q = 1$, however their $C_q(S)$ spectra differ markedly for other

[2]The shapes are taken from: www.lems.brown.edu/ dmc/.

q.

B. *Experiments for 3D case*

In this section we use measure $C_q(S)$ to calculate the compactness of a collection of $3D$ images [3]. Figure 5 shows some $3D$ shapes and Table II shows the computed $C_q(S)$ values for different values of q.

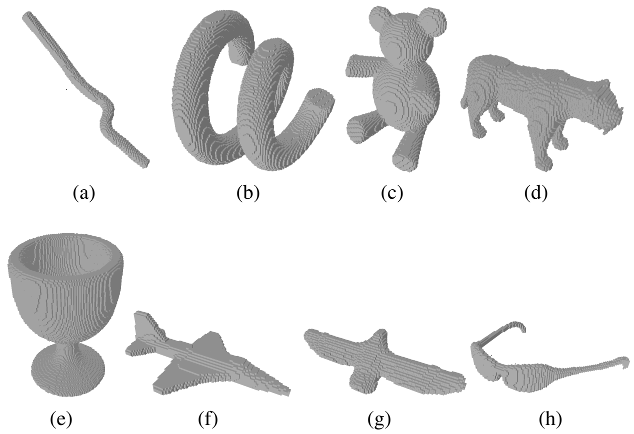


Fig. 5: A selection of $3D$ shapes to be analysed.

	$q = 0.1$	$q = 0.5$	$q = 1.0$	$q = 2.0$	$q = 5.0$
(a)	2.486	2.879	4.303	2.013	3.397
(b)	2.879	4.303	2.013	3.397	3.193
(c)	1.634	3.550	4.209	4.756	3.790
(d)	2.201	2.066	2.177	2.198	2.199
(e)	2.747	3.605	3.175	2.749	2.478
(f)	1.675	1.755	1.823	1.633	1.604
(g)	1.695	2.823	3.540	2.303	3.011
(h)	6.584	3.306	3.401	4.312	3.765

TABLE II: Measure $C_q(S)$ for shapes in Figure 5 for different values of q.

It can be observed in Figure 5 that different shapes have higher or lower $C_q(S)$ scores for different values of q. For example, shape (f) has the lowest $C_q(S)$ score for almost any value of q, however from the shapes in Figure 5 there is none which is consistently least compact for any value of q.

Using different values of q, shapes in 5 would be ordered differently. Therefore using different values of q allow for differentiation of shapes. For example, shapes (d) and (g) have very similar $C_{q=2.0}(S)$ scores (2.198 and 2.303), but their $C_{q=1.0}(S)$ (2.177 and 3.540) are distinctively different. This implies that they have similar spherical compactness but they differ in their octahedroness. Likewise, shapes (e) and (h) have very similar $C_{q=0.5}(S)$ and $C_{q=1.0}(S)$ scores, but are significantly different on their $C_{q=0.1}(S)$, $C_{q=2.0}(S)$ and $C_{q=5.0}(S)$ scores.

[3]The shapes are taken from the McGill database: http://www.cim.mcgill.ca/ shape/benchMark/. [12]

V. Conclusion

In this paper we have presented a generalised measure of compactness which is based on the Minkowski distance. We have show that the compactness measures previously presented in [7], [8] and [9] are indeed particular cases of the $C_q(S)$ compactness measure for specific values of q. We have shown that the calculation of $C_q(S)$ is straight forward and that it can be approximated numerically for any given shape. Section IV provides illustrative examples which demonstrate the behaviour of $C_q(S)$ applied to a set of $2D$ and $3D$ shapes.

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