

Compressed Sensing Performance Analysis via Replica Method using Bayesian framework

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Abstract

Compressive sensing (CS) is a new methodology to capture signals at lower rate than the Nyquist sampling rate when the signals are sparse or sparse in some domain. The performance of CS estimators is analyzed in this paper using tools from statistical mechanics, especially called replica method. This method has been used to analyze communication systems like Code Division Multiple Access (CDMA) and multiple input multiple output (MIMO) systems with large size. Replica analysis, now days rigorously proved, is an efficient tool to analyze large systems in general. Specifically, we analyze the performance of some of the estimators used in CS like LASSO (the Least Absolute Shrinkage and Selection Operator) estimator and Zero-Norm regularizing estimator as a special case of maximum a posteriori (MAP) estimator by using Bayesian framework to connect the CS estimators and replica method. We use both replica symmetric (RS) ansatz and one-step replica symmetry breaking (1RSB) ansatz, clamping the latter is efficient when the problem is not convex. This work is more analytical in its form. It is deferred for next step to focus on the numerical results.

1 Introduction

Recently questions like, *why go to so much effort to acquire all the data when most of what we get will be thrown away?*; *Can we not just directly measure the part that will not end up being thrown away?*, that were paused by Donoho [1] and others, triggered a new way of sampling or sensing called compact ("compressed") sensing (CS).

In CS the task is to estimate or recover a sparse or compressible vector $\mathbf{x}^0 \in \mathbb{R}^N$ from a measurement vector $\mathbf{y} \in \mathbb{R}^M$. These are related through the linear transform $\mathbf{y} = \mathbf{A}\mathbf{x}^0$. Here, \mathbf{x}^0 is a sparse vector and $M \ll N$. In the seminal papers [1] - [3], \mathbf{x}^0 is estimated from \mathbf{y} , by solving a convex optimization problem [4], [5]. Others have used greedy algorithms, like subspace pursuit (SP) [6], orthogonal matching pursuit (OMP) [7] to solve the problem. In this paper the focus is rather on the convex optimization methods. And we consider the noisy measurement system and the linear relation becomes

$$\mathbf{y} = \mathbf{A}\mathbf{x}^0 + \sigma_0 \mathbf{w}. \quad (1.1)$$

Here, \mathbf{y} and \mathbf{x}^0 are as in above where as the noise term, $\mathbf{w} \sim \mathcal{N}(0, \mathbf{I})$. There exists a large body of work on how to efficiently obtain an estimate for \mathbf{x}^0 . And the performances of such estimators are measured using metrics like Restricted Isometric Property (RIP) [8], Mutual Coherence (MC) [9], yet there is apparently no consensus on the bounds in using such metrics. The tool used in this paper gives performance bounds of large size CS systems [10].

Generally the linear model (1.1) is used to describe a multitude of linear systems like code division multiple access (CDMA) and multiple antenna systems like MIMO, to mention just a few. Tools from statistical mechanics have been employed to analyze large CDMA [11] and MIMO systems [12] [13], and on in this paper the same wisdom is applied to analyze the performance of estimators used in CS. Guo and et al in [10] used a Bayesian framework for statistical inference with noisy measurements and characterize the posterior distribution of individual elements of the sparse signal by describing the mean mean square error(MSE) exactly. To do so, they consider (1.1) in a large system and applied the decoupling principle using tools from statistical mechanics.

One can find also works that have used the tools from statistical mechanics to analyse CS system performances. To mention some, in [10] as stated above, Guo and et al used the tools to describe the minimum mean square error (MMSE) estimator, in [14] Rangan and others used the maximum a posterior(MAP) estimator of CS systems. These are referred as Replica MMSE claim and Replica MAP claim in [14].

In [16] - [20] authors have used Belief propagation and message passing algorithms for probabilistic reconstruction in CS using replica methods including RS. Especially, in [18] one finds excellent work about phase diagrams in CS systems while [21] generalizes replica analysis using free random matrices. Kabashima and et. al in [22], Ganguli and Sompolinsky in [23] and Takeda and Kabashima [24] - [26] have shown statistical mechanical analysis of the CS by considering the noiseless recovery problem and they indicated that RSB analysis is needed in the phase regimes where the RS solution is not stable. In this paper the performance of those CS estimators, considered as MAP estimator, is shown for the noisy problem by using the replica method including RS and RSB as in [27] - [29], where the RSB ansatz gives better solution when the replica symmetry (RS) solution is unstable. This work is kind of an extension of [29] from MIMO systems to the CS systems.

The paper is organized as follows. In section 2 the estimator in CS system are presented and redefined using the Bayesian framework, and based on that we present our basis of analysis in section 3 which is the replica method from the statistical physics and apply it on the different CS estimators which are presented generally as a MAP estimator. In section 4 we showed our analysis using a particular example, and section 5 presents conclusion and of future work.

2 Bayesian framework for Sparse Estimation

Beginning with a given vector of measurements $\mathbf{y} \in \mathbb{R}^M$ and measurement matrix $\mathbf{A} \in \mathbb{R}^{M \times N}$, assuming noisy measurement with $\mathbf{w} \in \mathbb{R}^M$ being i.i.d.

Gaussian random variables with zero mean and covariance matrix \mathbf{I} , estimating the sparse vector $\mathbf{x}^0 \in \mathbb{R}^N$ is the problem that we are considering where these variables are related by the linear model (1.1).

2.1 Sparse Signal Estimation

Various methods for estimating \mathbf{x}^0 may be used. The classical approach to solving inverse problems of such type is by least squares (LS) estimator in which no prior information is used and its closed form is

$$\hat{\mathbf{x}}^0 = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{y}, \quad (2.1)$$

which performs very badly for the CS estimation problem we are considering since it does not find the sparse solution. Another approach to estimate \mathbf{x}^0 is via the solution of the unconstrained optimization problem

$$\hat{\mathbf{x}}^0 = \min_{\mathbf{x} \in \mathbb{R}^N} \frac{1}{2} \|\mathbf{y} - \mathbf{A}\mathbf{x}^0\|_2^2 + u f(\mathbf{x}^0), \quad (2.2)$$

where $u f(\mathbf{x}^0)$ is a regularizing term, for some non-negative u . By taking $f(\mathbf{x}^0) = \|\mathbf{x}^0\|_p$, emphasis is made on a solution with LP norm, and $\|\mathbf{x}^0\|_p$ is defined as a penalizing norm. When $p = 2$, we get

$$\hat{\mathbf{x}}^0 = \min_{\mathbf{x}^0 \in \mathbb{R}^N} \frac{1}{2} \|\mathbf{y} - \mathbf{A}\mathbf{x}^0\|_2^2 + u \|\mathbf{x}^0\|_2. \quad (2.3)$$

This is penalizing the least square error by the L2 norm and this performs badly as well, since it does not introduce sparsity into the problem. When $p = 0$, we get the L0 norm, which is defined as

$$\|\mathbf{x}^0\|_0 = k \equiv \#\{i \in \{1, 2, \dots, N\} | x_i^0 \neq 0\},$$

the number of the non zero entries of \mathbf{x}^0 , which actually is a partial norm since it does not satisfy the triangle inequality property, but can be treated as norm by defining it as in [14], and get the L0 norm regularizing estimator

$$\hat{\mathbf{x}}^0 = \min_{\mathbf{x}^0 \in \mathbb{R}^N} \frac{1}{2} \|\mathbf{y} - \mathbf{A}\mathbf{x}^0\|_2^2 + u \|\mathbf{x}^0\|_0, \quad (2.4)$$

which gives the best solution for the problem at hand since it favors sparsity in \mathbf{x}^0 . Nonetheless, it is an NP- hard combinatorial problem. Instead, it has been a practice to approximate it using L1 penalizing norm to get the estimator

$$\hat{\mathbf{x}}^0 = \min_{\mathbf{x}^0 \in \mathbb{R}^N} \frac{1}{2} \|\mathbf{y} - \mathbf{A}\mathbf{x}^0\|_2^2 + u \|\mathbf{x}^0\|_1, \quad (2.5)$$

which is a convex approximation to the L0 penalizing solution 2.4. The best solution for estimate of the sparse vector \mathbf{x} is given by the zero-norm regularized estimator which is a hard combinatorial problem. These estimators, (2.3) - (2.5), can equivalently be presented as solutions to constrained optimization problem [1]- [3]. This constrained optimization version of (2.5) is known as the L1 penalized L2 minimization called LASSO (Least Absolute Shrinkage and Selection Operator) or BPDN(Basis Pursuit Denoising), which can be set

as Quadratic Programming (QP) and Quadratic Constrained Linear Programming (QCPL) optimization problems.¹ In the following subsection the above estimators are presented as a MAP estimator in Bayesian framework.

2.2 Bayesian framework for Sparse signal

Equivalently, the estimator of \mathbf{x}^0 in (2.2) can generally be presented as MAP estimator under the Bayesian framework. Assume a prior probability distribution for \mathbf{x} to be

$$p_u(\mathbf{x}) = \frac{e^{-uf(\mathbf{x})}}{\int_{\mathbf{x} \in \chi^N} e^{-uf(\mathbf{x})} d\mathbf{x}}, \quad (2.6)$$

where the cost function $f: \chi \rightarrow \mathbb{R}$ is some scalar-valued, non negative function with $\chi \subseteq \mathbb{R}$ and

$$f(\mathbf{x}) = \sum_{i=1}^N f(x_i). \quad (2.7)$$

such that for sufficiently large u , $\int_{\mathbf{x} \in \chi^n} \exp(-uf(\mathbf{x})) d\mathbf{x}$ is finite as in [14]. And let the assumed variance of the noise be given by

$$\sigma_u^2 = \frac{\gamma}{u}$$

where γ is system parameter which can be taken as $\gamma = \sigma_u^2 u$ where σ_u^2 is the assumed variance for each component of \mathbf{n} . Note that we incorporate the sparsity in the prior pdf via $f(\mathbf{x})$. By (1.1) the probability density function of \mathbf{y} given \mathbf{x} is given by

$$p_{\mathbf{y}|\mathbf{x}}(\mathbf{y} | \mathbf{x}; \mathbf{A}) = \frac{1}{(2\pi\sigma_u^2)^{N/2}} e^{-\frac{1}{2\sigma_u^2} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2}, \quad (2.8)$$

and prior distribution of \mathbf{x} by (2.6), the posterior distribution for the measurement channel (1.1) according to Bayes law is

$$p_{\mathbf{x}|\mathbf{y}}(\mathbf{x} | \mathbf{y}; \mathbf{A}) = \frac{e^{-u(\frac{1}{2\gamma} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2 + f(\mathbf{x}))}}{\int_{\mathbf{x} \in \chi^n} e^{-u(\frac{1}{2\gamma} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2 + f(\mathbf{x}))} d\mathbf{x}}. \quad (2.9)$$

Then the MAP estimator can be shown to be

$$\hat{\mathbf{x}}^{MAP} = \underset{\mathbf{x} \in \chi^n}{\operatorname{argmin}} \frac{1}{2\gamma} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2 + f(\mathbf{x}). \quad (2.10)$$

Now, as we choose different penalizing function in (2.10) we get the different estimators defined above in equations (2.3), (2.4), and (2.5) but this time under the Bayesian framework as a MAP estimator [14].

1. Linear Estimators: when $f(\mathbf{x}) = \|\mathbf{x}\|_2^2$ (2.10) reduces to

$$\hat{\mathbf{x}}_{Linear}^{MAP} = \mathbf{A}^T (\mathbf{A}\mathbf{A}^T + \gamma\mathbf{I})^{-1} \mathbf{y}, \quad (2.11)$$

which is the LMMSE estimator.

¹In this paper we consider the former and leave the later as they are equivalent algorithms.

2. LASSO Estimator: when $f(\mathbf{x}) = \|\mathbf{x}\|_1$ we get the LASSO estimator and (2.10) becomes

$$\hat{\mathbf{x}}_{Lasso}^{MAP} = \operatorname{argmin}_{\mathbf{x} \in \chi^n} \frac{1}{2\gamma} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2 + \|\mathbf{x}\|_1. \quad (2.12)$$

3. Zero-Norm regularization estimator: when $f(\mathbf{x}) = \|\mathbf{x}\|_0$, we get the Zero-Norm regularization estimator and (2.10) becomes

$$\hat{\mathbf{x}}_{Zero}^{MAP} = \operatorname{argmin}_{\mathbf{x} \in \chi^n} \frac{1}{2\gamma} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2 + \|\mathbf{x}\|_0. \quad (2.13)$$

Whether these minimization problems are solvable or not the replica analysis results can provide the asymptotic performances of all the above estimators via replica method as showed in [10], [14], [22], [23] and [24]. We apply RS ansatz as used by Müller and et al in [27] and RSB ansatz as used by Zaidel and et al [29] on vector precoding for MIMO. Actually, this work is an extension of the RSB analysis to MIMO systems done in [29] to the CS system.

3 A Statistical Physics Analysis

The performance of the Bayesian estimators like MMSE and MAP can be done using the pdf of the error vector. The error is random and it should be centered about zero for the estimator to perform well. Kay showed in that way (see section 11.6 in [30]) the performance analysis of MMSE estimator. We believe in general that inference for the asymptotic performance of MAP estimators is best done with statistical mechanical tools including RSB assumption and this is done in the sense of the mean square error (MSE).

The posterior distribution (2.9) is a sufficient statistics to estimate \mathbf{x}^0 [10] and the denominator is called the normalizing factor or evidence in Bayesian inference according to [31] and Partition function in statistical mechanics. Actually, it is this connection, which gives the ground to apply the tools, which are used in statistical mechanics. So the task of evaluating the above estimators for the sparse vector \mathbf{x}^0 can be translated to the statistical physics framework. And let us justify first how the analysis using statistical mechanical tool is able to do it.

Define the Gibbs-Boltzmann distribution as

$$p_{\mathbf{x}}(\mathbf{x}) = \frac{1}{\mathcal{Z}} e^{-\beta \mathcal{H}(\mathbf{x})} \quad (3.1)$$

where β is a constant known as the inverse temperature in the terminology of physical systems. For small β , the prior probability becomes flat, and for large β , the prior probability has sharp modes. \mathcal{H} , which is an expression of the total energy of the system, is called the Hamiltonian in physics literature and \mathcal{Z} is the partition function given by

$$\mathcal{Z} = \sum_{\mathbf{x}^N} e^{-\beta \mathcal{H}(\mathbf{x})} d\mathbf{x}. \quad (3.2)$$

Often the Hamiltonian can be given by a quadratic form like

$$\mathcal{H}(\mathbf{x}) = \mathbf{x}^T \mathbf{J} \mathbf{x}, \quad (3.3)$$

with \mathbf{J} being a Random matrix of dimension $N \times N$. Then the minimum average energy per component of \mathbf{x} can be given by

$$\mathcal{E} = \frac{1}{N} \min_{\mathbf{x} \in \chi^N} \mathcal{H}(\mathbf{x}) \quad (3.4)$$

For our system that we considered to address, which is given by (2.10) or equivalently by (2.2), the Hamiltonian becomes

$$\mathcal{H}(\mathbf{x}) = \frac{1}{2\sigma_u^2} (\mathbf{y} - \mathbf{A}\mathbf{x})^T (\mathbf{y} - \mathbf{A}\mathbf{x}) + uf(\mathbf{x}). \quad (3.5)$$

Compared to (3.3), the Hamiltonian in (3.5) has regularizing term in addition to the quadratic form, which is the energy of the error, in which the regularizing term $f(\mathbf{x})$ is accountable for addressing the problem in the CS. The Gibbs-Boltzman distribution is a solution to (2.10) or to (2.2) in general, after plugging (3.2) and (3.5) since they are equivalent problems. The normalizing factor (also called the partition function) of this distribution is central for calculating many important variables and we shall begin from this term to analyse the CS estimators performance.

Assuming that \mathbf{x}^0 and \mathbf{x} being drawn from the same discrete set (we shall later provide an example from such a set). The partition function of the posterior distribution given in (3.1) becomes

$$\mathcal{Z} = \sum_{\mathbf{x} \in \chi^N} e^{-\beta \left[\frac{1}{2\sigma_u^2} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2 + uf(\mathbf{x}) \right]}, \quad (3.6)$$

by using (3.2) and (3.5). The posterior distribution (2.10) depends on the predetermined random variables \mathbf{y} and \mathbf{A} called quenched states in physics literature [25], [26]. That is, we use fixed states $\mathbf{y} = \mathbf{A}\mathbf{x}^0 + \mathbf{w}$ instead of \mathbf{y} for the large system limit, as $N, M \rightarrow \infty$, while maintaining N/M fixed. We then calculate the n th moment of the partition function Z with respect to the predetermined variables, n replicas, hence the name replica method came from. The replicated partition function is then given by

$$\mathcal{Z}^n = \sum_{\{\mathbf{x}^a\}} e^{-\beta \left[\frac{1}{2\sigma_u^2} \sum_{a=1}^n (\|\mathbf{y} - \mathbf{A}\mathbf{x}^a\|_2^2) + \frac{\gamma}{\sigma_u^2} \sum_{a=1}^n f(\mathbf{x}^a) \right]}, \quad (3.7)$$

where $\sum_{\{\mathbf{x}^a\}} = \sum_{\mathbf{x}_1 \in \chi^N} \dots \sum_{\mathbf{x}_n \in \chi^N}$. And after substituting \mathbf{y} , it becomes

$$\mathcal{Z}^n = \sum_{\{\mathbf{x}^a\}} e^{-\beta \left[\frac{1}{2\sigma_u^2} \sum_{a=1}^n (\|\mathbf{A}(\mathbf{x}^0 - \mathbf{x}^a) + \mathbf{w}\|_2^2) + \frac{\gamma}{\sigma_u^2} \sum_{a=1}^n f(\mathbf{x}^a) \right]}. \quad (3.8)$$

Averaging over the noise \mathbf{n} first, we get

$$\int_{\mathbb{R}^M} \frac{d\mathbf{n}}{\pi^M} e^{-\frac{1}{2\sigma_0} (\mathbf{w}^T \mathbf{w})} \mathcal{Z}^n = \sum_{\{\mathbf{x}^a\}} e^{-\beta \left[\frac{1}{2} \text{Tr} \mathbf{J} \mathbf{L}(n) + \frac{\gamma}{\sigma_u^2} \sum_{a=1}^n f(\mathbf{x}^a) \right]}, \quad (3.9)$$

where $\mathbf{J} = \mathbf{A}^T \mathbf{A}$ and it is assumed to decompose into

$$\mathbf{J} = \mathbf{O} \mathbf{D} \mathbf{O}^{-1}, \quad (3.10)$$

and \mathbf{D} is a diagonal matrix while \mathbf{O} is $N \times N$ orthogonal matrix assumed to be drawn randomly from the uniform distribution defined by the Haar measure on the orthogonal group. For more clarity on this one can see — in [25]. And $L(n)$ is given by

$$\mathbf{L}(n) = -\frac{1}{\sigma_u^2} \sum_{a=1}^n (\mathbf{x}^0 - \mathbf{x}^a)(\mathbf{x}^0 - \mathbf{x}^a)^T + \frac{\sigma_0^2}{\sigma_u^2(\sigma_u^2 + n\sigma_0^2)} \left(\sum_{a=1}^n (\mathbf{x}^0 - \mathbf{x}^a) \right) \left(\sum_{b=1}^n (\mathbf{x}^0 - \mathbf{x}^b) \right)^T. \quad (3.11)$$

Further averaging what we get on the right side of (3.9) over the cross correlation matrix \mathbf{J} , by assuming the eigenvalue spectrum of \mathbf{J} to be self-averaging, we get

$$\begin{aligned} \mathbb{E}_{\mathbf{w}, \mathbf{J}} \{ \mathcal{Z}^n \} &= \mathbb{E}_{\mathbf{J}} \left(\sum_{\{\mathbf{x}^a\}} e^{-\beta \left[\frac{1}{2} \text{Tr} \mathbf{J} \mathbf{L}(n) + \frac{\gamma}{\sigma_u^2} \sum_{a=1}^n f(\mathbf{x}^a) \right]} \right) \\ &= \sum_{\{\mathbf{x}^a\}} e^{\frac{-\beta \gamma}{\sigma_u^2} \sum_{a=1}^n f(\mathbf{x}^a)} \mathbb{E}_{\mathbf{J}} \left(e^{-\beta \left[\frac{1}{2} \text{Tr} \mathbf{J} \mathbf{L}(n) \right]} \right) \end{aligned} \quad (3.12)$$

The inner expectation in (3.12) is the Harish -Chandra -Itzykson-Zuber integral (again see in [27] and [29] and the references therein). The plan here is to evaluate the fixed-rank matrices $\mathbf{L}(n)$ as $N \rightarrow \infty$. Further following the explanation in [29] (3.12) becomes

$$\mathbb{E}_{\mathbf{w}, \mathbf{J}} \{ \mathcal{Z}^n \} = \sum_{\{\mathbf{x}^a\}} e^{\frac{-\beta \gamma}{\sigma_u^2} \sum_{a=1}^n f(\mathbf{x}^a)} e^{-N \sum_{a=1}^n \int_0^{\lambda_a} R(-w) dw + o(N)} \quad (3.13)$$

where $R(w)$ is the R-transform of the limiting eigenvalue distribution of the matrix \mathbf{J} (see, definition 1 in [27] of R-transform or in [12] and [13] for better understanding of R-transform) and $\{\lambda_a\}$ denote the Eigenvalues of the $n \times n$ matrix $\beta \mathbf{Q}$, with \mathbf{Q} defined through

$$Q_{ab} = \frac{1}{N} \left[-\frac{1}{\sigma_u^2} \sum_{i=1}^N (x_i^0 - x_i^a)^T (x_i^0 - x_i^b) + \frac{\sigma_0^2}{\sigma_u^2(\sigma_u^2 + n\sigma_0^2)} \left(\sum_{i=1}^N (x_i^0 - x_i^a) \right)^T \left(\sum_{i=1}^N (x_i^0 - x_i^b) \right) \right], \quad (3.14)$$

for $a, b = 1, \dots, n$.

After applying replica trick, the average free energy can be given by

$$\begin{aligned} \beta \bar{\mathcal{F}} &= - \lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E}_{\mathbf{n}, \mathbf{R}} \{ \log \mathcal{Z} \} \\ &= - \lim_{N \rightarrow \infty} \frac{1}{N} \lim_{n \rightarrow 0} \frac{\partial}{\partial n} \log \mathbb{E}_{\mathbf{n}, \mathbf{R}} \{ (\mathcal{Z})^n \} \end{aligned} \quad (3.15)$$

and the energy of the error can be calculated from the average free energy as

$$\bar{\mathcal{E}} = \lim_{\beta \rightarrow \infty} \frac{1}{\beta} \bar{\mathcal{F}} \quad (3.16)$$

$$\begin{aligned} &= - \lim_{\beta \rightarrow \infty} \frac{1}{\beta} \lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E}_{\mathbf{n}, \mathbf{R}} \{ \log \mathcal{Z} \} \\ &= - \lim_{\beta \rightarrow \infty} \frac{1}{\beta} \lim_{n \rightarrow 0} \frac{\partial}{\partial n} \underbrace{\lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{E}_{\mathbf{n}, \mathbf{J}} \{ (\mathcal{Z})^n \}}_{\Xi_n}. \end{aligned} \quad (3.17)$$

where we get (3.16) by using one of the assumptions used in replica calculations, after interchanging the order of the limits we assumed we get the same result. Further, for Ξ_n we have

$$\Xi_n = - \lim_{N \rightarrow \infty} \frac{1}{N} \log \left(\sum_{\{\mathbf{x}^a\}} e^{\frac{-\beta\gamma}{\sigma_u^2} \sum_{a=1}^n f(\mathbf{x}^a)} \frac{\sum_{e=1}^n \int_0^{\lambda^a} R(-w) dw}{e^{\sum_{a=1}^n \int_0^{\lambda^a} R(-w) dw}} \right). \quad (3.18)$$

Since the additive exponential terms of order $\circ(N)$ have no effect on the results when taking saddle point integration in the limiting regime as $N \rightarrow \infty$ due to the factor $\frac{1}{N}$ outside the logarithm in (3.18) any such terms are dropped further for notational simplicity as in [29].

In order to find the summation in (3.18) we employed the procedure in [29] and the nN dimensional space spanned by the replicas is split into subshells, defined through $n \times n$ matrix \mathbf{Q}

$$S(\mathbf{Q}) = \{ \mathbf{x}^1, \dots, \mathbf{x}^n \mid (\mathbf{x}^0 - \mathbf{x}^a)^T (\mathbf{x}^0 - \mathbf{x}^b) = \frac{N}{\kappa_n} Q_{ab} \}. \quad (3.19)$$

The limit $N \rightarrow \infty$ able us to use saddle point integration. Hence we can have the following general result as similar to [29] but extended in this work with the term, which pertains to CS, where we have given the expression that helps to evaluate the performances of the CS estimators using equation (3.4).

Proposition 1 *The energy \mathcal{E} from (3.4), for any inverse temperature β , any structure of \mathbf{Q} consistent with (3.19), and any R -transform $R(\cdot)$ such that $R(\mathbf{Q})$ is well-defined, is given by*

$$\bar{\mathcal{E}} = - \lim_{n \rightarrow 0} \frac{1}{n} \text{Tr}[\mathbf{Q}R(-\beta\mathbf{Q})], \quad (3.20)$$

where \mathbf{Q} is the solution to the saddle point equation

$$\mathbf{Q} = \int \frac{\sum_{\{\bar{\mathbf{x}} \in \chi^n\}} (x^0 \mathbf{1} - \bar{\mathbf{x}})(x^0 \mathbf{1} - \bar{\mathbf{x}})^T e^{(x^0 \mathbf{1} - \bar{\mathbf{x}})^T \bar{\mathbf{Q}}(x^0 \mathbf{1} - \bar{\mathbf{x}}) - \frac{\beta\gamma}{\sigma_u^2} \bar{\mathbf{x}}}}{\sum_{\{\bar{\mathbf{x}} \in \chi^n\}} e^{(x^0 \mathbf{1} - \bar{\mathbf{x}})^T \bar{\mathbf{Q}}(x^0 \mathbf{1} - \bar{\mathbf{x}}) - \frac{\beta\gamma}{\sigma_u^2} \bar{\mathbf{x}}}} dF_{X^0}(x^0) \quad (3.21)$$

Proof 1 *See Appendix B.*

Further, to get specific results we need to assume simple structure onto the $n \times n$ cross correlation matrix \mathbf{Q} at the saddle point. So we assume two different assumptions for the entries of \mathbf{Q} called ansatz: replica symmetry(RS) and replica symmetric breaking (RSB) ansatz. Then compare the above limiting energy for the different estimators considered in this paper using the two types of ansatz for the CS system. That is the main purpose that we want to show in this paper. And we took the structures similar to [29] :

1. replica symmetry ansatz :

$$\mathbf{Q} = q_0 \mathbf{1}_{n \times n} + \frac{b_0}{\beta} \mathbf{I}_{n \times n} \quad (3.22)$$

2. one replica symmetry breaking ansatz :

$$\mathbf{Q} = q_1 \mathbf{1}_{n \times n} + p_1 \mathbf{I}_{\frac{n\beta}{\mu_1} \times \frac{n\beta}{\mu_1}} \otimes \mathbf{1}_{\frac{\mu_1}{\beta} \times \frac{\mu_1}{\beta}} + \frac{b_1}{\beta} \mathbf{I}_{n \times n} \quad (3.23)$$

Applying these assumptions we found some results as given in the following subsections. In the first subsection we assume the RS ansatz which can be considered as the extension of [27]. In the last two subsections we assume RSB ansatz as an extension of [29] to CS.

3.1 LASSO estimator with RS ansatz

Consider the LASSO estimator given in (2.12), which is equivalent to the solution of the main unconstrained optimization problem (2.2) in l_1 penalized sense. Its performance can be expressed in terms of the limiting energy penalty per component using two macroscopic variables q_0 and b_0 given by

$$q_0 = \int_{\mathbb{R}} \int_{\mathbb{R}} |x^0 - \Psi_1|^2 Dz dF_{X^0}(x^0), \quad (3.24)$$

$$(3.25)$$

$$b_0 = \frac{1}{f_0} \int_{\mathbb{R}} \int_{\mathbb{R}} \Re \left\{ x^0 - \Psi_1 z^* \right\} Dz dF_{X^0}(x^0), \quad (3.26)$$

where

$$\Psi_1 = \arg \min_{x \in \mathcal{X}} \left| -z f_0 + 2e_0(x^0 - x) - \frac{\gamma}{\sigma_u^2} \right|, \quad (3.27)$$

$$e_0 = \frac{1}{\sigma_u^2} R\left(\frac{-b_0}{\sigma_u^2}\right), \quad (3.28)$$

$$f_0 = \sqrt{2 \frac{q_0}{\sigma_u^4} R'\left(\frac{-b_0}{\sigma_u^2}\right)}, \quad (3.29)$$

and Dz is referring about integration over Gaussian measure, while dF_{X^0} refers to integration over the pdf of x^0 (See Appendix B). Under RS ansatz assumptions we then get the following statement.

Proposition 2 *Given the LASSO estimator in (2.12) and the macroscopic variables q_0 and b_0 , in addition given the conditions in proposition 1, the energy in (3.20) simplifies to*

$$\bar{\mathcal{E}}_{rs}^{lasso} = \frac{q_0}{\sigma_u^2} R\left(\frac{-b_0}{\sigma_u^2}\right) - \frac{b_0 q_0}{\sigma_u^4} R'\left(\frac{-b_0}{\sigma_u^2}\right) \quad (3.30)$$

Proof 2 *See Appendix B.*

3.2 LASSO estimator with 1RSB ansatz

Moving to the very purpose of the present paper, we use RSB ansatz instead of RS and we repeat what has been done in the above subsections. The limiting energy in this case involves four macroscopic variables like b_1 , p_1 , q_1 , and μ_1 , which can be given by the following fixed point equations as $n \rightarrow 0$ and $\beta \rightarrow \infty$, as showed in appendix D, and using the compact notation as in [29]. Let

$$\Delta(y, z) \equiv e^{-\mu_1 \min_{x \in \mathcal{X}} -2\Re\{(x^0-x)(f_1 z^* + g_1 y^*)\} + e_1(x^0-x)^2 - \frac{\gamma}{\sigma_u^2}|x|}, \quad (y, z) \in \mathfrak{R}^2 \quad (3.31)$$

and its normalized version

$$\tilde{\Delta}(y, z) = \frac{\Delta(y, z)}{\int_{\mathbb{C}} \Delta(\tilde{y}, z) d\tilde{y}} \quad (3.32)$$

$$b_1 + p_1 \mu_1 = \frac{1}{f_1} \int \int_{\mathbb{C}^2} \Re\{x^0 - (\Psi_2) z^*\} \tilde{\Delta}(y, z) Dy Dz dF_{X^0}(x^0) \quad (3.33)$$

$$b_1 + (q_1 + p_1) \mu_1 = \frac{1}{g_1} \int \int_{\mathbb{C}^2} \Re\{x^0 - (\Psi_2) y^*\} \tilde{\Delta}(y, z) Dy Dz dF_{X^0}(x^0) \quad (3.34)$$

$$q_1 + p_1 = \frac{1}{g_1} \int \int_{\mathbb{C}^2} |\Psi_2|^2 \tilde{\Delta}(y, z) Dy Dz dF_{X^0}(x^0) \quad (3.35)$$

where

$$\Psi_2 = \arg \min_{x \in \mathcal{X}} \left| -(f_1 z^* + g_1 y^*) + e_1(x^0 - x) - \frac{\gamma}{\sigma_u^2} \right|,$$

and

$$\begin{aligned} \int_{\frac{b_1}{\sigma_u^2}}^{\frac{b_1 + \mu_1 p_1}{\sigma_u^2}} R(-w) dw &= -R\left(-\frac{b_1 + \mu_1 p_1}{\sigma_u^2}\right) - \mu_1^2 \left((q_1 + p_1) g_1^2 + p_1 f_1^2 \right) \\ &+ \int \int_{\mathbb{C}} \log \left(\int_{\mathbb{C}} \Delta(y, z) Dy \right) Dz dF_{X^0}(x^0), \end{aligned} \quad (3.36)$$

where the other variables e_1 , f_1 , and g_1 , are given by

$$e_1 = \frac{1}{\sigma_u^2} R\left(\frac{-b_1}{\sigma_u^2}\right), \quad (3.37)$$

$$g_1 = \sqrt{\frac{1}{\mu_1 \sigma_u^2} \left[R\left(\frac{-b_1}{\sigma_u^2}\right) - R\left(\frac{-b_1 - \mu_1 p_1}{\sigma_u^2}\right) \right]}, \quad (3.38)$$

$$f_1 \xrightarrow{n \rightarrow 0} \frac{1}{\sigma_u^2} \sqrt{q_1 R'\left(\frac{-b_1 - \mu_1 p_1}{\sigma_u^2}\right)} \quad (3.39)$$

Then the following two statements are the extensions of the propositions in [29] to CS problems.

Proposition 3 *Given the LASSO estimator in (2.12) and suppose the random matrix \mathbf{J} satisfies the decomposability property (3.10). Then under some technical assumptions, including one-step replica symmetry breaking, and the*

macroscopic variables given by the above fixed point equations, the effective energy penalty per component converges in probability as $N, M \rightarrow \infty, N/M < \infty$, to

$$\begin{aligned} \bar{\mathcal{E}}_{1rsb}^{LASSO} &= \frac{1}{\sigma_u^2} (q_1 + p_1 + \frac{b_1}{\mu_1}) R\left(\frac{-b_1 - \mu_1 p_1}{\sigma_u^2}\right) - \frac{b_1}{\mu_1 \sigma_u^2} R\left(-\frac{b_1}{\sigma_u^2}\right) \\ &\quad + q_1 \left(\frac{b_1 + \mu_1 p_1}{\sigma_u^2}\right) R'\left(\frac{-b_1 - \mu_1 p_1}{\sigma_u^2}\right) \end{aligned} \quad (3.40)$$

Proof 3 See Appendices D.

3.3 Zero-Norm regularizing estimator with 1RSB ansatz

The LASSO estimation is considered as the convex relaxation of the Zero-Norm regularizing estimation. Since the latter is a non-convex problem its performance is better evaluated when we use RSB ansatz. So extending proposition (3) to this estimator we get the following statement.

Proposition 4 *Given the Zero-Norm regularizing estimator in (2.13) and suppose the random matrix \mathbf{J} satisfies the decomposability property (3.10). Then under some technical assumptions, including one-step replica symmetry breaking, the effective energy penalty per component converges in probability as $N, M \rightarrow \infty, N/M < \infty$, to*

$$\begin{aligned} \bar{\mathcal{E}}_{1rsb}^{zero-norm} &= \frac{1}{\sigma_u^2} (q_1 + p_1 + \frac{b_1}{\mu_1}) R\left(\frac{-b_1 - \mu_1 p_1}{\sigma_u^2}\right) - \frac{b_1}{\mu_1 \sigma_u^2} R\left(-\frac{b_1}{\sigma_u^2}\right) \\ &\quad + q_1 \left(\frac{b_1 + \mu_1 p_1}{\sigma_u^2}\right) R'\left(\frac{-b_1 - \mu_1 p_1}{\sigma_u^2}\right) \end{aligned} \quad (3.41)$$

Proof 4 See Appendix D.

4 Particular Example: Bernoulli-Gaussian Mixture Distribution

Assume the original vector $\mathbf{x}^0 \in \mathbb{R}^N$ follows a Bernoulli-Gaussian mixture distribution. So following the Bayesian framework analysis in section 3, let \mathbf{x} be composed of random variables with each component obeying the pdf

$$p(x) \sim \begin{cases} \mathcal{N}(0, 1) & \text{with probability } \rho \\ 0 & \text{with probability } 1 - \rho, \end{cases} \quad (4.1)$$

where $\rho = k/n$, with k being the number of non zero entries of \mathbf{x} . With out loss of generality, let $\rho = 0.1$, M/N and k/N vary between 0.2 and 1. Also lets assume that the entries of the measurement matrix \mathbf{A} follow i.i.d. Gaussian random variable of mean zero and variance $1/M$. In addition let σ_u^2 be such that the signal to noise ratio is $-10dB$.

We have simulated equations (2.7) and (2.8). Figure 1 shows MSE versus M/N of the two estimators, where we see that the l_2 penalizing estimator, LMMSE, is not as good as the l_1 penalizing estimator in general. Figure 2 shows MSE versus k/N of the two estimators and we see that LMMSE is not sensitive to the sparsity of the vector as compared to the l_1 penalizing estimator. Note that we have plotted the l_1 -penalizing estimator using different algorithms: LASSO, L1-LS, Log-Bar.

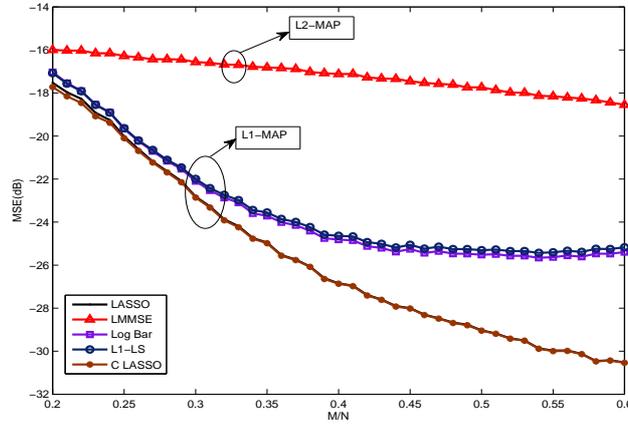


Figure 1: This figure shows the the normalized mean squared error for the different eastimators in (2.7) and (2.8) versus measurement ration M/N simulated using different algorithms like LASSO, LOG-BAR, L1-LS as L1 penalazing family and LMMSE for the the L2 penalaying.

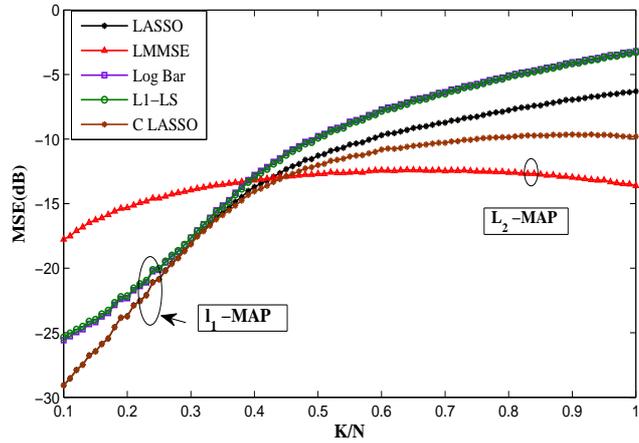


Figure 2: This figure shows the the normalized mean squared error for the different eastimators in (2.7) and (2.8) versus sparsity ratio k/N simulated using different algorithms like LASSO, LOG-BAR, L1-LS as l_1 penalazing family and LMMSE for the the l_2 penalaying for $M=50$ and $N=100$.

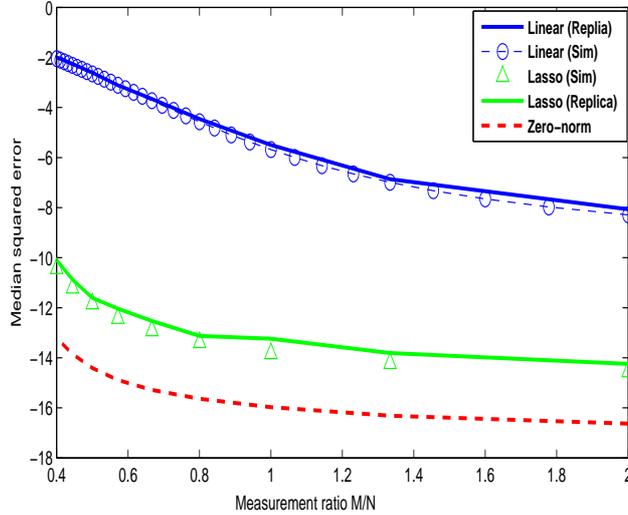


Figure 3: This figure shows the Median squared error against measurement ratio for the estimators in (2.7)-(2.9) as simulated by Rangan and others [14] plotted against M/N instead of N/M and the replica simulation points are included.

In both figures, we see that the least square estimator is not good for the compressive sensing problem. In addition, we also observed that simulating the l_0 penalizing estimator is hard. However, it is possible to apply statistical physics tools, including replica methods, to analyze the performances of all the estimators mentioned above, including zero norm estimator. In [14], median square error was used to compare the different estimators given by (2.11)-(2.13) as shown here in figure 3. What we do here is that we include 1RSB ansatz analysis of the performance of the CS estimators as each of them are presented here as a MAP estimator. Actually it is one of the conjectures made by Müller and others that the performance of MAP estimators is best done using one step RSB. And we showed it here via the minimized energy expressions as given in the propositions by the equations (3.1), (3.40), and (3.41).

4.1 Replica symmetry analysis

Considering the macroscopic variables given by (3.24) and (3.26) and plugging the assumed distributions above and simplifying it one more step, the fixed point equations become

$$q_0 = \frac{\rho^2}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \frac{zf_0 + \frac{\gamma}{\sigma_u^2}}{2e_0} \right|^2 e^{-\frac{x^0 + z^2}{2}} dz dx^0, \quad (4.2)$$

$$b_0 = \frac{\rho^2}{2\pi} \frac{1}{f_0} \int_{\mathbb{R}} \int_{\mathbb{R}} \Re \left\{ x^0(1 - z^*) + \left(\frac{zf_0 + \frac{\gamma}{\sigma_u^2}}{2e_0} \right) z^* \right\} e^{-\frac{x^0 + z^2}{2}} dz dx^0. \quad (4.3)$$

Using these macroscopic variables in we find the limiting energy numerically which is given under proposition 2 and the result is shown in figure 4.

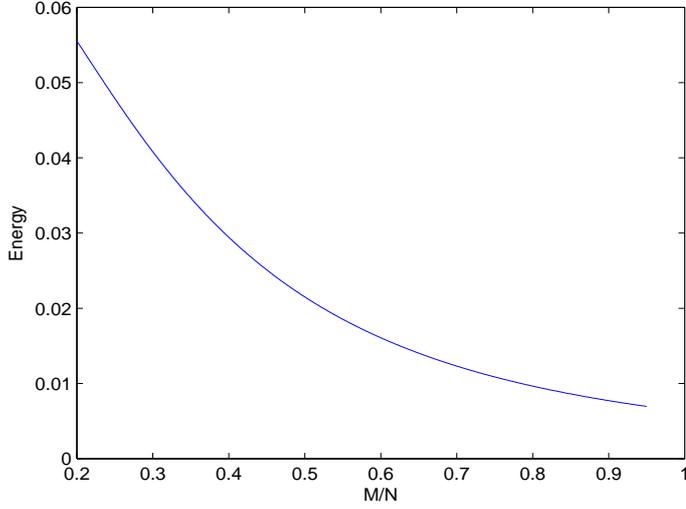


Figure 4: This figure shows the minimum energy for the error resulting from the RS ansatz for lasso versus the measurement ratio M/N .

4.2 Replica symmetry Breaking analysis

Considering the same Bernoulli-Gaussian mixture distribution (4.1) assumed in this section we consider the macroscopic variables which arises from one step replica symmetry breaking (1RSB) ansatz. Then the minimum energy per component as $M \rightarrow \infty, N \rightarrow \infty$, while M/N is finite ratio, which are given by (3.40) and (3.41) are dependent up on four macroscopic variables given by (3.33) -(3.36). The ther first are simplified further as follows:

We can further simplify (3.33)-(3.36) as follows

$$b_1 + p_1\mu_1 = \frac{1}{f_1} \int \int \int_{\mathbb{C}^2} \Re\{(x^0 - \Psi_2)z^*\} DyDz dF_X(x) dF_{X^0}(x^0) \quad (4.4)$$

$$b_1 + (q_1 + p_1)\mu_1 = \frac{1}{g_1} \int \int \int_{\mathbb{C}^2} \Re\{(x^0 - \Psi_2)z^*\} DyDz dF_X(x) dF_{X^0}(x^0) \quad (4.5)$$

$$q_1 + p_1 = \frac{1}{g_1} \int \int \int_{\mathbb{C}^2} |\Psi_2|^2 DyDz dF_X(x) dF_{X^0}(x^0) \quad (4.6)$$

It is possible to simplify these results further and give numerical results. But this is deferred for further work. We expect that the free energy from The RSB ansatz to be greater than the free energy from the RS ansatz for the Zero-Norm regularizing, which can be seen from the analytical terms which have more parameters in (3.41). However, for LASSO these free energy, hence the energy error, will be quite similar since for convex minimization problems there is one global minimum and RS ansatz is sufficient enough to produce the solution.

5 Conclusion

In this paper we have used the replica method to analyze the performance of the estimators used in compressed sensing which can be generalized as MAP estimators. And the performance of MAP estimators can well be shown using replica method including one-step replica breaking ansatz. It is a philosophical standpoint that 1RSB enough to analyze the estimators like MAP. We have only showed here for one particular example for the CS problem, i.e. for Bernoulli-Gaussian distribution. One may be interested to verify it using different examples. In addition we have only compared the estimators performance based on the free energy, but one can also use other metrics such as comparing the input/output distribution using replica analysis as it is done in [29]. The main result of this paper is analytical analysis for the performance of the estimators used in CS and many things can be extended including efficient algorithms in implementing the numerical analysis.

6 Acknowledgments

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A Important Definitions

A.1 Green's function

In Classical probability theory (CPT) one is concerned with the densities, moments and cumulants of elements of random matrices. Where as in Random matrix theory (RMT) also called (Free Random Variable calculus), one is engaged in finding the spectral densities, moments and cumulants (By Professor Maciej A. Nowak). As Fourier transform is the generating function for the moments in CPT, Green's function (also called Stieltjes transform) is the generating function for the spectral moments defined as

$$G(z) \equiv \frac{1}{N} \langle \text{Tr} \frac{1}{z \mathbf{1}_N - \mathbf{X}} \rangle \equiv \int \frac{\rho(\lambda)}{z - \lambda} d\lambda \equiv \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} M_n, \quad (\text{A.1})$$

where \mathbf{X} is $N \times N$ random matrix and $\mathbf{1}_N$ is of the same size unit matrix, λ are the eigenvalues, and M_n is the spectral moment. The integral is over the support set of the eigenvalues.

A.2 R-transform

The generating function for the cumulants of the CPT is given by the logarithm of the Fourier transform. In similar maner to the above section we can define the generating function for spectral cumulants. It is called the R-transform (Voiculescu,1986). It is given by

$$R(z) \equiv \sum_{n=1}^{\infty} C_n z^{n-1}, \quad (\text{A.2})$$

where C_n are the spectral cumulants of the random matrix \mathbf{X} . We can relate R-transform with Greens's function as follows:

$$G(R(z) + \frac{1}{z}) = z \quad \text{or} \quad R(G(z)) + \frac{1}{G(z)} = z. \quad (\text{A.3})$$

The spectral density of the matrix $\mathbf{J} = \mathbf{A}^T \mathbf{A}$ converges almost surely to the Marchenko-Pastur law as $M = \alpha N \rightarrow \infty$ [27]. And the R-transform of this matrix is given by

$$R(z) = \frac{1}{1 - \alpha z} \quad (\text{A.4})$$

and its derivative with respect to z becomes

$$R'(z) = \frac{\alpha}{(1 - \alpha z)^2}, \quad (\text{A.5})$$

where $\alpha = N/M$ is system load.

B Proof of propostion 1

The avarage energy penalty can be derived from the average free energy given in (3.15)

$$\begin{aligned} \bar{\mathcal{E}} &= \lim_{\beta \rightarrow \infty} \frac{1}{\beta} \bar{\mathcal{F}} = - \lim_{\beta \rightarrow \infty} \frac{1}{\beta} \lim_{N \rightarrow \infty} \frac{1}{N} \mathbb{E}_{\mathbf{n}, \mathbf{R}} \{ \log \mathcal{Z} \} \\ &= - \lim_{\beta \rightarrow \infty} \frac{1}{\beta} \lim_{n \rightarrow 0} \frac{\partial}{\partial n} \underbrace{\lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{E}_{\mathbf{n}, \mathbf{J}} \{ (\mathcal{Z})^n \}}_{\Xi_n}. \end{aligned} \quad (\text{B.1})$$

where Ξ_n is given by (3.18). Using (3.19) as the splitting of the space, we get

$$\Xi_n = \lim_{N \rightarrow \infty} \frac{1}{N} \log \int_{\mathbb{R}^{(n+1)^2}} e^{N\mathcal{L}} e^{N\mathcal{I}\{\mathbf{Q}\}} e^{-N\mathcal{G}\{\mathbf{Q}\}} D\mathbf{Q} \quad (\text{B.2})$$

where

$$D\mathbf{Q} = \prod_{a=0}^n dQ_{aa} \prod_{b=a+1}^n dQ_{ab} \quad (\text{B.3})$$

is the integration measure,

$$\mathcal{G}(\mathbf{Q}) = \sum_{a=0}^n \int_0^{\frac{\beta\gamma}{\sigma_u^2} \lambda_a(\mathbf{Q})} R(-w) dw \quad (\text{B.4})$$

$$= \text{Tr} \int_0^{\frac{\beta\gamma}{\sigma_u^2} \mathbf{Q}} R(-w) dw \quad (\text{B.5})$$

$$= \int_0^{\frac{\beta\gamma}{\sigma_u^2}} \text{Tr}[\mathbf{Q}R(-w\mathbf{Q})] dw \quad (\text{B.6})$$

$$\mathcal{L} = -\frac{\beta\gamma}{2N} \sum_{a=0}^n f(\mathbf{x}^a) \quad \text{and} \quad (\text{B.7})$$

$$e^{N\mathcal{I}\{\mathbf{Q}\}} = \sum_{\{\mathbf{x}^a\}} \prod_{a=0}^n \delta((\mathbf{x}^0 - \mathbf{x}^a)^T (\mathbf{x}^0 - \mathbf{x}^a) - NQ_{aa}) \prod_{b=a+1}^n \delta((\mathbf{x}^0 - \mathbf{x}^a)^T (\mathbf{x}^0 - \mathbf{x}^b) - NQ_{ab}) \quad (\text{B.8})$$

denotes probability weight of the subshell composed of Dirac-functions in the real line. This procedure is a change of integration variables in multiple dimensions where the integration of an exponential function over the replicas has been replaced by integration over the variables \mathbf{Q} . To evaluate $e^{NC} e^{N\mathcal{I}\{Q\}}$ we follow [12], [29] and represent the Dirac measure using the Fourier transform as

$$\delta((\mathbf{x}^0 - \mathbf{x}^b)^T (\mathbf{x}^0 - \mathbf{x}^a) - NQ_{ab}) = \int_{\mathcal{J}} e^{\tilde{Q}_{ab}((\mathbf{x}^0 - \mathbf{x}^b)^T (\mathbf{x}^0 - \mathbf{x}^a) - NQ_{ab})} \frac{d\tilde{Q}_{ab}}{2\pi}, \quad (\text{B.9})$$

where $a, b = 0, 1, \dots, n$ and this gives

$$\begin{aligned} e^{N\mathcal{L}} e^{N\mathcal{I}\{\mathbf{Q}\}} &= \sum_{\{\mathbf{x}^a\}} \int_{\mathcal{J}^{n^2}} e^{\sum_{a,b} \tilde{Q}_{ab}((\mathbf{x}^0 - \mathbf{x}^b)^T (\mathbf{x}^0 - \mathbf{x}^a) - NQ_{ab})} e^{\frac{-\beta\gamma}{\sigma_u^2} \sum_{a=1}^n f(\mathbf{x}^a)} \tilde{D}\tilde{\mathbf{Q}} \\ &= \int_{\mathcal{J}^{n^2}} e^{-N\text{Tr}(\tilde{\mathbf{Q}}\mathbf{Q})} \left(\sum_{\{\mathbf{x}^a\}} e^{\sum_{a,b} \tilde{Q}_{ab}(\mathbf{x}^0 - \mathbf{x}^b)^T (\mathbf{x}^0 - \mathbf{x}^a)} e^{\frac{-\beta\gamma}{\sigma_u^2} \sum_{a=1}^n f(\mathbf{x}^a)} \right) \tilde{D}\tilde{\mathbf{Q}} \end{aligned} \quad (\text{B.10})$$

where

$$\tilde{D}\tilde{\mathbf{Q}} = \prod_{a=0}^n \left(\frac{d\tilde{Q}_{aa}}{2\pi} \prod_{b=a+1}^n \frac{d\tilde{Q}_{ab}}{2\pi} \right) \quad (\text{B.11})$$

Assuming $f(\mathbf{x}^a) = \|\mathbf{x}^a\|_1 = \sum_{i=1}^N |x_i^a|$, which is the sparsity enforcer as described above in LASSO estimator, and after doing some rearrangements, the inner expectation of (B.10) can be given by

$$\sum_{\{\mathbf{x}^a\}} e^{\sum_{a,b} \tilde{Q}_{ab}(\mathbf{x}^0 - \mathbf{x}^b)^T (\mathbf{x}^0 - \mathbf{x}^a)} e^{\frac{-\beta\gamma}{\sigma_u^2} \sum_{a=1}^n f(\mathbf{x}^a)} = \prod_{i=1}^N \sum_{\{x_i^a \in \chi\}} e^{(\sum_{a,b} \tilde{Q}_{ab}(x_i^0 - x_i^b)^T (x_i^0 - x_i^a)) - \frac{\beta\gamma}{\sigma_u^2} \sum_{a=1}^n |x_i^a|} \quad (\text{B.12})$$

Now defining

$$M_i(\tilde{\mathbf{Q}}) = \sum_{\{x_i^a \in \chi\}} e^{\left(\sum_{a,b} \tilde{Q}_{ab}(x_i^0 - x_i^b)^T (x_i^0 - x_i^a) \right) - \frac{\beta\gamma}{\sigma_u^2} \sum_{a=1}^n |x_i^a|} \quad (\text{B.13})$$

we can get

$$e^{N\mathcal{L}} e^{N\mathcal{I}\{\mathbf{Q}\}} = \int_{\mathcal{J}^{n^2}} e^{-N\text{Tr}(\tilde{\mathbf{Q}}\mathbf{Q}) + \sum_{i=1}^N \log M_i(\tilde{\mathbf{Q}})} \tilde{D}\tilde{\mathbf{Q}}. \quad (\text{B.14})$$

Following the i.i.d. assumption for the component of the sparse vector \mathbf{x} , and applying the strong law of large numbers as $N \rightarrow \infty$ we get

$$\begin{aligned}
\log M(\tilde{\mathbf{Q}}) &= \frac{1}{N} \sum_{i=1}^N \log M_i(\tilde{\mathbf{Q}}) \\
&\rightarrow \int \log \sum_{\{x^a \in \chi\}} e^{\sum_{a,b} \tilde{Q}_{ab} (x^0 - x^b)^T (x^0 - x^a) - \frac{\beta\gamma}{\sigma_u^2} \sum_{a=1}^n |x^a|} \prod_{a=0}^n dF_X(x^a) \\
&= \int \log \sum_{\{\mathbf{x} \in \chi^n\}} e^{(x^0 \mathbf{1} - \tilde{\mathbf{x}})^T \tilde{\mathbf{Q}} (x^0 \mathbf{1} - \tilde{\mathbf{x}}) - \frac{\beta\gamma}{\sigma_u^2} \tilde{\mathbf{x}}} \prod_{a=0}^n dF_{X^0}(x^0) \quad (\text{B.15})
\end{aligned}$$

where, $\tilde{\mathbf{x}}$ is vector of dimension n . Next we apply the saddle point integration concept on the remaining part of (B.2), i.e., as $N \rightarrow \infty$ the integrand will be dominated by the exponential term with maximal exponent. Hence in (B.2) only the subshell that corresponds to this extremal value of the correlation between the vectors $\{\mathbf{x}^a\}$ is relevant for the calculation of the integral.

$$\begin{aligned}
&\int_{\mathbb{R}^{n^2}} e^{N\mathcal{L}} e^{N\mathcal{I}\{\mathbf{Q}\}} e^{-N\mathcal{G}\{\mathbf{Q}\}} D\mathbf{Q} \\
&= \int_{\mathbb{R}^{n^2}} \left(\int_{\mathcal{J}^{n^2}} e^{-N\text{Tr}(\tilde{\mathbf{Q}}\mathbf{Q}) + \sum_{i=1}^N \log M_i(\tilde{\mathbf{Q}})} \tilde{D}\tilde{\mathbf{Q}} \right) e^{-N\mathcal{G}\{\mathbf{Q}\}} D\mathbf{Q} \quad (\text{B.16})
\end{aligned}$$

Therefore, at the saddle point we have the following equations with partial derivatives being zero (see the proof in Appendix B of [29]):

$$\frac{\partial}{\partial \mathbf{Q}} \left[\mathcal{G}(\mathbf{Q}) + \text{Tr}(\tilde{\mathbf{Q}}\mathbf{Q}) \right] = \mathbf{0} \quad \text{and} \quad (\text{B.17})$$

$$\frac{\partial}{\partial \tilde{\mathbf{Q}}} \left[\log M(\tilde{\mathbf{Q}}) - \text{Tr}(\tilde{\mathbf{Q}}\mathbf{Q}) \right] = \mathbf{0}. \quad (\text{B.18})$$

And from the former we get

$$\tilde{\mathbf{Q}} = \beta R \left(-\frac{\beta\gamma}{\sigma_u^2} \mathbf{Q} \right) \quad (\text{B.19})$$

and from the later, using (B.15) we finally get

$$\mathbf{Q} = \int \frac{\sum_{\{\tilde{\mathbf{x}} \in \chi^n\}} e^{(x^0 \mathbf{1} - \tilde{\mathbf{x}})(x^0 \mathbf{1} - \tilde{\mathbf{x}})^T} e^{(x^0 \mathbf{1} - \tilde{\mathbf{x}})^T \tilde{\mathbf{Q}} (x^0 \mathbf{1} - \tilde{\mathbf{x}}) - \frac{\beta\gamma}{\sigma_u^2} \sum_{a=1}^n |x^a|}}{\sum_{\{\tilde{\mathbf{x}} \in \chi^n\}} e^{(x^0 \mathbf{1} - \tilde{\mathbf{x}})^T \tilde{\mathbf{Q}} (x^0 \mathbf{1} - \tilde{\mathbf{x}}) - \frac{\beta\gamma}{\sigma_u^2} \sum_{a=1}^n |x^a|}} dF_{X^0}(x^0) \quad (\text{B.20})$$

C Proof of proposition 2

Taking the same line of thought as we do for \mathbf{Q} , we can assume a natural replicated variables for the symmetric correlation matrix $\tilde{\mathbf{Q}}$ and the 1RSB as follows:

1. replica symmetry ansatz :

$$\tilde{\mathbf{Q}} = \frac{\beta^2 f_0^2}{2} \mathbf{1}_{n \times n} - \beta e_0 \mathbf{I}_{n \times n} \quad (\text{C.1})$$

2. one replica symmetry breaking ansatz :

$$\tilde{\mathbf{Q}} = \beta^2 f_1^2 \mathbf{1}_{n \times n} + \beta^2 g_1^2 \mathbf{I}_{\frac{n\beta}{\mu_1} \times \frac{n\beta}{\mu_1}} \otimes \mathbf{1}_{\frac{\mu_1}{\beta} \times \frac{\mu_1}{\beta}} - \beta e_1 \mathbf{I}_{n \times n} \quad (\text{C.2})$$

The variables $q_0, b_0, q_1, p_1, b_1, f_0, e_0, f_1, g_1, e_1$, and μ_1 are called the macroscopic variables and they are all functions of n . They all can be calculated from the saddle point equations that we shortly will derive. First let us try to prove proposition 2 using the ansatz in (3.22) and (C.1). We do it using equations (B.1), (C.3) and (B.16) and we apply the saddlepoint integration rule. What matters most becomes the argument of the exponential in (B.16). So we first find $\text{Tr}(\tilde{\mathbf{Q}}\mathbf{Q})$, $\mathcal{G}(\mathbf{Q})$, $\log M(\mathbf{Q})$ and in addition we will find the macroscopic parameters mentioned before since our limiting energy penalty expressions for the different estimators considered in this paper are calculated in terms of the macroscopic variables. Hence using (3.22) and (C.1) we get

$$\text{Tr}(\tilde{\mathbf{Q}}\mathbf{Q}) = n(q_0 + \frac{b_0}{\beta}) \left(\frac{\beta^2 f_0^2}{2} - \beta e_0 \right) + \frac{n(n-1)}{2} q_0 \beta^2 f_0^2 \quad (\text{C.3})$$

and using (B.13) and (C.1) again we get

$$M_i(\tilde{\mathbf{Q}}) = \sum_{\{x_i^a \in \mathcal{X}\}} e^{\left(\sum_{a,b} \tilde{Q}_{ab} (x_i^0 - x_i^b)(x_i^0 - x_i^a) \right) - \frac{\beta\gamma}{\sigma_u^2} \sum_{a=1}^n |x_i^a|} \quad (\text{C.4})$$

$$= \sum_{\{x_i^a \in \mathcal{X}\}} e^{\frac{\beta^2 f_0^2}{2} \left(\sum_{a=1}^n (x_i^0 - x_i^a) \right)^2 - e_0 \beta \sum_{a=1}^n (x_i^0 - x_i^a)^2 - \frac{\beta\gamma}{\sigma_u^2} \sum_{a=1}^n |x_i^a|} \quad (\text{C.5})$$

$$= \sum_{\{x_i^a \in \mathcal{X}\}} \int_{\mathbb{R}} e^{\beta \sum_{a=1}^n f_0 \Re\{(x_i^0 - x_i^a) z^*\} - e_0 (x_i^0 - x_i^a)^2 - \frac{\beta\gamma}{\sigma_u^2} |x_i^a|} Dz \quad (\text{C.6})$$

$$= \int \left(\sum_{\{x \in \mathcal{X}\}} e^{\beta f_0 \Re\{(x^0 - x_i^a) z^*\} + e_0 \beta (x^0 - x)^2 - \frac{\beta\gamma}{\sigma_u^2} |x|} \right)^n Dz. \quad (\text{C.7})$$

From (B.4) to (B.7) we apply completing the square on the exponential of the argument and the Hubbard-Stratonovich transform,

$$e^{-|x|^2} = \int_{\mathbb{C}} e^{2\Re\{xz^*\}} Dz, \quad (\text{C.8})$$

where Dz is Gaussian measure defined as before, to linearize the exponential argument. And we finally transformed the problem to a single integral and a single summation problem. To evaluate $\mathcal{G}(\mathbf{Q})$ we should first find the eigenvalues of the matrix $L(n)$. Under the RS ansatz the matrix $L(n)$ has three types of eigenvalues: $\lambda_1 = -(\sigma_u^2 + n\sigma_0^2)^{-1}(b_0 + n\beta q_0)$, $\lambda_2 = -(\sigma_u^2)^{-1}b_0$ and $\lambda_3 = 0$, and the numbers of degeneracy for each are 1, $n-1$, and $N-n$, respectively. Thus we get

$$\mathcal{G}(\mathbf{Q}) = \int_0^{\frac{(b_0 + n\beta q_0)}{\sigma_u^2 + n\sigma_0^2}} R(-w) dw + (n-1) \int_0^{\frac{b_0}{\sigma_u^2}} R(-w) dw \quad (\text{C.9})$$

The integral in (B.16) is dominated by the maximum argument of the exponential function. Therefore, the derivative of

$$\mathcal{G}(\mathbf{Q}) + \text{Tr}(\tilde{\mathbf{Q}}\mathbf{Q}) \quad (\text{C.10})$$

with respect to q_0 and b_0 must vanish as $N \rightarrow \infty$. Plugging (C.3) and (C.9) into (C.10) and taking the partial derivatives we get

$$\frac{\beta n}{\sigma_u^2 + n\sigma_0^2} R\left(\frac{-(b_0 + n\beta q_0)}{(\sigma_u^2 + n\sigma_0^2)}\right) + \frac{n(n-1)}{2} \beta^2 f_0^2 + n\beta\left(\frac{\beta f_0^2}{2} - e_0\right) = 0 \quad (\text{C.11})$$

$$\frac{1}{\sigma_u^2 + n\sigma_0^2} R\left(\frac{-(b_0 + n\beta q_0)}{(\sigma_u^2 + n\sigma_0^2)}\right) + \frac{1}{\sigma_u^2} (n-1) R\left(\frac{-b_0}{\sigma_u^2}\right) + n\left(\frac{\beta f_0^2}{2} - e_0\right) = 0, \quad (\text{C.12})$$

respectively. After algebraic simplification and solving for e_0 and f_0 we get

$$e_0 = \frac{1}{\sigma_u^2} R\left(\frac{-b_0}{\sigma_u^2}\right), \quad (\text{C.13})$$

$$f_0 = \sqrt{\frac{2}{n\beta} \left[\frac{1}{\sigma_u^2} R\left(\frac{-b_0}{\sigma_u^2}\right) - \frac{1}{\sigma_u^2 + n\sigma_0^2} R\left(\frac{-(b_0 + n\beta q_0)}{(\sigma_u^2 + n\sigma_0^2)}\right) \right]}. \quad (\text{C.14})$$

and with the limit for $n \rightarrow 0$

$$f_0 \xrightarrow{n \rightarrow 0} \sqrt{\frac{2}{\beta} \left[\frac{\sigma_0^2}{\sigma_u^4} R\left(\frac{-b_0}{\sigma_u^2}\right) + \frac{\beta q_0 \sigma_u^2 + b_0 \sigma_0^2}{\sigma_u^6} R\left(\frac{-b_0}{\sigma_u^2}\right) \right]}. \quad (\text{C.15})$$

By substituting (C.3) into (B.18) and doing the partial derivative of

$$\begin{aligned} & \log M(e_0, f_0) - \text{Tr}(\tilde{\mathbf{Q}}\mathbf{Q}) \\ &= \int \log \sum_{\{\tilde{\mathbf{x}} \in \chi^n\}} e^{(x^0 \mathbf{1} - \tilde{\mathbf{x}})^T \tilde{\mathbf{Q}}(x^0 \mathbf{1} - \tilde{\mathbf{x}}) - \frac{\beta \gamma}{\sigma_u^2} \tilde{\mathbf{x}}} dF_{X^0}(x^0) \\ & - \left(n(q_0 + \frac{b_0}{\beta}) \left(\frac{\beta^2 f_0^2}{2} - \beta e_0 \right) + \frac{n(n-1)}{2} q_0 \beta^2 f_0^2 \right) \end{aligned} \quad (\text{C.16})$$

$$\begin{aligned} &= \int \log \int \left(\sum_{\{x \in \chi\}} e^{\beta f_0 \Re\{(x^0 - x_i^a) z^*\} + e_0 \beta (x^0 - x)^2 - \frac{\beta \gamma}{\sigma_u^2} |x|} \right)^n Dz dF_{X^0}(x^0) \\ & - \left(n(q_0 + \frac{b_0}{\beta}) \left(\frac{\beta^2 f_0^2}{2} - \beta e_0 \right) + \frac{n(n-1)}{2} q_0 \beta^2 f_0^2 \right), \end{aligned} \quad (\text{C.17})$$

with respect to e_0 and f_0 and equating to zero we get,

$$q_0 = -\frac{b_0}{\beta} + \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\sum_{\{x \in \chi\}} (x^0 - x)^2 \zeta}{\sum_{\{x \in \chi\}} \zeta} Dz dF_{X^0}(x^0) \quad (\text{C.18})$$

$$b_0 = -\beta n q_0 + \frac{1}{f_0} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\sum_{\{x \in \chi\}} \Re\{(x^0 - x_i^a) z^*\} \zeta}{\sum_{\{x \in \chi\}} \zeta} Dz dF_{X^0}(x^0) \quad (\text{C.19})$$

where

$$\zeta = e^{\beta f_0 \Re\{(x^0 - x_i^a) z^*\} + e_0 \beta (x^0 - x)^2 - \frac{\beta \gamma}{\sigma_u^2} |x|}. \quad (\text{C.20})$$

So collecting the macroscopic variables in (C.13), (C.14), (C.18) and (C.19) and sending $n \rightarrow 0$ we have

$$e_0 = \frac{1}{\sigma_u^2} R\left(\frac{b_0}{\sigma_u^2}\right) \quad (\text{C.21})$$

$$f_0 \xrightarrow{n \rightarrow 0} \sqrt{\frac{2}{\beta} \left[\frac{\sigma_0^2}{\sigma_u^4} R\left(\frac{-b_0}{\sigma_u^2}\right) + \frac{\beta q_0 \sigma_u^2 + b_0 \sigma_0^2}{\sigma_u^6} R'\left(\frac{-b_0}{\sigma_u^2}\right) \right]} \quad (\text{C.22})$$

$$q_0 = -\frac{b_0}{\beta} + \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\sum_{\{x \in \chi\}} (x^0 - x)^2 \zeta}{\sum_{\{x \in \chi\}} \zeta} Dz dF_{X^0}(x^0), \quad (\text{C.23})$$

$$b_0 \xrightarrow{n \rightarrow 0} \frac{1}{f_0} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\sum_{\{x \in \chi\}} \Re\{(x^0 - x)z^*\} \zeta}{\sum_{\{x \in \chi\}} \zeta} Dz dF_{X^0}(x^0). \quad (\text{C.24})$$

And the fixed point equations (C.22), (C.18) and (C.19) further can be simplified via the saddle point integration rule in the limit $\beta \rightarrow \infty$ as

$$f_0 = \sqrt{2 \frac{q_0}{\sigma_u^4} R\left(\frac{-b_0}{\sigma_u^2}\right)} \quad (\text{C.25})$$

$$q_0 = \int_{\mathbb{R}} \int_{\mathbb{R}} \left| x^0 - \arg \min_{x \in \chi} \left| -z f_0 + 2e_0(x^0 - x) - \frac{\gamma}{\sigma_u^2} \right| \right|^2 Dz dF_{X^0}(x^0), \quad (\text{C.26})$$

$$b_0 = \frac{1}{f_0} \int_{\mathbb{R}} \int_{\mathbb{R}} \Re \left\{ x^0 - \arg \min_{x \in \chi} \left| -z f_0 + 2e_0(x^0 - x) - \frac{\gamma}{\sigma_u^2} \right| z^* \right\} Dz dF_{X^0}(x^0). \quad (\text{C.27})$$

Putting together the results above we have

$$\begin{aligned} \Xi_n &= \mathcal{I}\{Q\} + \mathcal{L} - \mathcal{G}(\mathbf{Q}) \\ &= -\mathcal{G}(\mathbf{Q}) + \log \mathbf{M}(\tilde{\mathbf{Q}}) - \text{Tr}(\tilde{\mathbf{Q}}\mathbf{Q}) \\ &= -\int_0^{\frac{(b_0+n\beta q_0)}{\sigma_u^2+n\sigma_0^2}} R(-w)dw - (n-1) \int_0^{\frac{b_0}{\sigma_u^2}} R(-w)dw \\ &\quad + \log M(e_0, f_0) - \left(n(q_0 + \frac{b_0}{\beta}) \left(\frac{\beta^2 f_0^2}{2} - \beta e_0 \right) + \frac{n(n-1)}{2} q_0 \beta^2 f_0^2 \right), \end{aligned} \quad (\text{C.28})$$

and the average free energy becomes

$$\begin{aligned} \beta \bar{\mathcal{F}} &= -\lim_{n \rightarrow 0} \frac{\partial}{\partial n} \lim_{N \rightarrow \infty} \frac{1}{N} \log \mathbb{E}_{\mathbf{n}, \mathbf{J}} \{ (\mathcal{Z})^n \} \\ &= \lim_{n \rightarrow 0} \frac{\partial}{\partial n} \underbrace{\left\{ \int_0^{\frac{(b_0+n\beta q_0)}{\sigma_u^2+n\sigma_0^2}} R(-w)dw + (n-1) \int_0^{\frac{b_0}{\sigma_u^2}} R(-w)dw \right\}}_{\Xi_n} \end{aligned} \quad (\text{C.29})$$

$$-\log M(e_0, f_0) + \left(n(q_0 + \frac{b_0}{\beta}) \left(\frac{\beta^2 f_0^2}{2} - \beta e_0 \right) + \frac{n(n-1)}{2} q_0 \beta^2 f_0^2 \right) \} \quad (\text{C.30})$$

$$\begin{aligned} &= \lim_{n \rightarrow 0} \left\{ \left[\frac{-(b_0 + n\beta q_0)}{\sigma_u^2 + n\sigma_0^2} \right] R \left(\frac{-(b_0 + n\beta q_0)}{\sigma_u^2 + n\sigma_0^2} \right) \right. \\ &+ \frac{-(b_0 + n\beta q_0)}{(\sigma_u^2 + n\sigma_0^2)} \left[- \frac{(\beta q_0(\sigma_u^2 + n\sigma_0^2) - (b_0 + n\beta q_0)\sigma_0^2)}{(\sigma_u^2 + n\sigma_0^2)^2} \right] R' \left(\frac{-(b_0 + n\beta q_0)}{(\sigma_u^2 + n\sigma_0^2)} \right) \\ &\left. + \int_0^{\frac{b_0}{\sigma_u^2}} R(-w) dw - \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\zeta^n \ln \zeta}{\zeta^n} DzdF_{X^0}(x^0) \right\} \quad (\text{C.31}) \end{aligned}$$

$$\begin{aligned} &= \frac{-b_0}{\sigma_u^2} R \left(\frac{-b_0}{\sigma_u^2} \right) + \frac{b_0(\beta q_0 \sigma_u^2 - b_0 \sigma_0^2)}{\sigma_u^6} R' \left(\frac{-b_0}{\sigma_u^2} \right) \\ &+ \int_0^{\frac{b_0}{\sigma_u^2}} R(-w) dw - \int_{\mathbb{R}} \int_{\mathbb{R}} \ln \zeta DzdF_{X^0}(x^0). \quad (\text{C.32}) \end{aligned}$$

Coming back to the main goal, the solution for the main unconstrained optimization problem (2.2) is given by the extremum of (3.5), it is calculated through the free energy by sending $\beta \rightarrow \infty$ as follows

$$\bar{\mathcal{E}}_{\text{rs}}^{\text{lasso}} = - \lim_{\beta \rightarrow \infty} \frac{1}{\beta} \lim_{n \rightarrow 0} \frac{\partial}{\partial n} \Xi_n \quad (\text{C.33})$$

$$\begin{aligned} &= \lim_{\beta \rightarrow \infty} \frac{1}{\beta} \left\{ \frac{-b_0}{\sigma_u^2} R \left(\frac{-b_0}{\sigma_u^2} \right) + \frac{b_0(\beta q_0 \sigma_u^2 - b_0 \sigma_0^2)}{\sigma_u^6} R' \left(\frac{-b_0}{\sigma_u^2} \right) + \int_0^{\frac{b_0}{\sigma_u^2}} R(-w) dw \right. \\ &\quad \left. - \int_{\mathbb{R}} \int_{\mathbb{R}} \ln \zeta DzdF_{X^0}(x^0) \right\} \quad (\text{C.34}) \end{aligned}$$

$$= \lim_{\beta \rightarrow \infty} R \left(\frac{-b_0}{\sigma_u^2} \right) \left(\frac{q_0}{\sigma_u^2} + \frac{b_0}{\beta \sigma_u^2} \right) + \frac{b_0 q_0}{\sigma_u^4} R' \left(\frac{-b_0}{\sigma_u^2} \right) \quad (\text{C.35})$$

$$- \lim_{\beta \rightarrow \infty} \frac{1}{\beta} \left\{ \int_{\mathbb{R}} \int_{\mathbb{R}} \ln \zeta DzdF_{X^0}(x^0) \right\} \quad (\text{C.36})$$

$$= \frac{q_0}{\sigma_u^2} R \left(\frac{-b_0}{\sigma_u^2} \right) - \frac{b_0 q_0}{\sigma_u^4} R' \left(\frac{-b_0}{\sigma_u^2} \right). \quad (\text{C.37})$$

This proves proposition 2. And to prove proposition ?? what we need is to use the zero norm regularizing term instead of the L1 norm, i.e. using $f(\mathbf{x}^a) = \|\mathbf{x}^a\|_0 = \frac{k}{N}$ in (B.10), and the result will be as in (??) which differ from (3.30) through the calculation of the macroscopic variables which depend on the distributions of the components of \mathbf{x} .

D Proof of proposition 3 and 4

Turning to LASSO estimator with RSB ansatz we first use (3.23) and (C.2) to get

$$\text{Tr}(\tilde{\mathbf{Q}}\mathbf{Q}) = n(q_1 + p_1 + \frac{b_1}{\beta})(\beta^2 f_1^2 + \beta^2 g_1^2 - \beta e_1) \quad (\text{D.1})$$

$$+ n\left(\frac{\mu_1}{\beta} - 1\right)(q_1 + p_1)(\beta^2 g_1^2 + \beta^2 f_1^2) + n\left(n - \frac{\mu_1}{\beta}\right)q_1\beta^2 f_1^2. \quad (\text{D.2})$$

To evaluate $\mathcal{G}(q_1, p_1, f_1, \mu_1)$ we should first find the eigenvalues of the matrix $\mathbf{L}(n)$. Under the RSB ansatz the matrix $\mathbf{L}(n)$ has four types of eigenvalues: $\lambda_1 = -(\sigma_u^2 + n\sigma_0^2)^{-1}(b_1 + \mu p_1 + \beta n q_1)$, $\lambda_2 = -(\sigma_u^2)^{-1}(b_1 + \mu p_1)$, $\lambda_3 = -(\sigma_u^2)^{-1}b_1$ and $\lambda_4 = 0$, and the numbers of degeneracy for each are 1, $n\beta/\mu - 1$, $n - n\beta/\mu$, and $N - n$, respectively. Hence

$$\begin{aligned} \mathcal{G}(q_1, p_1, f_1, \mu_1) &= \int_0^{\frac{b_1 + \mu_1 p_1 + \beta n q_1}{\sigma_u^2 + n\sigma_0^2}} R(-w)dw + \left(\frac{n\beta}{\mu_1} - 1\right) \int_0^{\frac{b_1 + \mu_1 p_1}{\sigma_u^2}} R(-w)dw \\ &\quad + \left(n - \frac{n\beta}{\mu_1}\right) \int_0^{\frac{b_1}{\sigma_u^2}} R(-w)dw \end{aligned} \quad (\text{D.3})$$

Further with entries of $\tilde{\mathbf{Q}}$ being RSB ansatz (B.15) will have more involved terms than the RS ansatz. i.e. ,

$$\begin{aligned} &\log M(q_1, p_1, f_1, \mu_1) \\ &= \int \log \sum_{\{\tilde{\mathbf{x}} \in \chi^n\}} e^{(x^0 \mathbf{1} - \tilde{\mathbf{x}})^T \tilde{\mathbf{Q}}(x^0 \mathbf{1} - \tilde{\mathbf{x}}) - \frac{\beta\gamma}{\sigma_u^2} \tilde{\mathbf{x}}} dF_{X^0}(x^0) \\ &= \int \log \sum_{\{\mathbf{x} \in \chi^n\}} e^{\beta^2 f_1^2 \left| \sum_{a=1}^n (x^0 - x_a) \right|^2 + \beta^2 g_1^2 \sum_{l=0}^{\frac{n\beta}{\mu} - 1} \left| \sum_{a=1}^{\frac{\mu}{\beta}} (x^0 - x_{a + \frac{l\mu_1}{\beta}}) \right|^2 - \beta e_1 \sum_{a=1}^n (x^0 - x_a)^2 - \frac{\beta\gamma}{\sigma_u^2} \sum_{a=1}^n |x_a^a|} \\ &\quad \cdot dF_{x^0}(x^0). \end{aligned} \quad (\text{D.4})$$

Using the Hubbard-Stratonovich transform (C.8) we can express (D.4) as in (c.f. [27], (66)- (70)) as follows

$$\begin{aligned} &\log M(q_1, p_1, f_1, \mu_1) \\ &= \int \log \sum_{\{\mathbf{x} \in \chi^n\}} \int_{\mathbb{C}} e^{\sum_{a=1}^n \left[2\beta f_1 \Re\{(x^0 - x_a)z^*\} - \beta e_1 |(x^0 - x_a)|^2 - \frac{\beta\gamma}{\sigma_u^2} |x_a^a| \right] + \beta^2 g_1^2 \sum_{l=0}^{\frac{n\beta}{\mu} - 1} \left| \sum_{a=1}^{\frac{\mu}{\beta}} (x^0 - x_{a + \frac{l\mu_1}{\beta}}) \right|^2} \\ &\quad \cdot Dz dF_{X^0}(x^0) \\ &= \int \log \int_{\mathbb{C}} \left[\int_{\mathbb{C}} \left(\sum_{\{\mathbf{x} \in \chi\}} \mathcal{K}(x, y, z) \right)^{\frac{\mu_1}{\beta}} Dy \right]^{\frac{n\beta}{\mu_1}} Dz dF_{X^0}(x^0) \end{aligned} \quad (\text{D.5})$$

where

$$\mathcal{K}(x, y, z) = e^{2\beta \Re\{(x^0 - x)(f_1 z^* + g_1 y^*)\} - \beta e_1 |(x^0 - x)|^2 - \frac{\beta\gamma}{\sigma_u^2} |x|}. \quad (\text{D.6})$$

Due to (B.17) the partial derivative of

$$\mathcal{G}(q_1, p_1, f_1, \mu_1) + \text{Tr}(\tilde{\mathbf{Q}}\mathbf{Q}) \quad (\text{D.7})$$

with respect to the macroscopic variables q_1 , p_1 , and b_1 vanishes as $N \rightarrow \infty$ by definition of the saddle point approximation. And plugging (D.3) and (??) in (D.7) and calculating the partial derivatives and setting them to zero and after

some algebraic manipulation we get the following set of equations

$$0 = n^2 \beta^2 f_1^2 + n\beta \mu_1 g_1^2 - n\beta e_1 + \frac{n\beta}{\sigma_u^2 + n\sigma_0^2} R\left(\frac{-b_1 - \mu_1 p_1 - \beta n q_1}{\sigma_u^2 + n\sigma_0^2}\right) \quad (\text{D.8})$$

$$0 = n\beta \mu_1 b_1^2 + n\beta \mu_1 g_1^2 - n\beta e_1 + \frac{(n\beta - \mu_1)}{\sigma_u^2} R\left(\frac{-b_1 - \mu_1 p_1}{\sigma_u^2}\right) + \frac{\mu_1}{\sigma_u^2 + n\sigma_0^2} R\left(\frac{-b_1 - \mu_1 p_1 - \beta n q_1}{\sigma_u^2 + n\sigma_0^2}\right) \quad (\text{D.9})$$

$$0 = n\beta f_1^2 + n\beta g_1^2 - n e_1 + \frac{(n - \frac{n\beta}{\mu_1})}{\sigma_u^2} R\left(\frac{-b_1}{\sigma_u^2}\right) + \frac{(\frac{n\beta}{\mu_1} - 1)}{\sigma_u^2} R\left(\frac{-b_1 - \mu_1 p_1}{\sigma_u^2}\right) \quad (\text{D.10})$$

$$+ \frac{1}{\sigma_u^2 + n\sigma_0^2} R\left(\frac{-b_1 - \mu_1 p_1 - \beta n q_1}{\sigma_u^2 + n\sigma_0^2}\right). \quad (\text{D.11})$$

Solving for e_1 , g_1 , f_1 we get

$$e_1 = \frac{1}{\sigma_u^2} R\left(\frac{-b_1}{\sigma_u^2}\right), \quad (\text{D.12})$$

$$g_1 = \sqrt{\frac{1}{\mu_1} \left[\frac{1}{\sigma_u^2} R\left(\frac{-b_1}{\sigma_u^2}\right) - \frac{1}{\sigma_u^2} R\left(\frac{-b_1 - \mu_1 p_1}{\sigma_u^2}\right) \right]}, \quad (\text{D.13})$$

$$f_1 = \sqrt{\frac{1}{n\beta} \left[\frac{1}{\sigma_u^2} R\left(\frac{-b_1 - \mu_1 p_1}{\sigma_u^2}\right) - \frac{1}{\sigma_u^2 + n\sigma_0^2} R\left(\frac{-b_1 - \mu_1 p_1 - n\beta q_1}{\sigma_u^2 + n\sigma_0^2}\right) \right]}, \quad (\text{D.14})$$

and further with the limits $n \rightarrow 0$

$$f_1 \xrightarrow{n \rightarrow 0} \sqrt{\frac{1}{\beta} \left[\frac{\sigma_0^2}{\sigma_u^4} R\left(\frac{-b_1 - \mu_1 p_1}{\sigma_u^2}\right) + \frac{(\sigma_u^2 \beta q_1 + \sigma_0^2 (b_1 + \mu_1 p_1))}{\sigma_u^6} R\left(\frac{-b_1 - \mu_1 p_1}{\sigma_u^2}\right) \right]}. \quad (\text{D.15})$$

and as $\beta \rightarrow \infty$ we can simplify it further as

$$f_1 \xrightarrow{n \rightarrow 0} \sqrt{\frac{q_1}{\sigma_u^4} R\left(\frac{-b_1 - \mu_1 p_1}{\sigma_u^2}\right)}. \quad (\text{D.16})$$

Also due to (B.18) the partial derivatives of

$$\log M(q_1, p_1, f_1, \mu_1) - \text{Tr}(\tilde{\mathbf{Q}}\mathbf{Q})$$

with respect to f_1 , g_1 , and e_1 , must also vanish as $N \rightarrow \infty$. This produces the following set of equations while taking $n \rightarrow 0$.

$$b_1 + p_1 \mu_1 = \frac{1}{f_1} \int \int_{\mathbb{C}^2} \frac{\left(\sum_{x \in \mathcal{X}} \mathcal{K}(x, y, z) \right)^{\frac{\mu_1}{\beta} - 1}}{\int_{\mathbb{C}} \left(\sum_{x \in \mathcal{X}} \mathcal{K}(x, y, z) \right)^{\frac{\mu_1}{\beta}} D\tilde{y}} \cdot \sum_{x \in \mathcal{X}} \Re\{xz^*\} \mathcal{K}(x, y, z) Dy Dz dF_{X^0}(x^0) \quad (\text{D.17})$$

$$b_1 + (q_1 + p_1)\mu_1 = \frac{1}{g_1} \int \int_{\mathbb{C}^2} \frac{\left(\sum_{x \in \mathcal{X}} \mathcal{K}(x, y, z)\right)^{\frac{\mu_1}{\beta} - 1}}{\int_{\mathbb{C}} \left(\sum_{x \in \mathcal{X}} \mathcal{K}(x, y, z)\right)^{\frac{\mu_1}{\beta}} D\tilde{y}} \cdot \sum_{x \in \mathcal{X}} \Re\{xy^*\} \mathcal{K}(x, y, z) Dz dF_{X^0}(x^0) \quad (\text{D.18})$$

$$q_1 + p_1 = -\frac{b_1}{\beta} + \frac{1}{g_1} \int \int_{\mathbb{C}^2} \frac{\left(\sum_{x \in \mathcal{X}} \mathcal{K}(x, y, z)\right)^{\frac{\mu_1}{\beta} - 1}}{\int_{\mathbb{C}} \left(\sum_{x \in \mathcal{X}} \mathcal{K}(x, y, z)\right)^{\frac{\mu_1}{\beta}} D\tilde{y}} \cdot \sum_{x \in \mathcal{X}} |x|^2 \mathcal{K}(x, y, z) Dy Dz dF_{X^0}(x^0). \quad (\text{D.19})$$

In addition when we take the partial derivative of

$$\mathcal{G}(q_1, p_1, f_1, \mu_1) + \text{Tr}(\tilde{\mathbf{Q}}\mathbf{Q}) - \log M(q_1, p_1, f_1, \mu_1) \quad (\text{D.20})$$

with respect of μ_1 is vanishes and yields at the limit as $n \rightarrow 0$

$$\begin{aligned} 0 &= \frac{1}{\mu_1^2} \int_{\frac{b_1}{\sigma_u^2}}^{\frac{b_1 + \mu_1 p_1}{\sigma_u^2}} R(-w) dw + \frac{p_1}{\mu_1^2} R\left(-\frac{b_1 + \mu_1 p_1}{\sigma_u^2}\right) + (q_1 + p_1)g_1^2 + p_1 f_1^2 \\ &+ \int \int_{\mathbb{C}} \left[\frac{1}{\mu_1^2} \log\left(\int_{\mathbb{C}} \left(\sum_{\{\mathbf{x} \in \mathcal{X}\}} \mathcal{K}(x, y, z)\right)^{\frac{\mu_1}{\beta}} Dy\right) \right. \\ &\left. - \int_{\mathbb{C}} \frac{\left(\sum_{x \in \mathcal{X}} \mathcal{K}(x, y, z)\right)^{\frac{\mu_1}{\beta}}}{\beta \mu_1^2 \int_{\mathbb{C}} \left(\sum_{x \in \mathcal{X}} \mathcal{K}(x, y, z)\right)^{\frac{\mu_1}{\beta}} D\tilde{y}} \cdot \log\left(\sum_{x \in \mathcal{X}} \mathcal{K}(x, y, z)\right) Dy \right] \\ &\quad \cdot Dz dF_{X^0}(x^0) \end{aligned} \quad (\text{D.21})$$

So as $\beta \rightarrow \infty$ these fixed point equations can be simplified as follows:

$$b_1 + p_1 \mu_1 = \frac{1}{f_1} \int \int_{\mathbb{C}^2} \Re\{(x^0 - \Psi_2)z^*\} Dy Dz dF_{X^0}(x^0) \quad (\text{D.22})$$

$$b_1 + (q_1 + p_1)\mu_1 = \frac{1}{g_1} \int \int_{\mathbb{C}^2} \Re\{(x^0 - \Psi_2)y^*\} Dy Dz dF_{X^0}(x^0) \quad (\text{D.23})$$

$$q_1 + p_1 = \frac{1}{g_1} \int \int_{\mathbb{C}^2} |\Psi_2|^2 Dy Dz dF_{X^0}(x^0) \quad (\text{D.24})$$

where

$$\Psi_2 = \arg \min_{x \in \mathcal{X}} \left| 2\Re\{(x^0 - x)(f_1 z^* + g_1 y^*)\} - e_1 |(x^0 - x)|^2 - \frac{\gamma}{\sigma_u^2} |x| \right|$$

Putting together the results again as in (C.28) and doing again the steps (B.34) to (B.38) for the RSB case

$$\bar{\mathcal{E}}_{\text{1rsb}}^{\text{lasso}} = - \lim_{\beta \rightarrow \infty} \frac{1}{\beta} \lim_{n \rightarrow 0} \frac{\partial}{\partial n} \Xi_n \quad (\text{D.25})$$

$$= - \lim_{\beta \rightarrow \infty} \frac{1}{\beta} \lim_{n \rightarrow 0} \frac{\partial}{\partial n} \{-\mathcal{G}(\mathbf{Q}) - \text{Tr}(\tilde{\mathbf{Q}}\mathbf{Q}) + \log M(\tilde{\mathbf{Q}})\} \quad (\text{D.26})$$

$$\begin{aligned} &= \lim_{\beta \rightarrow \infty} \frac{1}{\beta} \lim_{n \rightarrow 0} \frac{\partial}{\partial n} \left\{ \int_0^{\frac{b_1 + \mu_1 p_1 + \beta n q_1}{\sigma_u^2 + n \sigma_0^2}} R(-w) dw + \left(\frac{n\beta}{\mu_1} - 1\right) \int_0^{\frac{b_1 + \mu_1 p_1}{\sigma_u^2}} R(-w) dw \right. \\ &+ \left(n - \frac{n\beta}{\mu_1}\right) \int_0^{\frac{b_1}{\sigma_u^2}} R(-w) dw + \left[n(q_1 + p_1 + \frac{b_1}{\beta})(\beta^2 f_1^2 + \beta^2 g_1^2 - \beta e_1) \right. \\ &+ \left. n\left(\frac{\mu_1}{\beta} - 1\right)(q_1 + p_1)(\beta^2 g_1^2 + \beta^2 f_1^2) + n\left(n - \frac{\mu_1}{\beta}\right)q_1 \beta^2 f_1^2 \right] \\ &\left. - \log M(q_1, p_1, f_1, \mu_1) \right\} \quad (\text{D.27}) \end{aligned}$$

$$= \lim_{\beta \rightarrow \infty} \frac{1}{\beta} \left\{ \left(\frac{b_1 + \mu_1 p_1}{\sigma_u^2}\right) R\left(\frac{-b_1 - \mu_1 p_1}{\sigma_u^2}\right) \right. \quad (\text{D.28})$$

$$\begin{aligned} &+ \left(\frac{b_1 + \mu_1 p_1}{\sigma_u^2}\right) \frac{(\beta q_1 \sigma_u^2 - (b_1 + \mu_1 p_1) \sigma_0^2)}{\sigma_u^4} R'\left(\frac{-b_1 - \mu_1 p_1}{\sigma_u^2}\right) \\ &+ \frac{\beta}{\mu_1} \int_0^{\frac{b_1 + \mu_1 p_1}{\sigma_u^2}} R(-w) dw + \left(1 - \frac{\beta}{\mu_1}\right) \int_0^{\frac{b_1}{\sigma_u^2}} R(-w) dw \\ &+ \left[b_1(\beta f_1^2 + \beta g_1^2 - e_1) + \mu_1(q_1 + p_1)(\beta f_1^2 + \beta g_1^2 - \frac{\beta}{\mu_1} e_1) - \mu_1 q_1 \beta f_1^2\right] \\ &\left. - \frac{\beta}{\mu_1} \int \log \int_{\mathbb{C}} \int_{\mathbb{C}} \left(\sum_{\{\mathbf{x} \in \chi\}} \mathcal{K}(x, y, z)\right)^{\frac{\mu_1}{\beta}} Dy Dz dF_{X^0}(x^0) \right\} \quad (\text{D.29}) \end{aligned}$$

$$\begin{aligned} &= \frac{q_1}{\sigma_u^2} \left(\frac{b_1 + \mu_1 p_1}{\sigma_u^2}\right) R\left(\frac{-b_1 - \mu_1 p_1}{\sigma_u^2}\right) + \frac{1}{\mu_1} \int_0^{\frac{b_1 + \mu_1 p_1}{\sigma_u^2}} R(-w) dw - \frac{1}{\mu_1} \int_0^{\frac{b_1}{\sigma_u^2}} R(-w) dw \\ &+ \left[(b_1 + \mu_1(q_1 + p_1))(f_1^2 + g_1^2) - e_1(q_1 + p_1) - \mu_1 q_1 f_1^2\right] \end{aligned}$$

$$\begin{aligned} &- \lim_{\beta \rightarrow \infty} \frac{1}{\beta} \left\{ \frac{\beta}{\mu_1} \int \log \int_{\mathbb{C}} \int_{\mathbb{C}} \left(\sum_{\{\mathbf{x} \in \chi\}} \mathcal{K}(x, y, z)\right)^{\frac{\mu_1}{\beta}} Dy Dz dF_{X^0}(x^0) \right\} \quad (\text{D.30}) \end{aligned}$$

$$\begin{aligned} &= \frac{1}{\sigma_u^2} \left(q_1 + p_1 + \frac{b_1}{\mu_1}\right) R\left(\frac{-b_1 - \mu_1 p_1}{\sigma_u^2}\right) - \frac{b_1}{\mu_1 \sigma_u^2} R\left(-\frac{b_1}{\sigma_u^2}\right) \\ &+ q_1 \left(\frac{b_1 + \mu_1 p_1}{\sigma_u^2}\right) R'\left(\frac{-b_1 - \mu_1 p_1}{\sigma_u^2}\right) \quad (\text{D.31}) \end{aligned}$$

$$\begin{aligned} &= \frac{1}{\sigma_u^2} \left(q_1 + p_1 + \frac{b_1}{\mu_1}\right) R\left(\frac{-b_1 - \mu_1 p_1}{\sigma_u^2}\right) - \frac{b_1}{\mu_1 \sigma_u^2} R\left(-\frac{b_1}{\sigma_u^2}\right) \\ &+ q_1 \left(\frac{b_1 + \mu_1 p_1}{\sigma_u^2}\right) R'\left(\frac{-b_1 - \mu_1 p_1}{\sigma_u^2}\right) \quad (\text{D.32}) \end{aligned}$$

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