

# Performance Analysis of an OFDM System with Carrier Frequency Offset and Phase Noise

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**Abstract**—Orthogonal Frequency Division Multiplexing (OFDM) is sensitive to the carrier frequency offset (CFO) and phase noise, which destroys orthogonality and causes intercarrier interference (ICI). Previously, two methods were available for the analysis of the resultant degradation in performance. Firstly, the statistical average of the ICI could be used as a performance measure. Secondly, the bit error rate (BER) could be approximated by assuming the ICI to be Gaussian. However, a more precise analysis of the performance (i.e. BER or SER) degradation is desirable. In this paper, we propose a precise numerical technique for calculating the effect of the CFO and phase noise on the BER or symbol error rate (SER) in an OFDM system. In particular, closed form expressions for SER are derived using a Beaulieu series.

**Keywords**—Orthogonal Frequency Division Multiplexing, Carrier Frequency Offset, Phase Noise, Intercarrier Interference

## I. INTRODUCTION

OFDM has been accepted for several wireless LAN standards, as well as mobile multimedia applications [1]. It is, however, sensitive to CFO and phase noise. For a large number of subcarriers, these imperfections will destroy subcarrier orthogonality and introduce ICI. The CFO is caused by misalignment in carrier frequencies and/or Doppler shift [1–3] where as phase noise is due to instability of an oscillator [1, 2, 4–6]. Several methods have been developed to reduce these impairments [3, 6].

In these studies, the effect of these impairments are calculated in two ways. It may be approximately derived as a degradation in signal-to-noise ratio or the statistical average of the carrier-to-interference ratio [2–4]. Secondly, the bit error rate (BER) could be approximated by assuming the ICI to be Gaussian [5]. Alternatively, computer simulations may be used to obtain the performance degradation caused by the ICI. However, it is also both interesting and useful to know the precise correlation between the BER or SER and these impairments.

In this paper, we propose a precise technique for calculating the effect of the CFO and phase noise on the BER or SER in an OFDM system. The subcarriers are modulated with binary phase shift keying (BPSK), quaternary phase shift keying (QPSK) and 16-ary quadrature amplitude modulation (16QAM), which are common modulation formats used in OFDM applications. The BPSK case is solved using a series from Beaulieu. For the QPSK and 16QAM cases, we use an infinite series expression for the error function (again from Beaulieu) to express the average probability of error in terms of the 2D characteristic function (CHF) of the ICI. The technique readily achieves an accuracy in the region of 12 significant figures.

This paper is organized as follows. OFDM signalling, CFO

and phase noise models are presented in Section II. Exact expressions for the probability of error are presented in Section III. Numerical Results are reported in Section IV and the concluding remarks are in Section V.

## II. OFDM SIGNALLING

The complex baseband OFDM signal may be represented as

$$s(t) = \sum_{k=0}^{N-1} X_k e^{j\frac{2\pi}{T_o} kt} \text{ for } 0 \leq t \leq T_o. \quad (1)$$

where  $j^2 = -1$ ,  $N$  is the total number of subcarriers,  $X_k$  is the data symbol for the  $k$ -th subcarrier. The frequency separation between any two adjacent subcarriers is  $1/T_o$  where  $T_o$  is the OFDM symbol duration. For simplicity, we will not consider the cyclic prefix (so called guard interval) since it is eliminated in the receiver. However, we assume that there is no overlap between different OFDM symbols. In practice filtering can cause some degree of intersymbol interference, which will be neglected here.

We assume that  $s(t)$  is transmitted on an additive white Gaussian noise channel, and so the received signal is only affected by either CFO or phase noise. The received signal is expressed as

$$r(t) = s(t) \cdot e^{j\theta(t)} + n(t) \quad (2)$$

where  $n(t)$  is the Gaussian noise and  $\theta(t)$  is the time varying phase caused by either CFO or phase noise. We analyze these two impairments separately. In the first case,  $\theta(t) = 2\pi\Delta f t$  where  $\Delta f$  denotes the CFO [2, 3]. In the second case,  $\theta(t)$  is modelled as a Wiener process with zero mean and variance of  $2\pi\beta|t|$ , where  $\beta$  represents the two sided 3-dB line width of Lorentzian power density spectrum of the oscillator [2, 5].

The sampled signal for the  $k$ -th subchannel after the receiver fast Fourier transform processing can be written as

$$\begin{aligned} y_k &= \sum_{m=0}^{N-1} r[m] \cdot e^{-j\frac{2\pi}{N} km} + n_k \\ &= \sum_{m=0}^{N-1} e^{j\theta[m]} \sum_{l=0}^{N-1} X_l e^{j\frac{2\pi}{N} (l-k)m} + n_k \\ &= \sum_{l=0}^{N-1} X_l \sum_{m=0}^{N-1} e^{j\theta[m]} e^{j\frac{2\pi}{N} (l-k)m} + n_k \\ &= \sum_{l=0}^{N-1} X_l S_{l-k} + n_k \end{aligned} \quad (3)$$

where  $n_k$  is a complex Gaussian noise sample (with its real and imaginary components being independent and identically distributed with variance  $\sigma^2$ ). The sequence  $S_k$  (the ICI coefficients) is given by

$$S_k = \sum_{m=0}^{N-1} e^{j\theta[m]} \cdot e^{j\frac{2\pi}{N}km}. \quad (4)$$

The Eq. (3) can be rearranged as

$$y_k = X_k S_0 + \sum_{l=0, l \neq k}^{N-1} S_{l-k} X_l + n_k \text{ for } k = 0, 1, \dots, N-1. \quad (5)$$

The second right term in (5) is the ICI term attributable to either CFO or phase noise. When there is neither CFO nor phase noise in the channel,  $S_k$  reduces to the unit impulse sequence.

We assume the data symbols  $X_k$ s are independent and identically distributed random variables (RV's). For M-ary signaling,  $X_k$  is equally likely to assume one out of  $M$  levels. Without any loss in generality, we will consider the error rate for the 0-th subcarrier (i.e.  $k = 0$ ). So the problem at hand is to determine, for a given symbol sent on the 0-th subcarrier, the probability that an incorrect decision will be made.

### III. EXACT CALCULATION OF PROBABILITY OF ERROR

The techniques for exact calculation of SER for BPSK, QPSK and 16QAM schemes are presented in [7]. Interestingly, these techniques do not make any assumption on the statistics of  $S_k$ , although they are derived for CFO errors. Therefore, they can be used for exact calculation of SER considering the statistical nature of  $S_k$ . We briefly summarize the derivation for BPSK and refer the reader [7] for more detail.

#### A. BPSK Modulation

For BPSK modulation,  $X_k \in \{1, -1\}$  and without incurring a loss in generality, it will be sufficient to consider the real part of Eq. (5). The CHF of  $\Re(y_k)$  can be expressed as

$$\phi(\omega) = e^{j\omega X_k \Re\{S_0\} - \frac{\omega^2 \sigma^2}{2}} \prod_{l=0, l \neq k}^{N-1} \cos(\omega \Re\{S_{l-k}\}). \quad (6)$$

where  $\Re\{z\}$  denotes the real part of  $z$ . From the Bullied Series [8, 9], the cumulative distribution function (CDF) of an RV can be expressed using its CHF, i.e.,

$$Pr(X < x) = \frac{1}{2} - \sum_{n \in N_o} \frac{2\Im\{e^{-jn\omega_0 x} \phi(n\omega_0)\}}{n\pi} + \epsilon(x, \omega_0) \quad (7)$$

where  $\Im\{z\}$  denotes the imaginary part of  $z$ ,  $\omega_0 = 2\pi/T$ ,  $T$  is a parameter governing the sampling rate in the frequency domain, and  $\epsilon(x, \omega_0)$  is an error term. Note that the error term depends on both  $x$  and  $T$ . For a given  $x$ , the error can be arbitrarily reduced by increasing  $T$ . In practice, once a suitable  $T$  is selected, it is used for all possible values of  $x$ . The index set  $N_o = \{1, 3, \dots\}$  is the set of all positive odd integers. Without loss of generality, we consider the first subcarrier and the transmitted symbol  $X_0 = 1$ . Now, a decision error occurs if the real

part of  $y_0$  (5) is less than zero. Thus, the probability of a bit error is

$$P_b = \frac{1}{2} - \sum_{n \in N_o} \frac{\sin[n\omega_0 \Re\{S_0\}] e^{-\frac{1}{2}\omega_0^2 n^2 \sigma^2}}{n\pi} \prod_{l=1}^{N-1} \cos(n\omega_0 \Re\{S_l\}) + \epsilon(0, \omega_0). \quad (8)$$

Here, the error term can be made negligibly small by selecting a sufficiently large  $T$ . The numerical evaluation of (8) gives the exact probability of a bit error.

#### A.1 QPSK Modulation

The average probability of correct decision may be expressed as [7]

$$P_{c,av} = \frac{1}{4} + \frac{1}{\pi} \sum_{m \in N_o} \frac{\exp(-m^2 \omega_0^2 / 2) \Im\{\phi(\frac{m\omega_0}{\sigma}, 0)\}}{m} + \frac{1}{\pi} \sum_{n \in N_o} \frac{\exp(-n^2 \omega_0^2 / 2) \Im\{\phi(0, \frac{n\omega_0}{\sigma})\}}{n} + \frac{2}{\pi^2} \sum_{n \in N_o} \sum_{m \in N_o} \left( \frac{\exp(-(n^2 + m^2)\omega_0^2 / 2)}{nm} \times \Re\{\phi(\frac{-m\omega_0}{\sigma}, \frac{n\omega_0}{\sigma}) - \phi(\frac{m\omega_0}{\sigma}, \frac{n\omega_0}{\sigma})\} \right) \quad (9)$$

where

$$\phi(\omega_I, \omega_Q) = e^{j[\omega_I(\Re\{S_0\} - \Im\{S_0\}) + \omega_Q(\Im\{S_0\} + \Re\{S_0\})]} \prod_{l=1}^{N-1} (\cos(\omega_I \Re\{S_l\} + \omega_Q \Im\{S_l\}) \cos(\omega_I \Im\{S_l\} - \omega_Q \Re\{S_l\})). \quad (10)$$

Subsequently, the probability of symbol error is given by,

$$P_e = 1 - P_{c,av}. \quad (11)$$

#### A.2 16QAM Modulation

The probability of a correct decision can be given by [7]

$$P_{c,av} = \frac{1}{16} - \frac{1}{4\pi} \sum_{m \in N_o} \frac{\exp(-m^2 \omega_0^2 / 2) \Im\{\phi(\frac{m\omega_0}{\sigma}, 0)\}}{m} - \frac{1}{4\pi} \sum_{n \in N_o} \frac{\exp(-n^2 \omega_0^2 / 2) \Im\{\phi(0, \frac{n\omega_0}{\sigma})\}}{n} + \frac{1}{2\pi^2} \sum_{n \in N_o} \sum_{m \in N_o} \left( \frac{\exp(-(n^2 + m^2)\omega_0^2 / 2)}{nm} \times \Re\{\phi(\frac{-m\omega_0}{\sigma}, \frac{n\omega_0}{\sigma}) - \phi(\frac{m\omega_0}{\sigma}, \frac{n\omega_0}{\sigma})\} \right) \quad (12)$$

where  $\phi(\omega_I, \omega_Q)$  is defined as in (10). The probability of error can now be obtained using (11).

#### IV. NUMERICAL RESULTS

##### A. Carrier Frequency Offset

When there exists a frequency offset  $\Delta f$  between the transmitter and receiver,  $\theta(t)$  is deterministic and equals to  $2\pi\Delta ft$ . Then, the sequence  $S_k$  depends on the CFO and is given by [3]

$$S_k = \frac{\sin \pi(k + \varepsilon)}{N \sin \frac{\pi}{N}(k + \varepsilon)} \exp \left[ j\pi \left(1 - \frac{1}{N}\right)(k + \varepsilon) \right] \quad (13)$$

where  $\varepsilon$  is the normalized frequency offset which is the ratio between the carrier frequency offset and the adjacent subcarrier spacing. Since  $S_k$ 's are deterministic, we use the formulas in Section III to calculate the SER.

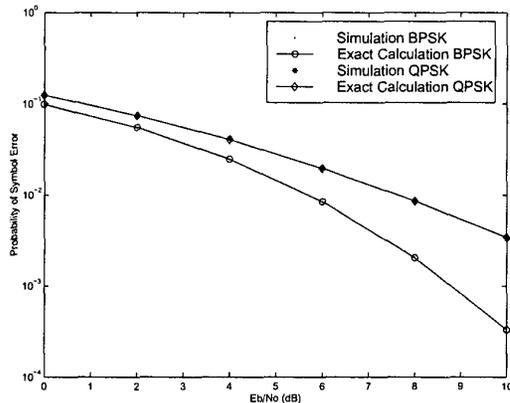


Fig. 1. Probability of Symbol Error for BPSK and QPSK, Normalized frequency Offset = 0.1

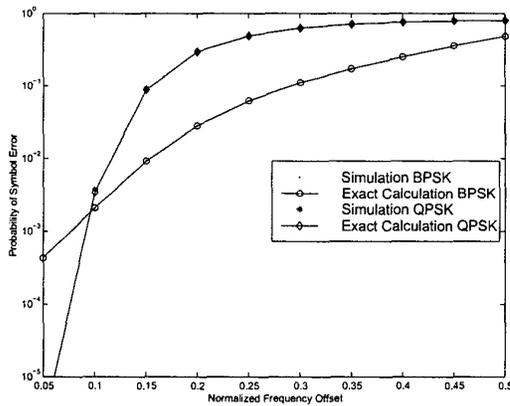


Fig. 2. Probability of Symbol Error with Normalized Frequency Offset,  $SNR = 8dB$  for BPSK and  $SNR = 10dB$  for QPSK

In Fig. 1 and Fig. 2, error rates are shown for an OFDM system with  $N = 128$  for BPSK, and QPSK modulation schemes. In the simulation results which follow,  $10^7$  random OFDM frames were generated to obtain each error rate point. The simulation results agree with those calculated using (8) and (11). Fig. 3 shows results for 16QAM OFDM. Again, our technique is precise.

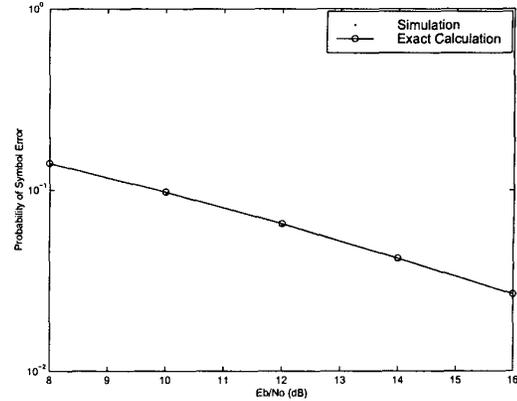


Fig. 3. Probability of Symbol Error for 16QAM, Normalized frequency Offset = 0.05

##### B. Phase Noise

In the case of phase noise,  $\theta(t)$  is modelled as a Wiener process, a continuous-path Brownian motion with zero mean and variance of  $2\pi\beta t$ . Therefore,  $\theta(t)$  is non-stationary. Consequently,  $S_k$ 's are random and the calculation of SER based on (8) and (11) are conditional probabilities of symbol error. Therefore, averaging (8) and (11) over  $S_k$  will give the exact SER. However, we need to derive the probability density function of  $S_k$ , which is rather difficult. The statistical properties of phase noise have been studied by several authors [10–12]. It is a complex problem and different approaches such as simulation techniques, characterization through moments and recursive method are reported in these studies.

We use a simple Monte Carlo method to calculate the exact SER. We generate  $S_k$  randomly (4) and calculate the corresponding SER using (8) and (11). This sample SER's are averaged to get the exact SER. These exact SERs are verified below by simulation. Note that we are interested in exact calculation of SER and thus, we avoid the use of probability density function of  $S_k$ , which can be approximately calculated as in [10–12]. However, exact formulas in (8) and (11) are not close-form solutions in the case of phase noise.

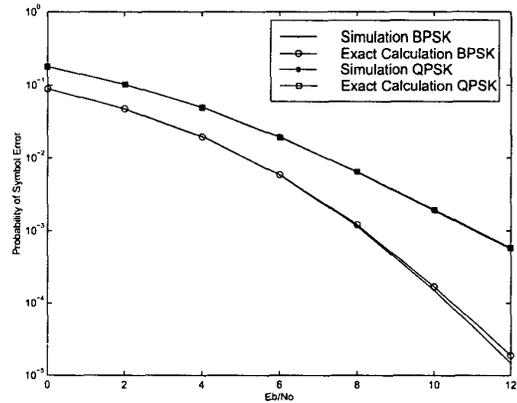


Fig. 4. Probability of Symbol Error for BPSK and QPSK,  $\gamma = 0.1$

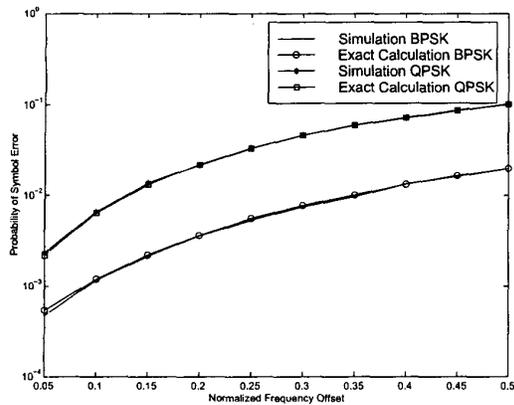


Fig. 5. Probability of Symbol Error with  $\gamma$ ,  $SNR = 8dB$

Fig. 4 shows the error rates of an OFDM system with  $N = 128$  for BPSK, and QPSK modulation schemes for  $\gamma = 0.1$  where  $\gamma = 2\pi\beta T_o$ . The simulation results agree with those calculated using (8) and (11). Fig. 5 also confirms the agreement between the simulation and exact calculation results when  $\gamma$  varies.

## V. CONCLUSION

In this paper, we have extended the numerical technique proposed in [7] to calculate the effect of phase noise on the BER or SER in an OFDM system. We considered BPSK, QPSK and 16QAM, which are common modulation formats used in OFDM applications. The proposed technique for exact calculation of SER is independent of the statistic of interference coefficients ( $S_k$ ). We analyzed this technique with deterministic coefficients for CFO and with random coefficients for phase noise. In both cases, this technique readily achieves high degree of accuracy. The accuracy does not depend on the amount of impairments or any other parameter.

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