Efficient Numerical Methods for Secrecy Capacity of Gaussian MIMO Wiretap Channel

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Abstract—This paper presents two different low-complexity methods for obtaining the secrecy capacity of multiple-input multiple-output (MIMO) wiretap channel subject to a sum power constraint (SPC). The challenges in deriving computationally efficient solutions to the secrecy capacity problem are due to the fact that the secrecy rate is a difference of convex functions (DC) of the transmit covariance matrix, for which its convexity is only known for the degraded case. In the first method, we capitalize on the accelerated DC algorithm, which requires solving a sequence of convex subproblems. In particular, we show that each subproblem indeed admits a water-filling solution. In the second method, based on the equivalent convex-concave reformulation of the secrecy capacity problem, we develop a so-called partial best response algorithm (PBRA). Each iteration of the PBRA is also done in closed form. Simulation results are provided to demonstrate the superior performance of the proposed methods.

Index Terms - MIMO, secrecy capacity, sum power constraint, convex-concave, closed form solution.

I. INTRODUCTION:

Wireless communication is an integral part of our modern life. Due to its broadcasting nature, information sent over wireless channels is vulnerable to security breach. Significant measures and techniques have been developed by both industry and academia to address this critical issue. Particularly, cryptography is a conventional method to ensure data security in wireless networks. In recent years, physical layer security has received growing interest as a promising alternative to addressing wireless security. While cryptographic methods are based on computational complexity and implemented in high network layers, physical layer security is concerned with exploiting distinguishing properties of wireless channels to achieve secure communication.

The wiretap channel (WTC), in which an eavesdropper aims to decode to the message exchanged by a pair of legitimate transceivers, represents a fundamental information-theoretic model for physical layer security. The secrecy capacity of the WTC was first studied by Wyner in [1]. Since Wyner's seminal paper, the WTC has been extended, covering various scenarios. In particular, the secrecy capacity of the Gaussian WTC was studied [2]. The use of multiple antennas at transceivers in contemporary wireless communications systems naturally gives rise to the so-called multiple-input multiple-output (MIMO) Gaussian WTC. The secrecy capacity of Gaussian MIMO WTC has received significant interests since the late 2000s. In this regard, there have been many significant results in the literature, which are discussed as follows.

The analytical solution for the Gaussian multiple-input single-out (MISO) WTC where both the eavesdropper and the legitimate receiver have a single antenna was proposed in [3]. When the channel state information is perfectly known, the secrecy capacity of MIMO WTC was characterized in [4], [5], [6]. Particularly, explicit expressions for optimal signaling for Gaussian MIMO WTC are possible under some special cases [7], [8], [9]. Power minimization and secrecy rate maximization for MIMO WTC was studied in [10] using a difference of convex functions algorithm (DCA). More recently, a lowcomplex solution for Gaussian MIMO WTC was proposed in [11] using the equivalent convex-concave reformulation of the secrecy capacity problem.

In this paper we consider the problem of finding the secrecy capacity-achieving input covariance for Gaussian MIMO WTC subject to a sum power constraint (SPC). The case of general linear transmit covariance constraints such as per-antenna power constraint is studied in [12]. In particular, we develop two low-complexity methods to solve the secrecy-capacity problem for the Gaussian MIMO WTC. The first method is an accelerated difference of convex functions algorithm (ADCA) [13] where each subproblem is found in closed form. In the second method, we propose an efficient iterative method to calculate the secrecy capacity, which is based on the equivalent concave-convex reformulation of the secrecy capacity problem. We refer to this proposed method as the partial best response algorithm (PBRA). The idea of PBRA is to find a saddle point of the concave-convex problem, for which efficient numerical methods are also derived. We remark that the method presented in [11] is a double-loop iterative algorithm, while the proposed PBRA in this paper only requires a single loop.

Notation: We use bold uppercase and lowercase letters to denote matrices and vectors, respectively. $\mathbb{C}^{M \times N}$ denotes the space of $M \times N$ complex matrices. To lighten the notation, **I** and **0** define identity and zero matrices respectively, of which the size can be easily inferred from the context. \mathbf{H}^{\dagger} and \mathbf{H}^{T} are Hermitian and ordinary transpose of **H**, respectively; $\mathbf{H}_{i,j}$ is the (i,j)-entry of **H**; $|\mathbf{H}|$ is the determinant of **H**; Furthermore, we denote the expected value of a random variable by $\mathbb{E}[.]$. For $\mathbf{x} \in \mathbb{R}^{N}$ $[\mathbf{x}]_{+} =$

 $[\max(x_1, 0), \max(x_2, 0), \cdots \max(x_N, 0)]$. The *i*th unit vector (i.e., its *i*th entry is equal to one and all other entries are zero) is denoted by \mathbf{e}_i . The notation $\mathbf{A} \succeq (\succ)\mathbf{B}$ means $\mathbf{A} - \mathbf{B}$ is positive semidefinite (definite). diag(\mathbf{x}) creates a diagonal matrix whose diagonal elements are taken from \mathbf{x} . $\|\mathbf{H}\|$ denotes the Frobenius norm of \mathbf{H} .

II. SYSTEM MODEL

A. MIMO Wiretap Channel Model

We consider a MIMO WTC, where Alice wants to transmit information to the legitimate receiver Bob in presence of Eve, the eavesdropper. The number of antennas at Bob, Alice and Eve is denoted by N_t , N_r , and N_e , respectively. $\mathbf{H}_b \in \mathbb{C}^{N_r \times N_t}$ is the channel matrix between Alice and Bob. The channel matrix between Bob and Eve is denoted by $\mathbf{H}_e \in \mathbb{C}^{N_e \times N_t}$. The received signals at Bob and Eve are respectively expressed as

$$\mathbf{y}_b = \mathbf{H}_b \mathbf{x} + \mathbf{z}_b \tag{1a}$$

$$\mathbf{y}_e = \mathbf{H}_e \mathbf{x} + \mathbf{z}_e \tag{1b}$$

where, $\mathbf{x} \in \mathbb{C}^{N_t \times 1}$ represents the transmitted signal ; $\mathbf{z}_b \in \mathbb{C}^{N_r \times 1} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I})$ and $\mathbf{z}_e \in \mathbb{C}^{N_e \times 1} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I})$ are the additive white Gaussian noise at Bob and Eve, respectively. In this paper \mathbf{H}_b and \mathbf{H}_e are assumed to be quasi-static and perfectly known at Alice and Bob. For a given input covariance matrix $\mathbf{X} = E\{\mathbf{xx}^{\dagger}\} \succeq \mathbf{0}$, where $E\{\cdot\}$ is the statistical expectation, the maximum secrecy rate (in nat/s/Hz) between Alice and Bob is given by [6]

$$C_{s}(\mathbf{X}) = \left[\underbrace{\ln |\mathbf{I} + \mathbf{H}_{b} \mathbf{X} \mathbf{H}_{b}^{\dagger}|}_{f_{b}(\mathbf{X})} - \underbrace{\ln |\mathbf{I} + \mathbf{H}_{e} \mathbf{X} \mathbf{H}_{e}^{\dagger}|}_{f_{e}(\mathbf{X})} \right]_{+}$$
(2)

The secrecy capacity under the sum power constraint (SPC) is written as

$$\begin{array}{l} \underset{\mathbf{X} \succeq \mathbf{0}}{\operatorname{maximize}} \ C_s(\mathbf{X}) \tag{3a} \end{array}$$

subject to
$$\operatorname{tr}(\mathbf{X}) \le P_0$$
 (3b)

where P_0 is the total transmit power. We remark that the problem (3) is non-convex in general, and thus, it is very difficult to find a globally optimal solution. However, if the channel is degraded, i.e. $\mathbf{H}_b^{\dagger}\mathbf{H}_b \succeq \mathbf{H}_e^{\dagger}\mathbf{H}_e$, (3) becomes convex but off-the-shelf solvers cannot be used to solve it. In this regard, the equivalent minimax reformulation of (3) presented in the next subsection is more numerically useful.

B. Minimax Reformulation

It is interesting to note that the secrecy capacity of MIMO WTC in (2) is equivalent to the following minimax optimization problem

$$C_s = \min_{\mathbf{Q} \in \mathcal{Q}} \max_{\mathbf{X} \in \mathcal{X}} f(\mathbf{Q}, \mathbf{X}) \triangleq \log \frac{|\mathbf{I} + \mathbf{Q}^{-1} \mathbf{H} \mathbf{X} \mathbf{H}^{\dagger}|}{|\mathbf{I} + \mathbf{H}_e \mathbf{X} \mathbf{H}_e^{\dagger}|}$$
(4)

where $\bar{\mathbf{H}} = [\mathbf{H}_b^{\mathsf{T}}, \mathbf{H}_e^{\mathsf{T}}]^{\mathsf{T}}$ and $\mathbf{Q} \in \mathbb{C}^{N_R \times N_E}$. The sets \mathcal{Q} and \mathcal{X} are defined as

$$\mathcal{X} = \{ \mathbf{X} | \mathbf{X} \succeq \mathbf{0}; \operatorname{tr}(\mathbf{X}) = P_0 \}$$
(5)

and

$$Q = \left\{ \mathbf{Q} | \mathbf{Q} \succeq \mathbf{0}; \mathbf{Q} = \left[\begin{array}{cc} \mathbf{I} & \bar{\mathbf{Q}} \\ \bar{\mathbf{Q}}^{\dagger} & \mathbf{I} \end{array} \right] \right\}.$$
(6)

Compared to (3), (4) is more tractable since the objective of (4) is concave with X for a given Q and convex with Q for a given X. In particular, we can compute the secrecy capacity and the optimal signaling by finding the saddle point of $f(\mathbf{Q}, \mathbf{X})$.

III. PROPOSED ALGORITHMS

In this section we present two low-complexity methods for finding the secrecy capacity of the MIMO WTC. The first method is a result of applying an ADCA to (3) and the second one is based on a finding a saddle point of (4).

A. ADCA for Solving (3)

To solve (3), we propose a simple but efficient method derived based on the obvious observation that $C_s(\mathbf{X})$ is a DC function, which naturally motivates the use of DCA. In this work we apply the ADCA presented in [13]. The idea is that from the current and previous iterates, denoted by \mathbf{X}_n and \mathbf{X}_{n-1} respectively, we compute an extrapolated point \mathbf{Z}_n using the Nesterov's acceleration technique: $\mathbf{X}_n + (t_k - t_k)$ $1)/t_{k+1}(\mathbf{X}_n - \mathbf{X}_{n-1})$, where t is the acceleration parameter. We remark that the specific t-update in Line 4 is a condition to guarantee the convergence of the iterative process as described in [14]. Since $C_s(\mathbf{X})$ is possibly non-convex for a general MIMO WTC, \mathbf{Z}_n can be a bad extrapolation and a monitor is required. Specifically, if \mathbf{Z}_n is better than one of the last q iterates, then \mathbf{Z}_n is considered a good extrapolation and thus will be used instead of \mathbf{X}_n to generate the next iterate. Thus, the ADCA is generally non-monotone. The algorithmic description of ADCA for solving (3) is outlined in Algorithm 1. Note that the subproblem in (7) is achieved by linearizing $f_e(\mathbf{X})$ around \mathbf{V}_n and by omitting the associated constants that do not affect the optimization. In Algorithm 1, q is any non-negative integer and γ_n is the minimum of the secrecy rate of the last q iterates. We remark that the case when q = 0reduces to the conventional DCA, which is exactly the same as the AO method in [15].

Algorithm 1 ADCA for solving (3) 1: Initialization: $\mathbf{W}_0 = \mathbf{X}_0 \in \mathcal{X}, t = \frac{1+\sqrt{5}}{2}, q$: integer. 2: for n = 1, 2, ... do 3: Update: $\mathbf{X}_n = \underset{\mathbf{X} \in \mathcal{X}}{\operatorname{arg\,max}} \underbrace{f_b(\mathbf{X}) - \operatorname{tr}\left(\nabla f_e(\mathbf{W}_{n-1})\mathbf{X}\right)}_{\overline{f}(\mathbf{X};\mathbf{W}_{n-1})}$ (7) where $\nabla f_e(\mathbf{X}) = \mathbf{H}_e^{\dagger} (\mathbf{I} + \mathbf{H}_e \mathbf{X} \mathbf{H}_e^{\dagger})^{-1} \mathbf{H}_e$ 4: $t_{n+1} = \frac{1+\sqrt{1+4t_n^2}}{2}$ 5: $\mathbf{Z}_n = \mathbf{X}_n + \frac{t_{n-1}}{t_{n+1}} (\mathbf{X}_n - \mathbf{X}_{n-1})$ 6: $\gamma_n = \min(C_s(\mathbf{X}_n), C_s(\mathbf{X}_{n-1}), \dots, C(\mathbf{X}_{\lfloor n-q \rfloor_+}))$ 7: $\mathbf{W}_n = \begin{cases} \mathbf{Z}_n & \text{if } C_s(\mathbf{Z}_n) \ge \gamma_n \text{ and } \mathbf{Z}_n \text{ is feasible} \\ \mathbf{X}_n & \text{otherwise} \end{cases}$ 8: end for 9: Output: \mathbf{X}_n To implement Algorithm 1, we need to efficiently solve (7), which is explicitly written as

$$\underset{\mathbf{X} \succ \mathbf{0}}{\operatorname{maximize}} \ln \left| \mathbf{I} + \mathbf{H}_{b} \mathbf{X} \mathbf{H}_{b}^{\dagger} \right| - \operatorname{tr} \left(\Phi_{n-1} \mathbf{X} \right)$$
(8a)

subject to
$$\operatorname{tr}(\mathbf{X}) = P_0$$
 (8b)

where $\Phi_{n-1} = \mathbf{H}_{e}^{\dagger} (\mathbf{I} + \mathbf{H}_{e} \mathbf{W}_{n-1} \mathbf{H}_{e}^{\dagger})^{-1} \mathbf{H}_{e}$. We now show that (8) admits a *water-filling solution* To begin with, let us form the *partial* Lagrangian function associated with (7) as

 $\mathcal{L}(\mathbf{X},\mu) = \ln |\mathbf{I} + \mathbf{H}_b \mathbf{X} \mathbf{H}_b^{\dagger}| - \operatorname{tr}(\bar{\mathbf{\Phi}}_{n-1} \mathbf{X}) + \mu P_0 \quad (9)$ where $\bar{\mathbf{\Phi}}_{n-1} = \mathbf{\Phi}_{n-1} + \mu \mathbf{I}_{N_t}$ and $\mu \ge 0$ is the Lagrangian multiplier. Let $\bar{\mathbf{X}} \triangleq \bar{\mathbf{\Phi}}_{n-1}^{1/2} \mathbf{X} \bar{\mathbf{\Phi}}_{n-1}^{1/2}$ and rewrite the Lagrangian function as a function of $\bar{\mathbf{X}}$ as

$$\mathcal{L}(\bar{\mathbf{X}},\mu) = \ln \left| \mathbf{I} + \mathbf{H}_b \bar{\mathbf{\Phi}}_{n-1}^{-1/2} \bar{\mathbf{X}} \bar{\mathbf{\Phi}}_{n-1}^{-1/2} \mathbf{H}_b^{\dagger} \right| - \operatorname{tr}(\bar{\mathbf{X}}).$$
(10)
to derive the dual function, the following lemma is in order

To derive the dual function, the following lemma is in order [11].

Lemma 1. For a given $\mu \geq 0$, let $\bar{\Phi}_{n-1}^{-1/2} \mathbf{H}_{b}^{\dagger} \mathbf{H}_{b} \bar{\Phi}_{n-1}^{-1/2} = \mathbf{V} \mathbf{\Sigma} \mathbf{V}^{\dagger}$ be the eigen value decomposition (EVD) of $\bar{\Phi}_{n-1}^{-1/2} \mathbf{H}_{b}^{\dagger} \mathbf{H}_{b} \bar{\Phi}_{n-1}^{-1/2}$, where $\mathbf{V} \in \mathbb{C}^{N_{t} \times N_{t}}$ is unitary, $\mathbf{\Sigma} = \text{diag}(\sigma_{1}, \sigma_{2}, \dots, \sigma_{r}, 0, \dots, 0)$, and r is the rank of $\bar{\Phi}_{n-1}^{-1/2} \mathbf{H}_{b}$. Then the solution to the problem $\max_{\bar{\mathbf{X}} \succeq \mathbf{0}} \mathcal{L}(\bar{\mathbf{X}}, \mu)$ is given by

$$\mathbf{X} = \bar{\mathbf{\Phi}}_{n-1}^{-1/2} \mathbf{V} \bar{\mathbf{\Sigma}} \mathbf{V}^{\dagger} \bar{\mathbf{\Phi}}_{n-1}^{-1/2}$$
(11)
where $\bar{\mathbf{\Sigma}} = \text{diag} \left([1 - \frac{1}{\sigma_1}]_+, \dots, [1 - \frac{1}{\sigma_r}]_+, 0, \dots, 0 \right).$

Next, to solve (8) we need to find the optimal value of μ which can be done by a bisection search. We skip the details here for the sake of brevity. The convergence proof of Algorithm 1 is provided in [12]. The idea is to show that the sequence $\{\gamma_n\}$ increasing and the feasible set \mathcal{X} is compact and convex. Thus, there exists a convergent subsequence, the accumulation point of which is then proved to be a stationary point.

B. Partial Best Response Method for Solving (4)

The second proposed method is an iterative algorithm to find the saddle point of the concave-convex problem in (4). Suppose $(\mathbf{X}_{n-1}, \mathbf{Q}_{n-1})$ has been computed at the *n*-th iteration. Then \mathbf{X}_n is found as

$$\mathbf{X}_{n} = \underset{\mathbf{X} \in \mathcal{X}}{\operatorname{arg\,max}} f(\mathbf{Q}_{n}, \mathbf{X})$$

=
$$\underset{\mathbf{X} \in \mathcal{X}}{\operatorname{arg\,max}} \log |\mathbf{Q}_{n-1} + \mathbf{H}\mathbf{X}\mathbf{H}^{\dagger}| - \log |\mathbf{I} + \mathbf{H}_{e}\mathbf{X}\mathbf{H}_{e}^{\dagger}|. \quad (12)$$

In words, \mathbf{X}_k is the best response to \mathbf{Q}_{n-1} as usual. Now given \mathbf{X}_n , due to the concavity of the term $\log |\mathbf{Q} + \mathbf{H}\mathbf{X}\mathbf{H}^{\dagger}|$, the following inequality holds

$$f(\mathbf{Q}, \mathbf{X}_{n}) \leq \log |\mathbf{Q}_{n-1} + \mathbf{H}\mathbf{X}_{n}\mathbf{H}^{\dagger}| + \operatorname{tr}(\boldsymbol{\Psi}_{n}(\mathbf{Q} - \mathbf{Q}_{n-1})) - \log(\mathbf{Q}) - \log |\mathbf{I} + \mathbf{H}_{e}\mathbf{X}_{n}\mathbf{H}_{e}^{\dagger}|, \forall \mathbf{Q} \in \mathcal{Q}.$$
$$\triangleq \bar{f}(\mathbf{Q}, \mathbf{X}_{n}). \tag{13}$$

where $\Psi_n = (\mathbf{Q}_{n-1} + \mathbf{H}\mathbf{X}_n\mathbf{H}^{\dagger})^{-1}$. Note that the above inequality is tight when $\mathbf{Q} = \mathbf{Q}_{n-1}$. Next, \mathbf{Q}_n is obtained as

$$\mathbf{Q}_{n} = \underset{\mathbf{Q} \in \mathcal{Q}}{\operatorname{arg\,min}} \bar{f}(\mathbf{Q}, \mathbf{X}_{n}) = \underset{\mathbf{Q} \in \mathcal{Q}}{\operatorname{arg\,min}} \operatorname{tr}(\boldsymbol{\Psi}_{n} \mathbf{Q}) - \log |\mathbf{Q}|.$$
(14)

That is to say, \mathbf{Q}_n is found be the best response to \mathbf{X}_n using *an upper bound* of the objective. The proposed solution for finding the secrecy capacity is summarized in Algorithm 2.

Algorithm 2 PBRA for solving (4)	
1: Input: $\mathbf{Q}_1 \in \mathcal{Q}, \ \epsilon_1 > 0$	
2: for $n = 1, 2$ do	
3: Update \mathbf{X}_n according to (12)	
4: Update \mathbf{Q}_{n+1} according to (14)	
5: end for	
6: Output: \mathbf{X}_n	

We remark that the iterative method presented in [11] also aims to find the saddle-point of (4). However, it is a doubleloop algorithm where a lower bound of $f(\mathbf{Q}_n, \mathbf{X})$ is used for the X-update. In contrast, Algorithm 2 is a single-loop one where the X-update is exact. The efficient methods for the Xupdate and Q-update are detailed in the following subsections.

1) X-update: To compute X_n as in Line 3 of Algorithm 2, we need to solve (12) which is a convex problem. Since the projection onto \mathcal{X} can be done in closed form, we can apply an accelerated projected gradient method (APGM) [16] to solve it efficiently, which is described as follows. To avoid confusion we use the superscript to denote the iteration count of the APGM. Suppose $Y^{(k)}$, the extrapolated point at iteration k, is available. The next iterate $X^{(k)}$ is found as

$$\mathbf{X}^{(k)} = p_{\mathcal{X}} \left(\mathbf{Y}^{(k)} + \frac{1}{\beta} \nabla f \left(\mathbf{Q}_{n-1}, \mathbf{Y}^{(k)} \right) \right)$$
(15)

where $\frac{1}{\beta}$ is a step size and $p_{\mathcal{X}}(\bar{\mathbf{X}})$ denotes the projection of $\bar{\mathbf{X}}$ onto \mathcal{X} . The gradient of $f(\mathbf{Q}_{n-1}, \mathbf{X})$ is given by

$$\nabla f(\mathbf{Q}_{n-1}, \mathbf{X}) = \left(\mathbf{H}^{\dagger}(\mathbf{Q}_{n-1} + \mathbf{H}\mathbf{X}\mathbf{H}^{\dagger})^{-1}\mathbf{H}\right) - \left(\mathbf{H}_{e}^{\dagger}(\mathbf{I} + \mathbf{H}_{e}\mathbf{X}\mathbf{H}_{e}^{\dagger})^{-1}\mathbf{H}_{e}\right).$$
(16)

For a given point $\bar{\mathbf{X}}$, the projection $p_{\mathcal{X}}(\bar{\mathbf{X}})$ is mathematically stated as

$$\underset{\mathbf{X} \succeq \mathbf{0}}{\operatorname{maximize}} \left\{ \left\| \mathbf{X} - \bar{\mathbf{X}} \right\| \mid \operatorname{tr}(\mathbf{X}) = P_0 \right\}$$
(17) which admits a closed-form solution as

$$\mathbf{X} = \mathbf{U} \operatorname{diag} \left(\left[\left[\bar{\boldsymbol{\sigma}} \right]_{\perp} - \tau \right]_{\perp} \right) \mathbf{U}^{\dagger}$$
(18)

where $\bar{\mathbf{X}} = \mathbf{U} \operatorname{diag}(\bar{\sigma}) \mathbf{U}^{\dagger}$ be the eigenvalue decomposition of $\bar{\mathbf{X}}$, $\bar{\sigma} = [\bar{\sigma}_1, \bar{\sigma}_2, \dots, \bar{\sigma}_r]$ where r is the rank of $\bar{\mathbf{X}}$, and τ is the unique number such that $\sum_{i=1}^r \max([\bar{\sigma}_i]_+ - \tau, 0) = P_0$.

The APGM for solving (12) is outlined in Algorithm 3. Note that a proper step size can be found by a backtracking line search as done in Lines (4)-(7). Starting from the step size of the previous iteration, the idea of the backtracking line search is to reduce it by a factor of θ until (7) is met. That is, we try to find a lower quadratic approximation of the objective at the current iterate. It is shown in [16] that 3 achieves the optimal convergence rate of $O(1/k^2)$.

2) **Q**-update: A closed-form solution is also possible for the **Q**-update. Specifically, we can partition Ψ_n into

$$\Psi_n = \begin{bmatrix} \Psi_{n,11} & \Psi_{n,12} \\ \Psi_{n,12}^H & \Psi_{n,22} \end{bmatrix}.$$
(19)

To lighten the notation, we will drop the subscript *n* onwards. (14) Now, let $\Psi_{12}\Psi_{12}^{\dagger} = \mathbf{U}_{\Psi}\bar{\Sigma}_{\Psi}\mathbf{U}_{\Psi}^{\dagger}$ be the eigenvalue decomAlgorithm 3 Accelerated projected gradient method for solving (12)

1: Input:
$$\mathbf{Y}^{(1)} = \mathbf{X}^{(0)} = \mathbf{X}_{n-1}, \eta_0 > 0, \theta > 1, \xi_1 = 1.$$

2: for $k = 1, 2, ...$ do
3: $\beta = \eta_{k-1}/\theta$
4: repeat
5: $\beta \leftarrow \theta\beta$
6: $\mathbf{X}^{(k)} = p_{\mathcal{X}} \left(\mathbf{Y}^{(k)} + \frac{1}{\beta} \nabla f \left(\mathbf{Q}_{n-1}, \mathbf{Y}^{(k)} \right) \right)$
7: until $f(\mathbf{Q}_{n-1}, \mathbf{X}^{(k)}) \ge f(\mathbf{Q}_{n-1}, \mathbf{Y}^{(k)}) + \langle \nabla f(\mathbf{Q}_{n-1}, \mathbf{Y}^{(k)}), (\mathbf{X}^{(k)} - \mathbf{Y}^{(k)}) \rangle - \frac{\beta}{2} \| \mathbf{X}^{(k)} - \mathbf{Y}^{(k)} \|^2$
8: $\xi_{k+1} = 0.5(1 + \sqrt{1 + 4\xi_k^2}); \eta_k = \beta$
9: $\mathbf{Y}^{(k+1)} = \mathbf{X}^{(k)} + \frac{\xi_{k-1}}{\xi_{k+1}} \left(\mathbf{X}^{(k)} - \mathbf{X}^{(k-1)} \right)$
10: end for

position of $\Psi_{12}\Psi_{12}^{\dagger}$ and $\bar{\Sigma}_{\Psi} = \text{diag}(\sigma_{\Psi_1}, \sigma_{\Psi_2}, \dots, \sigma_{\Psi_{N_r}})$. Then the optimal solution to (14) is given by

$$\bar{\mathbf{Q}}_{n+1} = -\mathbf{U}_{\Psi} \Xi \mathbf{U}_{\Psi}^{\dagger} \Psi_{12}$$
(20)

where

$$\Xi_{\Psi} = 2 \operatorname{diag}\left(\frac{1}{1+\sqrt{1+4\sigma_{\Psi_1}}}, \frac{1}{1+\sqrt{1+4\sigma_{\Psi_2}}}, \dots, \frac{1}{1+\sqrt{1+4\sigma_{\Psi_{N_r}}}}\right).$$
(21)

We refer the reader to [12] for the proof of (20).

The main idea behind the convergence proof of Algorithm 2 is show the monotonic decrease of the objective sequence $f(\mathbf{Q}_n, \mathbf{X}_n)$, which is due to the fact that the term $\log |\mathbf{Q} + \mathbf{H}\mathbf{X}\mathbf{H}^{\dagger}| - \log |\mathbf{I} + \mathbf{H}_e\mathbf{X}\mathbf{H}_e^{\dagger}|$ is jointly concave with \mathbf{Q} and \mathbf{X} . We refer the interested reader to [12] for further details.

IV. NUMERICAL RESULTS

In this section we provide numerical results to evaluate the proposed algorithms. We adopt the Kronecker model in our numerical investigation [17]. Specifically, the channel between Alice and Bob \mathbf{H}_b is modeled as $\mathbf{H}_b = \tilde{\mathbf{H}}_b \mathbf{R}_b^{1/2}$, where $\tilde{\mathbf{H}}_b$ is a matrix of i.i.d. complex Gaussian distribution with zero mean and unit variance and $\mathbf{R}_{h}^{1/2}$ the corresponding a transmit correlation matrix. Here we adopt the exponential correlation model whereby $[\mathbf{R}_b]_{i,j} = (re^{j\phi_b})^{|i-j|}$ for a given $r \in [0,1]$ and $\phi_b \in [0,2\pi)$. The channel between Alice and Eve is modeled as $\mathbf{H}_e = \gamma \tilde{\mathbf{H}}_e \mathbf{R}_e^{1/2}$ for a given $\gamma > 0$ and $\tilde{\mathbf{H}}_e$ and \mathbf{R}_e are generated in the same way. The purpose of introducing γ is to study the secrecy capacity of the MIMO WTC with respect to the relative average strength of H_b and H_e . For the simulation purpose we use $\phi_e = \pi/2$, $\gamma = 0.9$ and r = 0.9. The codes of all algorithms in comparison were written in MATLAB and executed in a 64-bit Windows PC with 16GB RAM and Intel Core i7, 3.20 GHz. Note that since the noise power is normalized to unity and thus P_0 is defined to be the signal to noise ratio (SNR) in this section. In all simulations results, the parameter q for Algorithm 1 is taken as q = 5.

In Fig. 1 we show the convergence results of proposed algorithms over two different SNRs 5 dB and 10 dB for a set of randomly generated channel where $(N_t, N_r, N_e) = (4, 3, 4)$. For Algorithm 1 we plot the secrecy rate $C_s(\mathbf{X}_n)$ where \mathbf{X}_n



Figure 1. Convergence results of iterative algorithms for different SNRs

Table I COMPARISON OF RUN-TIME (IN MILLISECONDS) BETWEEN THE PROPOSED METHODS AND [11, ALG. 2].

	$(N_t, N_r, N_e) = (4, 3, 2)$		$(N_t, N_r, N_e) = (4, 6, 8)$	
Algorithm	5dB	10dB	5dB	10dB
Algorithm 1	14.6	17.2	26.8	41.5
Algorithm 2	8.8	11.2	35.4	44.5
[11, Alg. 2]	52.5	55.8	217.1	371.0

is the solution returned at the *n*th iteration. For Algorithm 2 we plot the objective $f(\mathbf{Q}_n, \mathbf{X}_n)$ in (4). We also plot the convergence of the outer loop of [11, Algorithm 2] for comparison. It is clearly seen that the proposed algorithms converge very fast and all algorithms converge to the same objective. For Algorithm 2, we can also see that $f(\mathbf{Q}_n, \mathbf{X}_n)$ is indeed an upper bound of $C_s(\mathbf{X}_n)$ and it keeps decreasing until convergence as expected. We remark that while Algorithm 1 is developed based on a local optimization method, our extensive numerical results show that Algorithm 1 always achieves the same solution as Algorithm 2 which is optimal.

In Fig. 1 it also appears that all algorithms in comparison achieve similar convergence rate performance in terms of the required number of iterations. However, the complexity per iteration of each algorithm is different. To achieve a more meaningful comparison, we present their *average actual run time* in Table I. For this purpose, the stopping criterion of all algorithms is when the corresponding objective is not improved during the last 5 iterations. The average run time in Table I is obtained from 1000 random channel realizations. We can see that the proposed algorithms, i.e. Algorithms 1 and 2, outperform [11, Algorithm 2]; and 2 is slightly better than Algorithm 1 when $N_t > N_e$ and vice versa when $N_t < N_e$.

We now study how the secrecy capacity scales with the number of transmit antennas at Bob and Alice. Fig. 2 plots the average secrecy capacity for various numbers of antennas at Eve. The number of transmit antennas at Alice is $N_r = 4$. As can be seen in Fig. 2, the secrecy capacity increases with the number of receive antennas at Alice, which is expected.



Figure 2. Secrecy capacity as a function of N_r for different values of N_e . The number of transmit antennas is $N_t = 4$.



Figure 3. Impact of $N_e{\rm and}$ SNR on the secrecy capacity and asymptotic capacity at $N_t=6,\,N_r=4$

Simultaneously, we also observe that the secrecy capacity is reduced when the number of antennas at Eve increases. In particular, Eve can significantly decrease the secrecy capacity when N_e is much larger than N_t . This is because the null space of \mathbf{H}_b will increasingly intersect with the space spanned by \mathbf{H}_e .

Finally, Figure 3 plots the average secrecy capacity as a function of SNR for different numbers of antennas at Eve. The purpose is to understand the gap between the true secrecy capacity and the asymptotic capacity obtained in [5]. As expected, the true secrecy capacity converges to the asymptotic capacity when the SNR is sufficiently high. Again, we can observe the secrecy capacity decreases when the number of receive antennas at Eve increases.

V. CONCLUSION

In this paper, we have proposed two efficient numerical methods for computing the secrecy capacity and the optimal signaling of MIMO WTC. In the first method, the secrecy capacity problem is viewed as a DC program and we have applied an accelerated version of the celebrated DCA, referred to as the ADCA. In the second method, we have drawn on the convex-concave reformulation of the secrecy capacity problem and developed the PBRA in which each iteration is done in closed form. Numerical results have been provided to demonstrate that the proposed solutions can reduce the run time of a known solution by 5 times for the considered scenarios. Moreover, through extensive numerical experiments, we have observed that the ADCA, albeit inherently a local optimization method, always achieve the optimal solution. Our conjecture is that the proposed ADCA is indeed a global optimization method, the proof of which is left for future work.

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