A Frame-Theoretic Scheme for Robust Millimeter Wave Channel Estimation

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Abstract—We propose a new scheme for the robust estimation of the millimeter wave (mmWave) channel. Our approach is based on a sparse formulation of the channel estimation problem coupled with a frame theoretic representation of the sensing dictionary. To clarify, under this approach, the combined effect of transmit precoders and receive beamformers is modeled by a single frame, whose design is optimized to improve the accuracy of the sparse reconstruction problem to which the channel estimation problem is ultimately reduced. The optimized sensing dictionary frame is then decomposed via a Kronecker decomposition back into the precoding and beamforming vectors used by the transmitter and receiver. Simulation results illustrate the significant gain in estimation accuracy obtained over state of the art alternatives. As a bonus, the work offers new insights onto the sparse mmWave-multiple-input multiple-output (MIMO) channel estimation problem by casting the trade-off between correlation and variation range in terms of frame coherence and tightness.

Index Terms—mmWave channel estimation, Compressed Sensing, complex incoherent tight frames, basis pursuit denoising.

I. INTRODUCTION

The increasing demands in terms of higher rate, more access and lower latency at the physical link, coupled with the lack of spectral resources in conventional cellular systems is recently strongly motivating the development of millimeter wave (mmWave) communications [1].

In principle, the larger bandwidths and shorter wave lengths of mmWave systems enable the provision of higher communication rates [2], and respectively, the equipping of larger antennas arrays at transmitters and receivers favors the utilization of multiple-input multiple-output (MIMO) architectures [3], [4].

In practice, however, hardware costs and other implementation issues challenge the realization of mmWave systems, which therefore must be counter-acted via dedicatedly designed signal processing methods [5], [6].

In turn, previous work has demonstrated [7]–[12] that the efficacy of signal processing in ameliorating the hardware challenges of mmWave systems depends highly on the quality of *channel state information (CSI)*. In fact, although hybrid precoding with *partial CSI* has been well studied [13], the substantial performance losses resulting from imperfect/partial CSI only further motivate the quest for better methods for channel acquisition [7]–[12].

Retrieving complete and accurate mmWave CSI is challenging in practice due to the rapid variation and severe path-loss experienced under the high operating frequencies. In answer to this challenge, a sparse formulation of the mmWave MIMO channel estimation problem was proposed in [7] which allowed the use of Compressed Sensing (CS) [14] for the scant channel recovery problem [10], which posteriorly was improved by the introduction of a greedy orthogonal matching pursuit (OMP) recovery algorithm [15].

Recognizing that the efficacy of OMP in noisy systems is limited, as the method fails to exactly fit linear systems, an alternative solver to mitigate this problem has been proposed in our previous work [16] in which the basis pursuit denoising (BPDN) [17] has been leveraged as a more efficient solution. Furthermore, in [16] the sparsity of the problem was enhanced through a reweighted ℓ_1 -minimization formulation [18], which led to an efficient iterative BPDN $-\ell_1$ sparse recovery.

In this paper, we continue this trend and further contribute with a technique for joint channel estimation and training beamformer optimization. The generic optimization of training vectors is performed based on Frame Theory and its applicability to sparse recovery, [19, Ch.9], [20]. In turn, the measurement matrix selection problem is solved by a decoupled, flexible low-coherence tight frame design, with increased robustness compared to conventional random or optimized training vectors [21].

In the remainder of the paper the following notation is used:

- X, x and x represent a matrix, a vector, and a scalar;
- $\|\mathbf{X}\|_{F}, \|\mathbf{x}\|_{2}$ and $\|\mathbf{x}\|_{\infty}$ denote the Frobenius, Euclidean and ∞ -norms;
- **X**^T, **X**^H and **X**^{*} denote the transpose, complex conjugate transpose and conjugate of matrix **X**;
- $\mathbf{X} \otimes \mathbf{Y}$ is the Kronecker product of \mathbf{X} and \mathbf{Y} ;
- $diag(\mathbf{x})$ denotes the diagonal matrix with diagonal \mathbf{x} ;
- vec(X) is a column vector with all columns of X stacked;
- \mathbf{I}_N and $\mathbf{0}_N$ denote the *N*-sized identity and null matrices.

II. PROBLEM FORMULATION

A. Millimeter Wave Channel Estimation & Compressive Sensing

A downlink MIMO mmWave system formed of a base station (BS) with T transmit antennas and an user equipment (UE) with R receive antennas is considered. It is also assumed that the BS uses M_T training beamforming vectors to transmit a known training signal S, while the UE applies M_R combining vectors for each beamforming one in order to estimate the channel $\mathbf{H} \in \mathbb{C}^{R \times T}$. It follows that the receive signal matrix at the UE, denoted by $\mathbf{Y} \in \mathbb{C}^{M_{\mathsf{R}} \times M_{\mathsf{T}}}$, is given by

$$\mathbf{Y} = \mathbf{V}^{\mathrm{H}} \mathbf{H} \mathbf{U} \mathbf{S} + \mathbf{N},\tag{1}$$

where the precoding and combining (TX/RX beamforming) matrices are given by $\mathbf{U} \triangleq [\mathbf{u}_1, \cdots, \mathbf{u}_{M_T}] \in \mathbb{C}^{T \times M_T}$ and $\mathbf{V} \triangleq [\mathbf{v}_1, \cdots, \mathbf{v}_{M_R}] \in \mathbb{C}^{R \times M_R}$, respectively, $\mathbf{N} \in \mathbb{C}^{M_R \times M_T}$ denotes circularly symmetric complex additive white Gaussian noise (AWGN), and **H** is the mmWave channel matrix.

Following the usual sparse multipath scatter channel model [3], [7]-[12], we may rewrite **H** as

$$\mathbf{H} = \sqrt{\frac{TR}{L}} \sum_{l=1}^{L} \gamma_l \mathbf{a}_{\mathsf{r}}(\phi_l^{\mathsf{r}}) \mathbf{a}_{\mathsf{t}}^{\mathsf{H}}(\phi_l^{\mathsf{t}}), \qquad (2)$$

where *L* is the number of propagation paths, $\gamma_l \sim C\mathcal{N}(0, \sigma_{\gamma}^2)$ is the complex gain of the *l*-th path, and $\mathbf{a}_r(\phi_l^r)$ and $\mathbf{a}_t(\phi_l^r)$ are the array response vectors respectively at the receiver and transmitter, with corresponding angles of arrival (AoA) and angles of departure (AoD) denoted by ϕ_l^r , $\phi_l^t \in [0, 2\pi]$.

The channel matrix described by equation (2) can also be more compactly expressed as

$$\mathbf{H} = \mathbf{A}_{\mathbf{R}} \mathbf{H}_{\gamma} \mathbf{A}_{\mathbf{T}}^{\mathrm{H}},\tag{3}$$

with $\mathbf{A}_{R} \triangleq [\mathbf{a}_{r}(\phi_{1}^{r}), \cdots, \mathbf{a}_{r}(\phi_{L}^{r})], \mathbf{A}_{T} \triangleq [\mathbf{a}_{t}(\phi_{1}^{t}), \cdots, \mathbf{a}_{t}(\phi_{L}^{t})],$ and $\mathbf{H}_{\gamma} \triangleq \sqrt{\frac{TR}{L}} \operatorname{diag}(\gamma_{1}, \cdots, \gamma_{L}).$

For the sake of simplicity, identity signaling is assumed hereafter, such that the training transmit symbol matrix is given by $\mathbf{S} = \mathbf{I}_{M_{\rm T}}$, which in turn implies that equation (1) can be rewritten in a vectorized form as

$$\mathbf{y} \triangleq \operatorname{vec}(\mathbf{Y}) = (\mathbf{U}^{\mathrm{T}} \otimes \mathbf{V}^{\mathrm{H}})(\mathbf{A}_{\mathrm{T}}^{*} \otimes \mathbf{A}_{\mathrm{R}})\operatorname{vec}(\mathbf{H}_{\gamma}) + \operatorname{vec}(\mathbf{N}), \quad (4)$$

where $\mathbf{y} \in \mathbb{C}^{M_{\mathsf{R}}M_{\mathsf{T}} \times 1}$.

A sparse characterization of equation (4) can be obtained as follows [7], [8]. First, consider expanded versions of the scatter matrices \mathbf{A}_{R} and \mathbf{A}_{T} defined by $\hat{\mathbf{A}}_{R} \triangleq [\mathbf{a}_{r}(\theta_{0}), \cdots, \mathbf{a}_{r}(\theta_{G_{R}-1})]$ and $\hat{\mathbf{A}}_{T} \triangleq [\mathbf{a}_{t}(\theta_{0}), \cdots, \mathbf{a}_{t}(\theta_{G_{T}-1})]$ where the AoDs and AoAs $\theta_{g_{t}}$ and $\theta_{g_{t}}$ lay on a sufficiently fine quantization grid, *i.e.*

$$\theta_{g_{\rm t}} \triangleq \frac{2\pi g_{\rm t}}{G_{\rm T}}, \quad \text{and} \quad \theta_{g_{\rm r}} \triangleq \frac{2\pi g_{\rm r}}{G_{\rm R}},$$
(5)

with $g_t = \{0, \dots, G_T - 1\}, g_r = \{0, \dots, G_R - 1\}$, and respectively, $(G_T, G_R) \gg L.$

Next, expand also \mathbf{H}_{γ} into a sparse matrix $\hat{\mathbf{H}}_{\gamma}$, whose only non-zero entries are the *L* elements satisfying

$$[\hat{\mathbf{H}}_{\gamma}]_{i,j} = \gamma_{\ell} \quad \Longleftrightarrow \quad \begin{cases} \|\theta_i - \phi_{\ell}^{\mathrm{r}}\|_2 < \|\theta_{g_i \neq i} - \phi_{\ell}^{\mathrm{r}}\|_2, \\ \|\theta_j - \phi_{\ell}^{\mathrm{t}}\|_2 < \|\theta_{g_i \neq i} - \phi_{\ell}^{\mathrm{t}}\|_2, \end{cases}$$
(6)

for every $\ell = \{1, \cdots, L\}$.

And finally, obtain [16]

$$\mathbf{y} = \underbrace{(\mathbf{U}^{\mathrm{T}} \otimes \mathbf{V}^{\mathrm{H}})}_{\boldsymbol{\Phi}} \underbrace{(\hat{\mathbf{A}}_{\mathrm{T}}^{*} \otimes \hat{\mathbf{A}}_{\mathrm{R}})}_{\boldsymbol{\Psi}} \underbrace{\operatorname{vec}(\hat{\mathbf{H}}_{\gamma})}_{\mathbf{x}} + \underbrace{\operatorname{vec}(\mathbf{N})}_{\mathbf{n}}, \quad (7)$$

where the measurement matrix $\mathbf{\Phi} \triangleq (\mathbf{U}^T \otimes \mathbf{V}^H) \in \mathbb{C}^{M_T M_R \times TR}$, the sparse dictionary $\mathbf{\Psi} \triangleq (\hat{\mathbf{A}}_T^* \otimes \hat{\mathbf{A}}_R) \in \mathbb{C}^{TR \times G_T G_R}$, the *L*-sparse vector $\mathbf{x} \triangleq \operatorname{vec}(\hat{\mathbf{H}}_{\gamma}) \in \mathbb{C}^{G_T G_R \times 1}$ and the noise vector \mathbf{n} have been implicitly defined.

B. Previous Contributions

Under the assumption that the AoA and AoD angles ϕ_l^r and ϕ_l^t are known¹, and in light of the model expressed by equation (2), the mmWave channel estimation problem amounts to estimating the complex gains $\{\gamma_1, \dots, \gamma_L\}$. And under the further assumption that the precoding and combining matrices U and V are given, the vectorized expression of equation (7) enables the mmWave channel estimation problem to be solved as the sparse recovery CS optimization problem:

minimize
$$\|\mathbf{x}\|_{\ell_0}$$
, (8a)

subject to
$$\mathbf{y} = \underbrace{\mathbf{\Phi} \Psi}_{\Omega} \mathbf{x} + \mathbf{n},$$
 (8b)

where we have explicitly identified the *equivalent sensing* matrix $\mathbf{\Omega} \in \mathbb{C}^{M_{\mathrm{T}}M_{\mathrm{T}} \times G_{\mathrm{T}}G_{\mathrm{R}}}$ for future convenience.

The problem formulated above can be solved via the OMP [15] algorithm, *e.g.* as proposed in [7] and [10]. More recently, we have shown in [16] that the latter can be enhanced by relaxing the problem (8) to the associated ℓ_1 -norm formulation

$$\begin{array}{ll}
\text{minimize} & \|\mathbf{x}\|_{\ell_1}, \\
\end{array} \tag{9a}$$

subject to
$$\|\mathbf{y} - \mathbf{\Omega}\mathbf{x}\|_{\ell_2} \le \delta,$$
 (9b)

which can then be solved via BPDN, thus mitigating the noisy recovery limitations encountered by classical OMP.

In fact, the problem, can be even more accurately solved if the BPDN solver is further combined with the sparsityenhancing iterative ℓ_1 -reweighing scheme of [18], as also shown in [16, Alg. 1].

Another approach to further improve the performance of mmWave channel estimation that received comparatively less attention so far is to optimize the sensing matrix Ω , given a certain discrete angle dictionary Ψ , which is usually fixed by means of hardware/processing requirements.

Deriving a method to optimize Ω given Ψ , which in turn reduces to optimizing Φ , is the objective and the main contribution of this article, and the focus of the next section.

III. FRAME-THEORETICAL DESIGN OF PRECODING & BEAMFORMING MATRICES

CS is a direct application of a larger framework of linear projections, namely Frame Theory [19, Ch. 9], [22, Sect. 7.2]. In a general sense, a *frame* is defined as a set of N vectors $\mathbf{F} \triangleq [\mathbf{f}_1, \mathbf{f}_2, \dots, \mathbf{f}_N]$ over a Hilbert space \mathbb{H}^M (reduced to \mathbb{C}^M in the current setup) with M < N and

$$\alpha \|\mathbf{v}\|_2^2 \le \|\mathbf{F}^{\mathsf{H}}\mathbf{v}\|_2^2 \le \beta \|\mathbf{v}\|_2^2, \tag{10}$$

where (α, β) , $0 < \alpha \le \beta < \infty$, are the finite highest lower and lowest higher frame bounds, respectively [19].

A frame is *tight* iff $\alpha = \beta$, and *unit-norm* iff $||\mathbf{f}_i||_2 = 1$, $\forall i$. Unit-norm tight frames (UNTF) have both these properties so that $\alpha = \beta = \frac{N}{M} \triangleq \rho(\mathbf{F})$, where $\rho(\mathbf{F})$ is known as the redundancy of the frame [19, Ch. 1].

¹As literature on AoA estimation is vast we refrain from further discussion.

Similar to the restricted isometry property (RIP) in CS, the bounding property expressed by equation (10) offers a measure of how close a frame is to an orthogonal basis with respect to any projected vector in the spanned space. But another measure of the frame's similarity to an orthonormal basis is its *mutual coherence*, defined (for a unit-norm frame) as

$$\mu(\mathbf{F}) \triangleq \max_{i \neq j} \|\mathbf{f}_i^H \mathbf{f}_j\|_2 = \max_{\mathbf{G} \triangleq \mathbf{F}^H \mathbf{F}, i \neq j} |g_{ij}| \ge \sqrt{\frac{N-M}{M(N-1)}}, \quad (11)$$

where G is known as the *Gram operator*, and the lower bound on the right-hand side is the *Welch bound* [23] for $N \leq M^2$.

The performance of pursuit algorithms can be studied under concepts like coherence [24] and the restricted isometry property RIP [25]. As a result, there are two general requirements on the design of the measurement matrix Φ :

- 1) Φ must be highly incoherent in order to preserve the salient information of sparse vectors;
- 2) Ω must satisfy the RIP in order to afford robustness to the reconstruction.

We may remark at this stage that the dictionary Ψ in equation (7) can in fact be identified as a *harmonic frame*, sampled out of the discrete Fourier matrix of size $G_T \times G_R$, so that Ψ is an UNTF by construction [22]. Also interesting to notice is the fact that this frame admits a natural Kronecker-decomposable form $\Psi \triangleq (\hat{\mathbf{A}}_T^* \otimes \hat{\mathbf{A}}_R)$, as seen before.

In light of all the above, our goal is to design the optimized measurement matrix Φ as a *Kronecker-decomposable*, *normalized tight frame* with *low-coherence* and with a *RIP-compliant* associated sensing matrix $\Omega = \Phi \Psi$, which is addressed in the sequel.

A.QC-SIDCO: Measurement Matrix as a Low-coherence Frame

A low-coherence frame can be generated from a given unitnorm frame $\tilde{\mathbf{F}} \in \mathbb{C}^{M \times N}$ by iteratively decorrelating its vectors while constraining the feasibility region to an *M*-ball. This scheme, referred to as sequential iterative decorrelation via convex optimization (SIDCO), was originally introduced in [20] only for frames in \mathbb{R}^M . More recently, a strategy to generalize SIDCO to frames in \mathbb{C}^M , referred to as complex SIDCO (C-SIDCO), was reported by [26]. An explicit and complete mathematical formulation of C-SIDCO was, however, not given in [26]. A variation of the latter based on an explicit quadratic program is offered below.

Consider an existent unit-norm frame $\tilde{\mathbf{F}} \in \mathbb{C}^{M \times N}$. The strategy of C-SIDCO [26] is to minimize mutual coherence by iteratively solving the problem

$$\underset{\mathbf{f}_i \in \mathbb{C}^M}{\text{minimize}} \quad \|\mathbf{\tilde{F}}_i^{\text{H}} \mathbf{f}_i\|_{\infty},$$
 (12a)

subject to
$$\|\mathbf{f}_i - \tilde{\mathbf{f}}_i\|_2^2 \le T_i,$$
 (12b)

for all *i* vectors, where $\tilde{\mathbf{F}}_i$ denotes the $\tilde{\mathbf{f}}_i$ -pruned existent frame and the search *M*-ball radius of vector \mathbf{f}_i is given by

$$T_i \le 1 - \max_{j;j \ne i} |g_{ij}|^2,$$
 (13)

so that T_i is constrained to the largest *M*-ball such that the prospective solution \mathbf{f}_i cannot be collinear with other $\tilde{\mathbf{f}}_j, j \neq i$.

In order to circumvent the additional challenge of optimizing in the complex domain, the space \mathbb{C}^M is reinterpreted as \mathbb{R}^{2M} , with interlaced real and imaginary components.

The generically formulated C-SIDCO approach described by equation (12) can be explicitly reformulated as the quadratic program

$$\min_{\mathbf{x} \triangleq [\mathbf{f}_i; t_{\mathcal{R}}; t_{\mathcal{I}}] \in \mathbb{R}^{2M+2} } \mathbf{x}^{\mathrm{T}} \mathbf{Q} \mathbf{x},$$
 (14a)

subject to $\mathbf{A}_{\mathcal{R},1}\mathbf{x} \leq 0, \ \mathbf{A}_{\mathcal{R},2}\mathbf{x} \leq 0,$ (14b)

$$\mathbf{A}_{\mathcal{I},1}\mathbf{x} \le 0, \ \mathbf{A}_{\mathcal{I},2}\mathbf{x} \le 0, \tag{14c}$$

$$\mathbf{x}^{\mathrm{T}}\mathbf{B}\mathbf{x} - 2\mathbf{b}^{\mathrm{T}}\mathbf{x} + 1 - T_i \le 0, \quad (14\mathrm{d})$$

where

$$\mathbf{Q} \triangleq \begin{bmatrix} \mathbf{0}_{2M} & \mathbf{0}_{2M\times 2} \\ \mathbf{0}_{2\times 2M} & \mathbf{I}_2 \end{bmatrix} \in \mathbb{R}^{(2M+2)\times(2M+2)}, \quad (14e)$$

$$\mathbf{A}_{\mathcal{R},1} \triangleq \begin{bmatrix} \tilde{\mathbf{F}}_{i}^{\mathrm{T}} & -\mathbf{1}_{(N-1)\times 1} & \mathbf{0}_{(n-1)\times 1} \end{bmatrix} \in \mathbb{R}^{(N-1)\times (2M+2)},$$
(14f)

$$\mathbf{A}_{\mathcal{R},2} \triangleq \begin{bmatrix} -\tilde{\mathbf{F}}_{i}^{\mathrm{T}} & -\mathbf{1}_{(N-1)\times 1} & \mathbf{0}_{(n-1)\times 1} \end{bmatrix} \in \mathbb{R}^{(N-1)\times(2M+2)},$$
(149)

$$\mathbf{A}_{\mathcal{I},1} \triangleq \begin{bmatrix} \tilde{\mathbf{F}}_i^{\mathrm{T}} \mathbf{D}_M & \mathbf{0}_{(N-1)\times 1} & -\mathbf{1}_{(n-1)\times 1} \end{bmatrix} \in \mathbb{R}^{(N-1)\times(2M+2)},$$
(14b)

$$\mathbf{A}_{\mathcal{I},2} \triangleq \begin{bmatrix} -\tilde{\mathbf{F}}_i^{\mathsf{T}} \mathbf{D}_M & \mathbf{0}_{(N-1)\times 1} & -\mathbf{1}_{(n-1)\times 1} \end{bmatrix} \in \mathbb{R}^{(N-1)\times(2M+2)},$$
(14i)

$$\mathbf{D}_{M} \triangleq \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \otimes \mathbf{I}_{M} \in \mathbb{R}^{(2M+2) \times (2M+2)}$$
(14j)

$$\mathbf{B} \triangleq \begin{bmatrix} \mathbf{I}_{2M} & \mathbf{0}_{2M \times 2} \\ \mathbf{0}_{2 \times 2M} & \mathbf{0}_{2 \times 2} \end{bmatrix} \in \mathbb{R}^{(2M+2) \times (2M+2)}, \quad (14k)$$

$$\mathbf{b}^{\mathrm{T}} \triangleq \begin{bmatrix} \tilde{\mathbf{f}}_i^{\mathrm{T}} & 0 & 0 \end{bmatrix} \in \mathbb{R}^{2M+2}.$$
(141)

We remark that thanks to the replacement of the linear program adopted in [20, Alg. 1] with the quadratic program given by equation (14), and the direct interlacing from \mathbb{C}^M to \mathbb{R}^{2M} achieved in our formulation via the matrices $\mathbf{D}_M, \mathbf{A}_{\mathcal{R},1}, \mathbf{A}_{\mathcal{R},2}, \mathbf{A}_{\mathcal{I},1}$ and $\mathbf{A}_{\mathcal{A},2}$, C-SIDCO becomes here a simple extension of the original SIDCO algorithm, which converges (absolutely) to the local minimum coherence points since the original concept of *M*-ball packings is preserved.

For all the above, the explicit quadratic reformulation of C-SIDCO offered above is *original*, and can be directly coded on top of optimized standard quadratic solvers, unlike the formulation presented in [26].

We therefore refer to this realization of SIDCO as the quadratic complex SIDCO (QC-SIDCO) algorithm.

B. Beamformers: Decomposition of QC-SIDCO Φ

The frame obtained by the method described above is not strictly tight, unless an equiangular tight frame (ETF) is reached², which rarely happens in practice, since ETFs exist only for particular dimensions.

²ETFs are the closest frame analogies to orthogonal bases, attaining tightness and uniform lowest coherence, and therefore also the Welch bound [23].

Fortunately, tightness of the frame **F** constructed via the QC-SIDCO frame method can be posteriorly enforced by applying polar decomposition [27, Th. 2], which yields the UNTF closest, in Frobenius-norm sense, QC-SIDCO frame. For details we refer the reader to [27].

Finally, in order to uniformly distribute the sensing cost of Ω , the polar-decomposed QC-SIDCO-designed measurement matrix Φ is obtained by normalizing the latter frame \mathbf{F}^* , *i.e.*

$$\mathbf{\Phi} = \frac{\sqrt{TR} \mathbf{F}^*}{\|\mathbf{F}^*\|_{\mathrm{F}}}.$$
(15)

Returning to our original problem of mmWave channel estimation, however, we remark that the measurement matrix obtained as explained above still needs to be decomposed into precoding and combining beamforming matrices U and V in order for the channel estimation method to be practically implementable. This can be achieved by solving the problem

$$\underset{\mathbf{U}\in\mathbb{C}^{T\times M_{\mathrm{T}}},\mathbf{V}\in\mathbb{C}^{R\times M_{\mathrm{R}}}}{\mathrm{minimize}} \quad \|\boldsymbol{\Phi}-(\mathbf{U}^{\mathrm{T}}\otimes\mathbf{V}^{\mathrm{H}})\|_{\mathrm{F}}, \tag{16}$$

which reformulated by vectorization becomes

$$\underset{\mathbf{U}\in\mathbb{C}^{T\times M_{\mathrm{T}}},\mathbf{V}\in\mathbb{C}^{R\times M_{\mathrm{R}}}}{\mathrm{minimize}} \quad \|\boldsymbol{\varPhi}-\left(\mathrm{vec}(\mathbf{U}^{\mathrm{T}})\mathrm{vec}(\mathbf{V}^{\mathrm{H}})^{\mathrm{T}}\right)\|_{\mathrm{F}}, \quad (17)$$

where

$$\boldsymbol{\Phi}^{\mathrm{T}} \triangleq (18)$$

$$\left[\operatorname{vec}(\boldsymbol{\Phi}_{11}), \cdots, \operatorname{vec}(\boldsymbol{\Phi}_{M_{\mathrm{T}}1}), \cdots, \operatorname{vec}(\boldsymbol{\Phi}_{1T}), \cdots, \operatorname{vec}(\boldsymbol{\Phi}_{M_{\mathrm{T}}T})\right],$$

with

$$\boldsymbol{\Phi} = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{1T} & \Phi_{1T} \\ \Phi_{21} & \Phi_{22} & \Phi_{2T} & \Phi_{2T} \\ \Phi_{21} & \Phi_{22} & \Phi_{2T} & \Phi_{2T} \\ \Phi_{2T} & \Phi_{2T} & \Phi_{2T} & \Phi_{2T} \end{bmatrix} .$$
(19)

Notice that the matrix $\boldsymbol{\Phi}$ defined above is a rectangular matrix, such that as a result of this reshaping, the solution of equation (17) can be easily obtained via singular value decomposition (SVD), yielding

$$\operatorname{vec}(\mathbf{U}^{\mathrm{T}}) = \sqrt{\sigma_{\boldsymbol{\varPhi}}} \mathbf{u}_{\boldsymbol{\varPhi}} \quad \text{and} \quad \operatorname{vec}(\mathbf{V}^{\mathrm{H}}) = \sqrt{\sigma_{\boldsymbol{\varPhi}}} \mathbf{v}_{\boldsymbol{\varPhi}}, \quad (20)$$

where $\mathbf{u}_{\boldsymbol{\Phi}}$ and $\mathbf{v}_{\boldsymbol{\Phi}}$ are the dominant left and right singular vectors, and $\sigma_{\boldsymbol{\Phi}}$ the dominant singular value of $\boldsymbol{\Phi}$.

With knowledge of the vectorized forms $vec(\mathbf{U}^T)$, $vec(\mathbf{V}^T)$ as in equation (20), the precoding and combining matrices \mathbf{U} and \mathbf{V} are finally obtained such that $\mathbf{\Phi} \triangleq (\mathbf{U}^T \otimes \mathbf{V}^H)$, as desired for a practical implementation.

Given all the above, the proposed measurement matrix construction for application in the mmWave channel estimation problems described by equations (8) and (9) can therefore be summarized as follows:

- 1) Generate a unit-norm low-coherence frame **F** through the QC-SIDCO scheme described by equation (14);
- 2) Apply SVD and polar decomposition to **F**, [27, Th. 2], thus obtaining an UNTF with low-coherence **F**;
- 3) Obtain the ideal measurement matrix Φ from eq. (15);
- 4) Decompose Φ using equation (20), and reshape $\operatorname{vec}(\mathbf{U}^{\mathrm{T}}) \to \mathbf{U}$ and $\operatorname{vec}(\mathbf{V}^{\mathrm{H}}) \to \mathbf{V}$ accordingly for suboptimal but practical realizations.

IV. RESULTS AND ANALYSIS

In this section, the performance of the proposed mmWave channel estimation method employing the beamformers obtained as described above is evaluated numerically.

The simulation scenario is as follows. Both the transmitter and the receiver are assumed to have the same number of antennas T = R = 8, the number of training beamforming vectors are such that $M_T M_R = 16$, and a sparse channel model as described by equation (2) with L = 3 and $\sigma_{\gamma}^2 = 1$ was considered, with the BS/UE antenna subsystems assumed to be uniformly spaced linear antenna arrays, such that

$$\mathbf{a}_{\mathbf{r}}(\phi_{l}^{\mathbf{r}}) = \frac{1}{\sqrt{R}} \left[1, e^{j\frac{2\pi}{\lambda}d\sin(\phi_{l}^{\mathbf{r}})}, \cdots, e^{j\frac{2\pi}{\lambda}(R-1)d\sin(\phi_{l}^{\mathbf{r}})} \right]^{\mathsf{T}}, \quad (21)$$

$$\mathbf{a}_{t}(\phi_{l}^{t}) = \frac{1}{\sqrt{T}} \left[1, e^{j\frac{2\pi}{\lambda}d\sin(\phi_{l}^{t})}, \cdots, e^{j\frac{2\pi}{\lambda}(T-1)d\sin(\phi_{l}^{t})} \right]^{T}, \quad (22)$$

with an inter-antenna spacing d of half transmission wavelength λ , *i.e.* $d = \lambda/2$, and AoA/AoD uniformly and randomly distributed in the interval $[0, 2\pi]$.

Let us start our numerical evaluation of the proposed art by studying the impact of the proposed QC-SIDCO measurement matrix design. To this end, we first compare in Fig. 1 the *coherence profile* – defined as the distribution of inner-products $|g_{ij}| \triangleq ||\Phi_i^H \Phi_j||_2$ for all distinct column pairs (i, j) – corresponding to measurement matrices Φ obtained with the Parseval tight frame (PTF) construction approach of [21], against that achieved by the low-coherence quadratic complex SIDCO (QC-SIDCO) frames constructed as described in Section III. The empirical realizations have been fitted by families of Generalized Extreme Values (GEV), and respectively, Extreme Values (EV) distributions [28], *i.e* Fig. 1.

It can be observed that indeed the proposed QC-SIDCO approach is superior as it both reduces the frame coherence as defined in equation (11), yet also preserves the frame tightness, as defined in equation (10).



Fig. 1: Coherence profiles of measurement matrices Φ obtained with the PTF construction approach of [21], and with the low-coherence QC-SIDCO frame construction approach of Section III, with T=R=8 and $M_{\rm T}=M_{\rm R}=4$.

Next we compare the performance of the CS-based channel estimation algorithms employing the ideal measurement matrices highlighted above. To this end, a grid granularity of $G_{\rm T} = G_{\rm R} = 10$ was considered, and different methods were used to solve the channel estimation problem. In particular, the classical OMP algorithm of [15] was used to solve equation (8), and both the standard BPDN algorithm of [17] as well as our previously proposed BPDN- ℓ_1 variation given in [16, Alg. 1] (with maximum t = 4 iterations and tolerance $\epsilon = 0.1$) were used to solve equation (9).

The normalized mean square error (NMSE) $\mathbb{E}\left[\frac{\|\mathbf{H}-\hat{\mathbf{H}}\|_{F}}{\|\mathbf{H}\|_{F}}\right]$ was used as accuracy measure to compare the performance of the different mmWave channel estimation end schemes.



(b) Channel estimation via equation (9) solved by BPDN [17].

Fig. 2: Comparison of CS-based mmWave channel estimation algorithms employing measurement matrices obtained with the PTF construction approach of [21], and with the lowcoherence QC-SIDCO frame method of Section III. In Fig. 2, it can be seen that employing the measurement matrix proposed in Section III results in an improved estimation accuracy compared to the alternative of employing the state-of-the-art PTF construction approach of [21], regardless of whether the estimation problem itself is performed via OMP or BPDN, corresponding to equations (8) and (9), respectively.

Finally, in Fig. 3, the performance of mmWave channel estimation schemes based on the improved iterative BPDN ℓ_1 -reweighed sparse estimator proposed in [16] and employing PTF and QC-SIDCO is evaluated.

In this last comparison, however, we let $M_{\rm T}$, $M_{\rm R}$ assume different values while maintaining TR constant. Referring to equations (7) and (17), it can be seen that this impacts on the aspect ratio of the measurement matrices (*i.e.*, number of rows divided by number of columns), such that the results capture the robustness of the beamforming designs. Not only is the superiority of the QC-SIDCO method of Section III over the Parseval tight frame (PTF) approach of [21] once again confirmed, but also it is found that the relative gain obtained is robust against the noise power and the frame's aspect ratio.



Fig. 3: Sparse recovery performance given BPDN- ℓ_1 algorithm [16] – a comparison between the PTF optimized sensing matrix Ω and the proposed low-coherence QC-SIDCO tight frame measurement matrix Φ designs.

V. CONCLUSION

We discussed in this paper the problem of sparse recovery of mmWave-MIMO channels. In particular, we focused on the design of the training vectors for sparse channel recovery. We reformulated the problem in the context of Frame Theory and we proposed a novel tight low-coherence generic design associated with the Kronecker product of the transmit and receive beamformers. We also introduced a recovery mechanism for practical realizations based on the beamforming matrices by means of a vectorized SVD decomposition. Furthermore, we analyzed the proposed scheme against the state of the art and outlined its design advantages and robust performance gains for the estimation of mmWave-MIMO channels.

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