Reduced Complexity Calculation of LMMSE Filter Coefficients for GFDM

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A low-complexity algorithm for calculation of the LMMSE filter coefficients for GFDM in a block-fading multipath environment is derived in this letter. The simplification is based on the block circularity of the involved matrices. The proposal reduces complexity from cubic to squared order. The proposed approach can be generalized to other waveforms with circular pulse shaping.

Notation: Matrices are typeset in bold notation. $\otimes$ denotes the Kronecker product, $\mathbf{I}_N$ denotes the N-dimensional identity matrix and $\mathbf{F}_N$ is the unitary N-point DFT matrix. $\langle x\rangle_N$ denotes the remainder of x modulo N. $(\cdot)^\top$ and $(\cdot)^\mathrm{H}$ denote matrix transpose and hermitian conjugate, respectively. $\mathrm{diag}(\dots)$ returns a (block)-diagonal matrix with its arguments on the diagonal.

Motivation and Problem Description: Recently, several waveforms for 5G networks have been proposed [1]. Among them, waveforms utilizing circular pulse shaping [2, 3, 4] structure the signal into self-contained blocks that can be separated by a cyclic prefix (CP) to combat inter-block interference. Generalized Frequency Division Multiplexing (GFDM), the first 5G waveform that used circular pulse shaping, provides a flexible time and frequency grid that can be explored to provide low out of band radiation and robustness to time- and frequency misalignments [4]. For real-world implementations, low-complexity algorithms are always of concern. For GFDM, literature provides low-complexity descriptions for the linear GFDM modulator and demodulator [4] and the design of zero-forcing and LMMSE filters for additive white Gaussian noise (AWGN) channels [5]. However, no low-complexity implementation of the LMMSE demodulator for multipath environments is available. This letter presents an algorithm with significantly reduced complexity for calculating the LMMSE filter coefficients for GFDM.

The time domain signal of one GFDM block is given by

$$x[n] = \sum_{k=0}^{K-1}\sum_{m=0}^{M-1} d_{k,m} g_{k,m}[n] \tag{1}$$

$$\text{with } g_{k,m}[n] = g[\langle n - mK\rangle_N]\exp\left(j2\pi\frac{kM}{N}n\right), \tag{2}$$

where $N = KM$, $n = 0,\dots,N-1$, $g[n]$ denotes the prototype transmit filter and $d_{k,m}\in\mathcal{X}$ is the data symbol to be transmitted on the kth subcarrier and mth subsymbol taken from the complex-valued constellation $\mathcal{X}$. Eq. (1) is written in matrix form as $\vec{x} = \mathbf{A}\vec{d}$ with

$$\mathbf{A} = \left[\vec{g}_{0,0}, \vec{g}_{1,0}, \dots, \vec{g}_{K-1,0}, \vec{g}_{0,1}, \vec{g}_{1,1}\dots\vec{g}_{K-1,M-1}\right], \tag{3}$$

where the column vectors are $\vec{g}_{k,m} = (g_{k,m}[n])_{n=0,\dots,N-1}$, $\vec{d}$ contains $d_{k,m}$ in the appropriate order and $E[\vec{d}] = \mathbf{0}$, $E[\vec{d}\vec{d}^\mathrm{H}] = \mathbf{I}_N$. $g[n]$ is bandlimited within two subcarriers, i.e. $\mathbf{F}_N\vec{g}_{k,0}$ only has $2M$ nonzero elements centered around the index kM [4]. The signal is transmitted through a block-fading wireless multipath channel with impulse response $\vec{h}$. Assuming a CP between blocks that is longer than $\vec{h}$, the received time domain signal per block is given by

$$\vec{y} = \mathbf{H}\vec{x} = \mathbf{H}\mathbf{A}\vec{d} + \vec{w}, \tag{4}$$

where $\mathbf{H}$ is an $N\times N$ circulant matrix with the zero-padded channel impulse response $\vec{h}$ as its first column and $\vec{w}\sim\mathcal{CN}(0,\sigma_n^2\mathbf{I}_N)$ is AWGN. From (4), the LMMSE equalizer for $\vec{d}$ is given by $\vec{d}_\text{LMMSE} = \mathbf{W}^\mathrm{H}\vec{y}$, where $\mathbf{W}$ are the LMMSE filter coefficients given by

$$\mathbf{W} = \mathbf{HA}((\mathbf{HA})^\mathrm{H}(\mathbf{HA}) + \sigma_n^2\mathbf{I}_N)^{-1}. \tag{5}$$

Sec II. reviews properties of block circulant (BC) matrices. A low-complexity solution for computing $\mathbf{W}$ in (5) is developed in Sec. III and its complexity is evaluated in Sec IV. Sec V concludes this letter.

Block-circulant Matrices: Let $\mathbf{X}$ be an $N\times N$ BC matrix composed of M arbitrary submatrices $\{\mathbf{X}_m\}$ of size $K\times K$ each, i.e.

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_0 & \mathbf{X}_{M-1} & \mathbf{X}_{M-2} & \dots & \mathbf{X}_1 \\ \mathbf{X}_1 & \mathbf{X}_0 & \mathbf{X}_{M-1} & & \\ \mathbf{X}_2 & \mathbf{X}_1 & \mathbf{X}_0 & & \\ \vdots & & & \ddots & \vdots \\ \mathbf{X}_{M-1} & \mathbf{X}_{M-2} & \dots & & \mathbf{X}_0 \end{pmatrix}. \tag{6}$$

$\mathbf{X}$ is block-diagonalized by $\mathbf{Z} = \mathbf{F}_M\otimes\mathbf{I}_K$ [6] such that

$$\mathbf{ZXZ}^\mathrm{H} = \mathrm{diag}(\mathbf{D}_{\mathbf{X},0}, \mathbf{D}_{\mathbf{X},1}, \dots, \mathbf{D}_{\mathbf{X},M-1}), \tag{7}$$

where $\mathbf{D}_{X,u}$ is the uth submatrix of size $K\times K$ on the diagonal of $\mathbf{ZXZ}^\mathrm{H}$. Note that $\mathbf{Z}$ performs a discrete ZAK transform on its argument [5]. Let $\mathbf{X}_s$ be the first K colums of $\mathbf{X}$, i.e. $\mathbf{X}_s = \begin{pmatrix}\mathbf{X}_0^\top & \mathbf{X}_1^\top & \cdots & \mathbf{X}_{M-1}^\top\end{pmatrix}^\top$. Then [6],

$$\mathbf{D}_\mathbf{X} = \begin{pmatrix}\mathbf{D}_{\mathbf{X},0}^\top & \mathbf{D}_{\mathbf{X},1}^\top & \dots\mathbf{D}_{\mathbf{X},M-1}^\top\end{pmatrix}^\top = \mathbf{ZX}_s. \tag{8}$$

Note the similar behaviour of circulant matrices: Let $\mathbf{Y}$ be a circulant matrix with $\vec{y}$ in the first column, then $\mathbf{F}_N\mathbf{Y}\mathbf{F}_N^\mathrm{H} = \mathrm{diag}(\mathbf{F}_N\vec{y})$. Let

$$\mathbf{Z}_u = \vec{\omega}_u\otimes\mathbf{I}_K \tag{9}$$

$$\text{with } \vec{\omega}_u = \begin{pmatrix}1 & \omega^u & \omega^{2u} & \cdots & \omega^{(M-1)\cdot u}\end{pmatrix} \tag{10}$$

and $\omega = \exp(\frac{-j2\pi}{M})$ such that $\mathbf{D}_{\mathbf{X},u} = \mathbf{Z}_u\mathbf{X}_s$. Note that, according to above block diagonalization, the product and sum of two or the inverse of one BC matrix is again BC.

Reduced complexity Filter Calculation: From definition (3), $\mathbf{A}$ is BC with M blocks with size $K\times K$ and is hence block-diagonalized by $\mathbf{Z}$. Also, circularity of $\mathbf{H}$ implies block circularity. Accordingly, $\mathbf{G} = \mathbf{HA}$, $(\mathbf{G}^\mathrm{H}\mathbf{G} + \sigma_n^2\mathbf{I}_N)$ and $\mathbf{W}$ are all BC matrices. Hence, $\mathbf{W}$ is completely defined by its diagonalization $\mathbf{D}_\mathbf{W} = \mathbf{ZW}_s$. The M blocks of $\mathbf{D}_\mathbf{W}$ are given by the M equation systems

$$\mathbf{D}_{\mathbf{W},u} = ((\mathbf{D}_{\mathbf{G},u}^\mathrm{H}\mathbf{D}_{\mathbf{G},u} + \sigma_n^2\mathbf{I}_K)^{-1}\mathbf{D}_{\mathbf{G},u}^\mathrm{H})^\mathrm{H} \tag{11}$$

where $u = 0,\dots,M-1$ and $\mathbf{D}_{\mathbf{G},u} = \mathbf{Z}_u\mathbf{G}_s$. Now, $\mathbf{Z}_u^\mathrm{H}\mathbf{Z}_u = ((\vec{\omega}_u)^\mathrm{H}\vec{\omega}_u)\otimes\mathbf{I}_K$ is a circulant matrix since $((\vec{\omega}_u)^\mathrm{H}\vec{\omega}_u)_{i,j} = \omega^{u(i-j)}$ is circulant and accordingly $\mathbf{D}_u := \mathbf{F}_N\mathbf{Z}_u^\mathrm{H}\mathbf{Z}_u\mathbf{F}_N^\mathrm{H}$ is a diagonal matrix. Due to band-limitation of $g[n]$ only adjacent subcarriers overlap and $(\mathbf{F}_N\mathbf{G}_s)^\mathrm{H}\mathbf{D}_u\mathbf{F}_N\mathbf{G}_s$ is a tridiagonal matrix with periodic boundary conditions. Once $\mathbf{D}_\mathbf{W}$ is known, the first K columns of $\mathbf{W}$ in the time domain are given by

$$\mathbf{W}_s = \mathbf{Z}^\mathrm{H}\mathbf{D}_\mathbf{W}, \tag{12}$$

and remaining columns are given as circular shifts of $\mathbf{W}_s$. In addition, the ZAK domain $\mathbf{D}_\mathbf{W}$ can be also directly transformed into the frequency domain to readily employ a low-complexity receiver as in [7]. Furthermore, LMMSE filtering can even be directly performed in the ZAK domain by

$$\mathbf{Z}\vec{d}_\text{LMMSE} = \mathrm{diag}(\mathbf{D}_{\mathbf{W},0}, \mathbf{D}_{\mathbf{W},1}, \dots, \mathbf{D}_{\mathbf{W},M-1})^\mathrm{H}\mathbf{Z}\vec{y} \tag{13}$$

Complexity Analysis: In this section, the arithmetic complexity of the proposed algorithm is evaluated, considering one complex multiplication as one $\mathcal{O}(1)$ operation and neglecting other operations such as additions. The product $\mathbf{Z}\vec{x}$ is equivalent to K DFTs of length M each, and hence requires $KM\,\mathrm{ld}\,M$[1] operations, yielding $K^2M\,\mathrm{ld}\,M$ operations for $\mathbf{ZX}_s$.

To compute $\mathbf{F}_N\mathbf{G}_s = \mathbf{F}_N\mathbf{HA}_s$ we take advantage of the factorization

$$\mathbf{F}_N\mathbf{G}_s = \mathbf{F}_N\mathbf{H}\mathbf{F}_N^\mathrm{H}\mathbf{F}_N\mathbf{A}_s \tag{14}$$

where $\mathbf{F}_N\mathbf{A}_s$ is precalculated at the receiver. This is done by DFT for the first column. Other columns are given by circular shifts since $\mathbf{A}_s$ contains frequency shifts of $\vec{g}$. Considering band-limitation of $g[n]$, the multiplication of $\mathbf{F}_N\mathbf{A}_s$ with the diagonal matrix $\mathbf{F}_N\mathbf{H}\mathbf{F}_N^\mathrm{H}$ takes $2MK$ operations. The diagonal of $\mathbf{F}_N\mathbf{H}\mathbf{F}_N^\mathrm{H}$ is assumed to be available from previous channel estimation procedures. The product $\mathbf{G}_s^\mathrm{H}\mathbf{F}_N^\mathrm{H}\mathbf{D}_u\mathbf{F}_N\mathbf{G}_s$ requires $3K\cdot 4M$ operations due to the tridiagonal structure of the result and band-structure of $\mathbf{F}_N\mathbf{G}_s$.

[1] Assuming that the N-point DFT requires $N\,\mathrm{ld}\,N$ operations.

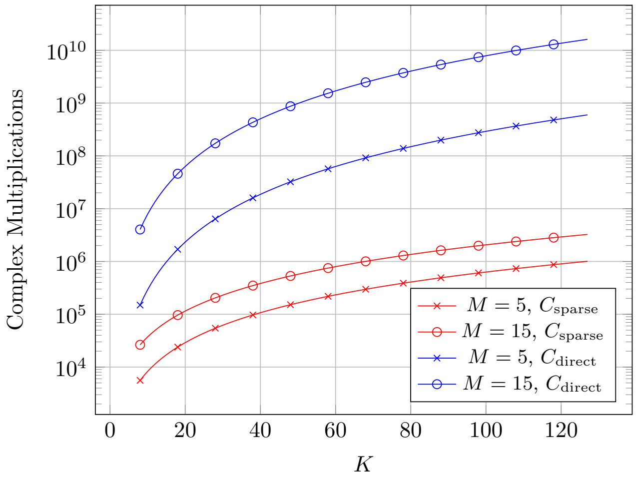

Fig. 1 *Number of floating point operations for proposed low-complexity LMMSE calculation.*

The tridiagonal system with periodic boundary conditions

$$(\mathbf{D}_{\mathbf{G},u}^{\mathrm{H}}\mathbf{D}_{\mathbf{G},u} + \sigma_n^2\mathbf{I}_K)^{-1}\mathbf{D}_{\mathbf{G},u} \tag{15}$$

is solved using the Thomas algorithm [8] with $2K$ operations for factorization and $5K$ operations for solving for each right hand side, resulting in $2K + 5K^2$ operations for the full linear system [9]. Finally, calculation of $\mathbf{Z}^{\mathrm{H}}\mathbf{D}_{\mathbf{W}}$ requires $K^2 M \,\mathrm{ld}\, M$ operations.

Hence, the number of complex multiplications C_{sparse} to solve (5) with the proposed method is given by

$$C_{\text{sparse}} = \underbrace{2MK}_{(a)} + M(\underbrace{3K \cdot 4M}_{(b)} + \underbrace{2K + 5K^2}_{(c)}) + \underbrace{K^2 M \,\mathrm{ld}\, M}_{(d)} \tag{16}$$

$$= K^2(5M + M \,\mathrm{ld}\, M) + K(12M^2 + 4M) \tag{17}$$

$$= \mathcal{O}(K^2 M \,\mathrm{ld}\, M + KM^2) \tag{18}$$

where (a) corresponds to calculation of $\mathbf{F}_N\mathbf{G}_s$, (b) respects $\mathbf{G}_s^{\mathrm{H}}\mathbf{F}_N^{\mathrm{H}}\mathbf{D}_u\mathbf{F}_N\mathbf{G}_S$, (c) describes the solution of the tridiagonal system and (d) accounts for $\mathbf{Z}^{\mathrm{H}}\mathbf{D}_W$. For comparison, direct application of a conventional Hermitian positive definite solver to (5) requires $C_{\text{direct}} = \frac{N^3}{3} + N \cdot 2N^2$ operations only for the solution step[2], where the first term corresponds to Cholesky decomposition and the second term refers to backward and forward substitution for N right-hand sides. An additional advantage is the reduced memory requirement of the proposed algorithm, as it suffices to store the MK^2 filter coefficients for $\mathbf{W}_s$ instead of M^2K^2 coefficients for $\mathbf{W}$.

Fig. 1 compares the number of complex multiplications required for the proposed technique and for conventional solving with Cholesky decomposition for different values of K and M. The number of required operations can be reduced by 4 orders of magnitude for 128 subcarriers.

Conclusion: A low-complexity approach for the calculation of LMMSE filter coefficients for block-fading multipath channels for GFDM has been presented. The proposal significantly reduces the complexity of the design from $\mathcal{O}(K^3M^3)$ to $\mathcal{O}(K^2 M \,\mathrm{ld}\, M + KM^2)$ which results in a complexity reduction of several orders of magnitude for reasonable system sizes. Since the technique exploits the block-circulant structure of the modulation matrix, it can be generalized to other multicarrier waveforms employing circular pulse shaping.

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[2] i.e. product of $\mathbf{HA}$ etc. is not considered

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