

Study of Half-Duplex Gaussian Relay Channels with Correlated Noises

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Abstract—In this paper, we consider a half-duplex Gaussian relay channel where the noises at the relay and destination are arbitrarily correlated. For this generalized relay channel, we first evaluate the cut-set bound as well as the achievable rates with three existing relay schemes: Decode-and-Forward (DF), Compress-and-Forward (CF), and Amplify-and-Forward (AF), with performance comparison under various channel settings. We observe that although DF completely disregards the noise correlation while the other two could exploit such extra information, none of the three relay schemes always outperforms the others over different correlation coefficients; however, the exploitation of noise correlation by CF and AF leads to more significant benefit when the source-relay channel is weak. It is further shown that a negative noise correlation is always helpful for AF. We also establish two capacity-achieving results under two special noise correlation coefficients, with one being achieved by DF and the other being achieved by direct link transmission (or a special case of CF), which correspond to the capacity results for the traditional Gaussian degraded relay channel and the Gaussian reversely-degraded one.

I. INTRODUCTION

The relay channel problem was introduced by van der Meulen [1], in which the source communicates with the destination via the help of a relay node. In [2], relay channels are thoroughly studied by presenting a max-flow min-cut upper bound and two lower bounds for the relay channel capacity. The strategies corresponding to these two lower bounds are called Decode-and-Forward (DF) and Compress-and-Forward (CF), respectively. A general lower bound is also discussed in [2] by combining the DF and CF schemes. In [3], the capacity bounds are established for Gaussian relay channels with correlated noises at the relay and the destination.

In the aforementioned work, the relay is assumed to receive and transmit under the full-duplex mode, which may be unrealistic due to practical restrictions such as the coupling of transmitted signals into the receiver path in the RF front-end. As such, more practical half-duplex relaying has been extensively studied. In [4], the half-duplex frequency-division AWGN relay channel is investigated. The counterpart time-division scheme is presented in [5], with the upper and lower bounds of the channel capacity established, where the relay node is operating in the receive mode for the first α fraction of each time slot and in the transmit mode for the remaining $1 - \alpha$ fraction. The authors show that the time-

division parameter α needs to be optimized numerically to maximize the information rates. A much simpler relay scheme, Amplify-and-Forward (AF), is discussed in [6], where the relay simply forwards an amplified version of the received signal in a half-duplex mode with $\alpha = 0.5$. Conventionally, most of the existing half-duplex relay channel work assumes uncorrelated noises in the relay and the destination.

In this paper, we investigate the half-duplex Gaussian relay channel with correlated noises, which may arise in practice when the relay and the destination are interfered by a common random source. The max-flow min-cut upper bound is first derived under this setting. The achievable rates of DF, CF, and AF are also established. In the DF strategy, the relay first decodes the information from the source and then re-encodes it before transmitting to the destination. Hence, DF completely disregards the correlation between the noises at the relay and the destination. In the CF strategy, the relay compresses the received signal (including the noise) and forwards it to the destination as the side-information, which may enable us to utilize the noise correlation. In AF, the relay only forwards a scaled version of the received signal without decoding, and the destination processes the signals from the source and relay via Maximum-Ratio Combining (MRC), for which the noise correlation could also be exploited. It is actually shown that the noise correlation always helps to improve the AF performance when $\rho_z \in [-1, 0]$. With general settings, we compare these three schemes and show that none of them always outperforms the others across different correlation coefficients. Moreover, we discuss two capacity-achieving cases where the achievable rates meet the upper bound, and they are in fact corresponding to the Gaussian degraded relay channel and reversely-degraded one established in [2].

The rest of the paper is organized as follows. The channel model is discussed in Section II. In Section III, we provide the upper bound for the half-duplex Gaussian relay channel with correlated noises. The achievable rates and special capacity-achieving cases are presented in Section IV. Numerical results are provided to compare different strategies against the upper bound in Section V. Finally, we conclude the paper in Section VI.

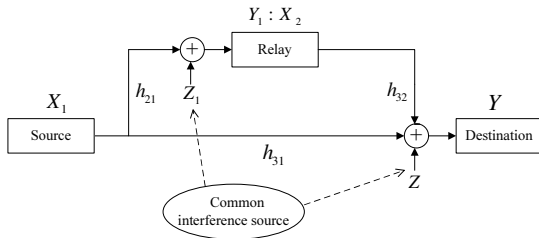


Fig. 1. The Relay Channel Model.

II. CHANNEL MODEL

Consider the Gaussian relay channel shown in Fig. 1, where the source conveys information to the destination with the help of the relay node. In practice, the full-duplex mode is not applicable since the relay cannot transmit and receive signals at the same time, unless with frequency-division. Hence, the time-division half-duplex mode is considered in this paper, where the relay listens to the source during the first α fraction of each time slot, and transmits in the remaining $1-\alpha$ fraction.

As shown in Fig. 1, the channel is supposed to be experiencing static fading with the channel gains denoted by h_{21} , h_{31} , and h_{32} , respectively. In the first α portion of the i th time slot, the transmitted signal at the source is denoted as $X_1^{(1)}(i)$ with an average power constraint $P_1^{(1)}$, such that the received signals at the relay and the destination are given by:

$$\begin{aligned} Y_1(i) &= h_{21}X_1^{(1)}(i) + Z_1(i), \\ Y^{(1)}(i) &= h_{31}X_1^{(1)}(i) + Z^{(1)}(i), \end{aligned}$$

, respectively, where $Z_1 \sim \mathcal{N}(0, N_1)$ and $Z^{(1)} \sim \mathcal{N}(0, N)$ are correlated AWGN noises with correlation coefficient

$$\rho_z = \frac{E\{Z_1 Z^{(1)}\}}{\sqrt{N_1 N}}.$$

In the remaining $1-\alpha$ portion, the transmitted signals at the source and the relay are denoted as $X_1^{(2)}(i)$ and $X_2(i)$, with average power constraints $P_1^{(2)}$ and P_2 , respectively. In this phase, the received signal at the destination is given by:

$$Y^{(2)}(i) = h_{31}X_1^{(2)}(i) + h_{32}X_2(i) + Z^{(2)}(i),$$

where $Z^{(2)} \sim \mathcal{N}(0, N)$ is the AWGN noise at the destination.

III. CAPACITY UPPER BOUND

In this section, we provide the max-flow min-cut upper bound for the half-duplex Gaussian relay channel with correlated noises. For convenience, we define the following function:

$$\Gamma(x) = \frac{1}{2} \log_2(1+x).$$

Also denote the received SNRs in different links (and in different half-duplex phases if applicable) as:

$$\begin{aligned} \gamma_{21} &= \frac{h_{21}^2 P_1^{(1)}}{N_1}, & \gamma_{32} &= \frac{h_{32}^2 P_2}{N}, \\ \gamma_{31}^{(1)} &= \frac{h_{31}^2 P_1^{(1)}}{N}, & \gamma_{31}^{(2)} &= \frac{h_{31}^2 P_1^{(2)}}{N}. \end{aligned}$$

Note that in this paper, all the results are based on a fixed power assignment $\{P_1^{(1)}, P_1^{(2)}, P_2\}$; the case with optimized power allocation is currently under investigation.

As an extension of the cut-set bound in [5], we have the following upper bound in the correlated-noise case.

Proposition 1 (Upper Bound): For the time-division half-duplex Gaussian relay channel with noise correlation defined in Section II, at a given α , the upper bound of the channel capacity is given by:

$$C^+ = \max_{0 \leq \rho_x \leq 1} \min\{C_1^+(\rho_x), C_2^+(\rho_x)\}, \quad (1)$$

where

$$\begin{aligned} C_1^+(\rho_x) &= \alpha \Gamma(\gamma_{31}^{(1)}) + \\ &\quad (1-\alpha) \Gamma(\gamma_{31}^{(2)} + \gamma_{32} + 2\rho_x \sqrt{\gamma_{31}^{(2)} \gamma_{32}}); \\ C_2^+(\rho_x) &= \alpha \Gamma\left(\frac{\gamma_{21} + \gamma_{31}^{(1)} - 2\rho_z \sqrt{\gamma_{21} \gamma_{31}^{(1)}}}{1 - \rho_z^2}\right) \\ &\quad + (1-\alpha) \Gamma((1 - \rho_x^2) \gamma_{31}^{(2)}). \end{aligned}$$

The proof is similar to that in [5].

IV. ACHIEVABLE RATES AND TWO CAPACITY-ACHIEVING CASES

In this section, we first provide the achievable rates for DF, CF, and AF schemes, based on the results in [5] and [6]. We then discuss two special cases where the capacity lower bound meets the upper bound.

A. Achievable Rate of Decode-and-Forward

In the DF scheme, the relay decodes the received signal from the source in the first phase, re-encodes it, and then transmits it to the destination in the second phase. The achievable rate for DF is given by the following proposition.

Proposition 2 (Achievable Rate for DF): For the time-division half-duplex Gaussian relay channel with noise correlation defined in Section II, at a given α , the achievable rate for the decode-and-forward strategy is given by:

$$R_{DF} = \max_{0 \leq \rho_x \leq 1} \min\{R_1(\rho_x), R_2(\rho_x)\}, \quad (2)$$

where

$$\begin{aligned} R_1(\rho_x) &= \alpha \Gamma(\gamma_{21}) + (1-\alpha) \Gamma((1 - \rho_x^2) \gamma_{31}^{(2)}); \\ R_2(\rho_x) &= \alpha \Gamma(\gamma_{31}^{(1)}) + \\ &\quad (1-\alpha) \Gamma(\gamma_{31}^{(2)} + \gamma_{32} + 2\rho_x \sqrt{\gamma_{31}^{(2)} \gamma_{32}}). \end{aligned}$$

We observe that the DF achievable rate is not a function of ρ_z , and it is the same as the case with uncorrelated noises. The reason is that the relay completely disregards the noise correlation by decoding the received signal in the first phase. The proof for Proposition 2 can be found in [5].

B. Achievable Rate of Compress-and-Forward

In the CF scheme, the relay compresses the received signal based on the Wyner and Ziv's rate distortion theory [7], and then transmits it to the destination as the side-information. We have the following proposition which gives the achievable rate for CF. The proposition is an extended version of Theorem 6 in [2] and Proposition 3 in [5]; so the proof is skipped here.

Proposition 3 (Achievable Rate for CF): For the time-division half-duplex Gaussian relay channel with noise correlation defined in Section II, at a given α , the achievable rate for the compress-and-forward strategy is given by:

$$R_{CF} = \alpha \Gamma \left(\frac{\gamma_{21} + \gamma_{31}^{(1)} + \gamma_{31}^{(1)} N_w / N_1 - 2\rho_z \sqrt{\gamma_{21} \gamma_{31}^{(1)}}}{(1 - \rho_z^2) + N_w / N_1} \right) + (1 - \alpha) \Gamma \left(\gamma_{31}^{(2)} \right),$$

where the quantization noise power N_w is given as

$$N_w = N_1 \frac{(1 - \rho_z^2) + \gamma_{21} + \gamma_{31}^{(1)} - 2\rho_z \sqrt{\gamma_{21} \gamma_{31}^{(1)}}}{\left(1 + \gamma_{31}^{(1)}\right) \left(\left(1 + \frac{\gamma_{32}}{1 + \gamma_{31}^{(2)}}\right)^{\frac{1-\alpha}{\alpha}} - 1 \right)}, \quad (3)$$

and we see that the CF achievable rate depends on the noise correlation coefficient ρ_z .

C. Achievable Rate of Amplify-and-Forward

In AF, the source keeps silent in the second phase, and the relay only forwards a scaled version of the received signal without decoding or compressing. The destination decodes the information at the end of each time slot by MRC. The time-division parameter α is fixed as 0.5, and the achievable rate for AF is given by the following proposition, which can be viewed as a generalization of Eq. (12) in [6].

Proposition 4 (Achievable Rate for AF): For the time-division half-duplex Gaussian relay channel with noise correlation defined in Section II, the achievable rate for the amplify-and-forward strategy is given by:

$$R_{AF} = \frac{1}{2} \Gamma \left(\gamma_{31}^{(1)} + \frac{\gamma_{32} \left(\sqrt{\gamma_{21}} - \rho_z \sqrt{\gamma_{31}^{(1)}} \right)^2}{1 + \gamma_{21} + \gamma_{32} (1 - \rho_z^2)} \right). \quad (4)$$

Based on the achievable rate of the AF scheme, we obtain the following theorem:

Theorem 1: For the time-division half-duplex Gaussian relay channel with noise correlation defined in Section II, the achievable rate of AF is a monotonically decreasing function over ρ_z when $\rho_z \in [-1, 0]$, which means that negative correlation always increases the achievable rate compared with the case of uncorrelated noises.

Proof: By taking the derivative over ρ_z of the item inside the Γ function of (4), it can be shown that the first order derivative is always negative when ρ_z is negative. Combined

with the monotonicity of Γ function, we can conclude that R_{AF} is strictly decreasing when ρ_z is between -1 and 0 . ■

D. Two Capacity-Achieving Cases

With the upper bound and the achievable rates for DF and CF, here we discuss two special capacity-achieving cases with one being achieved by DF and the other being achieved by direct transmission (or a special case of CF). We also show that these two cases are corresponding to the Gaussian degraded relay channel and the reversely-degraded one established in [2], respectively.

Theorem 2: If the correlation coefficient of noises is $\rho_z = \sqrt{\gamma_{31}^{(1)} / \gamma_{21}}$, the capacity of the relay channel defined in Section II is achieved by the DF strategy, which is given by:

$$C_1 = \max_{0 \leq \rho_x \leq 1} \min \{C_{11}(\rho_x), C_{12}(\rho_x)\}, \quad (5)$$

where

$$\begin{aligned} C_{11}(\rho_x) &= \alpha \Gamma(\gamma_{21}) + (1 - \alpha) \Gamma \left((1 - \rho_x^2) \gamma_{31}^{(2)} \right); \\ C_{12}(\rho_x) &= \alpha \Gamma \left(\gamma_{31}^{(1)} \right) + \\ &\quad (1 - \alpha) \Gamma \left(\gamma_{31}^{(2)} + \gamma_{32} + 2\rho_x \sqrt{\gamma_{31}^{(2)} \gamma_{32}} \right). \end{aligned}$$

Proof: By substituting $\rho_z = \sqrt{\gamma_{31}^{(1)} / \gamma_{21}}$ into Propositions 1 and 2, we observe that the achievable rate of DF meets the capacity upper bound. ■

Theorem 3: If the correlation coefficient of noises is $\rho_z = \sqrt{\gamma_{21} / \gamma_{31}^{(1)}}$, the capacity of the relay channel defined in Section II is achieved by direct link transmission from the source to the destination (or a special case of CF). The capacity is given by:

$$C_2 = \alpha \Gamma \left(\gamma_{31}^{(1)} \right) + (1 - \alpha) \Gamma \left(\gamma_{31}^{(2)} \right). \quad (6)$$

Proof: The achievability can be shown by setting \hat{Y}_1 as constant in Proposition 3, which means that the rate of CF strategy in this case is actually the direct link transmission rate.

The converse part can be proved by substituting $\rho_z = \sqrt{\gamma_{21} / \gamma_{31}^{(1)}}$ into Proposition 1:

$$\begin{aligned} C^+ &\leq \max_{0 \leq \rho_x \leq 1} C_2^+(\rho_x) \\ &= \max_{0 \leq \rho_x \leq 1} \left(\alpha \Gamma \left(\gamma_{31}^{(1)} \right) + (1 - \alpha) \Gamma \left((1 - \rho_x^2) \gamma_{31}^{(2)} \right) \right) \\ &= \alpha \Gamma \left(\gamma_{31}^{(1)} \right) + (1 - \alpha) \Gamma \left(\gamma_{31}^{(2)} \right). \end{aligned}$$

By combining both the achievability and converse parts, we can see that in this case, the capacity is achieved by the special CF scheme or the equivalent direct link transmission. ■

Remark: In fact, these two capacity-achieving cases are equivalent to the degraded relay channel and the reversely-degraded one in half-duplex mode, respectively. The detailed proof for such an equivalence is left to our journal version with the proof sketched as: Given the relay channel with

correlated noises as defined in this paper, after normalization over respective channel gains, the correlation coefficient of the remaining effective noises is equal to $\sqrt{\gamma_{31}^{(1)}/\gamma_{21}}$ for the channel to be degraded or equal to $\sqrt{\gamma_{21}/\gamma_{31}^{(1)}}$ for the channel to be reversely-degraded. The converse can also be established, i.e., given a noise correlation coefficient as $\sqrt{\gamma_{31}^{(1)}/\gamma_{21}}$ or $\sqrt{\gamma_{21}/\gamma_{31}^{(1)}}$, we could always construct a corresponding degraded Gaussian relay channel or a reversely-degraded one.

V. NUMERICAL RESULTS

In this section, we compare the achievable rates of DF, CF, and AF against the upper bound in several scenarios. The average power P_1 at the source and P_2 at the relay are both set to 1 (here $P_1^{(1)}$ and $P_2^{(2)}$ are both set equal to P_1 ; thus the average power constraint is satisfied). We also set $N = N_1 = 1$.

A. Scenario 1

First consider the scenario in which the source, relay, and destination are aligned on a line. For all the schemes, the time-division parameter α is set to 0.5 for a direct comparison with AF that has a fixed requirement of $\alpha = 0.5$. The distance between the source and the relay is d ($0 < d < 1$), and the distance between the source and the destination is 1. The channel amplitude gain is inversely proportional to the distance, which means:

$$h_{21} = \frac{1}{d}, h_{32} = \frac{1}{1-d}, h_{31} = 1, d \in (0, 1). \quad (7)$$

The upper bound, the CF rate, the DF rate, the AF rate, and the achievable rate of direct transmissions are plotted in Fig. 2 and Fig. 3 for different d values. The analysis of this scenario is similar to the full-duplex case [3]. When $d = 0.4$, DF is always the best which means that the exploitation of correlation noises is not necessary since the received SNR at the relay is high. However, when $d = 0.8$, CF outperforms DF for most of $\rho_z \in [-1, 1]$ since the SNR at the relay is relatively low such that the exploitation of noise correlation brings more benefits. Also when $\rho_z < 0$, *Theorem 1* is verified by showing that negative noise correlations help AF improve the performance, which is even better than that of DF for $\rho_z \in [-1, 0.15]$. Moreover, we also observe that the DF rate achieves the capacity upper bound when $\rho_z = \sqrt{\gamma_{31}^{(1)}/\gamma_{21}}$, which confirms *Theorem 2*.

B. Scenario 2

Here we discuss another channel scenario, where the link between the source and the destination is better than the link between the source and the relay:

$$h_{21} = r, h_{32} = 1 - r, h_{31} = 1, r \in (0, 1). \quad (8)$$

Fig. 4 and Fig. 5 show the achievable rates of CF, DF, AF, and direct transmission, against the capacity upper bound for different r values. It is shown that CF is always better than DF

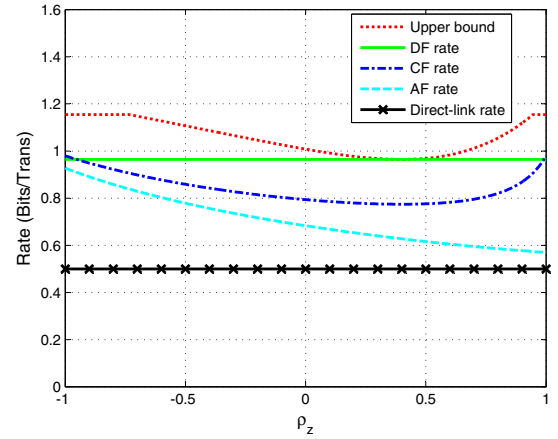


Fig. 2. Rate vs. ρ_z , $d = 0.4$, $\alpha = 0.5$.

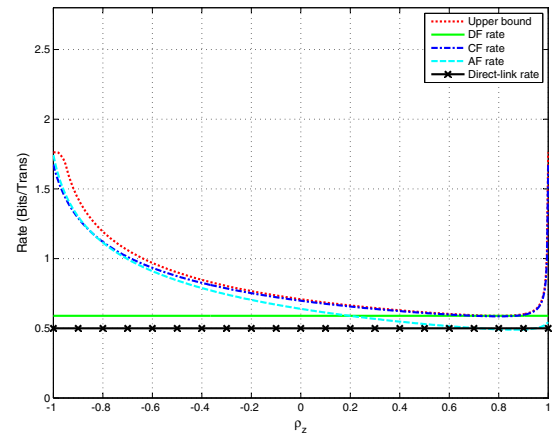


Fig. 3. Rate vs. ρ_z , $d = 0.8$, $\alpha = 0.5$.

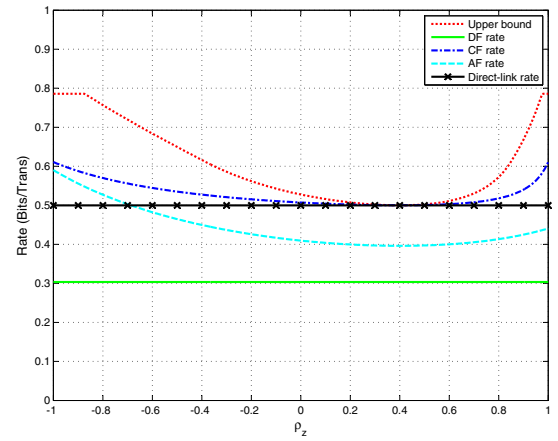


Fig. 4. Rate vs. ρ_z , $r = 0.4$, $\alpha = 0.5$.

in this scenario. Since the link between the source and the relay is weaker, there is no much gains by decoding and re-encoding the messages at the relay. The capacity-achieving points of CF in Figs. 4 and 5 confirm *Theorem 3*. Moreover, comparing Fig. 4 with Fig. 5, we see that CF achieves a higher rate when the link between the relay and the destination is stronger due

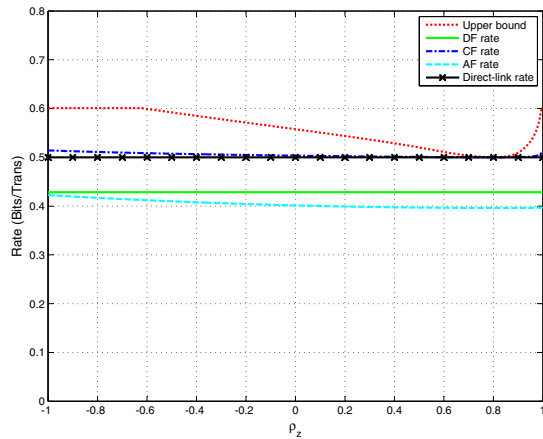


Fig. 5. Rate vs. ρ_z , $r = 0.8$, $\alpha = 0.5$.

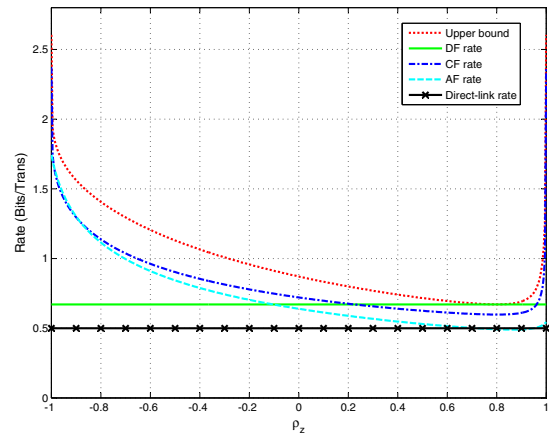


Fig. 7. Rate vs. ρ_z , $d = 0.8$, after optimizing over α .

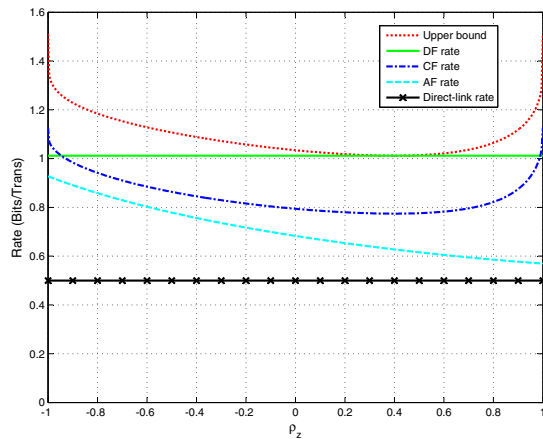


Fig. 6. Rate vs. ρ_z , $d = 0.4$, after optimizing over α .

to the finer quantization at the relay. When $r = 0.4$, AF also outperforms DF for all the $\rho_z \in [-1, 1]$.

C. Optimization over α

We now discuss the improved upper bound and achievable rates by optimizing over α for DF and CF under the first channel setting, with the results shown in Fig. 6 and Fig. 7.

Compared with the performance of $\alpha = 0.5$ in Fig. 2 and Fig. 3, we see that the upper bound and the achievable rates for DF and CF are increased by optimizing over α . When $d = 0.8$, CF is not dominant over DF any more for most of $\rho_z \in [-1, 1]$, which is different from the performance comparison at $\alpha = 0.5$. The range of ρ_z in which AF performs better than DF also shrinks after using optimal α in the DF scheme.

Fig. 6 and Fig. 7 show that the capacity-achieving points are the same as the case of fixed α , which means *Theorem 2* still holds for optimal α . Similarly, we can also show that *Theorem 3* holds at the optimal α under the second channel setting.

VI. CONCLUSION

In this paper, we established upper and lower bounds on the capacity of half-duplex Gaussian relay channels with arbitrarily correlated noises at the relay and the destination. The max-flow min-cut upper bound was first provided; then DF, CF, and AF strategies were studied to derive the respective achievable rates. We also discussed two capacity-achieving cases at special noise correlation coefficient values, which are equivalent to the degraded relay channel and the reversely-degraded one established in [2]. The performance comparison among different strategies was conducted via numerical examples; it was shown that negative noise correlations always help in AF and none of the strategies always outperforms the others over different correlation coefficients, although DF completely disregards the noise correlation information.

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