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A Wiener Measure Theoretic Approach to Pricing Extreme-Value-Related Derivatives

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ABSTRACT

Discretization schemes converge slowly when simulating extreme values for stochastic differential equations. Using a Wiener measure decomposition approach, this paper constructs an unbiased estimator for pricing extreme-value-related derivatives, such as barrier and lookback options, under a diffusion market model. A strong condition on the coefficients is needed in the derivation of the estimator. We also propose a truncation technique to remove this requirement and show that the truncation error decays exponentially. The numerical experiments reveal that this estimator is accurate and efficient.

1 INTRODUCTION

In option pricing applications, path dependence enters through extreme values of an underlying asset over the life of the option. Typical examples include barrier and lookback options. The payoffs of barrier options depends on whether or not the underlying price crosses a barrier and a standard lookback option gives the holder a right to buy/sell an asset at its lowest/highest price up to the maturity of the option.

To apply Monte Carlo recipe for pricing such derivatives, we have to simulate the running maximum or minimum to compute the payoffs along each sample path. The most straightforward way is through discretization. Simulate a time-discretized approximation to the underlying process over a time grid and take the maximum/minimum of the approximation as an approximation to the maximum/minimum of the original continuous-time model. However, the singular dynamic of the extreme values renders standard discretization procedures for SDE simulation inefficient. Asmussen, Glynn, and Pitman (1995) show that the error associated with the Euler scheme for simulating such values has both a strong and weak order of convergence of 1/2. This contrasts with the faster order 1 the Euler scheme can achieve for simulations of the process values at grid points.

Some ideas are suggested in the literature to address this difficulty. Andersen and Brotherton-Ratcliffe (1996) and Beaglehole, Dybvig, and Zhou (1997) use Brownian bridge interpolation for pricing lookback options. Baldi (1995) analyzes related techniques in a more general setting. Glasserman and Staum (2001) consider estimator for barrier options based on conditional Monte Carlo. Baldi, Caramellino, and Iovino (1999) develop approximations to one-step survival probabilities for reducing discretization error in a general class of barrier option simulation problems. All of these efforts can be viewed as corrections on basis of discretization schemes.

This paper explores how to construct an unbiased estimator for pricing extreme-value-related derivatives. The key observation is that we can decompose the probability measure defined by the modelling SDE with respect to the so called Wiener measure, which is defined by a standard Browinian motion. Simulate extreme values under the standard Brownian motion, which is shown to be easy in Asmussen, Glynn, and Pitman (1995), and then calculate weights for all the samples according to our measure decomposition. This procedure leads to an important sampling estimator for the option prices. A strong condition is needed to make sure our estimator is not biased. It turns out that many processes with financial interests do not satisfy it. The second contribution of the paper is that we propose a truncation method to circumvent this constraint. We show the error decays exponentially by choosing proper truncation parameters. The numerical experiments illustrate that the root of mean square error of our method can achieve the convergence rate of $t^{-1/2}$, which is the optimal rate associated with unbiased estimation.

The rest of the paper is organized as follows. Section 2 reviews some preliminary knowledge about extreme-value-related option and its simulation. In Section 3 we construct an unbiased price estimator using the Wiener measure decomposition under a technical condition. Section 4 is devoted to the discussion on how to relax the condition and the related truncation error analysis. We present some numerical examples in Section 5. All of the proofs are deferred to the Appendix.

2 PRELIMINARIES

Let S_t denote the price of a underlying asset. Suppose that it follows a general diffusion process in the risk neutral probability measure as follows:

$$dS_t = \mu(S_t)dt + \sigma(S_t)dW_t,\ S_0 = s, \tag{1}$$

where W_t is a standard Brownian motion, μ and σ are the drift and volatility coefficients, respectively. Both of them can be state dependent. σ is a positive definite function.

Many popular path-dependent options in the market have a payoff defined on the extreme values of S during the life time of the option. For instance, a down-and-out put option with a continuously monitored barrier offers the option owner the payoff of a European put option as long as the underlying asset price stays above a knock-out barrier, or equivalently, as long as the minimum asset price is above the barrier for its whole life. The payoff of a standard lookback put is determined by the difference between the running maximum and the spot price at the maturity of the option.

Let M_t and m_t denote the running maximum and minimum of S respectively, i.e.,

$$M_t = \max_{0\le u\le t} S_u \quad \text{and} \quad m_t = \min_{0\le u\le t} S_u.$$

In general, the (discounted) payoffs of extreme-value-related options are in the form of $g(M_T, S_T)$ or $g(m_T, S_T)$. Take the aforementioned knock-out and lookback options as examples. The payoff of the former one can be formulated as $(K-S_T)^+\mathbf{1}_{\{m_T>b\}}$, where T is the expiry date of the option, K is the strike price and b is the barrier, $b<S_0$. The latter option is with a payoff of $(M_T - S_T)$.

Some standard non-arbitrage arguments (see, e.g., Björk (1998) or Duffie (2001)) yield that the price of such extreme-value-related options should be equal to the expectation of discounted future payoffs:

$$p(s) = E[e^{-rT}g(M_T,S_T)|S_0 = s] \quad \text{or} \quad p(s) = E[e^{-rT}g(m_T,S_T)|S_0 = s].$$

To implement a Monte Carlo recipe for evaluating of the above expectations, the key step is to generate samples for the pair (M_T,S_T) or (m_T,S_T) under (1). A naïve approach is through discretization approximation. Fix a large integer n and let $h = T/n$ be the length of each time step. Then simulate a discrete process over the time grid $\{0,h,2h,\cdots,nh\}$ according to the following Euler scheme:

$$\hat{S}_i = \hat{S}_{i-1} + \mu(\hat{S}_{i-1})h + \sigma(\hat{S}_{i-1})\Delta W_i, \tag{2}$$

where $\Delta W_i = W_{(i+1)h} - W_{ih} \sim N(0,h)$. The continuous-time running maximum/minimum, M_T and m_T, can be approximated by the maximum/minimum of the Euler approximation $\hat{M}_T := \max_{1\le i\le N}\hat{S}_i$ and $\hat{m}_T := \min_{1\le i\le N}\hat{S}_i$, respectively. The Monte Carlo estimators for $E[g(M_T,S_T)]$ and $E[g(m_T,S_T)]$ are formed by sample averages

$$\frac{1}{M}\sum_{j=1}^{N} g(\hat{m}_T^j, \hat{S}_{nh}^j) \quad \text{and} \quad \frac{1}{M}\sum_{j=1}^{N} g(\hat{m}_T^j, \hat{S}_{nh}^j)$$

across M replications.

However, the simulation of extreme values turns out to be the bottleneck for the whole Monte Carlo method. Asmussen, Glynn, and Pitman (1995) show that the normalized discretization error

$$\frac{1}{\sqrt{h}}[\hat{M}_T^h - M_T]$$

has a limiting distribution as $h \to 0$. This result implies that the distribution of $\hat{M}_T^h$ converges to that of M_T at a rate $h^{1/2}$. In contrast, recall that the Euler scheme (2) has weak order of convergence 1, i.e.,

$$|E[g(\hat{S}_{nh})] - E[g(S_T)]| \le ch$$

for some constant c and all sufficiently small h (see, e.g., Kloeden and Platen (1992)). Thus, the part of $\hat{M}_T^h$ slows down the overall convergence rate of the estimator significantly.

3 A WIENER-MEASURE BASED ESTIMATOR

In this section, we propose an unbiased estimator for such extreme-value-related options on the basis of a Wiener measure decomposition of the distribution of S. From now on, we will only focus on the estimation of $E[g(M_T, S_T)|S_0 = s]$ because the treatment of $E[g(m_T, S_T)|S_0 = s]$ parallels.

In case of Browinian motion, the difficulty can be circumvented by generating the samples of (M_T, S_T) exactly. Simulate $S_T = W_T$ first. It is straightforward because W_T follows a normal distribution $N(0,T)$. Given $W_T = w$, the running maximum $M_T = \max_{0\le t\le T} W_t$ follows the so called Rayleigh distribution

$$G(x) = P[M_T \le x | W_T = w] = 1 - e^{-2x(x-w)/T}, \quad x \ge b$$

(Karatzas and Shreve (1991), Proposition 2.8.1). We can obtain a sample of M_T by substituting a uniform $U \sim U(0,1)$ in the inverse of G:

$$M_T := \frac{w + \sqrt{w^2 - 2T\log(1-U)}}{2}.$$

This exact simulation scheme does not work if we switch our attention to a general SDE given by (1). The explicit expression for the joint distributions of (M_T, S_T) is unknown at this time. But we still are able to combine this observation and the technique of importance sampling to produce an unbiased estimator. The essential step is to figure out the likelihood ratio between S and W.

3.1 Lamperti Transform and Wiener Measure Decomposition

Now assume that the functions $\mu(x)$ and $\sigma(x)$ are infinitely differentiable in x and there is a constant $c > 0$ such that $\sigma(x) > c$ for all $x \in (-\infty, \infty)$. Perform a transform defined as follows:

$$F(y) = \int_s^y \frac{1}{\sigma(u)} du.$$

This transform is known as the Lamperti transform in the literature (see, e.g., Florens (1999)). Apparently, F is a strictly increasing function because $\sigma(u) > 0$ for all u. Denote F^{-1} to be the inverse to F. Transform S into Y defined as $Y_t := F(S_t)$. Ito's lemma implies Y satisfies the following diffusion process:

$$dY_t = \alpha(Y_t)dt + dW_t, \quad Y_0 = 0, \tag{3}$$

where α is given by

$$\alpha(y) = \frac{\mu(F^{-1}(y))}{\sigma(F^{-1}(y))} - \frac{1}{2}\sigma'(F^{-1}(y)).$$

Simulation of (S_T, M_T) is equivalent to simulation of $(Y_T, \max_{0\le t\le T} Y_t)$. Once we have a pair of samples $(Y_T, \max_{0\le t\le T} Y_t)$, we can recover (S_T, M_T) by letting $S_T = F^{-1}(Y_T)$ and $M_T = F^{-1}(\max_{0\le t\le T} Y_t)$. The advantage of introducing Y is that we are able to find an explicit expression for the likelihood ratio of Y_T with respect to W_T. This helps us to build up a Wiener measure based estimator for extreme-value-related option prices.

Some technical assumptions are required to prevent Y from "exploding" (i.e., can reach the infinity boundary) in finite time. This prevention is especially necessary for the discussion regarding extreme-value-related options. The attainability of

$\pm\infty$ will make the definitions of $\max_{0\leq t\leq T} Y_t$ and $\min_{0\leq t\leq T} Y_t$ no sense. For this reason, introduce the following assumption to rule out this possibility:

Assumption 1 (Boundary Behavior) *There exist positive constants B and K such that $\alpha(y) \leq Ky$ for all $y > B$ and $\alpha(y) \geq Ky$ for all $y < -B$.*

With the restriction of Assumption 1, Y can not grow faster than linearly near infinity boundaries. Aït-Sahalia (2002) shows that under this assumption, (3) admits a weak solution $\{Y_t, t \geq 0\}$, unique in probability law for every initial value Y_0. In addition, he also proves $P[\lim_{n\to\infty} T_n = \infty] = 1$, where $T_n = \inf\{t \geq 0 : |Y_t| = n\}$, the first passage time of Y crossing $\pm n$. This precludes the explosion of Y in finite time. Note that this assumption is almost the best possible. It is easy to show that if $\alpha(y)$ is positive near $+\infty$ or negative near $-\infty$, and grows faster than linearly, then Y explodes. We refer interested readers to Chapter 5. 5 of Karatzas and Shreve (1991) for a detailed discussion on Feller's test for explosions.

With the help of the Lamperti transform, we have

Theorem 1 *Under Assumption 1, for any measurable function g,*

$$
\begin{aligned}
E[g(S_T, M_T)] &= E[g(F^{-1}(Y_T), F^{-1}(\max_{0\leq t\leq T} Y_t))] \\
&= E[g(F^{-1}(W_T), F^{-1}(\max_{0\leq t\leq T} W_t)) \cdot \exp(A(W_T)) \cdot E[\exp(-\int_0^T \phi(W_s)ds)|\theta_T, \max_{0\leq t\leq T} W_t, W_T]],
\end{aligned}
$$

where $\theta_T = \inf\{t \in [0,T] : W_t = \max_{0\leq t\leq T} W_t\}$, $A(y) = \int_0^u \alpha(u)du$ and $\phi(y) = (\alpha^2(y) + \alpha'(y))/2$.

The above theorem paves the way to implement an unbiased estimator to evaluate $E[g(S_T, M_T)]$. Let

$$
L(\theta_T, \max_{0\leq t\leq T} W_t, W_T) = E[\exp(-\int_0^T \phi(W_s)ds)|\theta_T, \max_{0\leq t\leq T} W_t, W_T]. \tag{4}
$$

We can form an important sampling estimator

$$
\frac{1}{M}\sum_{j=1}^{M} g(F^{-1}(W_T^j), F^{-1}(\max_{0\leq t\leq T} W_t^j)) \cdot \exp(A(W_T^j)) \cdot L(\theta_T^j, \max_{0\leq t\leq T} W_t^j, W_T^j),
$$

which is unbiased.

3.2 Simulation of θ_T, $\max_{0\leq u\leq T} W_u$ and W_T

The explicit knowledge of the joint distribution of $(\theta_T, \max_{0\leq u\leq T} W_u, W_T)$ facilitates us to design an exact simulation scheme. First, generate θ_T, the first time W attains its maximum over $[0,T]$. It should follow the arc-sine law:

$$
P[\theta_T \leq s] = \frac{2}{\pi} \arcsin\sqrt{\frac{s}{T}}, \quad 0 \leq s \leq T
$$

according to Problem 2.8.17, Karatzas and Shreve (1991). Sample it by setting $\theta_T = T\sin^2(\pi U/2)$ with $U \sim U(0,1)$.

Conditional on $\theta_T = \theta$, the closed-form expression for the distribution of $\max_{0\leq u\leq T} W_u$ is available too. It is given by

$$
P[\max_{0\leq u\leq T} W_u \leq b|\theta_T = \theta] = 1 - \exp(-\frac{b^2}{2\theta}),
$$

thanks again to Problem 2.8.17. Thus, a random number generator for $\max_{0\leq u\leq T} W_u$ is obtained if we let

$$
\max_{0\leq u\leq T} W_u = \sqrt{-2\theta\log(1-V)}
$$

for an independent $V \sim U(0,1)$. This can also be implemented as $\max_{0\leq u\leq T} W_u = \sqrt{-2\theta\log(V)}$ because V and $1-V$ have the same distribution.

Given $\theta_T = \theta$ and $\max_{0\le u\le T} W_u = b$, the distribution function of W_T is

$$P[W_T \le w | \theta_T = \theta, \max_{0\le u\le T} W_u = b] = (T-\theta)\cdot \exp(-\frac{(b-w)^2}{2(T-\theta)})$$

for all $w \le b$. Letting

$$W_T = b - \sqrt{2(T-\theta)(-\log(Z/(T-\theta)))},\ Z \sim U(0,1)$$

will produce a desired sample of W_T.

3.3 Brownian Meanders and Unbiased Estimator for L

Some preliminary knowledge about the Brownian motion path decomposition is needed before proceeding to form an unbiased estimator of L. For a standard Brownian motion, Williams (1970) and Denisov (1984) find a path decomposition at θ_T: given $\theta_T = \theta$, $\max_{0\le u\le T} W_u = b$ and $W_T = w$, the processes

$$\{b - W_{\theta-u},\ 0 \le u \le \theta\} \quad \text{and} \quad \{b - W_{\theta+u},\ 0 \le u \le T-\theta\}$$

are two independent *Brownian meanders*. As noted by Imhof (1984), the law of Brownian meanders can be further represented in terms of three independent Brownian bridges. Combining all of the above results, we can show that (see Proposition 2 in Asmussen, Glynn, and Pitman (1995))

$$\{W_u,\ 0 \le u \le \theta\} \stackrel{d}{=} b - \sqrt{(b(\theta-u)/\theta + B_u^{1,1})^2 + (B_u^{1,2})^2 + (B_u^{1,3})^2}, \tag{5}$$

and

$$\{W_u,\ \theta \le u \le T\} \stackrel{d}{=} b - \sqrt{((b-w)(u-\theta)/(T-\theta) + B_u^{2,1})^2 + (B_u^{2,2})^2 + (B_u^{2,3})^2}, \tag{6}$$

where $B^{1,i}, 1 \le i \le 3$ are three independent Brownian bridges from 0 to 0 over $[0,\theta]$ and $B^{2,i}, 1 \le i \le 3$ are three independent Brownian bridges from 0 to 0 over $[\theta, T]$.

Equations (5) and (6) provide us an approach to generate samples of $\{W_{t_1}, \cdots, W_{t_n}, W_{t_{n+1}}, \cdots, W_{t_m}\}$ for a collection of time instances: $0 \le t_1 < \cdots < t_n < \theta < t_{n+1} < \cdots < t_m \le T$ when we know $\theta_T = \theta$, $\max_{0\le u\le T} W_u = b$ and $W_T = w$. This task can be accomplished in two steps: generate two independent sets of $\{B_{t_1}^{1,i}, \cdots, B_{t_n}^{1,i}\}_{i=1,2,3}$ and $\{B_{t_{n+1}}^{2,i}, \cdots, B_{t_m}^{2,i}\}_{i=1,2,3}$ and then substitute them into (5) and (6), respectively. The simulation of Brownian bridges is a standard procedure in the literature. One may refer to Glasserman (2004) for a detailed description.

Evaluation of the conditional expectation L (cf. (4)) is the crucial step for the option pricing problem. In general, its closed-form expression is not available. Here we propose an unbiased estimator to it under an extra assumption about ϕ:

Assumption 2 *There exist two constants $k < K$ such that $k < \phi(x) < K$ for all $x \in (-\infty, \infty)$.*

Under this assumption, we have the following theorem, which lays out a foundation for the construction of an unbiased estimator for L.

Theorem 2 *Suppose that N is a Poisson random number with parameter $K-k$ and $0 \le \tau_1 < \cdots < \tau_N \le T$ are the order statistics of N independent uniform random numbers in $[0,T]$. Given $W_{\tau_1}, \cdots, W_{\tau_N}$, we have*

$$E[\exp(-\int_0^T (\phi(W_s)-k)ds)|W_t, 0 \le t \le T] = E[\prod_{i=1}^N \left(\frac{K-\phi(W_{\tau_i})}{K-k}\right)|W_t, 0 \le t \le T].$$

Following the above theorem, we obtain an algorithm for unbiased estimation of the option price:

1. Generate θ_T, $\max_{0\le u\le T} W_u$ and W_T.
2. Generate $N \sim Poisson(K-k)$ and N independent uniforms $u_1, \cdots, u_N \sim U(0,T)$.
3. Sort $u_1, \cdots, u_N$ to obtain the order statistics $\tau_1 < \cdots < \tau_N$.
4. Simulate $W_{\tau_1}, \cdots, W_{\tau_N}$ for given θ_T, $\max_{0\le u\le T} W_u$ and W_T.
5. Calculate $\hat{L} = \prod_{i=1}^N (K - \phi(W_{\tau_i}) / (K-k)$.

Repeat these 5 steps and we can form an important sampling estimator by taking average across all M samples

$$\frac{1}{M}\sum_{j=1}^{M} g(F^{-1}(W_T^j), F^{-1}(\max_{0\le t\le T} W_t^j)) \cdot \exp(A(W_T^j) - kT) \cdot \hat{L}^j.$$

4 TRUNCATION ERROR ANALYSIS

Unfortunately, many popular models in financial applications do not satisfy Assumption 2. In this section, a truncation method on the function ϕ is proposed to make it fit the preceding simulation algorithm. Of course, that would invite biases into simulation outcomes. However, the error will be negligible if we truncate it properly under some assumptions on the asymptotic behavior of α, the drift coefficient of the transformed process Y. More specifically, introduce the following assumption:

Assumption 3 *The drift function α satisfies either:*
(i). sublinear growth condition: there exists a constant $0 < \beta < 1$ such that $0 < \lim_{y\to+\infty} \alpha(y)/y^\beta < +\infty$ and $-\infty < \lim_{y\to-\infty} \alpha(y)/|y|^\beta < 0$.
or
(ii). mean-reverting condition: there exist positive constant β, E and C such that $\alpha(y) < -Cy^\beta$ for $y > E$ and $\alpha(y) > Cy^\beta$ for $y < -E$.
In addition, we require the function $\phi(y)$ has a lower bound.

Essentially, conditions (i) and (ii) limit the growth rate of $\alpha(y)$ near $\pm\infty$. It either grows slower than a linear function or mean reverts. This assumption is not strong on account of the fact that Y explodes when $\alpha(y)$ grows faster than linearly. Also, we can easily see that lower bound exists when α is a polynomial. Therefore, the assumption covers many processes in financial applications.

Let k be a lower bound of the function ϕ. Select two sufficiently large numbers U^-, U^+ and truncate the original function ϕ as follows:

$$\widetilde{\phi}(y) = \begin{cases} \phi(U^+), & y > U^+; \\ \phi(x), & -U^- \le y \le U^+; \\ \phi(-U^-), & y < -U^-, \end{cases} \tag{7}$$

where we make a convention that $U^+ = +\infty$ ($-U^- = -\infty$) if $\lim_{y\to+\infty} \phi(y) < +\infty$ ($\lim_{y\to-\infty} \phi(y) < +\infty$). The truncated $\widetilde{\phi}$ is bounded between k and $K := \max(\phi(U^+), \phi(-U^-), \max_{y\in[-U^-,U^+]} \phi(y))$. Implementation of the preceding algorithm with replacing ϕ by $\widetilde{\phi}$ will provide us an unbiased estimator for

$$\widetilde{p} = E[g(F^{-1}(W_T), F^{-1}(\max_{0\le t\le T} W_t)) \cdot \exp(A(W_T)) \cdot \exp(-\int_0^T \widetilde{\phi}(W_s)ds)].$$

The following theorem is regarding the error of our truncation.

Theorem 3 *Suppose that the payoff function g satisfies that $E[g^2(F^{-1}(W_T), F^{-1}(\max_{0\le t\le T} W_t))] < +\infty$. Under Assumption 3,*

$$|E[g(S_T, M_T)] - \widetilde{p}| \le C_1(\exp(-C_2(U^+)^2) + \exp(-C_3(U^-)^2))$$

for some positive constants C_1, C_2 and C_3.

5 NUMERICAL EXAMPLES

In this section we present several numerical examples to illustrate the efficiency of our method. Two kinds of models are considered: geometric Brownian motion and Ornstein-Uhlenbeck mean-reverting process. The former one has a constant function ϕ and hence the algorithm after Theorem 2 can be implemented directly. The function $\phi(y)$ of the latter one tends to infinity as $y \to +\infty$. We apply the aforementioned truncation technique on it. The numerical results reveal that our method is accurate. Use two options, lookback put and up-and-in call, as tests. Their payoffs are defined as

$$\max_{0\le t\le T} S_t - S_T \quad \text{and} \quad (S_T-K)^+\mathbf{1}_{\{\max_{0\le t\le T} S_t > b\}}$$

respectively.

5.1 Geometric Brownian motion

In the geometric Brownian motion

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \quad S_0 = s,$$

it is easy to verify that the Lamperti transform is given by $F(y) = \ln(y/s)/\sigma$ and the transformed process Y follows

$$dY_t = \left(\frac{\mu}{\sigma} - \frac{1}{2}\sigma\right)dt + dW_t, \quad Y_0 = 0.$$

Theorem 1 provides us the corresponding measure decomposition for the lookback put and up-and-in call prices under this model. They are

$$\begin{aligned}\text{Lookback} &= e^{-rT}E[\max_{0\le t\le T} S_t - S_T]\\ &= se^{-rT}E[(e^{\sigma \max_{0\le t\le T} W_t} - e^{\sigma W_T})\cdot \exp((\mu/\sigma-\sigma/2)W_T)\cdot\exp(-(\mu/\sigma-\sigma/2)^2 T)]\end{aligned}$$

and

$$\begin{aligned}\text{Up-and-In} &= e^{-rT}E[(S_T-K)^+\cdot\mathbf{1}_{\{\max_{0\le t\le T} S_t > b\}}]\\ &= e^{-rT}E[(se^{\sigma W_T}-K)^+\cdot\mathbf{1}_{\{s\exp(\sigma\max_{0\le t\le T} W_t)>b\}}\exp((\mu/\sigma-\sigma/2)W_T)\cdot\exp(-(\mu/\sigma-\sigma/2)^2 T)]\end{aligned}$$

Table 1 shows a comparison between our algorithm and the Euler scheme. Within comparable computational budget, our method yield better outcomes than the Euler scheme. The average reduction rate of RMSE is 0.3049 for our method when the computational budget increases 16 times, almost the same as the optimal 0.25. But the reduction rate for the Euler scheme is 0.6368 when the budget increases. Table 2 illustrates the results for the up-and-in call option. We can draw similar conclusion by the comparison.

Table 1: Lookback put option under the GBM. $r = 0.1, \sigma = 0.4, S_0 = 50, T = 1$. The true price is 14.9718. The RMSE is calculated based on 10 trials. Here we follow the Duffie-Glynn rule to allocate the budget for the Euler scheme. When the computational budget increases 16 times, generate $16^{1/3} \approx 2.5198$ times more steps within each path and $16^{2/3} \approx 6.3496$ times more paths because the convergence rate of $\max_{1\le i\le N}\hat{S}_i$ to M_T is in the order of $1/\sqrt{N}$.

Wiener Decomposition				Euler Scheme				
SampleNum	Price	Time(s)	RMSE	SampleNum	StepNum	Price	Time(s)	RMSE
256	15.0496	0.125162	0.7111	1625	40	12.8032	0.399455	2.1835
4096	14.9461	2.011741	0.0944	10321	102	13.4937	3.936076	1.4811
65536	14.9866	32.260256	0.0397	65536	256	14.059	42.793687	0.9134
1048576	14.9656	513.613986	0.0085	416128	645	14.3865	574.086363	0.5855
16777216	14.9715	8181.592503	0.0023	2642246	1625	14.6033	8582.284487	0.3686

Table 2: Up-and-in call option under the GBM. $r = 0.1, \sigma = 0.4, S_0 = 50, K = 50, b = 70, T = 1$. The true price is 9.2877. The RMSE is calculated based on 10 trials.

Wiener Decomposition				Euler Scheme				
SampleNum	Price	Time(s)	RMSE	SampleNum	StepNum	Price	Time(s)	RMSE
256	9.7103	0.135501	1.037	1625	40	8.9996	0.424713	0.5148
4096	9.3028	2.017299	0.3893	10321	102	9.0379	4.205301	0.3318
65536	9.2888	32.293416	0.0422	65536	256	9.1352	44.822045	0.1669
1048576	9.2939	519.651756	0.0192	416128	645	9.2034	579.509574	0.0878
16777216	9.2888	8269.052527	0.0053	2642246	1625	9.2409	8663.148775	0.0479

5.2 OU process

Consider a process

$$dS_t = a(b - S_t)dt + \sigma dW_t, \quad S_0 = s,$$

with a, b are two positive constants. Note that the drift is positive when $S_t < b$ and negative when $S_t > b$. Thus S is pulled toward level b, a property generally referred to as mean reversion. Such model is used to model short rates by Vasicek (1977). Its Lamperti transform is $F(y) = (y - s)/\sigma$ and it has an unbounded function

$$\phi(y) = \frac{a^2(b - (\sigma y + s))^2}{\sigma^2} - a.$$

As $y \to \pm\infty$, $\phi(y)$ tends to $+\infty$. Thus truncation is necessary.

Table 3 gives the outcomes of our experiment to price lookback put option under this model. It is worth pointing out that the purpose of this experiment is to test the algorithm's capability of dealing unbounded ϕ. In practice, there is rarely extreme-value-related options traded on basis of short rates.

Table 3: Lookback put option under the OU process. $a = 0.2, b = 0.05, \sigma = 0.1, r = 0.05, S_0 = 0.04, T = 1$. We truncate the process at $-U^- = -6, U^+ = 6$. No analytical expression for the option value is known. We use the estimator derived from the Wiener decomposition to approximate the true price. Simulate 5 million samples and the approximate price is 0.0728. The RMSE is calculated based on 10 trials.

Wiener Decomposition				Euler Scheme				
SampleNum	Price	Time(s)	RMSE	SampleNum	StepNum	Price	Time(s)	RMSE
640	0.073	2.780127	0.0021	10321	102	0.0674	3.778244	0.0054
10240	0.0727	33.83485	4.32×10^{-4}	65536	256	0.0695	44.065133	0.0033
163840	0.0729	531.035579	1.11×10^{-4}	416128	645	0.0706	583.816921	0.0022
2621440	0.0728	8642.666819	3.22×10^{-5}	2642246	1625	0.0714	8863.63471	0.0014

We can see that our algorithm outperforms the Euler scheme once again in terms of the reduction rate of RMSE. The RMSE of our method will shrink at an average speed of 0.259 as the computational budget increases 16 times. However, the Euler scheme decreases the RMSE at a rate of 0.6380.

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A PROOFS OF THE THEOREMS

Proof of Theorem 1. Denote Use a generalized Girsanov theorem (Karatzas and Shreve (1991), Exercise 5.5.28) that for every finite $T>0$, any $x,y\in\mathbf{R}, x<y$,

$$P[Y_T\in dx, M_T\in dy, \lim_{n\to+\infty} T_n>T] = E\left[\exp\left(\int_0^T \alpha(W_u)dW_u - \frac{1}{2}\int_0^T \alpha^2(W_u)du\right)\cdot \mathbf{1}_{\{W_T\in dx, \max_{0\le t\le T} W_t\in dy\}}\right], \tag{8}$$

where $T_n=\inf\{t\ge 0: |Y_t|=n\}$.

Applying Ito's lemma on $A(W_t)$ for any t will yield that

$$\int_0^t \alpha(W_u)dW_u = A(W_t)-A(x)-\frac{1}{2}\int_0^t \alpha'(W_u)du = A(W_t)-\frac{1}{2}\int_0^t \alpha'(W_u)du. \tag{9}$$

On the other hand, Assumptions 1 and 2 preclude the possibility that Y explodes in finite time. Therefore, $P[\lim_{n\to+\infty} T_n > T > T]=1$. Combining (8) and (9),

$$P[Y_T\in dx, M_T\in dy] = E\left[\exp\left(A(W_T)-\int_0^T \phi(W_u)du\right)\cdot \mathbf{1}_{\{W_T\in dx, \max_{0\le t\le T} W_t\in dy\}}\right],$$

where $\phi=(\alpha^2+\alpha')/2$. From this, we can easily complete the proof of Theorem 1. $\square$

Proof of Theorem 2. Note $N\sim Poisson(L-l)$ and $(\tau_1,\cdots,\tau_N)$ are the order statistics of N independent uniforms in $(0,T)$. Conditional on the whole sample path of $\{W_t, 0\le t\le T\}$,

$$E[\prod_{i=1}^N\left(\frac{K-\phi(W_{\tau_i})}{K-k}\right)|W_t, 0\le t\le T] \quad = \quad E\left[\sum_{N=0}^{+\infty}\frac{e^{-(K-k)T}((K-k)T)^N}{N!}\left(\frac{1}{T}\int_0^T\left[\frac{K-\phi(W_u)}{K-k}\right]du\right)^N|W_t, 0\le t\le T\right]$$

Some routine algebra shows the right hand side of the above equality is equal to

$$E[\exp(-\int_0^T(\phi(W_u)-k)du)|W_t, 0\le t\le T].\ \square$$

Proof of Theorem 3. We know that

$$\begin{aligned} & |E[g(S_T,M_T)]-\widetilde{p}| \\ \le\ & E[|g(F^{-1}(W_T),F^{-1}(\max_{0\le t\le T} W_t))|\cdot\exp(A(W_T))\cdot\exp(-\int_0^T\widetilde{\phi}(W_s)ds)\cdot|1-\exp(-\int_0^T(\phi(W_s)-\widetilde{\phi}(W_s))ds)|] \\ \le\ & E[|g(F^{-1}(W_T),F^{-1}(\max_{0\le t\le T} W_t))|^2]^{1/2}\cdot E[\exp(2A(W_T))\cdot\exp(-\int_0^T 2\widetilde{\phi}(W_s)ds)\cdot|1-\exp(-\int_0^T(\phi(W_s)-\widetilde{\phi}(W_s))ds)|^2]^{1/2}, \end{aligned} \tag{10}$$

where the last inequality holds because of Cauchy-Schwartz inequality.

By the condition of the theorem, $E[|g(F^{-1}(W_T),F^{-1}(\max_{0\le t\le T} W_t))|^2]^{1/2}<+\infty$. The truncation rule presented in Section 4 implies that $\phi\ge\widetilde{\phi}$ and that

$$\exp(-\int_0^T(\phi(W_s)-\widetilde{\phi}(W_s))ds)<1$$

if and only if $\max_{0\le t\le T} W_t>U^+$ or $\min_{0\le t\le T} W_t<-U^-$. Furthermore, $\widetilde{\phi}\ge l$, the lower bound of ϕ. So, the second term in (10) is bounded by

$$\exp(-lT)\cdot E[\exp(2A(W_T))\cdot\mathbf{1}_{\{\max_{0\le t\le T} W_t>U^+ \text{ or } \min_{0\le t\le T} W_t<-U^-\}}]^{1/2}.$$

When α satisfies condition (i) in Assumption 3, $A(y)$ grows at most at the rate of $|y|^{\beta+1}$. The expectation

$$E[\exp(2A(W_T))\cdot \mathbf{1}_{\{\max_{0\le t\le T} W_t > U^+ \text{ or } \min_{0\le t\le T} W_t < -U^-\}}] \le E[\exp(|W_T|^{\beta+1})\cdot \mathbf{1}_{\{\max_{0\le t\le T} W_t > U^+\}}] + E[\exp(|W_T|^{\beta+1})\mathbf{1}_{\{\min_{0\le t\le T} W_t < -U^-\}}].$$

The joint distribution of $(W_T, \max_{0\le t\le T} W_t)$ is known explicitly (see, e.g., Proposition 2.8.1, Karatzas and Shreve (1991)). Thus,

$$E[\exp(|W_T|^{\beta+1})\cdot \mathbf{1}_{\{\max_{0\le t\le T} W_t > U^+\}}] = \frac{2}{\sqrt{2\pi T^3}}\int_{U^+}^{+\infty}\int_{-\infty}^{b} \exp(|a|^{\beta+1})(2b-a)\exp\left(-\frac{(2b-a)^2}{2T}\right) dadb. \tag{11}$$

Under a change of variable $u = b-a$, the integral on the right hand side of (11) will be equal to

$$\begin{aligned} & \frac{2}{\sqrt{2\pi T^3}}\int_{U^+}^{+\infty}\int_{0}^{\infty} \exp(|b-u|^{\beta+1})(b+u)\exp\left(-\frac{(b+u)^2}{2T}\right) dudb \\ \le \quad & \int_{U^+}^{+\infty}\int_{0}^{\infty} \exp(c|u|^{\beta+1}+c|b|^{\beta+1})(b+u)\exp\left(-\frac{(b+u)^2}{2T}\right) dudb \end{aligned}$$

for a constant c. Note that the term $\exp\left(-\frac{(b+u)^2}{2T}\right)$ decays to zero faster than any other terms in the integral. We can show that the integral will be dominated by $C_1\exp(-C_2(U^+)^2)$ for sufficiently large U^+. Emulating the same arguments, we can obtain a similar upper bound for $E[\exp(|W_T|^{\beta+1})\cdot \mathbf{1}_{\{\min_{0\le t\le T} W_t < -U^-\}}]$.

When α satisfies condition (ii) in Assumption 3, $\alpha(y)$ will be positive if $y < -E$ and negative if $y > E$. So, $A(y) = \int_0^y \alpha(u)du$ has a global maximum, which yields that

$$E[\exp(2A(W_T))\cdot \mathbf{1}_{\{\max_{0\le t\le T} W_t > U^+ \text{ or } \min_{0\le t\le T} W_t < -U^-\}}] \le MP[\max_{0\le t\le T} W_t > U^+ \text{ or } \min_{0\le t\le T} W_t < -U^-].$$

Note that the right hand side of the above inequality decays exponentially as U^+ or $-U^-$ tend to ∞. In summary, we show that the theorem is true for both conditions of α. □

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