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On the complexity of the labeled domination problem in graphs

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Abstract

In 2008, a unified approach (labeled domination) to several domination problems (k-tuple domination, $\{k\}$-domination, and M-domination, among others) was introduced. The labeled domination problem is to find an L-dominating function of minimum weight in a graph. It is an NP-complete problem even when restricted to split graphs and bipartite graphs. On the other hand, it is known to be polynomial-time solvable for the class of strongly chordal graphs. In this paper, we state explicit formulas that relate the domination numbers considered. These relationships allow us to enlarge the family of graphs where the labeled domination problem is polynomial-time solvable to the class of graphs having cliquewidth bounded by a constant.

Keywords: computational complexity; labeled dominating function; k-tuple dominating function; $\{k\}$-dominating function

1. Introduction

Due to its large range of applications, many variations and extensions of the classical domination problem in graphs have been defined and studied. In fact, a remarkable aspect of the study of domination in graphs is related to the inclusion of additional constraints that model the problem under consideration (see Haynes et al., 1998 for a survey).

In general, these problems can be modeled by defining a set of functions from the vertex set of a graph to a subset of real numbers, satisfying the required extra constraints.

In this work, we focus on the k-tuple domination problem, $\{k\}$-domination problem, and two labeled domination problems: M-domination and L-domination.

All the graphs in this paper are finite and simple. Given a graph G, $V(G)$ and $E(G)$ denote its vertex and edge sets, respectively. For any $v \in V(G)$, $N_G(v)$ is the *open neighborhood* of v in G, $N_G[v]$ is the *closed neighborhood* of v in G (i.e., $N_G[v] = N_G(v) \cup \{v\}$) and $d_G(v) = |N_G(v)|$. The *weight* of a function f on $V(G)$ is $f(V(G)) = \sum_{v\in V(G)} f(v)$.

Let G be a graph. Given a nonnegative integer k, a function $f : V(G) \mapsto \{0, 1\}$ is a *k-tuple dominating function* of G if $f(N_G[v]) \geq k$ for all $v \in V(G)$. The minimum possible weight of a

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Table 1
Complexity table for k-DOM, $\{k\}$-DOM

Class	k-DOM (fixed $k \in \mathbb{Z}_+$)	$\{k\}$-DOM (fixed $k \in \mathbb{Z}_+$)
Strongly chordal	P (Liao and Chang, 2003)	P (Lee and Chang, 2008)
Doubly chordal	NP-c (Lee and Chang, 2008)	P (Lee and Chang, 2008)
Dually chordal	NP-c (Lee and Chang, 2008)	P (Lee and Chang, 2008)
Cographs	P (Lee and Chang, 2008)	P (Lee and Chang, 2008)
P_4-tidy	P (Dobson et al., 2011)	P (Argiroffo et al., 2015)
Bounded cliquewidth	P (Argiroffo et al., 2015)	P (Argiroffo et al., 2015)
Split	NP-c (Dobson et al., 2011)	NP-c (Argiroffo et al., 2015)
Planar	NP-c (Lee and Chang, 2008)	NP-c (Argiroffo et al., 2015)
Chordal	NP-c (Dobson et al., 2011)	NP-c (Argiroffo et al., 2015)
Bipartite	NP-c (Dobson et al., 2011)	NP-c (He and Liang, 2011)

"NP-c" and "P" mean NP-complete and polynomial, respectively.

k-tuple dominating function of G is called the *k-tuple domination number* of G and is denoted by $\gamma_{\times k}(G)$ (Harary and Haynes, 1996).

A function $f : V(G) \mapsto \{0, 1, \ldots, k\}$ is a *$\{k\}$-dominating function* of G if $f(N_G[v]) \geq k$ for all $v \in V(G)$. The minimum possible weight of a $\{k\}$-dominating function of G is called the *$\{k\}$-domination number* of G and it is denoted by $\gamma_{\{k\}}(G)$ (Bange et al., 1996).

As usual, these definitions induce the study of the following decision problems for a nonnegative fixed integer k:

k-**TUPLE DOMINATING FUNCTION** (k-DOM)
Instance: $G = (V, E)$, $j \in \mathbb{N}$
Question: Does G have a k-tuple dominating function of weight at most j?

$\{k\}$-**DOMINATING FUNCTION** ($\{k\}$-DOM)
Instance: $G = (V, E)$, $j \in \mathbb{N}$
Question: Does G have a $\{k\}$-dominating function of weight at most j?

These problems are NP-complete for general graphs and have been widely studied. The so far known computational complexity results involving k-DOM and $\{k\}$-DOM are summarized in Table 1.

One method for solving domination problems is the (vertex) labeling technique that has proved to be very efficient. In Liao and Chang (2002), a labeled domination was introduced: an M labeling function on a graph G assigns a label $M(v) = (t(v), k(v))$ to each vertex v of G, where $t(v) \in \{F, R\}$ and $k(v)$ is a nonnegative integer. An *M-dominating function* of G is a function $f : V(G) \mapsto \{0, 1\}$ such that:

1. if $t(v) \neq F$, then $f(v) = 1$,
2. $f(N_G[v]) \geq k(v)$ for all $v \in V(G)$.

The *M-domination number* of G, $\gamma_M(G)$, is the minimum weight of an M-dominating function of G. The associated decision problem can be formulated as follows:

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Table 2
Complexity table for M-DOM and L-DOM

Class	M-DOM	L-DOM (fixed $l \in \mathbb{N}$)
Strongly chordal	P (Liao and Chang, 2003)	P (Lee and Chang, 2008)
Doubly chordal	NP-c (Lee and Chang, 2008)	NP-c (Lee and Chang, 2008)
Dually chordal	NP-c (Lee and Chang, 2008)	NP-c (Lee and Chang, 2008)
Cographs	?	?
P_4-tidy	?	?
Bounded cliquewidth	?	?
Split	NP-c (Liao and Chang, 2003)	NP-c (Liao and Chang, 2003)
Planar	NP-c (Lee and Chang, 2008)	NP-c (Lee and Chang, 2008)
Chordal	NP-c (Liao and Chang, 2003)	NP-c (Lee and Chang, 2008)
Bipartite	NP-c (Liao and Chang, 2003)	NP-c (Lee and Chang, 2008)

"NP-c", "P," and "?" mean NP-complete, polynomial, and open problem, respectively.

***M*-DOMINATION PROBLEM** (M-DOM)
Instance: A graph G, a labeling M, $j \in \mathbb{N}$.
Question: Does G have an M-dominating function of weight at most j?

More recently, a unified approach to the concepts of $\{k\}$-dominating, k-tuple dominating, and M-dominating functions has been introduced (see Lee and Chang, 2008): let l, d, I_1 be integers and $l, d \geq 1$. Let $Y = \{I_1, I_1 + d, I_1 + 2d, \ldots, I_1 + (l-1)d\}$. Suppose that G is a graph and L is a labeling function that assigns to each $v \in V(G)$ a label $L(v) = (t(v), k(v))$, where $t(v) \in \{F\} \cup Y$ and $k(v)$ is an integer. An *L-dominating function* of G is a function $f : V(G) \mapsto Y$ satisfying the following two conditions:

1. if $t(v) \neq F$, then $f(v) = t(v)$,
2. $f(N_G[v]) \geq k(v)$ for all $v \in V(G)$.

The *L-domination number* of G, $\gamma_L(G)$, is the minimum weight of an L-dominating function of G.

Clearly, if k is a nonnegative integer, $Y = \{0, 1\}$ and every vertex $v \in V(G)$ has label $L(v) = (F, k)$, then an L-dominating function of G is a k-tuple dominating function of G.

Moreover, if k is a nonnegative integer, $Y = \{0, \ldots, k\}$ and every vertex $v \in V(G)$ has label $L(v) = (F, k)$, then an L-dominating function of G is a $\{k\}$-dominating function of G.

Finally, if $Y = \{0, 1\}$ and every vertex $v \in V(G)$ has label $L(v) = (t(v), k(v))$ with $t(v) \in \{F, 1\}$ and $k(v)$ is a nonnegative integer, then an L-dominating function of G is an M-dominating function of G.

The corresponding decision problem for L-domination can be formulated as follows:

LABELED DOMINATION PROBLEM (L-DOM)
Instance: A graph G, a labeling L, I_1, $j \in \mathbb{Z}$, $d, l \in \mathbb{N}$.
Question: Does G have an L-dominating function of weight at most j?

The already known complexity results involving M-DOM and L-DOM are summarized in Table 2.

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One of the main purposes of this work is to answer the unknown computational complexities concerning L-DOM that are included in Table 2, some of which were left as open questions in Lee and Chang (2008). Nevertheless, we also introduce an "intermediate" labeled domination—W-domination—that establishes a bridge between L-DOM and M-DOM. Our results reveal that W-domination is as general as L-domination itself, since the transformations performed in the reductions do not involve changes in the input graph. In this way and from a computational complexity point of view, we can say that L-domination is "as hard as" W-domination.

In addition, we develop polynomial-time reductions from L-DOM to one of its especial cases, k-DOM.

The paper is organized as follows. In Section 2, we study several relationships among the domination numbers involved. In Section 3, the relationships obtained allow us to enlarge the family of graphs where L-DOM (fixed l) and M-DOM are polynomial-time solvable to the class of graphs having cliquewidth bounded by a constant.

2. Relationships among the domination numbers

In this section, we relate the L-domination, the M-domination, and the k-tuple domination numbers. We proceed in two steps.

2.1. From L-domination to M-domination

In this first step, we introduce an intermediate labeled domination problem (having the same computational complexity than L-DOM, as Corollary 4) that allow us to relate L-DOM and M-DOM.

Theorem 1. *Let G be a graph and let L be the labeling function with parameters l, d, I_1, and $L(v) = (t(v), k(v))$ for all $v \in V(G)$.*

Let L' be the labeling function $L'(v) = (t'(v), k'(v))$ with parameters $I'_1 = 0, l' = l, d' = d$ and such that

1. *if $t(v) = F$, then $t'(v) = F$,*
2. *if $t(v) \neq F$, then $t'(v) = t(v) - I_1$,*
3. *$k'(v) = k(v) - I_1(d_G(v) + 1)$ for all $v \in V(G)$.*

Then

$$\gamma_L(G) = \gamma_{L'}(G) + I_1|V(G)|.$$

Proof. Consider the labeling function L' described above.

Let $f : V(G) \mapsto \{I_1, I_1 + d, \dots, I_1 + (l-1)d\}$ be an L-dominating function of G.

We define

$$f' : V(G) \mapsto \{0, d, \dots, (l-1)d\}$$
$$f'(v) = f(v) - I_1.$$

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We have:

1. if $t'(v) \neq F$, then $t(v) \neq F$ and $f(v) = t(v)$. Thus $f'(v) = f(v) - I_1 = t(v) - I_1 = t'(v)$.
2. $f'(N_G[v]) = f(N_G[v]) - I_1(d_G(v)+1) \geq k(v) - I_1(d_G(v)+1) = k'(v)$ for all $v \in V(G)$.

Then f' is an L'-dominating function of G and $f'(G) = f(G) - I_1|V(G)|$, implying

$$\gamma_{L'}(G) \leq \gamma_L(G) - I_1|V(G)|.$$

Analogously, it is easy to verify that if f' is an L'-dominating function of G then the function

$$\begin{aligned} f &: V(G) \mapsto \{I_1, I_1 + d, \ldots, I_1 + (l-1)d\} \\ f(v) &= f'(v) + I_1 \end{aligned}$$

is an L-dominating function of G and $f(G) = f'(G) + I_1|V(G)|$. Therefore $\gamma_L(G) \leq \gamma_{L'}(G) + I_1|V(G)|$ and the proof is complete. □

Now, let us introduce the following "intermediate labeling."

Definition 2. *Let k be a nonnegative integer and let W be a labeling function of a graph G such that $W(v) = (t(v), k(v))$, where $t(v) \in \{F\} \cup \{0, \ldots, k\}$ and $k(v)$ is an integer for all $v \in V(G)$. A W-dominating function of G is a function $f : V(G) \mapsto \{0, \ldots, k\}$ satisfying the following two conditions:*

1. *if $t(v) \neq F$, then $f(v) = t(v)$,*
2. *$f(N_G[v]) \geq k(v)$ for all $v \in V(G)$.*

The W-*domination number* $\gamma_W(G)$ is the minimum weight of a W-dominating function of G. Note that every W-dominating function is an L-dominating function with $I_1 = 0$, $l = k+1$, and $d = 1$. Also, note that the concept of W-domination generalizes $\{k\}$-domination, k-tuple domination, and M-domination. We introduce the associated decision problem:

W-**DOMINATION PROBLEM** (W-DOM)
Instance: A graph G, a labeling W, k, $j \in \mathbb{N}$.
Question: Does G have a W-dominating function of weight at most j?

The L-domination and W-domination numbers for a graph G are related as the following result shows.

Theorem 3. *Let G be a graph and let L be the labeling function with parameters l, d, $I_1 = 0$, and $L(v) = (t(v), k(v))$ for $v \in V(G)$. Let also W be the labeling function such that for each $v \in V(G) W(v) = (t'(v), k'(v))$, with parameter $k = l - 1$ and*

1. *if $t(v) = F$, then $t'(v) = F$,*
2. *if $t(v) \neq F$, then $t'(v) = \frac{t(v)}{d}$, and*
3. *$k'(v) = \left\lceil \frac{k(v)}{d} \right\rceil$ for all $v \in V(G)$.*

Then

$$\gamma_L(G) = d\gamma_W(G).$$

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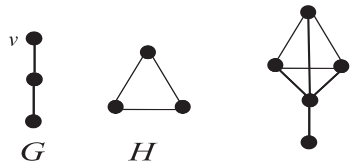

Fig. 1. Replacement in G of v by H.

Proof. Consider the labeling function W defined above. Let $f : V(G) \mapsto \{0, d, \ldots, (l-1)d\}$ be an L-dominating function of G. We define

$$f' : V(G) \mapsto \{0, \ldots, l-1\}$$
$$f'(v) = \frac{f(v)}{d}.$$

We have:

1. if $t'(v) \neq F$, then $t(v) \neq F$ and $f(v) = t(v)$. Therefore, $f'(v) = \frac{f(v)}{d} = \frac{t(v)}{d} = t'(v)$.
2. $f'(N_G[v]) = \frac{f(N_G[v])}{d} \geq \frac{k(v)}{d}$. Notice that $\frac{f(N_G[v])}{d}$ is integer for all $v \in V(G)$. Then, for all $v \in V(G)$

$$f'(N_G[v]) \geq \left\lceil \frac{k(v)}{d} \right\rceil = k'(v).$$

Hence f' is a W-dominating function of G and $f'(G) = \frac{f(G)}{d}$, implying

$$d\gamma_W(G) \leq \gamma_L(G).$$

Consider now a W-dominating function of G,
$f' : V(G) \mapsto \{0, \ldots, l-1\}$. We define
$f : V(G) \mapsto \{0, d, \ldots, (l-1)d\}$ as $f(v) = f'(v)d$. We have

1. If $t'(v) \neq F$, then $t(v) \neq F$ and $f'(v) = t'(v)$. Then $f(v) = f'(v)d = t'(v)d = t(v)$.
2. $f(N_G[v]) = f'(N_G[v])d \geq k'(v)d = \left\lceil \frac{k(v)}{d} \right\rceil d \geq \frac{k(v)}{d}d = k(v)$, for all $v \in V(G)$.

Hence f is an L-dominating function of G and $f(G) = f'(G)d$. We obtain

$$\gamma_L(G) \leq d\gamma_W(G).$$ □

Observe that in Theorems 1 and 3, the transformations performed do not involve changes in the input graph. As a consequence of these results we obtain the following.

Corollary 4. *L-DOM and W-DOM have the same computational complexity.*

In order to study the relationship between the W-domination and the M-domination numbers, let us consider the following known graph operation (see Fig. 1): for disjoint graphs G and H and $v \in V(G)$, $G[H/v]$ denotes the graph obtained by the *replacement* in G of v by H, that is,

$$V(G[H/v]) = (V(G) - \{v\}) \cup V(H)$$

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and

$$E(G[H/v]) = E(H) \cup$$
$$\{e : e \in E(G) \text{ and } e \text{ is not incident with } v\} \cup$$
$$\{uw : u \in V(H),\ w \in V(G) \text{ and } vw \in E(G)\}.$$

Now, let k be a nonnegative integer. Since every W-dominating function is nonnegative, we can consider $k(v) \geq 0$ for every v in its corresponding labeling function. We introduce the following graph transformation.

Definition 5 (Transformation G_W). *Given a graph G and a labeling function W such that $W(v) = (t(v), k(v))$, with $t(v) \in \{F\} \cup \{0, \ldots, k\}$ and $k(v) \geq 0$, we define the graph G_W obtained by replacing each vertex v in G by a graph G_v defined as follows:*

1. *if $t(v) = F$, then G_v is isomorphic to the complete graph on $k+1$ vertices K_{k+1} and $V(G_v) = \{v_1, \ldots, v_{k+1}\}$, and*
2. *if $t(v) \neq F$, then G_v is isomorphic to $K_{t(v)+1}$ and $V(G_v) = \{v_1, \ldots, v_{t(v)+1}\}$.*

Besides, for each vertex u in G_W, consider the label $M(u) = (t_M(u), k_M(u))$ with $t_M(u) \in \{F, R\}$ and $k_M(u)$ a nonnegative integer such that:

1. *if $t(v) = F$, then $t_M(v_1) = R, t_M(v_i) = F$ for $i = 2, \ldots, k+1$ and $k_M(v_i) = k(v) + d_G(v) + 1$ for $i = 1, \ldots, k+1$,*
2. *if $t(v) \neq F$, then $t_M(v_i) = R$ and $k_M(v_i) = k(v) + d_G(v) + 1$ for $i = 1, \ldots, t(v) + 1$.*

The W-domination number of G and the M-domination number of G_W are related as the following theorem shows.

Theorem 6. *For every graph G and labeling function W, $\gamma_W(G) = \gamma_M(G_W) - |V(G)|$.*

Proof. Let f be a W-dominating function of G and let G_W and M as in Definition 5.
We construct a function $f_M : V(G_W) \mapsto \{0, 1\}$ as follows:

1. if $t(v) \neq F$, then $f_M(v_i) = 1$ for all $i \in \{1, \ldots, t(v) + 1\}$,
2. if $t(v) = F$, then $f_M(v_i) = 1$ if $i \in \{1, \ldots, f(v) + 1\}$ and $f(v_i) = 0$ otherwise.

Let us prove that f_M is an M-dominating function of G_W.
Let $u \in V(G_W)$, then $u \in V(G_v)$ for some $v \in V(G)$.
If $t_M(u) = R$ and $t(v) = F$, then $u = v_1$ and from (2) above, $f_M(u) = 1$ since $f(v) + 1 \geq 1$. If $t_M(u) = R$ and $t(v) \neq F$, we have $u = v_i$ for some $i = 1, \ldots, t(v) + 1$ and then, from (1) above, $f_M(u) = 1$.

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Besides, for every $u \in V(G_W)$ we have,

$$
\begin{aligned}
f_M(N_{G_W}[u]) &= \sum_{\substack{x \in N_G[v] \\ t(x)=F}} f_M(V(G_x)) + \sum_{\substack{x \in N_G[v] \\ t(x)\neq F}} f_M(V(G_x)) \\
&= \sum_{\substack{x \in N_G[v] \\ t(x)=F}} (f(x)+1) + \sum_{\substack{x \in N_G[v] \\ t(x)\neq F}} (t(x)+1) \\
&= \sum_{\substack{x \in N_G[v] \\ t(x)=F}} (f(x)+1) + \sum_{\substack{x \in N_G[v] \\ t(x)\neq F}} (f(x)+1) \\
&= f(N_G[v]) + |N_G[v]| \geq k(v) + d_G(v) + 1 = k_M(u).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
f_M(V(G_W)) &= \sum_{\substack{v \in V(G) \\ t(v)=F}} f_M(V(G_v)) + \sum_{\substack{v \in V(G) \\ t(v)\neq F}} f_M(V(G_v)) \\
&= \sum_{\substack{v \in V(G) \\ t(v)=F}} (f(v)+1) + \sum_{\substack{v \in V(G) \\ t(v)\neq F}} (t(v)+1) \\
&= \sum_{\substack{v \in V(G) \\ t(v)=F}} (f(v)+1) + \sum_{\substack{v \in V(G) \\ t(v)\neq F}} (f(v)+1) \\
&= f(G) + |V(G)|.
\end{aligned}
$$

Therefore, $\gamma_W(G) + |V(G)| \geq \gamma_M(G_W)$.

Conversely, let f_M be an M-dominating function of G_W. Consider the function f in $V(G)$ such that $f(v) = f_M(V(G_v)) - 1$. From definition, it holds $f(v) \in \{0, \dots, k\}$ for all $v \in V(G)$.

Finally, it is easy to check that f is an L-dominating function of G and $f(G) = f_M(V(G_M)) - |V(G)|$. Then $\gamma_W(G) \leq \gamma_M(G_W) - |V(G)|$.

Hence, $\gamma_W(G) = \gamma_M(G_W) - |V(G)|$. □

As a consequence of Theorems 3 and 6 we have,

Corollary 7. *Let G be a graph and let L be the labeling function with parameters l, d, I_1, and $L(v) = (t(v), k(v))$ for all $v \in V(G)$. Then*

$$\gamma_L(G) = d\gamma_W(G) = d(\gamma_M(G_W) - |V(G)|).$$

Hence, if $\mathcal{F}$ is a graph class where M-DOM is polynomial-time solvable and $\mathcal{H}$ is a graph class such that $G_W \in \mathcal{F}$ for all $G \in \mathcal{H}$ and labeling W, then W-DOM (fixed k) is polynomial-time solvable for $\mathcal{H}$. From this fact and Corollary 4 we have the following result.

Corollary 8. *Let $\mathcal{F}$ be a graph class where M-DOM is polynomial-time solvable. If $\mathcal{H}$ is a graph class such that $G_W \in \mathcal{F}$ for all $G \in \mathcal{H}$ and labeling W, then L-DOM (fixed l) is polynomial-time solvable for $\mathcal{H}$.*

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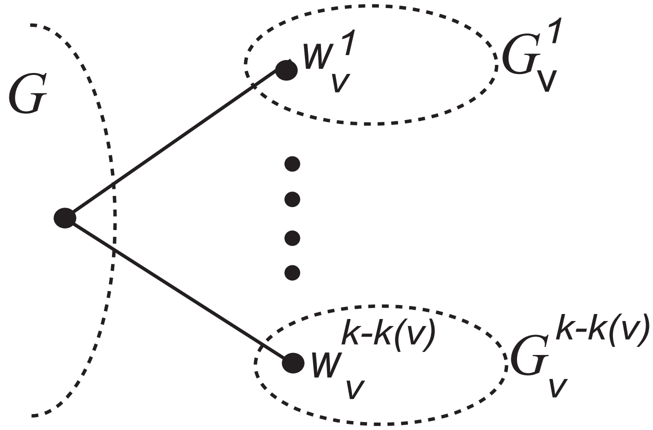

Fig. 2. Transformation $H(G)$ in Definition 10.

2.2. From M-domination to k-tuple domination

In this second step, in order to analyze the relationship between the M-domination and the k-tuple domination numbers, we consider some other graph transformations.

Definition 9 (Transformation G'). *Let G be a graph and let M be a labeling function of G such that $M(v) = (t(v), k(v))$ where $t(v) \in \{F, R\}$ and $k(v)$ is a nonnegative integer. If f is an M-dominating function of G, we define the graph G' whose vertex set is obtained by adding to $V(G)$ a new vertex u_v for each $v \in V(G)$ with $t(v) = R$, and $E(G') = E(G) \cup \bigcup_{v/t(v)=R} vu_v$, that is, we add a pendant vertex to each vertex $v \in V(G)$ with label $t(v) = R$.*

Consider the labeling $M' = (t'(u), k'(u))$ in G' such that $t'(u) = F$ for all $u \in V(G')$ and

1. $k'(u) = k(u)$ for every $u \in V(G)$, and
2. $k'(u_v) = 1$ for each $u_v \in V(G') - V(G)$.

It is not hard to prove that $\gamma_M(G) = \gamma_{M'}(G')$ (if $\gamma_M(G)$ exists).

Definition 10 (Transformation $H(G)$). *Let G be a graph and let M be a labeling function of G such that $M(v) = (t(v), k(v))$ where $t(v) \in \{F, R\}$ and $k(v)$ is a nonnegative integer. Let $k = \max\{2, \max\{k(v) : v \in V(G)\}\}$. We define a graph $H(G)$ by adding to each vertex $v \in V(G)$ the graphs G_v^i, isomorphic to K_k, and an edge vw_v^i for some $w_v^i \in V(G_v^i)$, for $i = 1, \dots, k - k(v)$. An example of this construction is given in Fig. 2.*

Theorem 11. *For every graph G,*

$$\gamma_M(G) = \gamma_{\times k}(H(G)) - (|V(H(G))| - |V(G)|).$$

Proof. Observe that

$$|V(H(G))| - |V(G)| = k(|V(G)|k - \sum_{v \in V(G)} k(v)).$$

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Let f_M be an M-dominating function of G. Consider $f : V(H(G)) \mapsto \{0, 1\}$ such that $f(w) = 1$ for all $w \in V(H(G)) - V(G)$ and for every $v \in V(G)$, $f(v) = f_M(v)$. Then, it is clear that $f(N_{H(G)}[w]) \geq k$ for all $w \in V(H(G)) - V(G)$. Besides, if $v \in V(G)$,

$$\begin{aligned} f(N_{H(G)}[v]) &= \sum_{i=1}^{k-k(v)} f(w_v^i) + \sum_{u \in N_G[v]} f(u) \\ &= k - k(v) + f_M(N_G[v]) \geq k - k(v) + k(v) = k. \end{aligned}$$

Therefore, f is a k-tuple dominating function of $H(G)$ and $f(H(G)) = f_M(V(G)) + |V(H(G))| - |V(G)|$. We conclude that

$$\gamma_M(G) \geq \gamma_{\times k}(H(G)) - (|V(H(G))| - |V(G)|).$$

On the other hand, as $k \geq 2$ from definition, every k-tuple dominating function g of $H(G)$ satisfies $g(w) = 1$ for all $w \in V(H(G)) - V(G)$.

Then, let f be a k-tuple dominating function of $H(G)$. We define f_M as the restriction of f to $V(G)$. In this case, we have $f_M(V(G)) = f(H(G)) - (|V(H(G))| - |V(G)|)$. Thus, $\gamma_M(G) \leq \gamma_{\times k}(H(G)) - (|V(H(G))| - |V(G)|)$.

We conclude that,

$$\gamma_M(G) = \gamma_{\times k}(H(G)) - (|V(H(G))| - |V(G)|). \qquad \square$$

Combining Corollaries 4 and 8 and Theorem 11, we obtain the following result.

Corollary 12. *For $k \in \mathbb{Z}_+$, assume that k-DOM is polynomial-time solvable for a graph class $\mathcal{F}$. If $\mathcal{H}$ is a graph class such that $H(G'_W) \in \mathcal{F}$ for all $G \in \mathcal{H}$ and labeling W, then L-DOM (fixed l) is polynomial-time solvable for $\mathcal{H}$.*

3. L-DOM for graphs with cliquewidth bounded by a constant

We close this paper by showing that the tractability of k-DOM and $\{k\}$-DOM for graphs with cliquewidth bounded by a constant can be extended to all labeled dominations studied in this work.

Let us briefly recall the notion of the cliquewidth of a graph.

To every graph G, it can be associated an algebraic expression built using the following operations (Courcelle et al., 1993):

1. creation of a vertex with label i ($i(v)$),
2. disjoint union ($\oplus$),
3. renaming label i to label j ($\rho_{i \to j}$),
4. connecting all vertices with label i to all vertices with label j, for $i \neq j$ ($\eta_{i,j}$).

If all the labels in the expression of G are in $\{1, \ldots, q\}$ for $q \in \mathbb{N}$, it is called a *q-expression* of G. It is clear that there is a $|V(G)|$-expression that defines G, for every graph G.

The *cliquewidth* of a graph G, denoted by $cwd(G)$, is defined by

$$cwd(G) = \min\{q : G \in \mathcal{C}(q)\},$$

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where $\mathcal{C}(q)$ is the graph class that can be defined by q-expressions. Clearly, if G has at least one edge then $cwd(G) \geq 2$. Besides, $cwd(K_n) \leq 2$ for every n and it is known that cographs, trees, and P_4-tidy graphs have cliquewidth at most 2, 3, and 4, respectively. Concerning distance-hereditary graphs, they have cliquewidth bounded by 3 (Golumbic and Rotics, 1999).

Many optimization problems that are NP-hard for more general graph classes may be solved efficiently on graphs of cliquewidth bounded by a constant (Courcelle et al., 2000). In particular, in Argiroffo et al. (2015) it is proved the following result.

Theorem 13 (Argiroffo et al., 2015). *Let $k \in \mathbb{Z}_+$, q be a constant and $\mathcal{C}(q)$ be a class of graphs of cliquewidth at most q. Then k-DOM and $\{k\}$-DOM on $\mathcal{C}(q)$ can be solved in polynomial time on $\mathcal{C}(q)$.*

Recall the following useful property concerning the cliquewidth of a graph (Courcelle et al., 2000):

$$cwd(G[H/v]) = \max\{cwd(G), cwd(H)\} \tag{1}$$

for every pair of disjoint graphs G and H and $v \in V(G)$.

In first place, we can prove the following result.

Lemma 14. *Let $\mathcal{G}$ be a family of graphs having cliquewidth bounded by a constant and consider the transformations in Definitions 5, 9, and 10. Then the graphs in $\mathcal{H} = \{H(G'_W) : G \in \mathcal{G}\}$ have cliquewidth bounded by a constant.*

Proof. Let $G \in \mathcal{G}$. Taking into account that $cwd(K_k) = 2$ for every $k \geq 2$, from (1) above we have $cwd(G_W) \leq \max\{2, cwd(G)\}$.

Let $q = cwd(G_W)$ and t be a q-expression defining G_W. In order to construct each clique K_k where $V(K_k) = \{w_1, \ldots, w_k\}$, let us consider the expression g as follows:

$$\begin{aligned}&\eta_{q+1,q+2}(q+2(w_k) \oplus \rho_{q+2\to q+1}(\eta_{q+1,q+2}(q+2(w_{k-1}) \oplus \rho_{q+2\to q+1}(\eta_{q+1,q+2}(q+2(w_{k-2}) \oplus \\ &\quad \ldots \rho_{q+2\to q+1}(\eta_{q+1,q+2}(q+2(w_3) \oplus \rho_{q+2\to q+1}(\eta_{q+1,q+2}(q+2(w_2) \oplus q+1(w_1))))) \ldots))))).\end{aligned}$$

Observe that vertices $w_1, \ldots, w_{k-1}$ have label $q+1$ and w_k has label $q+2$. Then the q-expression t must contain a unique subexpression of the form $i(v)$ corresponding to the initial label of v in the construction of G_W for each $v \in V(G_W)$.

By induction on the structure of t, it can be shown that the $(q+3)$-expression obtained by replacing in t for each $v \in V(G_W)$, the subexpression $i(v)$ by the expression

$$\eta_{i,q+3}((\eta_{i,q+2}(i(v) \oplus (g_1 \oplus \ldots \oplus g_{k-k(v)}))) \oplus \eta_{q+2,q+3}(q+3(u_v) \oplus (g_1 \oplus \ldots \oplus g_{k-1}))),$$

defines $H(G'_W)$, where $g_j = g$ for $j = 1, \ldots, k$ are the expressions defining G_v^j and u_v are the pendant vertices added in the construction of G'_W according to Definition 9. Hence $cwd(H(G'_W)) \leq q+3$. □

Hence, from Corollaries 4 and 12 and the lemma above, we obtain the following result.

Corollary 15. *Let $\mathcal{G}$ be a family of graphs having cliquewidth bounded by a constant. Then L-DOM (fixed l) is polynomial-time solvable for $\mathcal{G}$.*

Observe that as an immediate consequence of Corollary 15, we obtain that L-DOM (fixed l) is polynomial-time solvable for many well-known graph classes including cographs, P_4-tidy

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Table 3
Complexity table for M-DOM and L-DOM

Class	M-DOM	L-DOM (fixed $l \in \mathbb{N}$)
Strongly chordal	P (Liao and Chang, 2003)	P (Lee and Chang, 2008)
Doubly chordal	NP-c (Lee and Chang, 2008)	NP-c (Lee and Chang, 2008)
Dually chordal	NP-c (Lee and Chang, 2008)	NP-c (Lee and Chang, 2008)
Cographs	P	P
P_4-tidy	P	P
Bounded cliquewidth	P	P
Split	NP-c (Liao and Chang, 2003)	NP-c (Liao and Chang, 2003)
Planar	NP-c (Lee and Chang, 2008)	NP-c (Lee and Chang, 2008)
Chordal	NP-c (Liao and Chang, 2003)	NP-c (Lee and Chang, 2008)
Bipartite	NP-c (Liao and Chang, 2003)	NP-c (Lee and Chang, 2008)

"NP-c" and "P" mean NP-complete and polynomial, respectively.

graphs, distance hereditary graphs, and for any graph class having tree-width bounded by a constant.

To summarize, on the one hand we have enlarged the family of graph classes for which M-DOM and L-DOM (fixed l) are polynomial-time solvable. On the other hand, we were able to answer all the computational complexities left open in Table 2, as Table 3 shows.

We expect that the relations among the domination numbers shown in Section 2 can be even more exploited to extend the results on the computational complexities of the domination problems involved.

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