Quantity Flexible Contracts: Optimal Decisions with Information Updates¹

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Abstract

We study single and multi-period quantity flexible contracts involving one demand forecast update in each period and a spot market. We obtain the optimal order quantity at the beginning of a period and order quantities on contract and from the spot market at the then prevailing price after the forecast revision and before the demand materialization. The amount that can be purchased on contract is bounded by a given flexibility limit. We discuss the impact of the forecast quality and the level of flexibility on the optimal decisions and managerial insights behind the results.

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1 Introduction

Owing to economic globalization, product proliferation and technology progress, customer demands and market prices have become highly uncertain across many industry sectors. As a result, improving the reliability of forecasts for demands and prices has become very important for the survival of many companies. At the same time, various supply chain management tools and methods have become available to help companies streamline their supply chain operations. An important such tool is that of quantity flexibility contracts. Such a contract allows the buyer in a supply chain to postpone some of his purchases to a later date and at a favorable price after an improved forecast of the customer demand becomes available. Thus the contract provides the buyer with a cushion against demand uncertainty. The supplier on the other hand benefits by having a smoother production schedule as a result.

A number of papers dealing with quantity flexible contracts have appeared in the literature. Here we shall review them briefly, before developing our model of a quantity flexibility contract. Eppen and Iyer (1997) study a special form of the quantity flexible contract, which allows the retailer to return a portion of its purchase to the supplier. Bassok and Anupindi (1997) analyze a single-product periodic review inventory system with a minimum quantity contract, which stipulates that the cumulative purchase over the life of the contract must exceed a specified minimum quantity to qualify for a price discount. They demonstrate that the optimal inventory policy for the buyer is an order-up-to type and that the order-up-to level can be determined by a newsvendor model. Anupindi and Bassok (1998) extend this work to the case of multiple products. In this case, the supply contract requires that the total purchase amount in dollars exceed a specified minimum to obtain the price discount. Tsay (1999) studies incentives, causes of inefficiency, and possible ways of performance improvement in a quantity flexible contract. In particular, Tsay investigates order revisions in response to new demand information, where the information is a location parameter of the demand distribution. Tsay and Lovejoy (1998) investigate the quantity flexible contracts in more complex settings of multiple players, multiple demand periods, and demand forecast updates. They study issues relating to desired levels of flexibility and local and systemwide performances.

Similar to the structure of quantity flexible contracts, a form of "Take-or-Pay" provision has

been used in many long-term natural resources and energy supply contracts (Tsay 1999). A takeor-pay contract is an agreement between a buyer and a supplier, which often specifies a minimum quantity the buyer must purchase (take), and the maximum quantity the buyer can obtain (pay) over the contract period. Brown and Lee (1997) note that capacity reservation agreements, common in the semiconductor industry, have a similar structure. Brown and Lee examine how much capacity should be reserved as take and how much capacity should be reserved for the future as pay.

Related research has been carried out in the area of inventory management with demand forecast updates. It is possible to classify this research into three categories. The first category uses Bayesian analysis. Bayesian models were first introduced in the inventory literature by Dvoretzky, Kiefer, and Wolfowitz (1956). Eppen and Iyer (1997) analyze a quick response program in a fashion buying problem by using the Bayes Rule to update demand distributions. The use of time-series models in demand forecast updating characterizes the second category, which includes the papers by Johnson and Thompson (1975) and Lovejoy (1990). The third category is concerned with forecast revisions. This approach is developed and used by Hausmann (1969), Sethi and Sorger (1991), Heath and Jackson (1994), Donohue (2000), Yan, Liu, and Hsu (2003), Gurnani and Tang (1999), Barnes-Schuster, Bassok, and Anupindi (2002), and Gallego and Özer (2001). Sethi, Yan and Zhang (2001,2003), and others. We refer the readers to a more detailed review in Sethi, Yan and Zhang (2001) and references therein.

In this paper, we develop a model to analyze a quantity flexible contract involving multiple periods, rolling horizon demand and forecast updates. The contract permits the buyer to order at two distinct time instants, one at the beginning of a period and another at a time before the demand realizes at the end of the period. At the first instant, the buyer purchases q units of a product at price p. This gives him an option to purchase up to δq units of the same product at price $p_c > p$ at the second instant, where $0 < \delta \leq 1$ is known as the flexibility limit. Since the buyer may purchase any amount of the product in the spot market, the contract provides the buyer with both price and quantity protection against price and demand uncertainties. With a firm commitment of an amount q early on from the buyer, the contract is also appealing to the supplier. Quantity flexibility contracts of the type modelled here have been used by a number of major manufacturers, in particular, in the computer industry, where demand uncertainties, price fluctuations and dependence on parts suppliers are common. It is reported that since as early as 1990, Sun Microsystems, IBM, and Hewlett Packard had been adopting contracts with a flexibility limit for components and assemblies (Tsay and Lovejoy, 1998).

Our model differs from most of the existing models of quantity flexible contracts in the following ways: (i) we provide a model which takes both quantity flexible contract and spot market purchase into consideration; (ii) the contract has a flexibility level which specifies the maximum amount that can be purchased on contract; (iii) we model both speculative and reactive decisions, in particular, how both speculative and reactive decision are related to the information revisions, such as demand and price information updates; (iv) with stochastic comparison theory, we characterize the impact on the optimal policy and the expected profit of the quality of forecast updates; (v) we extend our results to the multiple period case.

The rest of the paper is organized as follows. In Section 2, we model a single-period contract and some fundamental structural results. We establish the existence of an optimal solution for our model. In Section 3, we provide explicit optimal solutions in some special cases. For the cases of worthless and perfect information updates, respectively, we obtain closed-form solutions. In Section 4, we use the stochastic comparison theory to establish results relating to the effect of the quality of information revisions on the optimal solution. The model is extended to allow for a finite number of periods in Section 5. The paper is concluded in Section 6. Proofs of all the results obtained in the paper are relegated to Appendix.

2 The Model

In this section we design a one-period, two-stage quantity flexible supply contract between a buyer and a supplier. In such a contract, the buyer has an option in the second stage to increase his first-stage order by up to a certain percentage of the initial purchase. The specifics of the contract are as follows.

Let t_1, t_2, t_3 denote the epochs representing the start of stage 1, start of stage 2, and the end of stage 2. At t_1 , the buyer purchases a quantity q of the product from the supplier at a unit price p. This decision is based on the following:

- (i) The available information about the uncertain customer demand to be realized at t_3 .
- (ii) The distribution of the spot market price at t_2 .
- (iii) The buyer will have an updated demand information at t_2 .
- (iv) The buyer has an option at t_2 to purchase from the supplier an additional quantity q_c not exceeding δq , $1 \ge \delta > 0$, at a unit price $p_c > p$. We call δ as the flexibility limit.
- (v) The buyer could purchase at t_2 any amount q_s of product in the spot market at the then prevailing price.
- (vi) The buyer loses revenue r for each unsatisfied customer demand at t_3 .
- (vii) The unsold product is salvaged at a value of s per unit.

To complete the statement of the buyer's problem, we introduce the following notation and assumptions. We model the market price as the random variable P taking value in the interval $[p_l, p_h], p_h \ge p_l > 0$. To avoid trivial cases, we assume

$$r > \max[p_h, p_c], \ s < \min[p_l, p]. \tag{1}$$

We let D denote the random demand and I denote the signal that updates the demand distribution at t_2 . So, what we have available at t_1 is the joint density $\phi_{D,I}(z,i)$, or simply $\phi(z,i)$, from which we can derive the marginal density g(i). The signal I is observed at t_2 . If the observed value of Iis i, then we obtain the updated demand density as the conditional density $h(z \mid i) = \phi(z,i)/g(i)$ for g(i) > 0. Let the distributions corresponding to the densities $\phi(z,i)$, g(i) and $h(z \mid i)$ be denoted by $\Phi(z,i)$, G(i), and $H(z \mid i)$, respectively. For convenience in exposition, we assume that the cumulative distribution $\Phi(z,i)$ and $H(z \mid i)$ are strictly increasing in z.

We can now define the optimal profit π^* of the buyer as

$$\pi^* = \max_{q \ge 0} \left\{ -pq + E\left(\max_{\substack{0 \le q_s \\ 0 \le q_c \le \delta q}} \Pi(q, q_s, q_c; I, P)\right) \right\},\tag{2}$$

where

$$\Pi(q, q_s, q_c; I, P) = E\left[\left\{r(D \land (q + q_s + q_c)) + s(q + q_s + q_c - D)^+ - p_c q_c - Pq_s\right\} | I, P\right],$$
(3)

and the notations $x^+ = \max\{0, x\}$ and $x \wedge y = \min\{x, y\}$. The term pq is the purchase cost of buying amount q at price p at t_1 . Similarly, p_cq_c is the cost of purchasing quantity q_c from the supplier at price p_c , and Pq_s represents the purchase cost of buying quantity q_s from the market at price P. The conditional expectation $\Pi(q, q_s, q_c; I, P)$ represents the buyer's profit at t_2 given the signal I and the market price P. Therefore, the buyer's problem is to determine the optimal purchasing decisions (q^*, q_s^*, q_c^*) that maximize the total expected profit. Clearly, q_s^* and q_c^* depend on q, I and P. In order to highlight the above dependence, we sometimes write these contingent decisions as $q_s^*(q, I, P)$ and $q_c^*(q, I, P)$, respectively. Note also that $\Pi(q, q_s, q_c; I, P)$ is a random variable dependent on I and P.

To solve the problem, we first determine the optimal $q_s^*(q, I, P)$ and $q_c^*(q, I, P)$ for any given q, I = i, and $P = p_s$. That is, we first solve the problem

$$\max_{\substack{0 \le q_s \\ 0 \le q_c \le \delta_q}} \Pi(q, q_s, q_c; i, p_s),\tag{4}$$

where

$$\Pi(q, q_s, q_c; i, p_s) = E\left(\left[r(D \land (q + q_s + q_c)) + s(q + q_s + q_c - D)^+ - p_c q_c - p_s q_s\right] \middle| i, p_s\right)$$

$$= r \int_0^{q+q_s+q_c} zh(z|i)dz + r(q + q_s + q_c) \int_{q+q_s+q_c}^{\infty} h(z|i)dz$$

$$+s \int_0^{q+q_s+q_c} [(q + q_s + q_c) - z]h(z|i)dz - p_c q_c - p_s q_s.$$
(5)

If the unit order cost p at time t_1 is larger than the highest possible market price p_h at time t_2 , i.e., if $p_h \leq p$, then it is obvious that the best strategy is to purchase all of the required quantity from the spot market, i.e., $q^* = 0$ and $q_c^* = 0$. To find out $q_s^*(0, i, p_s)$ in this case, we must solve

$$\max_{0 \le q_s} \left[r \int_0^{q_s} zh(z|i) dz + rq_s \int_{q_s}^\infty h(z|i) dz + s \int_0^{q_s} [q_s - z]h(z|i) dz - p_s q_s \right].$$

This is a newsvendor problem and its solution is $q_s^*(0, i, p_s) = H^{-1}((r - p_s)/(r - s)|i)$.

If $p_h < p_c$ or $\delta = 0$, then $q_c^* = 0$. Both of these cases also reduce to a newsvendor problem.

Thus, since $p < p_c$, it suffices to consider the following cases:

$$p < p_c \le p_l \le p_h; \quad p \le p_l \le p_c \le p_h; \quad p_l \le p < p_c \le p_h.$$
(6)

Strictly speaking, we should exclude the case $p = p_l = p_c \leq p_h$ from the middle case above.

In the next section, we take up the buyer's problem at time t_2 .

3 Contingent Order Quantities at t_2

In this section we solve for the contingent order quantities for every possible observation of the signal I and the market price P at t_2 . We also characterize monotonicity properties of the solutions with respect to these observations.

Proposition 3.1 For an observed value (i, p_s) of (I, P), we have the following solutions:

(i) If $p_s \leq p_c$, then

$$q_c^*(q, i, p_s) = 0, \quad q_s^*(q, i, p_s) = \left[H^{-1} \left(\frac{r - p_s}{r - s} \middle| i \right) - q \right]^+.$$

(ii) If $p_s > p_c$, then

$$q_{c}^{*}(q, i, p_{s}) = (\delta q) \wedge \left[H^{-1} \left(\frac{r - p_{c}}{r - s} \middle| i \right) - q \right]^{+},$$
$$q_{s}^{*}(q, i, p_{s}) = \left[H^{-1} \left(\frac{r - p_{s}}{r - s} \middle| i \right) - (1 + \delta)q \right]^{+}.$$

Let us explain the proposition in words. Statement (i) says that when the contract price p_c is higher than the prevailing market price p_s , then the buyer purchases nothing on the contract at time t_2 . Instead the buyer purchases the product from the spot market. The purchasing quantity is determined by the difference of the critical fractile of the updated demand distribution and the amount purchased at t_1 . The critical fractile is determined by the demand distribution, the sales price r, the salvage value s, and the spot market price p_s . When the market price p_s is higher than the contractual price p_c , then the buyer purchases on the contract first, and considers to purchase from the spot market only after exhausting the quantity flexibility provided in the contract. Note that the buyer can purchase δq at most. Therefore, the marginal purchase price can be the contract with the contract price p_c as the marginal purchasing price in the critical fractile calculation. Otherwise, in addition to exhausting the purchase option in the contract, the buyer purchases a desired additional amount from the spot market with the spot market price p_s as the marginal price in the critical fractile calculation.

With an assumption that the demand D is stochastically monotone in the signal observation i, we provide an explicit expression of the optimal purchase quantity with respect to i. Without

loss of generality, we assume D to be stochastically increasing. For the case of a stochastically decreasing demand, it is possible for us to redefine i so that the case of the stochastically decreasing demand can be translated to the case of a stochastically increasing demand.

Proposition 3.2 Let the demand D be stochastically increasing in i. Then, for an observed market price p_s , there exist information thresholds $\overline{I}(q, p_c)$, $\widehat{I}(q, p_c)$, $\overline{I}(q, p_s)$ and $\widehat{I}(q, p_s)$ defined by the relations

$$H^{-1}\left(\frac{r-p_{c}}{r-s}\middle|\bar{I}(q,p_{c})\right) = q, \ H^{-1}\left(\frac{r-p_{c}}{r-s}\middle|\hat{I}(q,p_{c})\right) = (1+\delta)q,$$
(7)

$$H^{-1}\left(\frac{r-p_{s}}{r-s}\middle|\bar{I}(q,p_{s})\right) = q, \ H^{-1}\left(\frac{r-p_{s}}{r-s}\middle|\hat{I}(q,p_{s})\right) = (1+\delta)q,$$
(8)

such that

(i) if $p_s \leq p_c$, then

$$\begin{aligned} q_c^*(q, i, p_s) &= 0, \\ q_s^*(q, i, p_s) &= \begin{cases} 0, & \text{if } i \leq \bar{I}(q, p_s), \\ H^{-1}\left(\frac{r - p_s}{r - s} \middle| i\right) - q, & \text{if } i > \bar{I}(q, p_s); \end{cases} \end{aligned}$$

(ii) if $p_c < p_s$, then $\overline{I}(q, p_c) \leq \hat{I}(q, p_c) \leq \hat{I}(q, p_s)$, and

$$\begin{aligned} q_c^*(q, i, p_s) &= \begin{cases} 0, & \text{if } i \leq \bar{I}(q, p_c), \\ H^{-1}\left(\frac{r-p_c}{r-s} \middle| i\right) - q, & \text{if } \bar{I}(q, p_c) < i < \hat{I}(q, p_c), \\ \delta q, & \text{if } i \geq \hat{I}(q, p_c), \end{cases} \\ q_s^*(q, i, p_s) &= \begin{cases} 0, & \text{if } \hat{I}(q, p_c) < i \leq \hat{I}(q, p_s), \\ H^{-1}\left(\frac{r-p_s}{r-s} \middle| i\right) - (1+\delta)q, & \text{if } i \geq \hat{I}(q, p_s). \end{cases} \end{aligned}$$

With the results obtained in Proposition 3.2, we can enhance our interpretation of the results in Proposition 3.1. The statement of Proposition 3.2(i) says that if the realized market price is lower than p_c , then there is no reason to buy on contract. Moreover, when the demand is low, indicated by a low value of *i*, then there is no need also to buy in the market. However, when the demand is high as indicated by a value of $i \ge \overline{I}(q, p_s)$, then we can compute the newsvendor amount, which is higher than the initial purchase quantity *q* because of the way we have defined $\overline{I}(q, p_s)$ in (8), and then buy the required difference in the market so that the total purchased quantity equals the newsvendor amount. In the case of Proposition 3.2(ii), the realized market price turns out to be higher than the contractual price p_c . So there will be some buying on contract if the demand turns out to be not too low. Indeed, when $i > \overline{I}(q, p_c)$, some quantity will be purchased on the contract, where as $i \leq \overline{I}(q, p_c)$ means very low demand, implying that the initial order is quite adequate and no amount needs to be purchased on contract. As *i* increases from $\overline{I}(q, p_c)$ to $\hat{I}(q, p_c)$, the newsvendor quantity increases from *q* to $(1 + \delta)q$, and the difference between the newsvendor quantity and the initial order quantity is purchased on the contract. Note that when $i = \hat{I}(q, p_c)$, the difference is exactly δq , the maximum quantity that can be purchased on the contract. As *i* increases further from $\hat{I}(q, p_c)$, the newsvendor quantity based on p_c increases beyond δq , but the amount purchase from the spot market to make up the difference. The reason is that the newsvendor quantity corresponding to the market price p_s , which is higher than p_c , is still smaller than $(1+\delta)q$ initially. But when *i* increases to $\hat{I}(q, p_s)$, the newsvendor quantity corresponding to p_s equals $(1+\delta)q$. As a result, as *i* increases beyond $\hat{I}(q, p_s)$, the difference between the newsvendor quantity and $(1+\delta)q$ is purchased in the spot market.

Remark 3.1 Statements (i) and (ii) of Proposition 3.2 imply that when the demand increases stochastically with i, then the optimal purchase quantity at time t_2 is nondecreasing in the observed signal.

Remark 3.2 When $\delta = 0$, (ii) of Propositions 3.1 and 3.2 is not needed. In the special case when $\delta = 0$ and P has a geometric distribution, Proposition 3.1 reduces to the results obtained by Gurnani and Tang (1999).

Remark 3.3 Note that (1) implies

$$r > \max\{EP, p_c\} \text{ and } s < \min\{EP, p\}.$$

$$\tag{9}$$

Regarding Proposition 3.1, since its proof is based on the classical newsboy problem, it can be easily shown that if $p_c < r \le p_s$, then $q_s^*(q, i, p_s) = 0$, and if $p_s \le s < p$, then $q_s^*(q, i, p_s) = \infty$. These are the cases that do not occur under (1), but occur under (9). Going along the lines of the proof of Proposition 3.2, we can show that Proposition 3.2 holds also for these cases.

4 Optimal Purchase Quantity at t_1

With the knowledge of the optimal reaction plan at t_2 derived in the previous section, it is possible to determine the purchase quantity q at time t_1 . This is done by substituting in (2) for q_s and q_c , their respective optimal quantities $q_c^*(q, I, P)$ and $q_s^*(q, I, P)$, and solving the optimization problem

$$\pi^* = \max_{q \ge 0} \left\{ -pq + E \Big[\Pi(q, q_s^*(q, I, p), q_c^*(q, I, p); I, P) \Big] \right\}.$$
 (10)

This is a problem of maximizing an objective function with a single variable q. For given values of the problem parameters and observations i and p_s , it can be easily solved numerically. One could also use the K-T theory to derive the first order conditions for a maximum. Such an approach was used by Brown and Lee (1997) on a related problem.

For a further mathematical analysis of the problem, we need to simplify the distributions of the random variables involved. To begin with, we shall assume that the market price is geometrically distributed. Specifically we make the following assumption.

Assumption A. The market price P has the value p_l with probability β and the value p_h with probability $(1 - \beta)$. The conditional distribution of D given i is a decreasing function of i.

It is clear from (10) that the initial order quantity q^* depends on several factors including δ . It is also easy to see that the "level" of flexibility is jointly determined by δ and q^* . The flexibility level increases as δ increases and as q^* increases. It is therefore important to know how q^* relates to δ . This is the subject of the following proposition.

Proposition 4.1 Under Assumption A, (i) the initial optimal order quantity q^* is non-increasing in δ and (ii) the optimal expected profit π^* is non-decreasing in δ .

Remark 4.1 From (31) and (32) we know that if $p_l \leq p_c$, then the initial optimal ordering quantity q^* can be uniquely solved by (31). In a similar way, by Proposition 3.2, we can also prove that if $p_c \leq p_l$, then the initial optimal ordering quantity q^* can be uniquely obtained by solving the equation

$$-p + \beta \int_{-\infty}^{\bar{I}(q,p_l)} \left[(s-r)H(q|i) + r \right] g(i)di + \beta p_l [G(\hat{I}(q,p_l)) - G(\bar{I}(q,p_l))] \\ + \beta \int_{\hat{I}(q,p_c)}^{\hat{I}(q,p_l)} \left\{ (1+\delta)[(s-r)H((1+\delta)q|i) + r] - p_c \delta \right\} g(i)di \\ + \beta \int_{\hat{I}(q,p_l)}^{\infty} \left[-p_c \delta + (1+\delta)p_l \right] g(i)di \\ + (1-\beta) \int_{-\infty}^{\bar{I}(q,p_c)} \left[(s-r)H(q|i) + r \right] g(i)di + (1-\beta)p_c [G(\hat{I}(q,p_c)) - G(\bar{I}(q,p_c))] \\ + (1-\beta) \int_{\hat{I}(q,p_c)}^{\hat{I}(q,p_h)} \left\{ (1+\delta)[(s-r)H((1+\delta)q|i) + r] - p_c \delta \right\} g(i)di \\ + (1-\beta) \int_{\hat{I}(q,p_c)}^{\infty} \left[-p_c \delta + (1+\delta)p_h \right] g(i)di = 0.$$
(11)

Remark 4.2 We call $dF(q^*, \delta)/d\delta$, the *flexibility value rate*. Using (35), we have

$$\frac{d^2 F(q^*, \delta)}{d\delta^2} = (p_h - p_c) [1 - G(\hat{I}(q^*, p_h))]
+ \int_{\hat{I}(q^*, p_c)}^{\hat{I}(q^*, p_h)} [(s - r) H((1 + \delta)q^*|i) + r - p_c] g(i) di
- (1 + \delta)(r - s) \left(q^* + (1 + \delta)\frac{dq^*}{d\delta}\right) \int_{\hat{I}(q^*, p_c)}^{\hat{I}(q^*, p_h)} h((1 + \delta)q^*|i)g(i) di.$$
(12)

Let $\widetilde{\delta}$ be the solution of

$$\frac{d^2 F(q^*,\delta)}{d\delta^2} = 0.$$

Then we know that the flexibility value rate is increasing on $[0, \tilde{\delta}]$ and non-increasing on $(\tilde{\delta}, \infty)$. Thus, $\tilde{\delta}$ is the critical number that makes the flexibility value rate to be the largest. Although the larger is the flexibility factor δ , the higher is the profit, when the buyer, however, considers the expense of flexibility, he may prefer to choose $\tilde{\delta}$ as the flexibility factor.

4.1 The Case of Worthless Information

The case of worthless information arises when the signal I observed at t_2 does not have any impact on the demand uncertainty. Mathematically, it means that the random variables I and D are independent. Hence, H(z|i) = H(z) and $\Phi(z,i) = G(i)H(z)$. From Proposition 3.1, $q_c^*(q,i,p_s)$ and $q_s^*(q,i,p_s)$ are independent of i. Therefore, in this subsection we denote them as $q_c^*(q,p_s)$ and $q_s^*(q,p_s)$, respectively. **Proposition 4.2** (Worthless Information). Assume that H(z|i) = H(z) and $\Phi(z,i) = G(i)H(z)$. Then the optimal solution is as follows.

(A) If $p \leq p_l$, the optimal order quantities are

$$q^* = H^{-1}\left(\frac{r-p}{r-s}\right), \ q_c^*(q^*, p_l) = q_s^*(q^*, p_l) = q_c^*(q^*, p_h) = q_s^*(q^*, p_h) = 0,$$

the optimal expected total order quantity is given by $H^{-1}((r-p)/(r-s))$, and the optimal expected profit is

$$(r-s)\int_0^{q^*} zh(z)dz.$$

(B) If $p > p_l$, then we have the following three subcases:

(B.1) when $\beta p_l + (1 - \beta)p_c \ge p$, the optimal order quantities are

$$q^* = H^{-1} \left(\frac{-p + \beta p_l + (1 - \beta)r}{(1 - \beta)(r - s)} \right),$$

$$q_c^*(q^*, p_l) = 0, \ q_s^*(q^*, p_l) = H^{-1} \left(\frac{r - p_l}{r - s} \right) - q^*,$$

$$q_c^*(q^*, p_h) = q_s^*(q^*, p_h) = 0,$$

the optimal expected total order quantity is

$$\beta H^{-1}\left(\frac{r-p_l}{r-s}\right) + (1-\beta)H^{-1}\left(\frac{-p+\beta p_l+(1-\beta)r}{(1-\beta)(r-s)}\right),$$

and the optimal expected profit is

$$(r-s)\left\{\beta\int_{0}^{q^{*}+q_{s}^{*}(q^{*},p_{l})}zh(z)dz+(1-\beta)\int_{0}^{q^{*}}zh(z)dz\right\};$$

(B.2) when $\beta p_l + (1-\beta)p_c , the optimal order quantities are$

$$\begin{split} q^* &= \frac{1}{1+\delta} H^{-1} \left(\frac{(1-\beta)(1+\delta)(r-p_c) - p + \beta p_l + (1-\beta)p_c}{(1-\beta)(1+\delta)(r-s)} \right), \\ q^*_c(q^*,p_l) &= 0, \quad q^*_s(q^*,p_l) = H^{-1} \left(\frac{r-p_l}{r-s} \right) - q^*, \\ q^*_c(q^*,p_h) &= \delta q^*, \quad q^*_s(q^*,p_h) = 0, \end{split}$$

the optimal expected total order quantity is

$$\beta H^{-1}\left(\frac{r-p_l}{r-s}\right) + (1-\beta)H^{-1}\left(\frac{(1-\beta)(1+\delta)(r-p_c) - p + \beta p_l + (1-\beta)p_c}{(1-\beta)(1+\delta)(r-s)}\right),$$

and the optimal expected profit is

$$(r-s)\left\{\beta\int_{0}^{q^{*}+q_{s}^{*}(q^{*},p_{l})}zh(z)dz+(1-\beta)\int_{0}^{(1+\delta)q^{*}}zh(z)dz\right\};$$

(B.3) when $p \ge \beta p_l + (1 - \beta)p_h + (1 - \beta)\delta(p_h - p_c)$, the optimal order quantities are

$$q^* = 0, \quad q_c^*(q^*, p_l) = 0, \quad q_s^*(q^*, p_l) = H^{-1}\left(\frac{r - p_l}{r - s}\right),$$
$$q_c^*(q^*, p_h) = 0, \quad q_s^*(q^*, p_h) = H^{-1}\left(\frac{r - p_h}{r - s}\right),$$

the optimal expected total order quantity is

$$\beta H^{-1}\left(\frac{r-p_l}{r-s}\right) + (1-\beta)H^{-1}\left(\frac{r-p_h}{r-s}\right),$$

and the optimal expected profit is

$$(r-s)\left\{\beta\int_{0}^{q_{s}^{*}(q^{*},p_{l})}zh(z)dz+(1-\beta)\int_{0}^{q_{s}^{*}(q^{*},p_{h})}zh(z)dz\right\}.$$

Let us provide intuitive insights into the various results obtained in Proposition 4.2. Case A addresses the situation when the initial unit order cost is less than the lowest possible market price. In this case, if the observed information is worthless, then the buyer gains nothing by delaying his purchase to t_2 . Thus, the entire purchase is made at t_1 , and nothing is purchased at t_2 . Indeed, in this case, the contract is of no value.

In Case B, we have (36). Clearly, $q_c^*(q^*, p_l) = 0$ in this case.

In (B.1), the expected relevant price at t_2 is clearly $\beta p_l + (1 - \beta)p_c$, and it is higher than the initial price p. Therefore, the buyer will buy a sufficiently large quantity q^* at the initial price p so that he would not need to buy any quantity at all when the market price is high. Moreover, q^* will not be too large to prohibit the buyer from taking advantage of buying in the market when the spot price is low.

We now consider (B.2) and (B.3). Note that since $\delta > 0$, the condition

$$p \ge \beta p_l + (1 - \beta)p_h + (1 - \beta)\delta(p_h - p_c)$$

$$\tag{13}$$

in (B.3) implies $p > \beta p_l + (1-\beta)p_c$. Thus, in both cases (B.2) and (B.3), $\beta p_l + (1-\beta)p_c$ is lower than the initial price p. In contrast to (B.1), it seems reasonable, therefore, to reduce or completely postpone the purchase to time t_2 in (B.2) and (B.3). The (B.3) condition (13), however, also implies $p > \beta p_l + (1 - \beta)p_h$. This says that the expected market price at t_2 is lower than the initial price p, which argues for a complete postponement of the purchase. Consequently, the initial purchase quantity is zero, and the entire respective newsvendor quantity is bought from the market depending on the prevailing market price at time t_2 .

This leaves us with (B.2), where we still have $p > \beta p_l + (1 - \beta)p_c$, but we do not have (13). In other words, the high market price p_h is not low enough for (13) to hold, and thus arguing perhaps for a reduction in the initial purchase amount rather than a complete postponement. Let us therefore consider an initial purchase of one unit at t_1 and δ unit at t_2 . Clearly, the purchase of δ unit at time t_2 will take place at p_l when the market price is low, and at p_c when the market price is high. Thus, the per unit expected cost of a reduced purchase at t_1 followed by an additional purchase up to the contracted amount is

$$[p + \beta \delta p_l + (1 - \beta) \delta p_c] / (1 + \delta).$$

On the other hand, a complete postponement of the purchase of a unit to time t_2 has the expected cost

$$[\beta p_l + (1 - \beta)p_h]$$

Thus, if

$$\frac{p+\beta\delta p_l+(1-\beta)\delta p_c}{1+\delta} < \beta p_l+(1-\beta)p_h,$$

i.e., if

$$p < \beta p_{l} + (1 - \beta)p_{h} + (1 - \beta)\delta(p_{h} - p_{c})$$

= $\beta p_{l} + (1 - \beta)p_{c} + (1 - \beta)(1 + \delta)(p_{h} - p_{c}),$ (14)

then it is better to reduce the initial purchase than to postpone it completely. This is precisely the result obtained in (B.2).

Remark 4.3 In order to get a simple representation for the optimal order quantities in terms of the observed information *i*, one usually assumes that $\Phi(z, i)$ is a bivariate normal distribution. Furthermore, if the correlation between *D* and *I* is zero, i.e., H(z) and G(i) are normal distributions, then, when $p > p_l$ and $\delta = 0$, (B.2) of Proposition 4.2 provides the same result as Part B of Proposition 4.1 of Gurnani and Tang (1999); when $p > p_l$ and $\delta = 0$, (B.3) of Proposition 4.2 implies Part A of Proposition 4.1 of Gurnani and Tang (1999); and when $p \le p_l$, our model reduces to that of Brown and Lee (1997) and Proposition 4.2 is similar to Theorem 4 of Brown and Lee for the case of worthless information.

Finally, comparing our results with Proposition 3.1 of Gurnani and Tang (1999), it is possible to show that the difference between the profit with the contract and that without the contract is

$$(1-\beta)(r-s)\int_{0}^{(1+\delta)q^{*}} zh(z)dz,$$
(15)

in (B.2) of Proposition 4.2. We denote this gap as the value of flexibility. Equation (15) indicates that the value of flexibility is positive under the specified condition of (B.2) even when demand information revision is worthless. In (B.1) and (B.3), the value of flexibility is zero.

4.2 The Case of Perfect Information Update

In this subsection we study the other extreme case when the information revision is perfect. The perfect information revision represents a scenario in which the demand D can be completely determined once I is observed. In other words, the demand D is a function of I, say, $\tau(I)$. Let $F(\cdot)$ denote the distribution function of D. From our assumptions in Section 1, we have $F(\cdot)$ to be strictly increasing on $[0,\infty)$. Parallel to Proposition 4.2, for the case of perfect information revision, we present the following proposition.

Proposition 4.3 (*Perfect Information*) Assume that $D = \tau(I)$.

(A) If $p_l \leq p_c$, then the optimal order quantity q^* at time t_1 is the solution of the equation

$$-p + \beta p_l + (1 - \beta)p_c + (1 - \beta)(1 + \delta)(p_h - p_c) + [s - \beta p_l - (1 - \beta)p_c]F(q) - (1 - \beta)(1 + \delta)(p_h - p_c)F((1 + \delta)q) = 0,$$
(16)

and $q^* = 0$ if there is no positive solution of (16). The optimal order quantities at t_2 are

$$q_{c}^{*}(q^{*}, i, p_{l}) = 0, \quad q_{s}^{*}(q^{*}, i, p_{l}) = [\tau(i) - q^{*}]^{+},$$

$$q_{c}^{*}(q^{*}, i, p_{h}) = [\tau(i) - q^{*}]^{+} \wedge (\delta q^{*}), \quad q_{s}^{*}(q^{*}, i, p_{h}) = [\tau(i) - (1 + \delta)q^{*}]^{+}, \quad (17)$$

the optimal expected total order quantity is $(q^* + ED)$, and the optimal expected profit is

$$rED - s \int_{0}^{q^{*}} z dF(z) - \beta p_{l} \int_{q^{*}}^{\infty} z dF(z) -(1 - \beta) \left\{ p_{c} \int_{q^{*}}^{(1+\delta)q^{*}} z dF(z) + p_{h} \int_{(1+\delta)q^{*}}^{\infty} z dF(z) \right\}.$$
 (18)

(B) If $p_l > p_c$, then the optimal order quantity q^* at t_1 is the solution of the equation

$$-p - \delta p_c + (1+\delta)[\beta p_l + (1-\beta)p_h] + (s - p_c)F(q) + (1+\delta)[p_c - \beta p_l - (1-\beta)p_h]F((1+\delta)q) = 0,$$
(19)

and $q^* = 0$ if there is no positive solution of (16). The optimal order quantities at t_2 are

$$q_s^*(q, i, p_l) = [\tau(i) - (1 + \delta)q^*]^+, \quad q_s^*(q^*, i, p_h) = [\tau(i) - (1 + \delta)q^*]^+,$$
$$q_c^*(q^*, i, p_l) = q_c^*(q^*, i, p_h) = [\tau(i) - q^*]^+ \wedge (\delta q^*).$$

The optimal expected total order quantity is $(q^* + ED)$ and the optimal expected profit is

$$rED - s \int_{0}^{q^{*}} z dF(z) - p_{c} \int_{q^{*}}^{(1+\delta)q^{*}} z dF(z)$$
$$- [\beta p_{l} + (1-\beta)p_{h}] \int_{(1+\delta)q^{*}}^{\infty} z dF(z).$$

Remark 4.4 If *I* is a normal random variable and D = aI + b with a > 0, then *D* is also a normal random variable, and $\Phi(z, i)$ is a bivariate normal distribution with the correlation coefficient of one. In particular, if $\delta = 0$ and $p_l < p$, then Case A of Proposition 4.3 provides the same results as Proposition 3.2 in Gurnani and Tang (1999).

Proposition 4.4 With the condition $p_l \leq p_c$, equation (16) has a solution $q^* > 0$ if, and only if, (14) holds.

Proposition 4.5 With the condition $p_l > p_c$, equation (19) has a solution $q^* > 0$ if, and only if

$$-p - \delta p_c + (1+\delta)[\beta p_l + (1-\beta)p_h] > 0.$$
(20)

Proposition 4.6 The flexibility value is either zero or a decreasing function of β in both the worthless and the perfect information cases.

5 The Impact of the Forecast Accuracy

In this section we investigate the impact of the forecast accuracy on the ordering decisions. We start with a related definition.

Definition 5.1 A random variable X is more increasing-convex than another random variable Y, denoted as $X \ge_{ic} Y$, if

$$E[\psi(X)] \ge E[\psi(Y)] \tag{21}$$

for all non-decreasing convex functions $\psi(\cdot)$.

Clearly, if E[X] = E[Y] and $X \ge_{ic} Y$, then we have

$$\operatorname{Var}(X) \ge \operatorname{Var}(Y). \tag{22}$$

Thus, when E(X) = E(Y), then X has a larger variance than Y if X is more increasing-convex than Y. For further discussion on the increasing-convex ordering relation, the reader is referred to Shaked and Shanthinkumar (1994) and Song (1994).

Consider two systems 1 and 2, which face demands D^1 and D^2 , respectively. We assume all other parameters to be the same for both systems. For simplicity, we also assume that both systems observe the same signal I in updating their respective demands. To be specific, the demands D^1 and D^2 , following the onion-layer peeling model of Sethi, Yan and Zhang (2001), can be written as

$$D^1 = \varphi^1(I, R^1)$$
 and $D^2 = \varphi^2(I, R^2),$

where R^1 and R^2 are independent random variables. Then, $\varphi^k(i, R^k)$ represents the updated demand based on the observed information *i* of *I* for system *k*, k = 1, 2. Furthermore, we say that the demand forecast for system 2 is more accurate under the increasing-convex ordering than the demand forecast for system 1, if $\varphi^1(i, R^1) \geq_{ic} \varphi^2(i, R^2)$ for each observed value *i*. It follows, therefore, that if $E[\varphi^1(i, R^1)] = E[\varphi^2(i, R^2)]$ and $\varphi^1(i, R^1) \geq_{ic} \varphi^2(i, R^2)$ for each *i*, then the variance of the updated demand of system 1 is larger than that of system 2 for each *i*. In this case, we can now prove the intuitive result that the expected profit of a system with more accurate forecast than another's is higher. **Proposition 5.1** If for each observed value *i* of *I*, $E[\varphi^1(i, R^1)] = E[\varphi^2(i, R^2)]$ and $\varphi^1(i, R^1) \ge_{ic} \varphi^2(i, R^2)$, then the expected profit for system 1 is lower than that for system 2, ceteris paribus.

In order to investigate the impact of the forecast accuracy on the optimal expected total order quantity, we introduce another definition to describe forecast accuracy.

Definition 5.2 Consider two non-negative random variables X and Y satisfying E[X] = E[Y]have distributions F_X and F_Y with densities f_X and f_Y . Suppose X and Y are either both continuous or both discrete. We say X is *more variable* than Y, denoted by $X \ge_{\text{var}} Y$, if

$$S(f_X - f_Y) = 2 \quad with \ sign \ sequence \ +, -, +, \tag{23}$$

that is, there exist $0 < \alpha_1 < \alpha_2 < \infty$ such that $f_X(t) - f_Y(t) > 0$ when $t \in (0, \alpha_1)$, $f_X(t) - f_Y(t) < 0$ when $t \in (\alpha_1, \alpha_2)$, and $f_X(t) - f_Y(t) > 0$ when $t \in (\alpha_2, \infty)$. Here the notation S(f(t)) means the number of sign changes of a function $f(\cdot)$ as t increases from 0 to ∞ .

For further discussion on the property of more variability, see Song (1994) and Whitt (1985). Note that (23) implies

$$S(F_X - F_Y) = 1$$
 with sign sequence $+, -$.

Furthermore, from E[X] = E[Y] and (23), it is possible to show that

$$E(X - E[X])^{2} > E(Y - E[Y])^{2}.$$
(24)

See also Song (1994) and Ross (1996). As the variance measures the deviation of a random variable from its mean, so (24) motivates why X is known to be more variable than Y if X and Y satisfy (23).

Let T^k be the total quantity ordered by system k, k = 1, 2. Note that T_1 and T_2 are random variables. We have the following proposition.

Proposition 5.2 If $\varphi^1(i, R^1) \geq_{\text{var}} \varphi^2(i, R^2)$ and $\delta = 0$, then there is a positive θ such that (i) when $\max\left\{\frac{r-p_l}{r-s}, \frac{r-p_h}{r-s}\right\} \leq \theta$, we have $E[T^1] \leq E[T^2]$; and (ii) when $\min\left\{\frac{r-p_l}{r-s}, \frac{r-p_h}{r-s}\right\} \geq \theta$, we have $E[T^1] \geq E[T^2]$.

Remark 5.1 There are many commonly used demand distributions having the relationship $\varphi^1(i, R^1) \geq_{\text{var}} \varphi^2(i, R^2)$. For example, $\varphi^1(i, R^1)$ is uniform $(a_1 + i, b_1 + i)$ and $\varphi^2(i, R^2)$ is uniform $(a_2 + i, b_2 + i)$ with $a_1 < a_2, b_1 > b_2$ and $a_1 + b_1 = a_2 + b_2$.

6 Multi-Period Problems

In this section we extend our analysis to the case of N periods, $1 \leq N < \infty$. To obtain the notation in this case, we add the superscript m to the notation of Section 2 to refer to period m, $1 \leq m \leq N$. The only exception is s, which we replace by s^N . In addition, we need to define $x^m \geq 0$ to denote the ending inventory in period m after the demand in that period is satisfied to the extent possible. Any shortage in period m is lost. Thus, the inventory x^m , which is also the beginning inventory in period (m + 1), provides a coupling between period m and period m + 1. If the excess inventory x^m were to be salvaged at the end of each period, there would be no such coupling, and consequently the N-period problem would be just a sequence of N independent one-period problems. What makes our extension a genuine multi-period problem is the presence of the excess ending inventory in period m that can be used to satisfy demands in later periods.

The sequence in which various events occur is shown in Figure 1. To complete the statement of the problem, we need some additional assumptions. These are as follows:

- (i) The excess inventory x^m at the end of period m, $1 \le m \le N 1$, incurs a unit holding cost of h^m in carrying the inventory to the next period and x^N has the unit salvage value s^N .
- (ii) D^m , $1 \le m \le N$, are independent demands and I^m , $1 \le m \le N$, are independent forecast updating signals.
- (iii) We replace the assumption (1) by

$$r^m > \max[p_h^m, p_c^m, h^m], 1 \le m \le N, \text{ and } s^N < \min[p_l^N, p^N].$$
 (25)

As before, (25) is imposed to avoid trivial cases.

With x^{m-1} as the initial inventory level in period m, the profit obtained in period $m, 1 \le m \le N-1$ is given by

$$-p^{m}q^{m} + E\left[\Pi^{m}(x^{m-1}, q^{m}, q^{m}_{s}, q^{m}_{c}; I^{m}, P^{m})\right],$$



Figure 1: A timeline for the execution of a quantity flexible contract

where

$$\Pi^{m}(x^{m-1}, q^{m}, q_{s}^{m}, q_{c}^{m}; I^{m}, P^{m})$$

$$= E \left\{ \left[r^{m}(D^{m} \wedge (x^{m-1} + q^{m} + q_{c}^{m} + q_{s}^{m})) - h^{m}(x^{m-1} + q^{m} + q_{c}^{m} + q_{s}^{m} - D^{m})^{+} - p_{c}^{m}q_{c}^{m} - P^{m}q_{s} \right] | (I^{m}, P^{m}) \right\}$$

$$(26)$$

The profit obtained in the last period is

$$-p^{N}q^{N} + E\left[\Pi^{N}(x^{N-1}, q^{N}, q_{s}^{N}, q_{c}^{N}; I^{N}, P^{N})\right],$$

where

$$\Pi^{N}(x^{N-1}, q^{N}, q_{s}^{N}, q_{c}^{N}; I^{N}, P^{N})$$

$$= E\left\{\left[r^{N}(D^{N} \wedge (x^{N-1} + q^{N} + q_{c}^{N} + q_{s}^{N})) + s^{N}(x^{N-1} + q^{N} + q_{c}^{N} + q_{s}^{N} - D^{N})^{+} - p_{c}^{N}q_{c}^{N} - P^{N}q_{s}\right]\right|(I^{N}, P^{N})\right\}.$$

$$(27)$$

Let $F^{N}(x^{N-1})$ be the maximum profit in period N with the initial inventory level x^{N-1} , that is,

$$F^{N}(x^{N-1}) = \max_{q^{N} \ge 0} \left\{ -p_{h}q^{N} + E\left(\max_{\substack{0 \le q_{s}^{N} < \infty \\ 0 \le q_{c}^{N} \le \delta^{N}q^{N}}} \Pi^{N}(x^{N-1}, q^{N}, q_{s}^{N}, q_{c}^{N}; I^{N}, P^{N})\right) \right\}.$$
 (28)

Similarly, let $F^m(x^{m-1})$ be the maximum profit from period m to the last period with the initial inventory level x^{m-1} . Then

$$F^{m}(x^{m-1}) = \max_{q^{m} \ge 0} \left\{ -p^{m}q^{m} + E \Big[\max_{\substack{0 \le q_{c}^{m} < \infty \\ 0 \le q_{c}^{m} \le \delta^{m}q^{m}}} \Pi^{m}(x^{m-1}, q^{m}, q_{s}^{m}, q_{c}^{m}; I^{m}, P^{m}) + E(F^{m+1}(x^{m-1} + q^{m} + q_{c}^{m} + q_{s}^{m} - D^{m}) | (I^{m}, P^{m})) \Big] \right\}.$$
(29)

It is easy to verify that $F^m(x^{m-1})$ is concave. Based on the concavity of $F^m(x^{m-1})$, similar to Proposition 3.1, we also have the following result.

Proposition 6.1 There are Q^{m*} , $Q_c^{m*}(i^m, p_l^m)$, $Q_s^{m*}(i^m, p_l^m)$, $Q_c^{m*}(i^m, p_h^m)$ and $Q_s^{m*}(i^m, p_h^m)$ such that:

(A.1) If $p_l^m \leq p_c^m$, then the optimal order at the beginning of period m is

$$q^{m*} = (Q^{m*} - x^{m-1})^+,$$

and the optimal reaction at time t^m is to order all additional required product from the spot market if the market price is low, i.e., p_l^m . Otherwise, order additional product on contract, and to order an appropriate quantity from the spot market only when some additional quantity beyond the quantity flexibility bound is needed. Specifically,

$$\begin{split} q_c^{m*}(q^{m*}, i^m, p_l^m) &= 0, \quad q_s^{m*}(q^{m*}, i^m, p_l^m) = \left[Q_c^{m*}(i^m, p_l^m) - q^{m*} - x^{m-1}\right]^+, \\ q_s^{m*}(q^{m*}, i^m, p_h^m) &= (\delta^m q^{m*}) \wedge \left[Q_c^{m*}(i^m, p_h^m) - q^{m*} - x^{m-1}\right]^+, \\ q_s^{m*}(q^{m*}, i^m, p_h^m) &= \left[Q_s^{m*}(i^m, p_h^m) - (1 + \delta^m)q^{m*} - x^{m-1}\right]^+. \end{split}$$

(A.2) If $p_c^m \leq p_l^m$, then the optimal order at the beginning of period m is

$$q^{m*} = (Q^{m*} - x^{m-1})^+,$$

and the optimal reaction at time t^m is to order additional product on contract and to order an appropriate quantity from the spot market only when some additional quantity beyond the quantity flexibility bound is needed. Specifically,

$$q_c^{m*}(q^{m*}, i^m, p_l^m) = (\delta^m q^{m*}) \wedge \left[Q_c^{m*}(i^m, p_l^m) - q^{m*} - x^{m-1}\right]^+,$$

$$\begin{split} q_s^{m*}(q^{m*}, i^m, p_l^m) &= \left[Q_s^{m*}(i^m, p_l^m) - (1 + \delta^m)q^{m*} - x^{m-1}\right]^+, \\ q_c^{m*}(q^{m*}, i^m, p_h^m) &= (\delta^m q^{m*}) \wedge \left[Q_c^{m*}(i^m, p_h^m) - q^{m*} - x^{m-1}\right]^+, \\ q_s^*(q^{m*}, i^m, p_h^m) &= \left[Q_s^{m*}(i^m, p_h^m) - (1 + \delta^m)q^{m*} - x^{m-1}\right]^+. \end{split}$$

Remark 6.1 Suppose that for each period, the market prices are iid, and demands are iid and the demand forecasts are also iid. That is, for all m,

$$\beta^m = \beta$$
, $G^m(\cdot) = G(\cdot)$ and $H^m(\cdot|\cdot) = H(\cdot|\cdot)$.

Then the optimal purchase quantities are myopic. Specifically, there are pairs

$$(Q^*, Q_c^*(i, p_l), Q_s^*(i, p_l), Q_c^*(i, p_h), Q_s^*(i, p_h))$$

such that for all m,

$$Q^{m*} = Q^*, \quad Q^{m*}_c(i, p_l) = Q^*_c(i, p_l), \quad Q^{m*}_s(i, p_l) = Q^*_s(i, p_l),$$
$$Q^{m*}_c(i, p_h) = Q^*_c(i, p_h), \quad Q^{m*}_s(i, p_h) = Q^*_s(i, p_h).$$

7 Concluding Remarks

We have studied single and multi-period quantity flexible contracts that allow an initial order at the beginning of a period, a forecast revision in the middle of the period, and further purchases on contract and in the spot market before the demand is realized at the end of the period. The additional purchase quantity on the contract at a contractual price is limited by the specified flexibility limit. Any amount, however, can be purchased on the spot market at the then prevailing market price. The initial purchase quantity at a given price is based on the demand distribution, the market price distribution, the contractual price and the flexibility level, and the possibility of a forecast revision before additional final purchases. We provide optimal initial orders and the optimal feedback quantities to be purchased following the demand forecast revisions. We provide intuitive interpretations of our results. We examine the impact of the information quality and the flexibility on the optimal decisions. We measure the value of flexibility and provide conditions when this value is positive. We would like to mention that in our study of the problem of optimal management and design of flexible contracts, we have made a number of simplifications. In our multi-period model, the inventory carry-over from one period to the next provides the dynamics. On the other hand, we treat the case of lost sales, which does not carry over from one period to the next. There is no fixed cost in our model, either in ordering or in exercising the contract. Any one of these would make the model considerably more complex. Convexity of the value function would be lost. Whether (s, S)-type policies would be optimal in this case remains to be seen. Finally, we allow only two possible market prices, high and low, which are geometrically distributed. It would be of interest to extend the model to allow for a market price having a general probability distribution defined over a range of prices.

Appendix

In this appendix, we provide proofs of all of the propositions in the paper.

Proof of Proposition 3.1: We only prove (ii). As for (i), q_c^* is obvious, and the proof of q_s^* follows easily in the same way as the proof of (ii). First note that

$$\max_{\substack{0 \le q_s < \infty \\ 0 \le q_c \le \delta q}} \Pi(q, q_s, q_c; i, p_s)$$

=
$$\max_{\substack{0 \le q_s < \infty \\ 0 \le q_c \le \delta q}} \left\{ r \int_0^{q+q_s+q_c} zh(z|i)dz + r(q+q_s+q_c) \int_{q+q_s+q_c}^{\infty} h(z|i)dz + s \int_0^{q+q_s+q_c} [q+q_s+q_c-z]h(z|i)dz - p_cq_c - p_sq_s \right\}.$$

It follows from simple calculations that $[(1+\delta)q] \wedge \left[H^{-1}\left(\frac{r-p_s}{r-s}\middle|i\right) \lor q\right]$ maximizes

$$-(r-s)\int_{0}^{t} (t-z)h(z|i)dz + (r-p_{c})t + p_{c}q$$

on the interval $[q, (1+\delta)q]$, and $[(1+\delta)q] \vee H^{-1}\left(\frac{r-p_s}{r-s}\middle|i\right)$ maximizes

$$-(r-s)\int_0^t (t-z)h(z|i)dz + (r-p_s)t + p_s\left[(1+\delta)q\right] \wedge \left[H^{-1}\left(\left.\frac{r-p_s}{r-s}\right|i\right) \lor q\right]$$

on the interval $((1 + \delta)q, \infty)$. If $p_s > p_c$, then for any given $q_s \ge 0$ and $q_c \ge 0$, and any $\varepsilon > 0$,

$$r \int_{0}^{q+q_{s}+q_{c}} zh(z|i)dz + r(q+q_{s}+q_{c}) \int_{q+q_{s}+q_{c}}^{\infty} h(z|i)dz$$

$$+s \int_{0}^{q+q_{s}+q_{c}} (q+q_{s}+q_{c}-z)h(z|i)dz - p_{c}(q_{c}-\varepsilon) - p_{s}(q_{s}+\varepsilon)$$

$$< r \int_{0}^{q+q_{s}+q_{c}} zh(z|i)dz + r(q+q_{s}+q_{c}) \int_{q+q_{s}+q_{c}}^{\infty} h(z|i)dz$$

$$+s \int_{0}^{q+q_{s}+q_{c}} [q+q_{s}+q_{c}-z]h(z|i)dz - p_{s}(q_{s}+q_{c}).$$

Consequently, $(q_c^*(q, i, p_s), q_s^*(q, i, p_s))$ also maximize the function

$$r \int_{0}^{q+q_{s}+q_{c}} zh(z|i)dz + r(q+q_{s}+q_{c}) \int_{q+q_{s}+q_{c}}^{\infty} h(z|i)dz + s \int_{0}^{q+q_{s}+q_{c}} (q+q_{s}+q_{c}-z)h(z|i)dz - p_{c}q_{c} - p_{s}q_{s}$$

of (q_c, q_s) on the region $[0, \delta q] \times [0, \infty)$. Therefore, the proof of (ii) is completed.

Proof of Proposition 3.2: The inequalities relations among the information thresholds are obvious from (7) and (8). Remaining parts of Statements (i) and (ii) follow directly from the corresponding results (i) and (ii) in Proposition 3.1, respectively, when D is stochastically increasing in I.

Proof of Proposition 4.1: We prove the results only in the case $p_l < p_c < p_h$, as similar analysis can be carried out for the others. It follows from Proposition 3.2 that

$$\begin{split} -pq + E\left(\max_{\substack{0 \le q_{8} \le \infty \\ 0 \le q_{c} \le \delta q}} \Pi(q, q_{s}, q_{c}; I, P)\right) \\ &= -pq + \beta \int_{-\infty}^{\overline{I}(q,p_{l})} \left[(s-r) \int_{0}^{q} (q-z)h(z|i)dz + rq \right] g(i)di \\ &+ \beta \int_{\overline{I}(q,p_{l})}^{\infty} \left\{ (s-r) \int_{0}^{H^{-1}((r-p_{l})/(r-s)|i)} \left[H^{-1} \left(\frac{r-p_{l}}{r-s} \middle| i \right) - z \right] h(z|i)dz \\ &+ (r-p_{l})H^{-1} \left(\frac{r-p_{l}}{r-s} \middle| i \right) + p_{l}q \right\} g(i)di \\ &+ (1-\beta) \int_{-\infty}^{\overline{I}(q,p_{c})} \left[(s-r) \int_{0}^{q} (q-z)h(z|i)dz + rq \right] g(i)di \\ &+ (1-\beta) \int_{\overline{I}(q,p_{c})}^{\widehat{I}(q,p_{c})} \left\{ (s-r) \int_{0}^{H^{-1}((r-p_{c})/(r-s)|i)} \left[H^{-1} \left(\frac{r-p_{c}}{r-s} \middle| i \right) - z \right] h(z|i)dz \\ &+ (r-p_{c})H^{-1} \left(\frac{r-p_{c}}{r-s} \middle| i \right) + p_{c}q \right\} g(i)di \end{split}$$

$$+ (1 - \beta) \int_{\hat{I}(q,p_c)}^{\hat{I}(q,p_h)} \left\{ (s - r) \int_0^{(1+\delta)q} \left[(1 + \delta)q - z \right] h(z|i) dz + (r - p_c)(1 + \delta)q + p_c q \right\} g(i) di$$

$$+(1-\beta)\int_{\hat{I}(q,p_h)}^{\infty} \left\{ (s-r)\int_{0}^{H^{-1}((r-p_h)/(r-s)|i)} \left[H^{-1}\left(\frac{r-p_h}{r-s}\Big|i\right) - z \right] h(z|i)dz +(r-p_h)H^{-1}\left(\frac{r-p_h}{r-s}\Big|i\right) - p_c\delta q + p_h(1+\delta)q \right\} g(i)di.$$
(30)

Denote the above expression as $F(q, \delta)$. Then using calculus, one can show that

$$\frac{\partial F(q,\delta)}{\partial q} = -p + \beta \int_{-\infty}^{\bar{I}(q,p_l)} \left[(s-r)H(q|i) + r \right] g(i)di + \beta p_l [1 - G(\bar{I}(q,p_l))] \\
+ (1-\beta) \int_{-\infty}^{\bar{I}(q,p_c)} \left[(s-r)H(q|i) + r \right] g(i)di + (1-\beta)p_c [G(\hat{I}(q,p_c)) - G(\bar{I}(q,p_c))] \\
+ (1-\beta) \int_{\hat{I}(q,p_c)}^{\hat{I}(q,p_h)} \left\{ (1+\delta)[(s-r)H((1+\delta)q|i) + r] - p_c \delta \right\} g(i)di \\
+ (1-\beta) \int_{\hat{I}(q,p_h)}^{\infty} [-p_c \delta + (1+\delta)p_h]g(i)di.$$
(31)

Furthermore,

$$\frac{\partial^2 F(q,\delta)}{\partial q^2} = \beta \int_{-\infty}^{\bar{I}(q,p_l)} (s-r)h(q|i)g(i)di
+ (1-\beta)(s-r) \left\{ \int_{-\infty}^{\bar{I}(q,p_c)} h(q|i)g(i)di + \int_{\hat{I}(q,p_c)}^{\hat{I}(q,p_h)} (1+\delta)^2 h((1+\delta)q|i)g(i)di \right\}$$
(32)

and

$$\frac{\partial^2 F(q,\delta)}{\partial q \partial \delta} = (1-\beta) \Big\{ (p_c - p_h) G(\hat{I}(q, p_c)) + \int_{\hat{I}(q, p_c)}^{\hat{I}(q, p_h)} [(s-r) H((1+\delta)q|i) + (r-p_h) + (s-r)(1+\delta)^2 q h((1+\delta)q|i) \Big] g(i) di \Big\}.$$
(33)

By the definitions of $\hat{I}(q, p_h)$ and $\hat{I}(q, p_c)$, we know that for $i \in [\hat{I}(q, p_c), \hat{I}(q, p_h)]$,

$$(s-r)H((1+\delta)q|i) + r - p_h \le 0.$$
(34)

Hence, the result (i) follows from (32) and (33).

If $q^* > 0$, then

$$\left. \frac{\partial F(q,\delta)}{\partial q} \right|_{q=q^*} = 0.$$

Note that q^* depends on δ . Therefore,

$$\frac{dF(q^*,\delta)}{d\delta} = \left(\frac{\partial F(q,\delta)}{\partial q} \Big|_{q=q^*} \right) \frac{dq^*}{d\delta} + \frac{\partial F(q^*,\delta)}{\partial \delta} \\
= (1+\delta)(p_h - p_c)[1 - G(\hat{I}(q^*,p_h))] \\
+ (1+\delta) \int_{\hat{I}(q^*,p_c)}^{\hat{I}(q^*,p_h)} [(s-r)H((1+\delta)q^*|i) + r - p_c] g(i)di.$$
(35)

Similar to (34), we have that for $i \in [\hat{I}(q, p_c), \hat{I}(q, p_h)]$,

$$(s-r)H((1+\delta)q|i) + r - p_c \ge 0.$$

Consequently, the result (ii) follows from (35).

Proof of Proposition 4.2: We prove only (B.1) and (B.2), since the other results in the proposition can be established similarly. Since $p > p_l$ in Case B, then in view of $p < p_c$ and (6), we have

$$p_l$$

Thus,

$$H^{-1}\left(\frac{r-p_h}{r-s}\right) \le H^{-1}\left(\frac{r-p_c}{r-s}\right) \le H^{-1}\left(\frac{r-p}{r-s}\right) < H^{-1}\left(\frac{r-p_l}{r-s}\right).$$
(37)

It suffices therefore to show that when $\beta p_l + (1 - \beta)p_c \ge p$, q^* given in (B.1) is a maximizer of the function

$$V(q) \triangleq -pq + E\left[\Pi(q, q_s^*(q, P), q_c^*(q, P); I, P)\right];$$

and when $\beta p_l + (1 - \beta)p_c < p$ and $[-p + \beta p_l + (1 - \beta)p_c + (1 + \delta)(1 - \beta)(p_h - p_c)] > 0$, q^* given in (B.2) is a maximizer of V(q).

First we look at the proof of (B.1). The proof is divided into three subcases:

Case B.1.1. $q \ge H^{-1}((r-p_l)/(r-s))$. By Proposition 3.1,

$$q_c^*(q, p_l) = q_s^*(q, p_l) = q_c^*(q, p_h) = q_s^*(q, p_h) = 0.$$

Then,

$$V(q) = -pq + s \int_0^q (q-z)h(z)dz + r \int_0^q zh(z)dz + rq[1 - H(q)].$$

This implies that

$$\frac{dV(q)}{dq} = -p + sH(q) + r[1 - H(q)]$$

= $r - p - (r - s)H(q)$
< $r - p - (r - p_l) < 0.$

Hence, V(q) is decreasing in $[H^{-1}((r-p_l)/(r-s)), \infty)$.

Case B.1.2. $H^{-1}((r - p_c)/(r - s)) \le q \le H^{-1}((r - p_l)/(r - s))$. It follows from Proposition 3.1 that

$$V(q) = -pq + V_l(r, s, p_l) + \beta p_l q + (1 - \beta) \left[s \int_0^q (q - z)h(z)dz + r \int_0^q zh(z)dz + rq(1 - H(q)) \right],$$

where

$$V_{l}(r,s,p_{l}) = \beta \left\{ -(r-s) \int_{0}^{H^{-1}((r-p_{l})/(r-s))} \left[H^{-1} \left(\frac{r-p_{l}}{r-s} \right) - z \right] h(z) dz + (r-p_{l}) H^{-1} \left(\frac{r-p_{l}}{r-s} \right) \right\}.$$

Therefore,

$$\frac{dV(q)}{dq} = -p + \beta p_l + (1 - \beta)[r - rH(q) + sH(q)]$$

$$\leq -p + \beta p_l + (1 - \beta)p_c.$$

This implies that V(q) is increasing on the interval $[H^{-1}((r-p_c)/(r-s)), q^*]$ and decreasing on the interval $[q^*, H^{-1}((r-p_l)/(r-s))]$.

Case B.1.3. $q < H^{-1}((r - p_c)/(r - s))$. Proceeding as in Case B.1.2, we can show that V(q) is increasing on the interval $[0, H^{-1}((r - p_c)/(r - s))]$.

Combining Cases 1-3 completes the proof for (B.1).

Now we look at (B.2). It's proof is also divided into seven cases.

Case B.2.1. $q < H^{-1}((r - p_h)/(r - s))$ and $(1 + \delta)q < H^{-1}((r - p_h)/(r - s))$. V(q) can be written as

$$V(q) = -pq + V_l(r, s, p_l) + \beta p_l q$$

$$+(1-\beta)\Big\{-(r-s)\int_{0}^{H^{-1}((r-p_{h})/(r-s))}\Big[H^{-1}\left(\frac{r-p_{h}}{r-s}\right)-z\Big]h(z)dz\\+rH^{-1}\left(\frac{r-p_{h}}{r-s}\right)-p_{c}\delta q-p_{h}\left[H^{-1}\left(\frac{r-p_{h}}{r-s}\right)-(1+\delta)q\right]\Big\}.$$

Using the condition $[-p + \beta p_l + (1 - \beta)p_c + (1 + \delta)(1 - \beta)(p_h - p_c)] \ge 0$, we see that

$$\frac{dV(q)}{dq} = -p + \beta p_l + (1 - \beta) \Big[-p_c \delta + p_h (1 + \delta) \Big] \ge 0.$$

So V(q) is increasing for q satisfying $q < H^{-1}((r-p_h)/(r-s))$ and $(1+\delta)q < H^{-1}((r-p_h)/(r-s))$.

Case B.2.2. $q < H^{-1}((r - p_h)/(r - s)), (1 + \delta)q \ge H^{-1}((r - p_h)/(r - s))$ and $(1 + \delta)q < H^{-1}((r - p_c)/(r - s))$. Under this case, V(q) can be written as

Consequently,

$$\frac{dV(q)}{dq} = -p + \beta p_l + (1 - \beta) \Big[-(r - s)(1 + \delta)H((1 + \delta)q) + r(1 + \delta) - p_c \delta \Big].$$

In the following, if $(1+\delta)H^{-1}((r-p_h)/(r-s)) \ge H^{-1}((r-p_c)/(r-s))$, we go to Cases B.2.3 and B.2.5–B.2.7, and if $(1+\delta)H^{-1}((r-p_h)/(r-s)) < H^{-1}((r-p_c)/(r-s))$, we go to Cases B.2.4–B.2.7.

Case B.2.3. $q < H^{-1}((r-p_h)/(r-s))$ and $(1+\delta)q \ge H^{-1}((r-p_c)/(r-s))$. We have

$$V(q) = -pq + V_l(r, s, p_l) + \beta p_l q + (1 - \beta) \Big\{ -(r - s) \int_0^{H^{-1}((r - p_c)/(r - s))} \Big[H^{-1} \left(\frac{r - p_c}{r - s} \right) - z \Big] h(z) dz + r H^{-1} \left(\frac{r - p_c}{r - s} \right) - p_c \Big[H^{-1} \left(\frac{r - p_c}{r - s} \right) - q \Big] \Big\}.$$

Then,

$$\frac{dV(q)}{dq} = -p + \beta p_l + (1 - \beta)p_c < 0.$$

Case B.2.4. $H^{-1}((r-p_h)/(r-s)) \le q \le H^{-1}((r-p_c)/(r-s))$, and $(1+\delta)q < H^{-1}((r-p_c)/(r-s))$. We have

$$V(q) = -pq + V_l(r, s, p_l) + \beta p_l q + (1 - \beta) \Big\{ -(r - s) \int_0^{(1+\delta)q} \Big[(1 + \delta)q - z \Big] h(z) dz + r(1 + \delta)q - p_c \delta q \Big\}.$$

Consequently,

$$\frac{dV(q)}{dq} = -p + \beta p_l + (1 - \beta) \Big[-(r - s)(1 + \delta)H((1 + \delta)q) + r(1 + \delta) - p_c \delta \Big].$$

Case B.2.5. $H^{-1}((r-p_h)/(r-s)) \le q \le H^{-1}((r-p_c)/(r-s))$, and $(1+\delta)q \ge H^{-1}((r-p_c)/(r-s))$. We have

$$V(q) = -pq + V_l(r, s, p_l) + \beta p_l q + (1 - \beta) \Big\{ -(r - s) \int_0^{H^{-1}((r - p_c)/(r - s))} \Big[H^{-1} \left(\frac{r - p_c}{r - s} \right) - z \Big] h(z) dz + r H^{-1} \left(\frac{r - p_c}{r - s} \right) - p_c \Big[H^{-1} \left(\frac{r - p_c}{r - s} \right) - q \Big] \Big\}.$$

Then,

$$\frac{dV(q)}{dq} = -p + \beta p_l + (1 - \beta)p_c < 0.$$

Case B.2.6. $H^{-1}((r-p_c)/(r-s)) < q \le H^{-1}((r-p_l)/(r-s))$. We have

$$V(q) = -pq + V_l(r, s, p_l) + \beta p_l q + (1 - \beta) \Big\{ - (r - s) \int_0^q \Big[q - z \Big] h(z) dz + rq \Big\}.$$

Consequently,

$$\frac{dV(q)}{dq} = -p + \beta p_l + (1 - \beta) \Big[-(r - s)H(q) + r \Big] < -p + \beta p_l + (1 - \beta)p_c < 0.$$

Case B.2.7. $q > H^{-1}((r - p_l)/(r - s))$. We have

$$V(q) = -pq + \beta \Big\{ -(r-s) \int_0^q \Big[q-z \Big] h(z) dz + rq \Big\} \\ + (1-\beta) \Big\{ -(r-s) \int_0^q \Big[q-z \Big] h(z) dz + rq \Big\}.$$

Consequently,

$$\frac{dV(q)}{dq} = -p + \beta \Big[-(r-s)H(q) + r \Big] + (1-\beta) \Big[-(r-s)H(q) + r \Big] < -p + \beta p_l + (1-\beta)p_c < 0.$$

According to $(1+\delta)H^{-1}((r-p_h)/(r-s)) \ge H^{-1}((r-p_c)/(r-s))$ or $(1+\delta)H^{-1}((r-p_h)/(r-s)) < H^{-1}((r-p_c)/(r-s))$, (B.2) follows from Cases B.2.1–B.2.3 and B.2.5–B.2.7 or Cases B.2.1–B.2.2 and B.2.4–B.2.7, respectively.

Proof of Proposition 4.3 The proof of the proposition is similar to the proof of Proposition 4.2.

Proof of Proposition 4.4: Setting q = 0 in (16) and using (14) and the fact that F(0) = 0, we obtain

$$-p + \beta p_{l} + (1 - \beta)p_{c} + (1 - \beta)(1 + \delta)(p_{h} - p_{c})$$

+[s - \beta p_{l} - (1 - \beta)p_{c}]F(0) - (1 - \beta)(1 + \delta)(p_{h} - p_{c})F(0)
= -p + \beta p_{l} + (1 - \beta)p_{c} + (1 - \beta)(1 + \delta)(p_{h} - p_{c}) > 0. (38)

In view of $\lim_{q\to\infty} F(q) = 1$ and assumption (1), we have

$$-p + \beta p_{l} + (1 - \beta)p_{c} + (1 - \beta)(1 + \delta)(p_{h} - p_{c}) + [s - \beta p_{l} - (1 - \beta)p_{c}] \lim_{q \to \infty} F(q) - (1 - \beta)(1 + \delta)(p_{h} - p_{c}) \lim_{q \to \infty} F((1 + \delta)q) = s - p < 0.$$
(39)

Taking the derivative of the LHS of (16) with respect to q, we obtain

$$d\left\{ [s - \beta p_l - (1 - \beta) p_c] F(q) - (1 - \beta) (1 + \delta) (p_h - p_c) F((1 + \delta) q) \right\} / dq$$

= $[s - \beta p_l - (1 - \beta) p_c] \cdot \frac{dF(q)}{dq} - (1 - \beta) (1 + \delta)^2 (p_h - p_c) \cdot \frac{dF(x)}{dx} \Big|_{x = (1 + \delta)q}.$ (40)

Since $p_l \leq p_c$ and $s < p_l$ as assumed in (1), we have

$$s - \beta p_l - (1 - \beta)p_c \le s - \beta p_l - (1 - \beta)p_l = s - p_l < 0.$$

Thus, the derivative in (40) is strictly negative. The proposition follows from (38)-(40). \Box

Proof of Proposition 4.5 : Setting q = 0 in (19) and using F(0) = 0 and (20), we have

$$-p - \delta p_{c} + (1+\delta)[\beta p_{l} + (1-\beta)p_{h}] + (s - p_{c})F(0) + (1+\delta)[p_{c} - \beta p_{l} - (1-\beta)p_{h}]F(0) - p - \delta p_{c} + (1+\delta)[\beta p_{l} + (1-\beta)p_{h}] > 0$$
(41)

and

$$-p - \delta p_{c} + (1+\delta)[\beta p_{l} + (1-\beta)p_{h}] + (s - p_{c}) \lim_{q \to \infty} F(q) + (1+\delta)[p_{c} - \beta p_{l} - (1-\beta)p_{h}] \lim_{q \to \infty} F((1+\delta)q)\Big|_{q=\infty}$$

= $-p + s < 0.$ (42)

Furthermore, taking the derivative of the LHS of (19) with respect to q and using the facts $s , and the condition <math>p_l > p_c$, we obtain

$$d\left\{-p - \delta p_{c} + (1+\delta)[\beta p_{l} + (1-\beta)p_{h}] + (s - p_{c})F(q) + (1+\delta)[p_{c} - \beta p_{l} - (1-\beta)p_{h}]F((1+\delta)q)\right\} / dq$$

= $(s - p_{c}) \cdot \frac{dF(q)}{dq} + (1+\delta)^{2}[p_{c} - \beta p_{l} - (1-\beta)p_{h}] \cdot \frac{dF(x)}{dx}\Big|_{x=(1+\delta)q} < 0,$ (43)

The proposition follows from (41)-(43).

Proof of Proposition 4.6: First consider the case of worthless information. Using Proposition 3.1 of Gurnani and Tang (1999) and Proposition 4.2, we know that the flexibility value is zero if any one of the following conditions hold.

(i)
$$p \leq p_l$$
;

(ii)
$$p > p_l$$
 and $\beta p_l + (1 - \beta)p_c \ge p;$

(iii) $p > p_l$ and $p \ge \beta p_l + (1 - \beta)p_h + (1 - \beta)\delta(p_h - p_c)$.

Thus we only need to prove the proposition in case (B.2) of Proposition 4.2, i.e., when $p > p_l$ and $\beta p_l + (1 - \beta)p_c . In this case the flexibility value is$ $obtained in (15), which is clearly decreasing in <math>\beta$.

Now consider the case of perfect information. We must treat the following four cases:

(A.1)
$$p_l \leq p_c, \ p \leq \beta p_l + (1 - \beta) p_h;$$

(A.2) $p_l \leq p_c, \ \beta p_l + (1 - \beta) p_h
(A.3) $p_l \leq p_c, \ \beta p_l + (1 - \beta) p_h + (1 - \beta) \delta(p_h - p_c) \leq p;$
(B) $p_l > p_c.$$

The optimal solutions in the first three cases (A.1), (A.2), and (A.3) are given in Proposition 4.3(A) and the optimal solution in case B is given in Proposition 4.3(B). In (A.3), we know from Proposition 4.4 that $q^* = 0$, which implies that the flexibility value is zero. Below we will provide the details of the proof only in case (A.1), since the proofs in cases (A.2) and (B) follow in the same way.

In case (A.1), if there is no contract, then we would have $\delta = 0$. Then the condition of the case implies that the inequality (14) is satisfied with $\delta = 0$. By Proposition 4.4, therefore, the optimal q^* in Proposition 4.3(A) would be given by solving (16) with $\delta = 0$, which we write as

$$q_0^* = F^{-1} \left(\frac{-p + \beta p_l + (1 - \beta) p_h}{-s + \beta p_l + (1 - \beta) p_h} \right).$$
(44)

Moreover from (17), the optimal order quantity at time t_2 regardless of the market price would be

$$[\tau(i) - q_0^*]^+, \quad i = 1, 2.$$

By (44) we have

$$-p + \beta p_l + (1 - \beta)p_c + (1 - \beta)(1 + \delta)(p_h - p_c) + [s - \beta p_l - (1 - \beta)p_c]F(q_0^*) + (1 - \beta)(1 + \delta)(p_c - p_h)F((1 + \delta)q_0^*) = (p_h - p_c)(1 - \beta)[F(q_0^*) - (1 + \delta)F((1 + \delta)q_0^*)] < 0.$$
(45)

Thus, the solution q^* given by (16) with $\delta > 0$ is smaller than q_0^* ordered in the absence of a contract. In other words, the buyer purchases less at t_1 when he has a contract ($\delta > 0$). Then, from Proposition 4.3(A) and equation (18), the difference of the expected profits with and without

the contract is

$$(1-\beta)\left\{-p_{c}\int_{q^{*}}^{(1+\delta)q^{*}}zdF(z)+p_{h}\mathrm{sign}((1+\delta)q^{*}-q_{0}^{*})\int_{((1+\delta)q^{*})\wedge q_{0}^{*}}^{((1+\delta)q^{*})\vee q_{0}^{*}}zdF(z)\right\}$$
$$+(s-\beta p_{l})\int_{q^{*}}^{q_{0}^{*}}zdF(z),$$
(46)

where

$$\operatorname{sign}(x) = \begin{cases} 1, & \text{if } x > 0, \\ -1, & \text{if } x < 0, \\ 0, & \text{if } x = 0. \end{cases}$$

From (46), we know that under (A.1), the contract improves the buyer's expected profit. Furthermore, the smaller the value of β is, the larger is the value of flexibility.

This completes the proof.

Proof of Proposition 5.1: First we consider the case $p_c < p_l < p_h$. Let $\Pi^k(q, q_c, q_s; I, P)$ be the conditional expected profit, as defined in (3), of system k at time t_2 given I and P. If we could show that for any given $q \ge 0$ and any observed value (i, p_s) of (I, P),

$$\max_{\substack{0 \le q_s \le \infty\\0 \le q_c \le \delta q}} \Pi^1(q, q_s, q_c; i, p_s) \le \max_{\substack{0 \le q_s \le \infty\\0 \le q_c \le \delta q}} \Pi^2(q, q_s, q_c; i, p_s),$$
(47)

then

$$E\left(\max_{\substack{0 \le q_s < \infty\\0 \le q_c \le \delta q}} \Pi^1(q, q_s, q_c; I, P)\right) \le E\left(\max_{\substack{0 \le q_s < \infty\\0 \le q_c \le \delta q}} \Pi^2(q, q_s, q_c; I, P)\right),\tag{48}$$

and, in turn,

$$\max_{q\geq 0} \left\{ -pq + E\left(\max_{\substack{0\leq q_s<\infty\\0\leq q_c\leq\delta q}} \Pi^1(q, q_s, q_c; I, P)\right) \right\}$$
$$\leq \max_{q\geq 0} \left\{ -pq + E\left(\max_{\substack{0\leq q_s<\infty\\0\leq q_c\leq\delta q}} \Pi^2(q, q_s, q_c; I, P)\right) \right\}.$$

Thus, we have the proposition if we prove (47). To this end, it is sufficient to show that for any given $q \ge 0$, $q_c \ge 0$, and q_s , $0 \le q_s \ge \delta_q$,

$$\Pi^{1}(q, q_{s}, q_{c}; i, p_{s}) \leq \Pi^{2}(q, q_{s}, q_{c}; i, p_{s}).$$
(49)

To prove (49), let $H^k(z|i)$ and $h^k(z|i)$ be the conditional distribution and the conditional density of D^k for system k given i, respectively. That is, $H^k(z|i)$ and $h^k(z|i)$ are distribution and density of $\varphi^k(i, R^k)$, respectively. Note that by (5),

$$\Pi^{k}(q, q_{s}, q_{c}; i, p_{s}) = \int_{0}^{q+q_{c}+q_{s}} -(r-s)[q+q_{c}+q_{s}-z]h^{k}(z|i)dz +r(q+q_{c}+q_{s}) - p_{s}q_{s} - p_{c}q_{c}$$
$$= -(r-s)\int_{q+q_{c}+q_{s}}^{\infty} \left[z - (q+q_{c}+q_{s})\right]h^{k}(z|i)dz +(r-s)E[\varphi^{k}(i, R^{k})] + s(q+q_{c}+q_{s}) - p_{s}q_{s} - p_{c}q_{c}.$$
(50)

Note that $[z - (q + q_c + q_s)]^+$ is a non-decreasing convex function of z. Hence, in view of our assumptions $E[\varphi^1(i, R^1)] = E[\varphi^2(i, R^2)]$ and $\varphi^1(i, R^1) \ge_{ic} \varphi^2(i, R^2)$, we have

$$\Pi^{1}(q, q_{s}, q_{c}; i, p_{s}) = -(r-s) \int_{q+q_{c}+q_{s}}^{\infty} \left[z - (q+q_{c}+q_{s}) \right] h^{1}(z|i) dz + (r-s) E[\varphi^{1}(i, R^{1})] + s(q+q_{c}+q_{s}) - p_{s}q_{s} - p_{c}q_{c} \leq -(r-s) \int_{q+q_{c}+q_{s}}^{\infty} \left[z - (q+q_{c}+q_{s}) \right] h^{2}(z|i) dz + (r-s) E[\varphi^{2}(i, R^{2})] + s(q+q_{c}+q_{s}) - p_{s}q_{s} - p_{c}q_{c} = \Pi^{2}(q, q_{s}, q_{c}; i, p_{s}).$$
(51)

This proves (49) as required. Proceeding along these lines, we can show that the proposition holds in the other cases as well. $\hfill \Box$

Proof of Proposition 5.2: Let $q_s^{*k}(q, i, p_s)$ be the optimal order quantity by system k at time t_2 , when the observed value of (I, P) is (i, p_s) , k = 1, 2. It follows from Proposition 4.11 of Song (1994) that for fixed q and I, there exists a $\theta(I)$ such that when

$$\max\left\{\frac{r-p_l}{r-s}, \frac{r-p_h}{r-s}\right\} \le \theta,\tag{52}$$

then

$$q_s^{*1}(q, i, p_l) \le q_s^{*2}(q, i, p_l), \quad q_s^{*1}(q, i, p_h) \le q_s^{*2}(q, i, p_h),$$
(53)

and when

$$\min\left\{\frac{r-p_l}{r-s}, \frac{r-p_h}{r-s}\right\} \ge \theta,\tag{54}$$

then

$$q_s^{*1}(q,i,p_l) \ge q_s^{*2}(q,i,p_l), \quad q_s^{*1}(q,i,p_h) \ge q_s^{*2}(q,i,p_h).$$
(55)

Let q^{*k} be the optimal order quantity by system k at t_1 , k = 1, 2. If $q^{*k} > 0$, then $q^{*k} > 0$ must be the solution of the following equation in q:

$$-p + \beta p_l + (1 - \beta) p_h + \beta \int_{-\infty}^{I_l^k(q)} \left[(s - r) H^k(q|i) + (r - p_l) \right] g(i) di + (1 - \beta) \int_{-\infty}^{I_h^k(q)} \left[(s - r) H^k(q|i) + (r - p_h) \right] g(i) di = 0,$$
(56)

where $I_l^k(q)$ and $I_h^k(q)$ are defined by

$$H^{k}(q|I_{l}^{k}(q)) = \frac{r-p_{l}}{r-s}$$
 and $H^{k}(q|I_{h}^{k}(q)) = \frac{r-p_{h}}{r-s}$

From the monotonicity of $H^k(q|i)$ and (52), we have

$$I_l^1(q) \ge I_l^2(q), \quad I_h^1(q) \ge I_h^2(q),$$
(57)

and with (54) we have

$$I_l^1(q) \le I_l^2(q), \quad I_h^1(q) \le I_h^2(q).$$
 (58)

By $\varphi^1(i, R^1) \geq_{\text{var}} \varphi^2(i, R^2)$, if (52) holds for any $i \leq I_h^1(q)$, then

$$H^1(q|i) \ge H^2(q|i).$$
 (59)

Therefore,

$$\int_{-\infty}^{I_l^{1}(q)} \left[(s-r)H^1(q|i) + (r-p_l) \right] g(i)di$$

$$\leq \int_{-\infty}^{I_l^{2}(q)} \left[(s-r)H^2(q|i) + (r-p_l) \right] g(i)di$$
(60)

and

$$\int_{-\infty}^{I_h^{1}(q)} \left[(s-r)H^1(q|i) + (r-p_h) \right] g(i)di$$

$$\leq \int_{-\infty}^{I_h^{2}(q)} \left[(s-r)H^2(q|i) + (r-p_h) \right] g(i)di.$$
(61)

Thus, the result $q^{*1} \leq q^{*2}$ follows directly from

$$0 = -p + \beta p_{l} + (1 - \beta) p_{h} + \beta \int_{-\infty}^{I_{l}^{2}(q^{*2})} \left[(s - r) H^{2}(q^{*2}|i) + (r - p_{l}) \right] g(i) di$$

+ $(1 - \beta) \int_{-\infty}^{I_{h}^{2}(q^{*2})} \left[(s - r) H^{2}(q^{*2}|i) + (r - p_{h}) \right] g(i) di$
> $-p + \beta p_{l} + (1 - \beta) p_{h} + \beta \int_{-\infty}^{I_{l}^{1}(q^{*2})} \left[(s - r) H^{1}(q^{*2}|i) + (r - p_{l}) \right] g(i) di$
+ $(1 - \beta) \int_{-\infty}^{I_{h}^{1}(q^{*2})} \left[(s - r) H^{1}(q^{*2}|i) + (r - p_{h}) \right] g(i) di.$

The first part of the proposition is proved. The second part can be proved in a similar way. \Box

Proof of Proposition 6.1: The proof is similar to the proof of Proposition 3.2. \Box

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