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Γ-robust linear complementarity problems with ellipsoidal uncertainty sets

Vanessa Krebs[a,b], Michael Müller[a] and Martin Schmidt[c,*]

[a] *Friedrich-Alexander-Universität Erlangen-Nürnberg, Discrete Optimization, Cauerstr. 11, Erlangen 91058, Germany*
[b] *Energie Campus Nürnberg, Fürther Str. 250, Nürnberg 90429, Germany*
[c] *Department of Mathematics, Trier University, Universitätsring 15, Trier 54296, Germany*
E-mail: vanessa.krebs@fau.de [Krebs]; mueller.mwkg@posteo.de [Müller]; martin.schmidt@uni-trier.de [Schmidt]

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Abstract

We study uncertain linear complementarity problems (LCPs), that is, problems in which the LCP vector q or the LCP matrix M may contain uncertain parameters. To this end, we use the concept of Γ-robust optimization applied to the gap function formulation of the LCP. Thus, this work builds upon Krebs and Schmidt (2020). There, we studied Γ-robustified LCPs for ℓ_1- and box-uncertainty sets, whereas we now focus on ellipsoidal uncertainty sets. For uncertainty in q or M, we derive conditions for the tractability of the robust counterparts. For these counterparts, we also give conditions for the existence and uniqueness of their solutions. Finally, a case study for the uncertain traffic equilibrium problem is considered, which illustrates the effects of the values of Γ on the feasibility and quality of the respective robustified solutions.

Keywords: robust optimization; linear complementarity problems; ellipsoidal uncertainty sets; traffic equilibrium problems

1. Introduction

Linear complementarity problems (LCPs) are a powerful tool in mathematical optimization with many applications in, for example, game theory, traffic modeling, economics, or energy markets but also within mathematics and optimization itself. For an overview of LCPs, we refer the reader to the seminal book by Cottle et al. (2009). As it is the case for most likely all other fields of optimization, the main branch of research on LCPs deals with the case of certain problem data, that is, both the matrix M and the vector q describing the specific LCP at hand are considered to be certain. However, in many practical applications these data are not known exactly and thus subject to uncertainty. This is, for example, the case for the future demand in energy markets modeled using

*Corresponding author.

LCPs or for the demand and travel times in traffic equilibrium models that are also often studied using complementarity problems.

There are two major fields of optimization under uncertainty: stochastic (Kall and Wallace, 1994; Birge and Louveaux, 2011) and robust optimization (Ben-Tal et al., 2009; Bertsimas et al., 2011). In stochastic optimization, uncertainties are described by probability distributions and, based on that, one usually optimizes expected values or considers chance-constrained models. In robust optimization, it is assumed that such distributions are not available and optimization of expected values is replaced by optimizing the worst case that might appear with respect to given uncertainty sets. Both routes can also be followed for uncertain LCPs. Accordingly, in the stochastic case, one usually minimizes the expected value of the LCP's gap function, which is called expected residual minimization in the literature; cf., for example, Chen and Fukushima (2005), Chen et al. (2009, 2012), and Lin and Fukushima (2006) and the references therein. In this paper, we instead consider the minimization of the worst-case gap of the LCP, that is, we follow a robust approach to uncertain LCPs. For this setting, much less literature is available. To the best of our knowledge, the first paper on robust LCPs is Wu et al. (2011), where the authors apply the concept of strict robustness. The same concept has also been studied in Xie and Shanbhag (2014, 2016), where the authors consider strict robustness for uncertain LCPs for the case of different uncertainty sets like box or ellipsoidal uncertainties and focus on questions of tractability of the respective robust counterparts. The first and, to the best of our knowledge, only application of the results in Xie and Shanbhag (2014, 2016) is given in Mather and Munsing (2017), where the general theory is applied to Cournot–Bertrand equilibria on power networks. A related study of robustified market equilibrium problems can be found in the recent paper by Kramer et al. (2018).

One criticism often raised with respect to strictly robust optimization is that it typically leads to highly conservative solutions because they are explicitly hedged against the worst-case scenario. Thus, alternative robustness concepts have been developed starting with the Γ-approach introduced in Bertsimas and Sim (2004) that we also study in this paper in the context of uncertain LCPs. The Γ-approach for uncertain LCPs has also been studied in the recent paper by Krebs and Schmidt (2020), where ℓ_1- and box-uncertainty sets have been considered. The present paper is an extension of the latter work and studies Γ-robustified LCPs with ellipsoidal uncertainty sets.

Our contribution is the following. First, following Krebs and Schmidt (2020), we review the concept of Γ-robust LCPs and afterward analyze in which cases we can reformulate them as a tractable, that is, convex, optimization problem. To this end, we consider two different cases separately: uncertainties in the LCP vector q and uncertainties in the LCP matrix M. Moreover, we consider the related concept of ρ-robustness that has been introduced in Wu et al. (2011). For the tractable counterparts mentioned above, we also derive conditions for the existence and uniqueness of solutions. Finally, we apply the concept of Γ-robustified LCPs with ellipsoidal uncertainty sets to the well-studied case of traffic equilibrium problems; see, for example, Dafermos (1980), Facchinei and Pang (2003), and Patriksson (2015).

The remainder of the paper is structured as follows: In Section 2, we review LCPs, their gap function formulation, and state the uncertain and robustified LCP with different variants of ellipsoidal uncertainty sets that we afterward study. Then, in Section 3, we derive counterparts for these uncertainty sets, where we analyze uncertainty in the LCP vector q and in the LCP matrix M separately. Further, we consider the concept of ρ-robustness. For the tractable counterparts, we study existence and uniqueness of solutions in Section 4. Finally, a case study for the classic traffic equilibrium problem is given in Section 5 before we close the paper with a conclusion and the discussion of some future research perspectives in Section 6.

2. Problem statement

We consider the LCP(q, M), which is the problem to find a point $x \in \mathbb{R}^n$ satisfying

$$0 \leq x \perp Mx + q \geq 0, \tag{1}$$

where $M \in \mathbb{R}^{n\times n}$ is a given matrix and $q \in \mathbb{R}^n$ is a given vector, or to show that no such point exists. Here and in what follows, we use the standard $\perp$-notation, which abbreviates

$$0 \leq a \perp b \geq 0 \iff 0 \leq a, b \geq 0,\ a^\top b = 0$$

for vectors $a, b \in \mathbb{R}^n$. In the following, let $\mathcal{X} := \{x \in \mathbb{R}^n : x \geq 0,\ Mx + q \geq 0\}$ be the set of feasible points of (1). The gap function formulation of the LCP (1) is the quadratic optimization problem

$$\min_{x\in\mathbb{R}^n} \quad g(x) := x^\top(Mx + q) \quad \text{s.t.} \quad x \geq 0,\ Mx + q \geq 0. \tag{2}$$

Here, the objective function g is the so-called gap function of the LCP (1). Obviously, the objective function g of Problem (2) is bounded from below by zero on its polyhedral feasible set $\mathcal{X}$, which ensures the existence of a minimizer by the theorem of Frank–Wolfe (Frank and Wolfe, 1956) if the feasible set is not empty.

In this paper, we consider the situation in which the entries in the problem's data M and q are uncertain, that is, we have $M(u_1)$ and $q(u_2)$ with $u_i \in \mathcal{U}_i$, $i = 1, 2$, and $\mathcal{U}_i$ are given uncertainty sets. Taking these uncertainty sets into account means that we have an infinite family of complementarity problems

$$\{0 \leq x \perp M(u_1)x + q(u_2) \geq 0\}_{(u_1,u_2)\in\mathcal{U}_1\times\mathcal{U}_2} \tag{3}$$

instead of the single nominal LCP (1). We call Problem (3) an uncertain LCP (ULCP). In this uncertain setting, there exists a lot of work, for example, Chen and Fukushima (2005) and Chen et al. (2009, 2012) focusing on the minimization of the expected gap function. In this paper, our focus, however, is not on the minimization of the expected gap function. Instead, we consider worst-case minima, that is, the robust case. Thus, we study the problem

$$\min_x \left\{ \sup_{(u_1,u_2)\in\mathcal{U}_1\times\mathcal{U}_2} g(x; u_1, u_2) : x \in \mathcal{X}(u_1, u_2) \text{ for all } (u_1, u_2) \in \mathcal{U}_1 \times \mathcal{U}_2 \right\}, \tag{4}$$

that is, the robust feasible set is given by

$$\mathcal{X}(u_1, u_2) := \{x \in \mathbb{R}^n : x \geq 0,\ M(u_1)x + q(u_2) \geq 0\}.$$

Note that this can be seen as the feasible set of a semi-infinite optimization problem; see, for example, Reemtsen and Rückmann (1998). To be more specific, we consider the Γ-robust setting. This means, there are at most $\Gamma_M \in \{1, \dots, n^2\}$ many values in $M(u_1)$ and $\Gamma_q \in \{1, \dots, n\} =: [n]$ many values in $q(u_2)$, which are uncertain and can thus realize in a worst-case way. In the paper by Krebs and Schmidt (2020), we already considered Problem (4) for uncertainty realizing in q or M and for box- and ℓ_1-norm uncertainty. Here, we consider the case of ellipsoidal uncertainty sets. To this end, we define the ellipsoidal uncertainty set as

$$\mathcal{U}^2 := \{u \in \mathbb{R}^n : \|u\|_2 \leq 1\}.$$

Applying the Γ-approach to ellipsoidal uncertainty leads to the set

$$\mathring{\mathcal{U}}^2_\Gamma := \{u \in \mathbb{R}^n : \|u\|_2 \leq 1 \ \wedge \ |\{l \in [n] : u_l \neq 0\}| \leq \Gamma\}.$$

Moreover, we also consider the convex version

$$\mathcal{U}^2_\Gamma := \{u \in \mathbb{R}^n : \|u\|_2 \leq 1 \ \wedge \ \|u\|_1 \leq \Gamma\}.$$

In the recent paper by Kurtz (2018), a slightly different definition for the convex ellipsoidal uncertainty set is introduced. There, the set

$$\mathcal{U}^2_{\Gamma,\lambda} := \left\{ u \in \mathbb{R}^n : \|u\|_2 \leq 1 \ \wedge \ \sum_{l \in [n]} \left| \frac{u_l}{\sqrt{\lambda_l}} \right| \leq \Gamma \right\}$$

is used, where $\lambda = (\lambda_l)_{l \in [n]}$ denotes the axis-length of the ellipsoid in direction l. We will use this concept later. With these definitions and the inequality $\|u\|_1 \leq \sqrt{n}\|u\|_2$, we obtain $\|u\|_1 \leq \sqrt{n}$ for $u \in \mathcal{U}^2$.

This means that for $\Gamma \geq \sqrt{n}$ one has $\mathcal{U}^2_\Gamma = \mathcal{U}^2$ and, thus, the uncertainty set $\mathcal{U}^2_\Gamma$ can only be smaller than $\mathcal{U}^2$ in cases where $\Gamma < \sqrt{n}$ holds, that is, for problems in which only a few worst-case deviations from the nominal values occur at the same time.

3. Tractable counterparts

In this section, we derive tractable counterparts of Γ-robust LCPs for uncertainties in q and M separately. We start with uncertain LCP vector q in Section 3.1 and afterward study the case of uncertain M in Section 3.2.

3.1. Uncertain LCP vector q

In this section, we consider uncertainties in the vector q, that is, we consider the uncertain LCP

$$0 \leq x \perp Mx + q(u) \geq 0, \quad u \in \mathcal{U},$$

and the parameterization

$$q(u) := q^0 + \sum_{l\in[m]} u_l q^l$$

for $u \in \mathcal{U}$, $q^0, q^l \in \mathbb{R}^n$, $l \in [m]$. Here, the vector q^0 contains the nominal values. In this case, the robust counterpart (4) can be written as

$$\min_{x\geq 0} \quad x^\top(Mx + q^0) + \max_{u\in\mathcal{U}} \sum_{l\in[m]} u_l x^\top q^l \tag{5a}$$

$$\text{s.t.} \quad 0 \leq M_{i,\cdot}x + q_i^0 + \min_{u\in\mathcal{U}} \sum_{l\in[m]} u_l q_i^l, \quad i \in [n]. \tag{5b}$$

Note that the minimization problems in (5b) do not depend on the variables x. For a shorter notation, we define

$$p_i := \min_{u\in\mathcal{U}} \sum_{l\in[m]} u_l q_i^l, \quad i \in [n]. \tag{6}$$

In a first step, we consider the nonconvex ellipsoidal uncertainty set $\mathring{\mathcal{U}}_\Gamma^2$ and state a reformulation of Problem (5).

Theorem 1. *Let the uncertainty set be $\mathring{\mathcal{U}}_\Gamma^2$. Then, Problem (5) is equivalent to*

$$\min_{z\geq 0} \quad x^\top(Mx + q^0) + \mu \tag{7a}$$

$$s.t. \quad M_{i,\cdot}x + q_i^0 + p_i \geq 0, \quad i \in [n], \tag{7b}$$

$$\mu^2 \geq \Gamma\alpha + \sum_{l\in[m]} \beta_l, \tag{7c}$$

$$\alpha + \beta_l \geq (x^\top q^l)^2, \quad l \in [m], \tag{7d}$$

where $z := (x^\top, \alpha, \beta^\top, \mu)^\top$.

Proof. As $\max\{u^\top v : u \in \mathcal{U}^2\} = \|v\|_2$ holds for $v \in \mathbb{R}^n$ and since at most Γ components of the uncertain vector u are nonzero in $\mathring{\mathcal{U}}_\Gamma^2$, we obtain

$$\max_{u\in\mathring{\mathcal{U}}_\Gamma^2} \sum_{l\in[m]} u_l x^\top q^l = \sqrt{\max_{\{L\subseteq[m]:|L|\leq\Gamma\}} \sum_{l\in L} (x^\top q^l)^2}. \tag{8}$$

By introducing a nonnegative variable μ, we can rewrite Problem (5) as

$$\min_{x\geq 0,\mu\geq 0} \quad x^\top(Mx + q^0) + \mu \tag{9a}$$

$$\text{s.t.} \quad M_{i,\cdot}x + q_i^0 + p_i \geq 0, \quad i \in [n], \tag{9b}$$

$$\mu^2 \geq \max_{\{L \subseteq [m] : |L| \leq \Gamma\}} \sum_{l \in L} (x^\top q^l)^2. \tag{9c}$$

In a second step, we reformulate the inner maximization problem in (9c) to

$$\max_{z \in \{0,1\}^m} \quad \sum_{l \in [m]} (x^\top q^l)^2 z_l \quad \text{s.t.} \quad \sum_{l \in [m]} z_l \leq \Gamma.$$

Now, we use its tight convex relaxation (see, for example, Proposition 1 in Sim (2004)) and the corresponding dual problem

$$\min_{\alpha \geq 0, \beta \geq 0} \quad \alpha\Gamma + \sum_{l \in [m]} \beta_l \quad \text{s.t.} \quad \alpha + \beta_l \geq (x^\top q^l)^2, \quad l \in [m],$$

to obtain the claim of the theorem. □

Note that the Reformulation (7) contains inner minimization problems (see the definition of p_i in (6)) in the Constraints (7b). Since p_i, $i \in [n]$, does not depend on the variables x, it can be computed before solving Problem (7). Using the fact that at most Γ many parameters realize in a worst-case way and since $\max\{u^\top v : u \in \mathcal{U}^2\} = \|v\|_2$ holds for $v \in \mathbb{R}^n$, we obtain

$$p_i = -\sqrt{\max_{\{L \subseteq [m] : |L| \leq \Gamma\}} \sum_{l \in L} (q_i^l)^2} \tag{10}$$

for each $i \in [n]$.

Remark 1. Unfortunately, as Constraint (7c) is nonconvex quadratic, the optimization problem (7) is, in general, not tractable.

In a second step, we consider the convex ellipsoidal uncertainty set $\mathcal{U}_\Gamma^2$ and give a reformulation of Problem (5). In this case, we obtain a tractable convex optimization problem.

Theorem 2. *Let the uncertainty set be $\mathcal{U}_\Gamma^2$. Then, Problem (5) can be reformulated as*

$$\min_{z \geq 0} \quad x^\top (Mx + q^0) + \alpha\Gamma + \|\beta\|_2 \tag{11a}$$

$$\text{s.t.} \quad M_{i,\cdot} x + q_i^0 + p_i \geq 0, \quad i \in [n], \tag{11b}$$

$$\alpha + \beta_l \geq |x^\top q^l|, \quad l \in [m], \tag{11c}$$

where $z := (x^\top, \alpha, \beta^\top)^\top$.

Proof. First, we rewrite the inner optimization problem in the objective function (5a). For that reason, we mention that with $(u_i)_{i \in [m]} \in \mathcal{U}_\Gamma^2$, the vector $(u_1, \ldots, u_{i-1}, -u_i, u_{i+1}, \ldots, u_m)^\top$ for each $i \in [m]$ is contained in $\mathcal{U}_\Gamma^2$, too. This is why we can replace the maximization problem

$$\max_{u \in \mathcal{U}_\Gamma^2} \sum_{l \in [m]} u_l x^\top q^l$$

with

$$\max_{u \in \mathcal{U}^2, u \geq 0} \sum_{l \in [m]} u_l |x^\top q^l| \quad \text{s.t.} \quad \sum_{l \in [m]} u_l \leq \Gamma. \tag{12}$$

Now, we state its dual problem and then use strong duality to reformulate (5a). We are able to do this because Slater's condition is satisfied for Problem (12) due to the fact that $\Gamma > 0$ holds. The Lagrangian of (12) with multipliers $\alpha \geq 0$ and $\omega_l \geq 0$, $l \in [m]$, reads

$$\begin{aligned}\mathcal{L}(u, \alpha, \omega) &= \sum_{l \in [m]} u_l |x^\top q^l| - \alpha \left(\sum_{l \in [m]} u_l - \Gamma \right) + \sum_{l \in [m]} \omega_l u_l \\ &= \sum_{l \in [m]} u_l (|x^\top q^l| - \alpha + \omega_l) + \alpha \Gamma.\end{aligned}$$

Then, the dual problem of (12) is given by

$$\min_{\alpha \geq 0, \omega \geq 0} \phi(\alpha, \omega) \tag{13}$$

with

$$\phi(\alpha, \omega) := \max_{u \in \mathcal{U}^2} \mathcal{L}(u, \alpha, \omega).$$

As $\max\{u^\top v : u \in \mathcal{U}^2\} = \|v\|_2$ holds for $v \in \mathbb{R}^m$, one has

$$\phi(\alpha, \omega) = \sqrt{\sum_{l \in [m]} (|x^\top q^l| - \alpha + \omega_l)^2} + \alpha \Gamma.$$

Now, we can rewrite Problem (13) as

$$\min_{\alpha \geq 0, \omega \geq 0, \beta} \quad \|\beta\|_2 + \alpha \Gamma \quad \text{s.t.} \quad \beta_l = |x^\top q^l| - \alpha + \omega_l, \quad l \in [m]. \tag{14}$$

In a last step, we note that we can restrict us to the case $\beta \geq 0$: Assume that (α, ω, β) is an optimal solution of (14) with $\beta_l < 0$ for at least one index $l \in [m]$. From feasibility of (14), we obtain $|x^\top q^l| - \alpha < 0$. This is a contradiction to optimality since choosing ω_l such that $\beta_l = |x^\top q^l| - \alpha + \omega_l = 0$ would lead to a solution with lower objective function value. We also note that we can eliminate the slack variables ω. Thus, the dual problem reads

$$\min_{\alpha \geq 0, \beta \geq 0} \|\beta\|_2 + \alpha \Gamma \quad \text{s.t.} \quad \beta_l \geq |x^\top q^l| - \alpha, \quad l \in [m].$$

Using strong duality, due to the fact that Slater's condition is satisfied, the claim follows. □

If we assume that the matrix M is positive semidefinite, we obtain a tractable robust counterpart.

Corollary 1. *Let M be positive semidefinite. Then, the robust counterpart (11) is equivalent to the convex (and tractable) optimization problem*

$$\min_{z\geq 0} \quad x^\top(Mx+q^0)+\alpha\Gamma+\mu \tag{15a}$$

$$\text{s.t.} \quad M_{i,\cdot}x+q_i^0+p_i\geq 0, \quad i\in[n], \tag{15b}$$

$$\alpha+\beta_l\geq x^\top q^l, \quad l\in[m], \tag{15c}$$

$$\alpha+\beta_l\geq -x^\top q^l, \quad l\in[m], \tag{15d}$$

$$\mu\geq \|\beta\|_2, \tag{15e}$$

where $z:=(x^\top,\alpha,\beta^\top,\mu)^\top$.

Proof. The Constraints (15c) and (15d) are reformulations of (11c). □

Remark 2. Note that each constraint $i\in[n]$ in (11b) and (15b), respectively, contains an inner optimization problem denoted by p_i given in (6). In the case of the uncertainty set $\mathcal{U}_\Gamma^2$, it is an convex optimization problem because $\mathcal{U}_\Gamma^2$ is a convex set and we have a linear objective function.

Next, we briefly consider the uncorrelated ellipsoidal uncertainty specified in the following assumption.

Assumption 1. *The uncertainty vector is given by* $q(u):=q^0+u$ *with* $u\in\{u\in\mathbb{R}^n : u^\top\Lambda u\leq 1\}$ *for a positive definite diagonal matrix* $\Lambda\in\mathbb{R}^{n\times n}$.

Corollary 2. *Suppose that Assumption 1 holds. Then, the robust counterpart (5) reads*

$$\min_{z\geq 0} \quad x^\top(Mx+q^0)+\alpha\Gamma+\mu, \tag{16a}$$

$$\text{s.t.} \quad M_{i,\cdot}x+q_i^0-\lambda_i^{-\frac{1}{2}}\geq 0, \quad i\in[n], \tag{16b}$$

$$\alpha+\beta_i\geq x_i\lambda_i^{-\frac{1}{2}}, \quad i\in[n], \tag{16c}$$

$$\mu\geq\sqrt{\sum_{i\in[n]}\beta_i^2}, \tag{16d}$$

where λ_i, $i\in[n]$, *are the eigenvalues of* Λ *and* $z:=(x^\top,\alpha,\beta^\top,\mu)^\top$.

Proof. With $q^i:=\lambda_i^{-1/2}e^i$, $i\in[n]$, and unit vectors e^i, we can write $q(u)=q^0+\sum_{i\in[n]}u_iq^i$ and $u\in\mathcal{U}_\Gamma^2$. Then, the claim follows by Corollary (1) and

$$p_i=-\max_{u\in\mathcal{U}_\Gamma^2}-\sum_{l\in[n]}u_lq_i^l=-q_i^i=-\lambda_i^{-\frac{1}{2}}.$$
□

Remark 3. The mathematical model of the robust counterpart (5) for the uncertainty set $\mathcal{U}_{\Gamma,\lambda}^2$, as discussed in Kurtz (2018), is given in the master thesis by Müller (2019). There it is shown that it can be formulated as a tractable optimization problem, which is achieved by using the same techniques as used in this section. As this uncertainty set only leads to more complicated notation, we decided to present the results for the notationally simpler uncertainty sets in the section for the ease of better reading.

We close this section by a brief discussion about ρ-robustness as it is introduced in Wu et al. (2011). The proof of the following theorem is analogous to the one of Theorem 3.12 given in Krebs and Schmidt (2020) in case of ρ-robustness. We consider the uncertain set $\mathcal{U}_\Gamma^2$ and an uncertain vector q; cf. Theorem 2.

Theorem 3. *Let M be positive semidefinite. Then, x is a ρ-robust LCP solution if and only if there exist $\alpha \in \mathbb{R}$ and $\beta_l \in \mathbb{R}$, $l \in [m]$, that satisfy*

$$\begin{aligned}
&x^\top(Mx + q^0) + \alpha\Gamma + \|\beta\|_2 \le \rho, \\
&\qquad M_{i,\cdot}x + q_i^0 + p_i \ge 0, \quad i \in [n], \\
&\qquad \alpha + \beta_l - |x^\top q^l| \ge 0, \quad l \in [m], \\
&\qquad \alpha \ge 0, \\
&\qquad x_i \ge 0, \quad i \in [n], \\
&\qquad \beta_l \ge 0, \quad l \in [m].
\end{aligned}$$

3.2. Uncertain LCP matrix M

In this section, we consider uncertainty in the LCP matrix M, that is, we consider the uncertain LCP

$$0 \le x \perp M(u)x + q \ge 0, \quad u \in \mathcal{U},$$

and the parameterization

$$M(u) := M^0 + \sum_{l\in[m]} u_l M^l$$

for $u \in \mathcal{U}$ and positive semidefinite matrices $M^0, \ M^l \in \mathbb{R}^{n\times n}$, $l \in [m]$. Here, the matrix M^0 contains the nominal values. As it was the case for uncertainty in q in the last section, we analyze the robust counterpart

$$\min_{x\ge 0} \quad x^\top(M^0x + q) + \max_{u\in\mathcal{U}} \sum_{l\in[m]} u_l x^\top M^l x \tag{17a}$$

$$\text{s.t.} \quad M_{i,\cdot}^0 x + q_i + \min_{u\in\mathcal{U}} \sum_{l\in[m]} u_l M_{i,\cdot}^l x \ge 0, \quad i \in [n], \tag{17b}$$

for the different uncertainty sets $\mathcal{U}$ introduced in Section 2. First, we consider the nonconvex ellipsoidal uncertainty set $\mathring{\mathcal{U}}_\Gamma^2$.

Theorem 4. *Let the uncertainty set be $\mathring{\mathcal{U}}^2_\Gamma$. Then, Problem (17) can be reformulated as*

$$\min_{z\geq 0, \eta\geq 0, \mu\geq 0} \quad x^\top(M^0x + q) + \eta \tag{18a}$$

$$s.t. \quad \eta^2 \geq \alpha\Gamma + \sum_{l\in[m]} \beta_l, \tag{18b}$$

$$\alpha + \beta_l \geq (x^\top M^l x)^2, \quad l \in [m], \tag{18c}$$

$$M^0x + q - \mu \geq 0, \tag{18d}$$

$$\mu_i^2 = \gamma_i\Gamma + \sum_{l\in[m]} \delta_{i,l}, \quad i \in [n],\ l \in [m], \tag{18e}$$

$$\gamma_i + \delta_{i,l} \geq (M^l_{i,\cdot}x)^2, \quad i \in [n],\ l \in [m], \tag{18f}$$

where $z := (x^\top, \alpha, \beta^\top, \gamma^\top, \delta^\top)^\top$.

Proof. With the same strategies as they are used for the case of uncertainty in q, we reformulate Problem (17). In a first step, we consider the inner maximization problem in the objective function. Again we use the fact that

$$\max_{u\in\mathcal{U}^2} u^\top(x^\top M^1 x, \ldots, x^\top M^m x)^\top = \|(x^\top M^1 x, \ldots, x^\top M^m x)^\top\|_2$$

holds. As at most Γ components of u are nonzero in $\mathring{\mathcal{U}}^2_\Gamma$, we have

$$\max_{u\in\mathring{\mathcal{U}}^2_\Gamma} \sum_{l\in[m]} u_l x^\top M^l x = \sqrt{\max_{\{L\subseteq[m]:|L|\leq\Gamma\}} \sum_{l\in L} (x^\top M^l x)^2}.$$

Hence, we consider

$$\min_{x\geq 0, \eta\geq 0} \quad x^\top(M^0x + q) + \eta \tag{19a}$$

$$\text{s.t.} \quad \max_{\{L\subseteq[m]:|L|\leq\Gamma\}} \sum_{l\in L} (x^\top M^l x)^2 \leq \eta^2, \tag{19b}$$

$$M^0_{i,\cdot}x + q_i + \min_{u\in\mathring{\mathcal{U}}^2_\Gamma} \sum_{l\in[m]} u_l M^l_{i,\cdot}x \geq 0, \quad i \in [n], \tag{19c}$$

instead of Problem (17). With the same argumentation as in the previous proofs, we state and use the dual problem to reformulate the inner maximization problem in (19b). This means, we use the dual problem

$$\min_{\alpha\geq 0, \beta\geq 0} \quad \alpha\Gamma + \sum_{l\in[m]} \beta_l \quad \text{s.t.} \quad \alpha + \beta_l \geq (x^\top M^l x)^2, \quad l \in [m],$$

of

$$\begin{aligned}\max_{z} \quad & \sum_{l\in[m]} (x^\top M^l x)^2 z_l,\\ \text{s.t.} \quad & \sum_{l\in[m]} z_l \leq \Gamma,\\ & 0 \leq z_l \leq 1, \quad l \in [m],\end{aligned}$$

which is the convex relaxation of the maximization problem in Constraint (19b). Now, we use the same techniques for the reformulation of the inner minimization problems in the Constraints (19c). We can replace these constraints by

$$M^0_{i,\cdot}x + q_i - \mu_i \geq 0, \quad \mu_i^2 = \max_{\{L\subseteq[m]:|L|\leq\Gamma\}} \sum_{l\in L} (M^l_{i,\cdot}x)^2, \quad \mu_i \geq 0, \quad i \in [n], \tag{20}$$

because

$$\min_{u\in\mathring{\mathcal{U}}^2_\Gamma} \sum_{l\in[m]} u_l M^l_{i,\cdot}x = -\max_{u\in\mathring{\mathcal{U}}^2_\Gamma} \sum_{l\in[m]} u_l\left(-M^l_{i,\cdot}x\right) = -\sqrt{\max_{\{L\subseteq[m]:|L|\leq\Gamma\}} \sum_{l\in L} (M^l_{i,\cdot}x)^2}$$

holds. Using again a dual problem of the inner maximization problem in (20) with dual variables γ_i and $\delta_{i,l}$, $i \in [n]$, $l \in [m]$, we obtain

$$\begin{aligned} M^0_{i,\cdot}x + q_i - \mu_i &\geq 0, \quad i \in [n],\\ \mu_i^2 &= \gamma_i\Gamma + \sum_{l\in[m]} \delta_{i,l}, \quad i \in [n],\ l \in [m],\\ \gamma_i + \delta_{i,l} &\geq (M^l_{i,\cdot}x)^2, \quad l \in [m],\\ \gamma_i &\geq 0, \quad i \in [n],\\ \delta_{i,l} &\geq 0, \quad i \in [n],\ l \in [m]\end{aligned}$$

for the Constraints (19c) and the claim follows. □

As it was the case for the uncertainty set $\mathring{\mathcal{U}}^2_\Gamma$ and uncertainty in q, Model (18) is not convex. We see in the following that we obtain a tractable reformulation of the robust counterpart (17) for the convex uncertainty set $\mathcal{U}^2_\Gamma$.

Theorem 5. *Let the uncertainty set be $\mathcal{U}^2_\Gamma$. Then, Problem (17) is equivalent to*

$$\min_{z\geq 0} \quad x^\top(M^0x + q) + \alpha\Gamma + \|\beta\|_2 \tag{21a}$$

$$s.t. \quad M^0_{i,\cdot}x + q_i - \gamma_i\Gamma - \|\delta_i\|_2 \geq 0, \quad i \in [n], \tag{21b}$$

$$\alpha + \beta_l \geq x^\top M^l x, \quad l \in [m], \tag{21c}$$

$$\gamma_i + \delta_{i,l} \geq |M^l_{i,\cdot} x|, \quad i \in [n], \ l \in [m], \tag{21d}$$

where $z := (x^\top, \alpha, \beta^\top, \gamma^\top, \delta^\top)^\top$.

Proof. First, we rewrite the inner optimization problem in the objective function (17a). As all matrices M^l, $l \in [m]$, are positive semidefinite, we can restrict ourselves to the case $u \geq 0$. Then, this problem is equivalent to

$$\max_{u \in \mathcal{U}^2, u \geq 0} \sum_{l \in [m]} u_l x^\top M^l x \quad \text{s.t.} \quad \sum_{l \in [m]} u_l \leq \Gamma.$$

With the same arguments as in the proof of Theorem 2, we replace this problem with its dual problem

$$\min_{\alpha \geq 0, \beta \geq 0} \|\beta\|_2 + \alpha\Gamma \quad \text{s.t.} \quad \alpha + \beta_l \geq x^\top M^l x, \quad l \in [m].$$

Again, with the same arguments as in the proof of Theorem 2, we can replace each inner minimization problem in (17b) with

$$-\max_{u \in \mathcal{U}^2, u \geq 0} \sum_{l \in [m]} u_l | - M^l_{i,\cdot} x| \quad \text{s.t.} \quad \sum_{l \in [m]} u_l \leq \Gamma$$

and its dual problem

$$-\min_{\gamma_i \geq 0, \delta_i \geq 0} \|\delta_i\|_2 + \gamma_i \Gamma \quad \text{s.t.} \quad \gamma_i + \delta_{i,l} \geq |M^l_{i,\cdot} x|, \quad l \in [m].$$

Note that we make use of strong duality. We are able to do so, as Slater's condition is satisfied with the same argument as in proof of Theorem 2. □

With analogous arguments as used for the tractability in Corollary 1, we obtain the following corollary.

Corollary 3. *The robust counterpart (21) is equivalent to the convex optimization problem*

$$\min_{z \geq 0} \quad x^\top (M^0 x + q) + \alpha\Gamma + \|\beta\|_2 \tag{22a}$$

$$\text{s.t.} \quad M^0_{i,\cdot} x + q_i - \gamma_i \Gamma \geq \sqrt{\sum_{l \in [m]} \delta^2_{i,l}}, \quad i \in [n], \tag{22b}$$

$$\alpha + \beta_l \geq x^\top M^l x, \quad l \in [m], \tag{22c}$$

$$\gamma_i + \delta_{i,l} \geq M^l_{i,\cdot} x, \quad l \in [m], i \in [n], \tag{22d}$$

$$\gamma_i + \delta_{i,l} \geq -M^l_{i,\cdot} x, \quad l \in [m], i \in [n], \tag{22e}$$

where $z := (x^\top, \alpha, \beta^\top, \gamma^\top, \delta^\top)^\top$.

Before we close this section, we consider the robust counterpart for the uncertainty set $\mathcal{U}^2_{\Gamma,\lambda}$ discussed in Kurtz (2018). As the following result (in contrast to the case of uncertain q) is not given in Müller (2019), we also state and proof the result here.

Theorem 6. *Let the uncertainty set be $\mathcal{U}^2_{\Gamma,\lambda}$. Then, Problem (17) can be reformulated as the tractable optimization problem*

$$\begin{aligned}\min_{z\geq 0} \quad & x^\top(M^0x+q)+\alpha\Gamma+\|\beta\|_2\\ s.t. \quad & \beta_l+\frac{\alpha}{\sqrt{\lambda_l}}\geq x^\top M^l x, \quad l\in[m],\\ & M^0_{i,\cdot}x+q_i+\hat{p}_i\geq 0, \quad i\in[n],\end{aligned}$$

with

$$\hat{p}_i:=\min_{u\in\mathcal{U}^2_{\Gamma,\lambda}}\sum_{l\in[m]}u_l M^l_{i,\cdot}x$$

for $i\in[n]$ and $z:=(x^\top,\alpha,\beta^\top)^\top$.

Proof. To prove this theorem, we proceed as in the proof of Theorem 2. Here, we have the optimization problem

$$\max_{u\in\mathcal{U}^2,u\geq 0}\quad\sum_{l\in[m]}u_l x^\top M^l x\quad\text{s.t.}\quad\sum_{l\in[m]}\frac{u_l}{\sqrt{\lambda_l}}\leq\Gamma$$

instead of (12). Its Lagrangian reads

$$\mathcal{L}(u,\alpha,\omega):=\sum_{l\in[m]}u_l\left(x^\top M^l x-\frac{\alpha}{\sqrt{\lambda_l}}+\omega_l\right)+\alpha\Gamma$$

and in this case, Problem (13) is given by

$$\min_{z\geq 0}\quad x^\top(M^0x+q)+\alpha\Gamma+\|\beta\|_2\quad\text{s.t.}\quad\beta_l+\frac{\alpha}{\sqrt{\lambda_l}}\geq x^\top M^l x,\quad l\in[m],$$

where $z:=(x^\top,\alpha,\beta^\top)^\top$. We are able to use strong duality because Slater's condition is always satisfied. Each $\hat{p}_i$, $i\in[n]$, is given by a convex optimization problem, which completes the proof. □

As it was the case in the section before, we close this section by a brief discussion about ρ-robustness. Again, the proof of the following theorem is analogous to the one of Theorem 4.10 given in Krebs and Schmidt (2020). Here, we consider the uncertain set $\mathcal{U}^2_\Gamma$, an uncertain matrix M, and obtain the following theorem.

Theorem 7. *Let M^0, M^l, $l\in[m]$, be positive semidefinite. Then, x is a ρ-robust LCP solution if and only if there exist $\alpha\in\mathbb{R}$, $\beta_l\in\mathbb{R}$, $l\in[m]$, $\gamma_i\in\mathbb{R}$, $i\in[n]$, and $\delta_{i,l}\in\mathbb{R}$, $i\in[n]$, $l\in[m]$, that satisfy*

$$x^\top(M^0x+q)+\alpha\Gamma+\|\beta\|_2\leq\rho$$

$$
\begin{aligned}
M^0_{i,\cdot}x + q_i - \gamma_i\Gamma - \|\delta_i\|_2 &\geq 0, \quad i \in [n],\\
\alpha + \beta_l - x^\top M^l x &\geq 0, \quad l \in [m],\\
\gamma_i + \delta_{i,l} - |M^l_{i,\cdot}x| &\geq 0, \quad i \in [n],\ l \in [m],\\
\alpha &\geq 0,\\
x_i, \gamma_i &\geq 0, \quad i \in [n],\\
\beta_l &\geq 0, \quad l \in [m],\\
\delta_{i,l} &\geq 0, \quad i \in [n],\ l \in [m].
\end{aligned}
$$

4. Existence and uniqueness

In this section, we investigate the existence and uniqueness of solutions for the tractable counterparts (15) and (22) established in the previous section. Up to this point, we only know that the objective function of these optimization problems are bounded from below by zero on the respective feasible sets. For a quadratic function that is bounded from below on a nonempty and convex polyhedron, the Frank–Wolfe theorem implies the existence of a minimum. If we consider, for example, the tractable robust counterpart (15), we face a quadratic objective, which is convex (as M is positive semidefinite), linear constraints, and a single second-order cone constraint. Thus, the question arises if there exists a Frank–Wolfe type theorem guaranteeing that an infimum is attained for this model. To see that this is the case, we have to review the concepts of quasi-Franke-and-Wolfe sets and recession cones of sets and functions. First, we introduce quasi-convex functions and quasi-Franke-and-Wolfe sets as given in Martinez-Legaz et al. (2018). A function $f : \mathbb{R}^n \to \mathbb{R}$ is called quasi-convex on a convex set $\mathcal{C} \subseteq \mathbb{R}^n$, if the sublevel sets of $f : \mathcal{C} \to \mathbb{R}$ are convex. Further, a convex set $\mathcal{C} \subseteq \mathbb{R}^n$ is called a quasi-Frank-and-Wolfe set, if every quadratic function f, which is quasi-convex and bounded from below on $\mathcal{C}$, attains its infimum on $\mathcal{C}$.

Let us briefly review the literature on quasi-Frank-and-Wolfe sets. If we consider the set $\mathcal{C} := \{x \in \mathbb{R}^n : f_i(x) \leq 0,\ i \in [m]\}$ with $m \in \mathbb{N}$ and (quasi-)convex functions $f_i : \mathbb{R}^n \to \mathbb{R}$, $\mathcal{C}$ is convex due to the fact that all sublevel sets of (quasi-)convex functions are convex. If all functions f_i, $i \in [m]$, are convex quadratic or linear, then $\mathcal{C}$ is a quasi-Frank-and-Wolfe set as shown in Luo and Zhang (1999). An extension to convex polynomial functions f_i is given in Belousov and Klatte (2002) and an extension for f_i being a (quasi-)convex polynomial is given in Obuchowska (2006). Unfortunately, the property of $\mathcal{C}$ being a quasi-Frank-and-Wolfe set does no longer hold if all f_i define general second-order cone constraints, that is, if $f_i(x) = \|A^i x + b^i\|_2 - x^\top c^i - d^i$ with $A^i \in \mathbb{R}^{n\times n}$, $b^i, c^i, d^i \in \mathbb{R}^n$ for all $i \in [m]$. This is shown in Martinez-Legaz et al. (2018) using the following example.

Example 1. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x) := (x_1 - 1)^2 - x_2 + x_3$ and define $\mathcal{C} := \{x \in \mathbb{R}^3 : \sqrt{x_1^2 + x_2^2} \leq x_3\}$. Then, f does not attain its infimum on $\mathcal{C}$. First, note that f is bounded from below by zero on $\mathcal{C}$ and for the set of points $x_\lambda := (1, \lambda, \sqrt{1+\lambda^2})$, $\lambda \in \mathbb{R}$, it holds $f(x_\lambda) \to 0$ for $\lambda \to \infty$. This implies that $\inf_{x\in\mathcal{C}} f(x) = 0$. Second, this infimum cannot be attained because the function value of f is strictly positive on $\mathcal{C}$. This follows from the fact that for $x \in \mathcal{C}$ with $x_1 = 0$, we have $x_3 \geq x_2$, and if $x_1 \neq 0$, it holds $x_3 > x_2$.

Consequently, the question arises if we can guarantee the existence of solutions for the specific optimization problems (15) and (22). For the following existence theorems, we need the notions of recession cones of sets and functions. The recession cone $\mathcal{R}_{\mathcal{C}}$ of a nonempty convex set $\mathcal{C}$ is given by

$$\mathcal{R}_{\mathcal{C}} := \{d : x + \lambda d \in \mathcal{C} \text{ for all } x \in \mathcal{C} \text{ and for all } \lambda \geq 0\}.$$

An element $d \in \mathcal{R}_{\mathcal{C}}$ is called a direction of recession of $\mathcal{C}$. If the set $\mathcal{C}$ is additionally closed, then one can show that d is a direction of recession if and only if there exists a vector $x \in \mathcal{C}$ such that $x + \lambda d \in \mathcal{C}$ for all $\lambda \geq 0$. For a closed proper convex function $f : \mathbb{R}^n \to (-\infty, \infty]$, the recession cone of the nonempty level sets $\{x \in \mathbb{R}^n : f(x) \leq r\}$, $r \in \mathbb{R}$, is the recession cone of f and is denoted by $\mathcal{R}_f$. Now, we are able to state a characterization for the existence of solutions of an optimization problem.

Theorem 8 (Proposition 3.2.2 in Bertsekas (2009))**.** *Let $\mathcal{C}$ be a closed convex subset of $\mathbb{R}^n$ and let $f : \mathbb{R}^n \to (-\infty, \infty]$ be a closed convex function such that $\mathcal{C} \cap \operatorname{dom}(f) \neq \emptyset$. Then, the set of minimizing points of f over $\mathcal{C}$ is nonempty and compact if and only if $\mathcal{C}$ and f have no common nonzero direction of recession.*

In the following, we consider again the setting from Example 1 and show that the set of minimizers is empty.

Example 2. Let f and $\mathcal{C}$ be defined as in Example 1. The recession cone $\mathcal{R}_{\mathcal{C}}$ of $\mathcal{C}$ is equal to $\mathcal{C}$. Now, we show that $0 \neq d := (0, 1, 1)^\top$ is a direction of recession of f and $\mathcal{C}$. To this end, let $\lambda \geq 0$. Next, we show $x + \lambda d \in \mathcal{C}$ for all $x \in \mathcal{C}$. It holds

$$x_1^2 + (x_2 + \lambda)^2 = x_1^2 + x_2^2 + 2\lambda x_2 + \lambda^2.$$

Due to $\sqrt{x_1^2 + x_2^2} \leq x_3$, the right-hand side is at most $x_3^2 + 2\lambda x_2 + \lambda^2$. Using $x_2 \leq x_3$ for all $x \in \mathcal{C}$, we obtain $x_3^2 + 2\lambda x_3 + \lambda^2 = (x_3 + \lambda)^2$. This means, we have $x + \lambda d \in \mathcal{C}$ and, thus, $d \in \mathcal{R}_{\mathcal{C}}$. As f is constant in direction d, the vector d is a common nonzero direction of recession. Using Theorem 8, the set of minimizers of f over $\mathcal{C}$ is empty.

Using Theorem 8, we can show that our robust counterpart (15) has a solution under mild assumptions.

Theorem 9. *Suppose that Problem (15) is feasible and that M is positive semidefinite. Further assume that there exists an $l \in [m]$ for each $i \in [n]$ with $q_i^l \neq 0$. Then, there exists a solution of Problem (15).*

Proof. Let $y := (x^\top, \alpha, \beta^\top, \mu)^\top \in \mathbb{R}^{n+m+2}$ and $\mathcal{C}$ be the set of feasible points of Problem (15), that is,

$$\mathcal{C} := \{y \in \mathbb{R}^{n+m+2}_{\geq 0} : y \text{ satisfies Constraints (15b) –(15e)}\}.$$

Further, we define $f(y) := x^\top(Mx + q^0) + \alpha\Gamma + \mu$, which is the objective function of (15). Note the following two facts:

(a) f is bounded from below by zero on $\mathcal{C}$.
(b) For all directions of recession $d := (d_x^\top, d_\alpha, d_\beta^\top, d_\mu)^\top \in \mathcal{R}_\mathcal{C}$ it holds $d \geq 0$. Otherwise, if there exists a component $d_i < 0$, we can choose for each $y \in \mathcal{C}$ a $\lambda \geq 0$ such that $y_i + \lambda d_i < 0$. This implies that $y + \lambda d \notin \mathcal{C}$, which contradicts $d \in \mathcal{R}_\mathcal{C}$.

In order to apply Theorem 8, we have to show that $\mathcal{R}_f \cap \mathcal{R}_\mathcal{C} = \{0\}$ holds with

$$R_f := \{d \in \mathbb{R}^{n+m+2} : f(y + \lambda d) \leq r \; \forall \lambda \geq 0 \text{ and } y \in \mathbb{R}^{n+m+2} \text{ with } f(y) \leq r\}.$$

There exists a vector $y \in \mathcal{C}$ satisfying $f(y) \leq r$ for an $r \in \mathbb{R}$. Otherwise, all feasible points have an unbounded, positive function value and $\mathrm{dom}(f) \cap \mathcal{C} = \emptyset$. This means, there exists a feasible point at which the objective function is defined. In the following, let $d \in \mathcal{R}_\mathcal{C}$ be a nonzero direction of recession. Then, we have to show that $f(y + \lambda d) \to \infty$ holds for $\lambda \to \infty$. We have

$$f(y + \lambda d) = (x + \lambda d_x)^\top (M(x + \lambda d_x) + q^0) + (\alpha + \lambda d_\alpha)\Gamma + (\mu + \lambda d_\mu). \tag{23}$$

Now, we consider the two cases $d_x = 0$ and $d_x \neq 0$. First, let $d_x = 0$. If, additionally, $(d_\alpha, d_\mu) = 0$, there exists an $l \in [m]$ with $d_{\beta_l} \neq 0$ because $d \neq 0$ holds. So, $d_{\beta_l} > 0$ follows from Fact (b). Constraint (15e) implies $\mu + \lambda d_\mu \geq \|\beta + \lambda d_\beta\|_2$ for all $\lambda \geq 0$. As there exists for each value $s \in \mathbb{R}$ a scalar $\lambda_s \geq 0$ such that $s < \|\beta + \lambda_s d_\beta\|_2$ holds, the left-hand side has to increase with increasing values of λ. Thus, it is not possible that $d_\mu = 0$. Hence, we can assume that $(d_\alpha, d_\mu) \neq 0$. In this case, Equation (23) reads $f(y + \lambda d) = f(y) + \lambda(d_\alpha \Gamma + d_\mu)$ and with the preceding arguments we have $f(y + \lambda d) \to \infty$ for $\lambda \to \infty$. Second, let $d_x \neq 0$. Then, there exists an index $i \in [n]$ such that $d_{x_i} \neq 0$ holds. Using the assumption of the theorem, there exists some index $l \in [m]$ such that $d_{x_i} q_i^l \neq 0$. Condition (15c) and (15d) imply $(\alpha + \beta_l) + \lambda(d_\alpha + d_{\beta_l}) \geq |x^\top q^l + \lambda d_x^\top q^l|$ and with increasing λ, the right-hand side increases, and so the left-hand side must increase as well. Thus, $d_\alpha + d_{\beta_l} > 0$ holds. If $d_{\beta_l} > 0$, it follows from Constraint (15e) that $d_\mu \neq 0$. Hence, we obtain $(d_\alpha, d_\mu) \neq 0$ and $(\alpha + \lambda d_\alpha)\Gamma + (\mu + \lambda d_\mu) \to \infty$ for $\lambda \to \infty$. The first term on the right-hand side in (23) is nonzero because d is a direction of recession, thus, $y + \lambda d$ satisfies Constraint (15b), and each p_i is due to its Definition (6) nonpositive. This proves the claim. □

The analogous result for the robust counterpart (22) can be shown using the same techniques as in the last proof.

Theorem 10. *Suppose that Problem (22) is feasible, that $M^0, \dots, M^m$ are positive semidefinite, and that at least one M^i, $i \in [m]$, is positive definite. Then, there exists a solution of Problem (22).*

Up to this point, we have addressed the existence of solutions. In a next step, we also consider the uniqueness of solutions.

Theorem 11. *Suppose that the matrix M in Problem (15) is positive definite. Then, Problem (15) is feasible and the solution is unique in x if it exists.*

Proof. As M is positive definite, there exists a vector $x \geq 0$ satisfying $Mx > 0$. Thus we can choose a $\lambda \geq 0$ sufficiently large such that $\lambda Mx \geq -q^0 - p$ holds. Here, $p \in \mathbb{R}^n$ is the vector containing all p_i given in (6). Note that p can be computed a priori. Then, we define $\hat{x} := \lambda x$. Hence, $\alpha = 0$, $\beta_l := \hat{x}^\top q^l$, $l \in [m]$, and $\mu = \|\beta\|_2$ is a feasible solution of Problem (15). The uniqueness of the x-variables is a direct consequence of Mangasarian (1988). □

Using Mangasarian (1988), we also obtain the analogous statement for Problem (22).

Proposition 1. *Suppose that the matrix M^0 in Problem (22) is positive definite. Then, the solution of Problem (22) is unique in x if it exists.*

In the light of Example 4.11 in Krebs and Schmidt (2020), we do not expect uniqueness of the other variables (besides x) in Problem (15) and (22).

Let us close this section with a final remark. Here, we studied the existence and uniqueness of solutions for the case of the uncertainty set $\mathcal{U}_\Gamma^2$. We think that the consideration of existence and uniqueness for the case of the uncertainty set $\mathring{\mathcal{U}}_\Gamma^2$ would require different techniques and is, thus, out of scope of this paper but is a reasonable topic of future research.

5. Case study: The uncertain traffic equilibrium problem

In this section, we exemplarily apply Γ-robustifications to the well-studied traffic equilibrium problem (TEP). We first review a standard complementarity modeling for the TEP in Section 5.1, consider uncertain data in Section 5.2, and finally discuss some numerical results in Section 5.3.

5.1. Complementarity modeling of deterministic traffic equilibrium problems

In this section, we review the modeling of the deterministic TEP as a complementarity problem. We consider a network modeled by a graph $G := (N, A)$, where N denotes the set of nodes and A the set of arcs. Further, the subset $O \subseteq N$ is the set of origin nodes and $D \subseteq N$ is the set of destination nodes with $O \cap D = \emptyset$. For an origin-destination (OD) pair $p \in P \subseteq O \times D$, the set $R(p)$ contains the set of cycle-free routes in G for the node pair p and h_r is the flow on route $r \in R := \cup_{p \in P} R(p)$, that is, R is the set of all routes. The travel costs $t_r(h)$ along route $r \in R$ depend on the flow $h := (h_r)_{r \in R}$ and the minimal costs for an OD pair $p \in P$ is denoted by τ_p. The function $d_p(\tau)$ models the demand between the OD pair $p \in P$ for $\tau := (\tau_p)_{p \in P}$. In what follows, we consider the so-called additive model, that is, the total costs on a route are given as the sum of the costs on each arc of the route. This means, if f_a denotes the flow on arc $a \in A$ and $c_a(f)$ is the respective cost function, then

$$t_r(h) = \sum_{a \in A} \Theta_{a,r} c_a(f)$$

holds with $f := \Theta h$, $\Theta := (\Theta_{a,r})_{a \in A, r \in R} \in \{0, 1\}^{|A| \times |R|}$ and

$$\Theta_{a,r} := \begin{cases} 1, & \text{if } a \in r, \\ 0, & \text{else.} \end{cases}$$

Based on Wardrop's principle, one is interested in equilibria in which each driver minimizes her travel costs (Wardrop, 1952; Wardrop and Whitehead, 1952). This means we are interested in the solution of

$$0 \le h_r \perp t_r(h) - \tau_p \ge 0, \quad r \in R(p),\ p \in P. \tag{24}$$

A flow is called feasible if it satisfies the demand, that is, if

$$\sum_{r \in R(p)} h_r - d_p(\tau) = 0, \quad p \in P,\ \tau_p \ge 0 \tag{25}$$

holds. We now want to rewrite (25) as a complementarity constraint, which is possible under some mild assumptions. For a proof of the following theorem, see Section 1.4.5 in Facchinei and Pang (2003).

Theorem 12. *Suppose that the cost and demand functions are nonnegative and that the flow $h \ge 0$ satisfies*

$$\sum_{r \in R(p)} h_r t_r(h) = 0 \Rightarrow h_r = 0 \text{ for all } r \in R(p) \tag{26}$$

for each OD pair $p \in P$. Then, (24) and (25) are satisfied if and only if (24) and

$$0 \le \tau_p \perp \sum_{r \in R(p)} h_r - d_p(\tau) \ge 0 \tag{27}$$

are satisfied for all $p \in P$.

Note that also (26) has a natural interpretation: It means that, for any OD pair, the demand is only over-satisfied if the minimal travel costs are zero. Using Theorem 12, we can thus state the TEP as the (nonlinear) complementarity system

$$0 \le \begin{pmatrix} h \\ \tau \end{pmatrix} \perp \begin{pmatrix} t(h) - \Delta^\top \tau \\ \Delta h - d(\tau) \end{pmatrix} \ge 0,$$

where $\Delta_{p,r} \in \{0, 1\}$ for all $p \in P$, $r \in R$, and

$$\Delta_{p,r} := \begin{cases} 1, & \text{if } r \in R(p), \\ 0, & \text{else.} \end{cases}$$

If the cost and demand functions are affine, we obtain an LCP. Being more specific, if the costs are of the form $t(h) = Th + t$ and if the demand is given by $d(\tau) = D\tau + d$ with matrices T and D and vectors t and d of suitable dimensions, the above complementarity system can be written in the form of (1) with

$$x := \begin{pmatrix} h \\ \tau \end{pmatrix}, \quad M := \begin{bmatrix} T & -\Delta^\top \\ \Delta & -D \end{bmatrix}, \quad q := \begin{pmatrix} t \\ -d \end{pmatrix}.$$

If, in addition, the arc flow costs $c(f) := (c_a(f))_{a\in A}$ are affine, that is, $c(f) = Cf + c$ for a matrix C and a vector c of suitable dimension, then the path-cost function can be written (in the additive model) as $t(h) = \Theta^\top c(f) = \Theta^\top(C\Theta h + c)$.

5.2. Uncertain demand and costs

In the last section, we reviewed the standard complementarity formulation for the deterministic TEP. In what follows, we consider the cases of uncertain demand and costs. First, we investigate the TEP with affine cost and demand functions. Moreover, the affine part d of the demand function $d(\tau)$ is considered to be uncertain and the uncertainty is parameterized by $d(u) = d^0 + \sum_{p\in P} u_p d^p$ for $U \in \mathcal{U} \subseteq \mathbb{R}^{|P|}$ and vectors $d^p \in \mathbb{R}^{|P|}$ for all OD pairs $p \in P$. Then, the parameters for the uncertain model in Formulation (5) are

$$M = \begin{bmatrix} T & -\Delta^\top \\ \Delta & -D \end{bmatrix}, \quad q^0 = \begin{pmatrix} t \\ -d^0 \end{pmatrix}, \quad q^p = \begin{pmatrix} 0 \\ -d^p \end{pmatrix}.$$

Further, we make the structural assumption that d^p is given by $d^p = \bar{d}_p e^p$, where $\bar{d}_p \geq 0$ and e^p is the pth unit vector for $p \in P$. In this case, Theorem 2 states that the robust counterpart is given by

$$\min_{z\geq 0} \quad h^\top(Th + t) - \tau^\top(D\tau + d^0) + \alpha\Gamma + \|\beta\|_2 \tag{28a}$$

$$\text{s.t.} \quad T_{r,\cdot}h + t_r - \Delta_{r,\cdot}^\top \tau \geq 0, \quad r \in R, \tag{28b}$$

$$\Delta_{p,\cdot}h - (D_{p,\cdot}\tau + d_p^0) - \bar{d}_p \geq 0, \quad p \in P, \tag{28c}$$

$$\alpha + \beta_p \geq \tau_p \bar{d}_p, \quad p \in P, \tag{28d}$$

with $z := (h^\top, \tau^\top, \alpha, \beta^\top)^\top$. This problem is convex if T is positive semidefinite and if D is negative semidefinite. These assumptions are reasonable in many practical applications. Consider, for instance, the setting in which arc flow costs only depend on the arc's flow or in which the demand of an OD-pair does not depend on the demands of other OD-pairs. In this case, T and D are diagonal matrices and the mentioned definiteness of the matrices correspond to the monotonicity of minimal travel costs and demand.

Next, we consider uncertain costs. To this end, let the affine uncertain arc-cost functions be given as $c(f, u) = C(u)f + c$, where the uncertain matrix $C(u)$ is parameterized by $C(u) = C^0 + \sum_{a\in A} u_a C^a$ with positive semidefinite matrices C^0 and C^a for $a \in A$. By using $t(h) = T(u)h + t$, the respective uncertain path-cost matrix can be obtained via $t(h) = \Theta^\top C(u)\Theta + \Theta^\top c$ and is thus given by $T(u) = \Theta^\top C(u)\Theta$. Moreover, using the notation $T^0 = \Theta^\top C^0\Theta$ and $T^a = \Theta^\top C^a \Theta$ for $a \in A$, we obtain $T(u) = T^0 + \sum_{a\in A} u_a T^a$. Note that T^0 and T^a are positive semidefinite for all $a \in A$ if C^0 and C^a, $a \in A$, are positive semidefinite. Defining

$$M^0 := \begin{bmatrix} T^0 & -\Delta^\top \\ \Delta & -D \end{bmatrix}, \quad M^a := \begin{bmatrix} T^a & 0 \\ 0 & 0 \end{bmatrix} \text{ for } a \in A, \quad q := \begin{pmatrix} t \\ -d \end{pmatrix},$$

we obtain for $u \in \mathcal{U}^2_\Gamma$ the convex robust counterpart

$$\min_{z\geq 0} \quad h^\top(T^0 h + t) - \tau^\top(D\tau + d^0) + \alpha\Gamma + \|\beta\|_2 \tag{29a}$$

$$\text{s.t.} \quad T^0_{r,\cdot} h - \Delta^\top_{r,\cdot}\tau_r + t_r - \gamma_r\Gamma \geq \sqrt{\sum_{a\in A}\delta^2_{r,a}}, \quad r \in R, \tag{29b}$$

$$\Delta_{p,\cdot} h - D_{p,\cdot}\tau + d_p \geq 0, \quad p \in P, \tag{29c}$$

$$\alpha + \beta_a \geq h^\top T^a h, \quad a \in A, \tag{29d}$$

$$\gamma_r + \delta_{r,a} \geq |T^a_{r,\cdot} h|, \quad a \in A, r \in R, \tag{29e}$$

where $z := (h^\top, \tau^\top, \alpha, \beta^\top, \gamma^\top, \delta^\top)^\top$.

Before we discuss the computational results, we also briefly consider the combination of uncertain demand and uncertain costs. In this case, we can easily combine the two given robust counterparts (28) and (29). This is possible because we consider the independent uncertainty model $M(u_1)x + q(u_2) \geq 0$ with $u_1 \in \mathcal{U}^1$ and $u_2 \in \mathcal{U}^2$ instead of $(u_1, u_2) \in \mathcal{U}$. Thus, the combined robust counterpart model reads

$$\min_{z\geq 0} \quad h^\top(T^0 h + t) - \tau^\top(D\tau + d^0) + \alpha\Gamma + \|\beta\|_2 + \hat{\alpha}\hat{\Gamma} + \|\hat{\beta}\|_2 \tag{30a}$$

$$\text{s.t.} \quad \Delta_{p,\cdot} h - (D_{p,\cdot}\tau + d^0_p) - \bar{d}_p \geq 0, \quad p \in P, \tag{30b}$$

$$\hat{\alpha} + \hat{\beta}_p \geq \tau_p \bar{d}_p, \quad p \in P, \tag{30c}$$

$$T^0_{r,\cdot} h - \Delta^\top_{r,\cdot}\tau_r + t_r - \gamma_r\Gamma \geq \sqrt{\sum_{a\in A}\delta^2_{r,a}}, \quad r \in R, \tag{30d}$$

$$\alpha + \beta_a \geq h^\top T^a h, \quad a \in A, \tag{30e}$$

$$\gamma_r + \delta_{r,a} \geq |T^a_{r,\cdot} h|, \quad a \in A, \; r \in R, \tag{30f}$$

with $z := (h^\top, \tau^\top, \alpha, \beta^\top, \gamma^\top, \delta^\top, \hat{\alpha}, \hat{\beta})^\top$.

5.3. Computational results

For completeness, we stated all three robustified versions of the TEP in the last section. In this section, we exemplarily study the case of uncertain arc-flow costs numerically and highlight some effects of the LCP's robustification. To this end, we consider the 5-node network given in Fig. 1, which is also studied in Xie and Shanbhag (2016). The uncertain arc-flow costs are given in the second column of Table 1 with $u \in [-1, 1]^7$; $u = 0$ corresponds to the nominal case. The nominal arc-flow cost matrix C^0 is given as the diagonal matrix of the flow factors and, for instance, the matrix C^{a_1} is given by diag$(0.01125, 0, \dots, 0)$ and so on. The cost vector c is given by $(3, 5, 6, 4, 6, 4, 1)^\top$. The certain demand is $d^0 = (200, 220)^\top$ and the route-cost matrices $T^0, T^{a_1}, \dots, T^{a_7}$ are given as shown in the previous section.

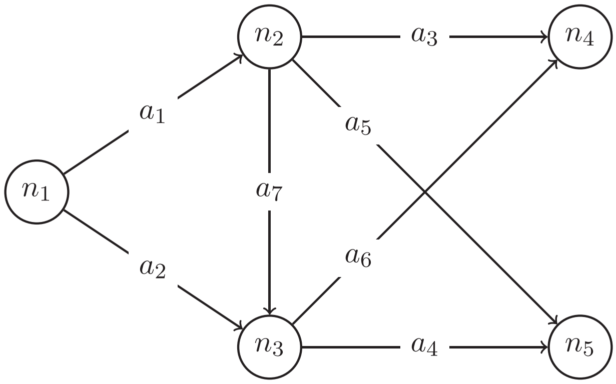

Fig. 1. 5-node network.

Table 1
Uncertain arc-cost model calibration as well as nominal flows ($\Gamma = 0$) and the Γ-robust ($\Gamma \in \{1, 2, 3\}$) arc flows in percentage with respect to the nominal flow for the 5-node network in Fig. 1

Arc	Uncertain cost model	Nominal flow	$\Gamma = 1$	$\Gamma = 2$	$\Gamma = 3$
a_1	$(1 + u_{a_1}) \cdot 0.01125f + 3$	269.20	0.98	1.01	1.02
a_2	$(1 + u_{a_2}) \cdot 0.01875f + 5$	150.80	1.03	0.99	0.96
a_3	$(1 + u_{a_3}) \cdot 0.04500f + 6$	77.32	0.85	0.95	0.93
a_4	$(1 + u_{a_4}) \cdot 0.03000f + 4$	134.68	1.08	1.01	1.03
a_5	$(1 + u_{a_5}) \cdot 0.04500f + 6$	85.32	0.88	0.98	0.96
a_6	$(1 + u_{a_6}) \cdot 0.03000f + 4$	122.68	1.09	1.03	1.04
a_7	$(1 + u_{a_7}) \cdot 0.00750f + 1$	106.55	1.17	1.07	1.14

Table 2
Objective function values for the uncertain TEP on the 5-node network in Fig. 1 for different values of Γ

Γ	0	1	2	3
Objective value	0	$24.18 \cdot 10^2$	$32.83 \cdot 10^2$	$33.17 \cdot 10^2$

The robust counterpart (for different values of Γ) is the SOCP (29), which we set up using Python 3.6.8 and that is solved with Gurobi 8.1.0. The case $\Gamma = 0$ corresponds to the nominal model and the case $\Gamma = 3$ corresponds to the strictly robust solution with ellipsoidal uncertainty since $\sqrt{7} < 3$; see the remarks at the end of Section 2. As nothing would change for larger Γ, we only consider the cases $\Gamma \in \{0, 1, 2, 3\}$. Table 2 shows the objective function values of the solved SOCPs. The results show the characteristic behavior of the Γ approach: As expected, lower values of Γ lead to less conservative solutions. Figure 2 shows the arc flows in the solutions for different Γ and in Table 1 we also show the nominal arc-flow and how the robust flows deviate from the nominal situation.

Next, we study the feasibility of the different solutions in some "worst-case" scenarios. To this end, we denote by (h^0, τ^0) the solution of the nominal problem and by (h^i, τ^i) the solution of the Γ-robust problem with $\Gamma = i$. The nominal solution can become infeasible in scenarios with lower

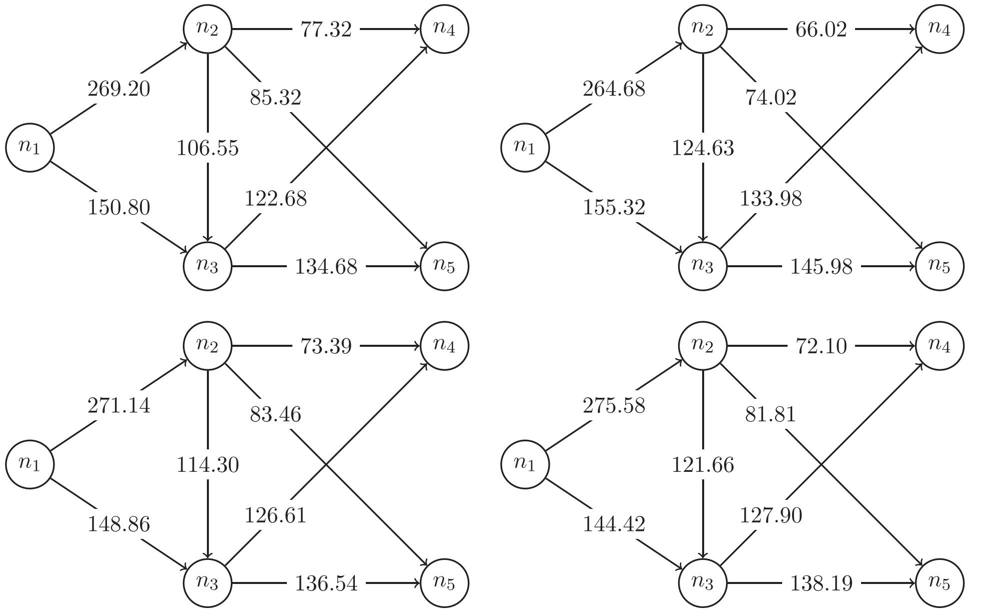

Fig. 2. 5-node arc flow. Top: $\Gamma = 0$ (left) and $\Gamma = 1$ (right). Bottom: $\Gamma = 2$ (left) and $\Gamma = 3$ (right).

Table 3
Feasibility of robust and nominal solution in certain scenarios

arcs	u	(h^0, τ^0)	(h^1, τ^1)	(h^2, τ^2)	(h^3, τ^3)
a_1	-1	inf	$8.42 \cdot 10^2$	$12.43 \cdot 10^2$	$11.96 \cdot 10^2$
a_1, a_4	-0.7	inf	inf	$11.00 \cdot 10^2$	$10.51 \cdot 10^2$

arc cost as Equation (29b) might be violated. This is shown in Table 3. Regarding the worst-case u, let us briefly discuss what “worst-case” means with respect to u. The table gives information on the feasibility of solutions. For this case, the worst-case values for u are the minimal ones. Interestingly, this differs from worst-case values for u in terms of costs, where the maximal values are worse. Here, we fixed the variables h and τ to the values of the nominal solution (h^0, τ^0) and consider the scenario in which costs on arc a_1 are lowered. It turns out that the nominal solution then is infeasible. In case of fixing the variables to the Γ-robust solutions for $\Gamma \in \{1, 2, 3\}$, the problem still remains feasible. For scenarios in which more than Γ many arcs costs are lowered, obviously, the robust solution can also become infeasible as feasibility is only guaranteed in dependence of Γ. This is illustrated in the second row of Table 3.

Table 4 shows how the nominal and Γ-robust solutions perform in the worst-case cost scenario with respect to arc a_1. Before we discuss the numerical results, let us briefly comment on the different interpretations of objective function values in the nominal and the robustified setting. The nominal objective is of the form $h^\top(Th + t) - \tau^\top d$ (as $D = 0$ in our model). The first term represents flow multiplied with costs and the second term represents demand multiplied with minimal costs per OD-pair. In an equilibrium, all flows between the same OD-pair have the same (minimal)

Table 4
Comparison for the scenario of worst-case cost on route a_1

Solution	Objective	Sum of all route costs
(h^0, τ^0)	$8.15 \cdot 10^2$	106.24
$(h^0, \tau^0_{\min})$	$24.40 \cdot 10^2$	106.24
(h^1, τ^1)	$24.18 \cdot 10^2$	106.61
(h^2, τ^2)	$28.96 \cdot 10^2$	106.55
(h^3, τ^3)	$29.03 \cdot 10^2$	106.93

costs τ. Thus, the terms are equal and add up to zero. In contrast, in the robust setting, the second term represents demand times minimal costs of all possible scenarios. Hence, the robust objective (for example, $\Gamma = 1$) tries to find (h^1, τ^1) that minimizes the difference of worst-case costs of flow h^1 and the variables τ^1 multiplied with demand, where τ^1 are variables representing the minimal costs of all possible scenarios in dependence of h^1. To make the second (negative) term as high as possible, it is preferable to change the flow in a way that the τ^1-variables (representing the minimal costs of all possible scenarios) are larger as long as the costs for the worst-case scenarios do not increase. Thus, the robust objective aims at a flow that has low worst-case cost but also high best-case cost.[1] Let us now turn back to Table 4. One might expect that the robust solution for $\Gamma = 1$ yields a better objective value in this scenario than the nominal solution. This is not the case, even though the sum of all route costs do not differ greatly. In the equilibrium of the nominal problem, $\tau^0 = (15.8679, 15.5079)^\top$ represents the minimal route cost between each origin-destination pair. In the robust setting, due to Equation (29b), $\tau^1 = (11.9123, 11.9123)^\top$ is forced to be the minimal route cost of all possible scenarios (in dependence of h^1) because of guaranteed feasibility that is hedged against all possible realization of uncertainty; see also Equation (17b). This results in larger objective values for the robust solutions in higher cost scenarios as the cost term of the objective becomes large while the τ-dependent term is forced to be small. If we replace τ^0 by the minimal route cost $\tau^0_{\min} = (11.8276, 11.8276)^\top$ of all scenarios[2] for the flow h^0, we see in Table 4 that the objective for (h^1, τ^1) is actually smaller than for $(h^0, \tau^0_{\min})$.

6. Conclusion

In this paper, we studied uncertain linear complementarity problems and used the combination of Γ-robustifications together with different variants of ellipsoidal uncertainty sets for modeling the uncertainty in the parameters of the LCP vector q or the LCP matrix M. We derived conditions for

[1]In the light of this discussion, let us briefly revisit Table 1. Actually, the scenarios in which costs for a_3 or a_5 are doubled are scenarios with lower total route costs compared to the worst case of doubling cost on a_1 due to the high flow on the route. On the other hand, reducing costs on a_4 and a_6 leads to the lowest route-costs. With the "low worst-case/high best-case" compromise in mind, it thus makes sense to shift flow from $a3$ and $a5$, which does not affect the worst case too much to a_4 and a_6, which increases the best-case cost.

[2]We restrict ourselves here to the scenarios in which a single route is affected. Thus, only one component of u is nonzero as we want to compare the solution to the $\Gamma = 1$ case.

the tractability of the robust counterparts and also provided results for the existence and uniqueness of solutions for these counterparts. Finally, a case study for the well-known traffic equilibrium problem is given that highlights some effects of Γ-robustifications applied to this problem.

It turns out that there are at least two issues of robust LCPs that may lead to interesting questions for future research. First, as for the cases of strict robustness (Xie and Shanbhag, 2016) or Γ-robustness using ℓ_1- as well as box-uncertainty sets (Krebs and Schmidt, 2020), it cannot be expected that robust LCP solutions exist in the pure sense of robust optimization. Second, the case study revealed that it is not easy to interpret solutions of robustified LCPs. Both aspects strongly lead to the question about other robustification concepts that lead to existing robust LCP solutions that have a clear-cut interpretation for practical problems.

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