ON ASTHENO-KÄHLER METRICS

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ABSTRACT. A Hermitian metric on a complex manifold of complex dimension n is called *astheno-Kähler* if its fundamental 2-form F satisfies the condition $\partial \overline{\partial} F^{n-2} = 0$. If n = 3, then the metric is *strong KT*, i.e. F is $\partial \overline{\partial}$ -closed. By using blow-ups and the twist construction, we construct simply-connected astheno-Kähler manifolds of complex dimension n > 3. Moreover, we construct a family of astheno-Kähler (non strong KT) 2-step nilmanifolds of complex dimension 4 and we study deformations of strong KT structures on nilmanifolds of complex dimension 3.

Finally, we study the relation between astheno-Kähler condition and (locally) conformally balanced one and we provide examples of locally conformally balanced astheno-Kähler metrics on \mathbb{T}^2 -bundles over (non-Kähler) homogeneous complex surfaces.

1. INTRODUCTION

Let (M, J, g) be a Hermitian manifold of complex dimension n. By [17] there exists a one-parameter family of canonical Hermitian connections

$$\nabla^t = t\nabla^C + (1-t)\nabla^0,$$

where ∇^C and ∇^0 denote the *Chern connection* and the first canonical connection respectively. This family includes for t = -1 the Bismut connection ∇^B considered by J.M. Bismut in [4].

If the fundamental 2-form $F(\cdot, \cdot) = g(J \cdot, \cdot)$ is closed, then the metric g is Kähler and any connection ∇^t in the above family coincides with the Levi-Civita connection. In the literature, weaker conditions on F have been studied and they involve the closure with respect to the $\partial\overline{\partial}$ -operator of the (k, k)-form $F^k = F \wedge \cdots \wedge F$. Some of these conditions are characterized by some properties of either the Chern or Bismut connection.

More precisely, if $\partial \overline{\partial} F = 0$, then the Hermitian structure (J, g) is said to be strong Kähler with torsion and g is called strong KT (see e.g. [15]). In this case the Hermitian structure is characterized by the condition that the Bismut connection has skew-symmetric torsion. Strong KT metrics have been recently studied by many authors and they have also applications in type II string theory and in 2dimensional supersymmetric σ -models [15, 39, 23]. Moreover, they have also links with generalized Kähler structures (see for instance [15, 19, 22, 2, 11, 12]). New simply-connected strong KT examples have been recently constructed by A. Swann

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in [40] via the twist construction, by reproducing the 6-dimensional examples found previously in [18].

If $\partial \overline{\partial} F^{n-2} = 0$, then in the terminology of J. Jost and S.-T. Yau ([26, 30]) the Hermitian metric g on M is said to be *astheno-Kähler*. Therefore, on a complex surface any Hermitian metric is automatically astheno-Kähler and in complex dimension n = 3 the notion of astheno-Kähler metric coincides with that one of strong KT. For n > 3, as far as we know, not many results and examples of astheno-Kähler manifolds are known.

Some rigidity theorems concerning compact astheno-Kähler manifolds have been showed in [26, Theorem 6] and in [30], where, in particular, a generalization to higher dimension of the Bogomolov's Theorem on VII_0 surfaces is proved (see [30, Corollary 3]). Astheno-Kähler structures on Calabi-Eckmann manifolds have been constructed in [34].

In [13] we proved that the blow-up of a complex manifold M at a point or along a compact complex submanifold Y is still strong KT, as in the Kähler case (see for example [5]). In Section 2 we will show that the results of [13] about resolutions of strong KT orbifolds can be extended to Hermitian orbifolds satisfying the conditions

(1)
$$\partial \overline{\partial} F = 0, \quad \partial \overline{\partial} F^2 = 0.$$

We will show that these manifolds satisfy $\partial \overline{\partial} F^k = 0$ for all k > 1 and therefore they are a proper subset of the astheno-Kähler manifolds.

As an application, we will construct a simply-connected example of compact astheno-Kähler manifold satisfying previous conditions. Moreover, we will show that other 8-dimensional examples may be obtained by applying the twist construction of [40] to astheno-Kähler manifolds with torus action.

In complex dimension 3 invariant astheno-Kähler structures on *nilmanifolds*, i.e. on compact quotients of nilpotent Lie groups by uniform discrete subgroups, were studied in [11] showing that the existence of such a structure depends only on the left-invariant complex structure on the Lie group. In Section 2 we will construct a family of astheno-Kähler 2-step nilmanifolds of complex dimension 4, showing that in general, for n > 3, there is no relation between the astheno-Kähler and strong KT condition (Theorem 2.7) and that is not anymore true that if (J, g) is astheno-Kähler, then any other J-Hermitian metric \tilde{g} is astheno-Kähler.

By the classification obtained in [11] one of the strong KT nilmanifolds is the *Iwasawa manifold*. In contrast with the Kodaira-Spencer stability theorem [29] and the case of complex surfaces, in [13] we proved that on the Iwasawa manifold the condition strong KT is not stable under small deformations of the complex structure. Deformations of complex structures on nilmanifolds have been studied in [38, 7, 32, 8] and recently S. Rollenske proved in [36, 37] that, in the generic case, small deformations of invariant complex structures on nilmanifolds are again invariant. This result can be applied to the strong KT 6-dimensional nilmanifolds and then we have that any small deformation and deformation in large of an invariant strong KT complex structure J_0 on a 6-dimensional nilmanifold is still invariant.

By using the results of [27, 41] we will prove that the space of deformations of a strong KT complex structure J_0 on a 6-dimensional nilmanifold for which there exists a strong KT metric is parametrized generically by a real algebraic hypersurface of degree 4 in \mathbb{C}^4 through the origin (Theorem 3.6). Furthermore, we show that the origin is non singular (respectively singular) according to the fact that J_0 is non abelian (respectively abelian).

If F^{n-1} is $\partial \overline{\partial}$ -closed or equivalently if its Lee form is co-closed, then the Hermitian metric g is called *standard* or a *Gauduchon metric* [16]. The Hermitian structure is said to be *balanced* if its Lee form vanishes and *conformally balanced* if its Lee form is exact. Astheno-Kähler and strong KT metrics on compact complex manifolds cannot be balanced for n > 2 unless they are Kähler (see [33, 1]). Moreover, by [23, 35] a conformally balanced strong KT structure on a compact manifold of complex dimension n whose Bismut connection has (restricted) holonomy contained in SU(n) is necessarily Kähler.

We will prove a similar result for the astheno-Kähler metrics (Theorem 4.1) and we will show that any non-Kähler compact homogeneous complex surface admits a non-trivial compact \mathbb{T}^2 -bundle M carrying an astheno-Kähler metric whose Lee form is closed. In the case of the \mathbb{T}^2 -bundle over the secondary Kodaira surface we will obtain a "locally conformal solution" of the Strominger's system considered in [39].

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2. Astheno-Kähler manifolds

Let (M, J) be a complex manifold of complex dimension n. Following Jost and Yau (see [26]), we recall the following

Definition 2.1. A Hermitian metric g on (M, J) is said to be astheno-Kähler if its fundamental 2-form form F satisfies the condition

$$\partial \overline{\partial} F^{n-2} = 0.$$

Thus, by definition, any Hermitian metric on a complex surface is astheno-Kähler and in complex dimension 3, an astheno-Kähler structure means a strong KT metric.

Remark 2.2. The product of two strong KT manifolds is still strong KT. This property is not true anymore for astheno-Kähler metrics. Indeed, for instance the product metric on the product of the Hopf surface and the Kodaira-Thurston surface is strong KT but it is not astheno-Kähler.

If n > 3, then the condition $\partial \overline{\partial} F^{n-2} = 0$ is equivalent to

$$d(c \wedge F^{n-3}) = 0$$

where c = -JdF is the torsion 3-form of the Bismut connection.

Note that, in general, if a Hermitian manifold (M, J, g) satisfies the conditions (1) then one has $\partial \overline{\partial} F^k = 0$, for any $k \ge 1$ and in particular g is astheno-Kähler, strong KT and standard. This follows by

$$\partial \overline{\partial} F^k = k \, \partial \left(\overline{\partial} F \wedge F^{k-1} \right) = k \left(\partial \overline{\partial} F \wedge F - (k-1) \overline{\partial} F \wedge \partial F \right) \wedge F^{k-2}, \quad k > 1 \, .$$

Indeed, if (1) holds, then $\partial F \wedge \overline{\partial} F = 0$ and therefore any F^k is $\partial \overline{\partial}$ -closed.

Following [16] we recall that a Hermitian metric g on (M, J) is said to be *standard* if F^{n-1} is $\partial\overline{\partial}$ -closed. Then, if n = 4 a Hermitian metric which is at the same time strong KT and astheno-Kähler metric, it must be also standard.

A necessary condition for the existence of astheno-Kähler metrics on compact complex manifolds was found in [26, Lemma 6], proving that any holomorphic 1-form must be *d*-closed.

We will provide a compact complex 3-dimensional manifold satisfying the previous condition on holomorphic 1-forms, with no astheno-Kähler metrics.

We start to note that, by using similar methods to those ones used in [10, Theorem 2.2] and in [41, Prop. 21] in the context of strong KT geometry, it is possible to show that if M is a compact quotient $M = \Gamma \backslash G$ of a simply-connected Lie group G by a uniform discrete subgroup Γ , endowed with an invariant complex structure J and having no invariant astheno-Kähler J-Hermitian metrics, then M does not admit any astheno-Kähler J-Hermitian metric at all.

Example 2.3. Let us consider the 6-dimensional nilpotent real Lie algebra \mathfrak{g} with structure equations

$$(0, 0, 0, 0, 0, 0, e^{12} + e^{34}),$$

where, with this notation, we mean that the dual space of \mathfrak{g} is generated by $\{e^1, \ldots, e^6\}$ satisfying

$$\begin{cases} de^i = 0, \quad i = 1, \dots, 5, \\ de^6 = e^{12} + e^{34}, \end{cases}$$

where e^{ij} stands for $e^i \wedge e^j$. Let G be the simply-connected Lie group whose Lie algebra is \mathfrak{g} and set

$$\eta^j = e^{2j-1} + ie^{2j}, \qquad j = 1, 2, 3$$

Then $\{\eta^1, \eta^2, \eta^3\}$ are complex (1, 0)-forms that define a left-invariant rational complex structure J on the nilmanifold $M = \Gamma \backslash G$, where Γ is a co-compact discrete subgroup of G such that $J(\Gamma) \subset \Gamma$.

In view of [11, Theorem 3.2], there are no strong KT metrics on (M, J). On the other hand, by [7, Theorem 2], it turns out that the Dolbeault cohomology group $H^{1,0}_{\overline{\partial}}(M)$ is spanned by $\{\eta^1, \eta^2\}$. Therefore, any holomorphic 1-form on M is *d*-closed.

2.1. Examples by blow-ups and resolutions. The proof of the result by [13] about the blow-up of a strong KT manifold at a point or along a compact complex submanifold can be adapted to the class of Hermitian manifolds whose fundamental 2-form F satisfies the conditions (1), since in both cases the new fundamental 2-form on the blow-up is obtained by adding a *d*-closed form to a $\partial\overline{\partial}$ -closed form. The Hermitian manifolds satisfying (1) are exactly equivalent to those which satisfy $\partial\overline{\partial}F^k = 0$ for all $k \geq 1$ and therefore such manifolds are a proper subset of the astheno-Kähler manifolds.

Then one can prove the following

Proposition 2.4. Let (M, J, g) be an astheno-Kähler manifold of complex dimension n such that its fundamental 2-form F satisfies (1). Then both the blow-up \tilde{M}_p of M at a point $p \in M$ and the blow-up \tilde{M}_Y of M along a compact complex submanifold Y admit an astheno-Kähler metric satisfying (1) too.

Thus by Proposition 2.4 it is possible to construct new examples of astheno-Kähler manifolds by blowing-up a given astheno-Kähler manifold M (satisfying (1)) at one or more points or along a compact complex submanifold.

Moreover, one may resolve singularities of a complex orbifold endowed with a special

astheno-Kähler metric (satisfying (1)). We recall that orbifolds are a special class of singular manifolds and they have been used by Joyce in [25] to construct compact manifolds with special holonomy, in [6, 9] to obtain non-formal symplectic compact manifolds and in [13] to construct new examples of strong KT manifolds.

One may give the following

Definition 2.5. A Hermitian metric g on an n-dimensional complex orbifold (M, J) is said to be astheno-Kähler if the fundamental 2-form F of g satisfies

$$\partial \overline{\partial} F^{n-2} = 0$$
.

An astheno-Kähler resolution of an astheno-Kähler orbifold (M, J, g) is the datum of a smooth complex manifold (\tilde{M}, \tilde{J}) endowed with a \tilde{J} -Hermitian astheno-Kähler metric \tilde{g} and of a map $\pi : \tilde{M} \to M$, such that

- i) π : M̃ \ E → M \ S is a biholomorphism, where S is the singular set of M and E = π⁻¹(S) is the exceptional set;
- ii) $\tilde{g} = \pi^* g$ on the complement of a neighborhood of E.

As in [13], we can apply Hironaka Resolution of Singularities Theorem [21], for which the singularities can be resolved by a finite number of blow-ups and we may use the previous results about blow-ups to prove the following

Theorem 2.6. Let (M, J) be a complex orbifold of complex dimension n endowed with a J-Hermitian astheno-Kähler metric g satisfying (1). Then there exists an astheno-Kähler resolution of (M, J, g) satisfying also (1).

We may apply the previous theorem to the complex orbifold, quotient of the standard complex torus by an involution. Let $\mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ be the standard torus and denote by (x_1, \ldots, x_{2n}) global coordinates on \mathbb{R}^{2n} . Consider the complex structure J on \mathbb{T}^{2n} defined by

(2)
$$\begin{cases} \eta^1 = dx_1 + i \left(f(x_n, x_{2n}) dx_n + dx_{n+1} \right), \\ \eta^j = dx_j + i dx_{n+j}, \quad j = 2, \dots, n, \end{cases}$$

where $f = f(x_n, x_{2n})$ is a \mathcal{C}^{∞} , \mathbb{Z}^{2n} -periodic and even function. Let σ be the *J*-holomorphic involution $\sigma : \mathbb{T}^{2n} \to \mathbb{T}^{2n}$ induced by

$$\sigma\left((x_1,\ldots,x_{2n})\right)=(-x_1,\ldots,-x_{2n}).$$

Thus, $(M = \mathbb{T}^{2n}/\langle \sigma \rangle, J)$ is a complex orbifold with singular set

$$S = \left\{ x + \mathbb{Z}^{2n} \mid x \in \frac{1}{2} \mathbb{Z}^{2n} \right\},\$$

which consists of 256 points for n = 4. Since $\sigma^*(\eta^j) = -\eta^j$, $j = 1, \ldots, n$, the natural Hermitian metric and the corresponding fundamental 2-form on \mathbb{T}^{2n}

$$g = \frac{1}{2} \sum_{j=1}^{n} \left(\eta^{j} \otimes \overline{\eta}^{j} + \overline{\eta}^{j} \otimes \eta^{j} \right), \quad F = \frac{i}{2} \sum_{j=1}^{n} \eta^{j} \wedge \overline{\eta}^{j}$$

are both σ -invariant and by [13] g is strong KT. For n > 3, a direct computation gives $\partial \overline{\partial} F^2 = -2\overline{\partial} F \wedge \partial F = 0$, i.e. the metric g is also astheno-Kähler. According to Theorem 2.6, now we may resolve the singularities of $\mathbb{T}^{2n}/\langle \sigma \rangle$ in order to obtain a simply-connected astheno-Kähler manifold \tilde{M} . More precisely, for any singular point $p \in S$, we take the blow-up at p. As in [25] we deduce that the astheno-Kähler resolution \tilde{M} of the orbifold $\mathbb{T}^{2n}/\langle \sigma \rangle$ is simply-connected.

2.2. Examples by twist construction. We recall that in general, given a manifold M with a torus action and a principal torus bundle P with connection θ , if the torus action lifts to P commuting with the principal action, then one may construct the twist W of the manifold, as the quotient of P by the torus action (see [40]). Moreover, if the lifted torus action preserves the principal connection θ , then tensors on M can be transferred to tensors on W if their pullbacks to P coincide on $\mathcal{H} = \text{Ker } \theta$. A differential form α on M is \mathcal{H} -related to a differential form α_W on W, $\alpha \sim_{\mathcal{H}} \alpha_W$, if their pull-backs to P coincide on \mathcal{H} .

By applying the twist construction of [40, Prop. 4.5] to 8-dimensional astheno-Kähler manifolds with torus action, one can get new simply-connected astheno-Kähler examples.

Let (N^6, J) be a 6-dimensional simply-connected compact complex manifold with a *J*-Hermitian structure *g* which is strong KT and standard. Consider the product $M^8 = N^6 \times \mathbb{T}^2$, where \mathbb{T}^2 is a 2-torus with an invariant Kähler structure. Then M^8 is astheno-Kähler and strong KT with torsion *c* supported on N^6 .

Assume that there are two linearly independent integral closed (1,1)-forms $\Omega_i \in \Lambda^{1,1}_{\mathbb{Z}}(N^6)$, i = 1, 2, with $[\Omega_i] \in H^2(N^6, \mathbb{Z})$. If

$$\sum_{i,j=1}^{2} \gamma_{ij} \Omega_i \wedge \Omega_j = 0$$

for some positive definite matrix $(\gamma_{ij}) \in M_2(\mathbb{R})$, then by [40, Prop. 4.5] there is a compact simply connected \mathbb{T}^2 -bundle \tilde{W} over N^6 whose total space is strong KT. The manifold \tilde{W} is the universal covering of the twist W of $N^6 \times \mathbb{T}^2$, where the Kähler flat metric over $\mathbb{T}^2 = \mathbb{C}/\mathbb{Z}^2$ is given by the matrix (γ_{ij}) with respect to the standard generators with a compatible complex structure and topologically W is a principal torus bundle over N^6 with Chern classes $[\Omega_i]$. Under the additional condition

(3)
$$c \wedge \Omega_j = 0, \qquad j = 1, 2,$$

we will prove that the total space is astheno-Kähler. By [40, Prop. 4.2], W has torsion 3-form c_W such that

$$c_W \sim_{\mathcal{H}} c - a^{-1}\Omega \wedge \xi^b,$$

$$dc_W \sim_{\mathcal{H}} dc + \sum_{i,j=1}^2 \gamma_{ij}\Omega_i \wedge \Omega_j,$$

where $a^{-1}\Omega = (\Omega_1, \Omega_2)$ and ξ is the standard action of the torus on the \mathbb{T}^2 -factor. Denote by $F = F_{N^6} + F_{\mathbb{T}^2}$ and F_W respectively the fundamental 2-form associated to the Hermitian structure (J, g) on M^8 and the induced Hermitian structure (J_W, g_W) on W. Then, since $F_W \sim_{\mathcal{H}} F$ we have that also $F_W^2 \sim_{\mathcal{H}} F^2$. Therefore, by [40, Prop. 4.2]

$$c_W \wedge F_W \sim_{\mathcal{H}} (c - a^{-1}\Omega \wedge \xi^b) \wedge F_{\mathcal{H}}$$

By using again [40, Cor. 3.6], it follows that

$$d(c_W \wedge F_W) \sim_{\mathcal{H}} d((c - a^{-1}\Omega \wedge \xi^b) \wedge F) - a^{-1}\Omega \wedge i_{\xi}((c - a^{-1}\Omega \wedge \xi^b) \wedge F),$$

where i_{ξ} denotes the contraction by ξ . Now, $i_{\xi}c = 0$ and $i_{\xi}F = J\xi^b$ and thus

(4)
$$\begin{aligned} d(c_W \wedge F_W) &\sim_{\mathcal{H}} & (dc + \sum_{i,j=1}^2 \gamma_{ij} \Omega_i \wedge \Omega_j) \wedge F - (c - a^{-1} \Omega \wedge \xi^b) \wedge dF \\ &+ a^{-1} \Omega \wedge (c - a^{-1} \Omega \wedge \xi^b) \wedge i_{\mathcal{E}} F . \end{aligned}$$

Observe that

(5)
$$a^{-1}\Omega \wedge a^{-1}\Omega \wedge \xi^b \wedge i_{\xi}F = \sum_{i,j=1}^2 \gamma_{ij}\Omega_i \wedge \Omega_j = 0,$$

and that $dF_{N^6} \wedge a^{-1}\Omega = 0$, since $a^{-1}\Omega$ is of type (1, 1). Therefore, by the assumption (3), by (5) and by the astheno-Kählerianity of the metric g on $M^8 = N^6 \times \mathbb{T}^2$, it follows that the right hand side of (4) vanishes. Hence \tilde{W} is strong KT and astheno-Kähler.

2.3. 8-dimensional nilmanifolds. We will construct a family of astheno-Kähler (non strong KT) 2-step nilmanifolds of real dimension 8, showing that in higher real dimension than 6 there is in general no relation between astheno-Kähler and strong KT structures. Let $\{\eta^1, \ldots, \eta^4\}$ be the set of complex forms of type (1,0), such that

(6)
$$\begin{cases} d\eta^{j} = 0, \ j = 1, 2, 3, \\ d\eta^{4} = a_{1} \eta^{1} \wedge \eta^{2} + a_{2} \eta^{1} \wedge \eta^{3} + a_{3} \eta^{1} \wedge \overline{\eta}^{1} + a_{4} \eta^{1} \wedge \overline{\eta}^{2} + a_{5} \eta^{1} \wedge \overline{\eta}^{3} \\ + a_{6} \eta^{2} \wedge \eta^{3} + a_{7} \eta^{2} \wedge \overline{\eta}^{1} + a_{8} \eta^{2} \wedge \overline{\eta}^{2} + a_{9} \eta^{2} \wedge \overline{\eta}^{3} + a_{10} \eta^{3} \wedge \overline{\eta}^{1} \\ + a_{11} \eta^{3} \wedge \overline{\eta}^{2} + a_{12} \eta^{3} \wedge \overline{\eta}^{3}, \end{cases}$$

where $a_j \in \mathbb{C}, j = 1, \ldots, 12$.

Then the complex forms $\{\eta^1, \ldots, \eta^4\}$ span the dual of a 2-step nilpotent Lie algebra \mathfrak{n} , depending on the complex parameters a_1, \ldots, a_{12} and define an integrable almost complex structure J on \mathfrak{n} . Let N be the simply connected Lie group with Lie algebra \mathfrak{n} . Then, for any $a_1, \ldots, a_{12} \in \mathbb{Q}[i]$, by the nilpotency of N, in view of Malcev's theorem [31], there exists a uniform discrete subgroup Γ of N such that $M = \Gamma \setminus N$ is a compact nilmanifold.

Theorem 2.7. Let $a_1, \ldots, a_{12} \in \mathbb{Q}[i]$ and $(M = \Gamma \setminus N, J)$ be the corresponding compact complex nilmanifold of real dimension 8. Then the Hermitian metric

$$g = \frac{1}{2} \sum_{j=1}^{4} \eta^{j} \otimes \overline{\eta}^{j} + \overline{\eta}^{j} \otimes \eta^{j}$$

is astheno-Kähler if and only if

(7)
$$\begin{aligned} |a_1|^2 + |a_2|^2 + |a_4|^2 + |a_5|^2 + |a_6|^2 + |a_7|^2 + |a_9|^2 + \\ |a_{10}|^2 + |a_{11}|^2 &= 2\Re \mathfrak{e} \left(a_3 \overline{a}_8 + a_3 \overline{a}_{12} + a_8 \overline{a}_{12} \right). \end{aligned}$$

If, in addition $a_8 = 0$ and $|a_4|^2 + |a_{11}|^2 \neq 0$, then the astheno-Kähler metric g is not strong KT. Moreover, if $a_8 = 0$, the astheno-Kahler metric g is strong KT if and only if $a_1 = a_4 = a_6 = a_7 = a_9 = a_{11} = 0$.

Proof. The fundamental 2-form of (J, g) is given by

$$F = \frac{i}{2} \sum_{j=1}^{4} \eta^{j} \wedge \overline{\eta}^{j} \,.$$

A straightforward computation yields

$$\begin{aligned} \partial \overline{\partial} F^2 &= -\frac{1}{2} \partial \overline{\partial} \left(\sum_{i < j} \eta^{i \overline{i} j \overline{j}} \right) \\ &= \frac{1}{2} (|a_1|^2 + |a_2|^2 + |a_4|^2 + |a_5|^2 + |a_6|^2 + |a_7|^2 + |a_9|^2 + |a_{10}|^2 + |a_{11}|^2 \\ &- 2 \Re \mathfrak{e} \left(a_3 \overline{a}_8 + a_3 \overline{a}_{12} + a_8 \overline{a}_{12} \right) \right) \eta^{123\overline{123}}. \end{aligned}$$

where, for instance, $\eta^{i\bar{i}j\bar{j}}$ denotes the wedge product $\eta^i \wedge \eta^{\bar{i}} \wedge \eta^j \wedge \eta^{\bar{j}}$. Hence $\partial \overline{\partial} F^2 = 0$ if and only if (7) holds.

The last part of the Theorem can be easily showed by a direct computation. \Box

As an application of the last result, we explicitly construct an astheno-Kähler metric which is not strong KT. Take

$$a_1 = a_2 = a_5 = a_6 = a_7 = a_8 = a_9 = a_{10} = 0, a_3 = a_4 = a_{11} = a_{12} = 4$$

and set $\eta^j = e^{2j-1} + ie^{2j}$, $j = 1, \dots, 4$. Then **n** has structure equations

(8)
$$\begin{cases} de^{i} = 0, \quad i = 1, \dots, 6, \\ de^{7} = 4(e^{13} + e^{24} - e^{35} - e^{46}), \\ de^{8} = 4(e^{23} - e^{14} + e^{45} - e^{36} - 2e^{12} - 2e^{56}) \end{cases}$$

Let $M = \Gamma \setminus G$ be the associated compact nilmanifold. Then, the Hermitian metric $g = \sum_{i=1}^{8} e^{j} \otimes e^{j}$ is an astheno-Kähler metric on M, that is not strong KT, according to Theorem 2.7.

If we take

$$a_1 = a_2 = a_4 = a_6 = a_7 = a_8 = a_9 = a_{11} = 0, a_3 = a_5 = a_{10} = a_{12} = 2$$

and set $\eta^j = e^{2j-1} + ie^{2j}$, j = 1, ..., 4, then we get a Hermitian metric satisfying conditions (1).

Remark 2.8. In real dimension six, by [11] the existence of a strong KT structure on a nilpotent Lie algebra depends only on the complex structure on the nilpotent Lie algebra. The same property is not anymore true for a astheno-Kähler structure on a nilpotent Lie algebra of real dimension eight. Indeed, for the nilpotent Lie algebra defined by (6) with the coefficients $a_j, j = 1, ..., 12$, satisfying the condition (7), the *J*-Hermitian metric given by

$$\frac{1}{2}[2(\eta^1\otimes\overline{\eta}^1+\overline{\eta}^1\otimes\eta^1)+3(\eta^2\otimes\overline{\eta}^2+\overline{\eta}^2\otimes\eta^2)+4(\eta^3\otimes\overline{\eta}^3+\overline{\eta}^3\otimes\eta^3)+5(\eta^4\otimes\overline{\eta}^4+\overline{\eta}^1\otimes\eta^4)]$$

is not any more astheno-Kähler.

3. Deformations of strong KT complex structures on 6-dimensional Nilmanifolds

We will say that a complex structure J on a nilmanifold $\Gamma \setminus G$ is *invariant* if it arises from a corresponding left-invariant complex structure on the Lie group G. We recall that a complex structure J on a Lie algebra \mathfrak{g} is called *abelian*, if and only if [JX, JY] = [X, Y], for any $X, Y \in \mathfrak{g}$ (see [3]) and it is *bi-invariant* if J commutes with the adjoint representation.

By using the result of [11] together with the results about "symmetrization" of non-invariant structures obtained in [10, 41], one has the following

Theorem 3.1. Let $M^6 = \Gamma \setminus G$ be a 6-dimensional nilmanifold with an invariant complex structure J. Then there exists a J-Hermitian strong KT metric g if and only if J has a basis $(\omega^1, \omega^2, \omega^3)$ of (1, 0)-forms such that

(9)
$$\begin{cases} d\omega^{1} = 0 \\ d\omega^{2} = 0 \\ d\omega^{3} = A\omega^{\overline{12}} + B\omega^{\overline{22}} + C\omega^{1\overline{1}} + D\omega^{1\overline{2}} + E\omega^{12} \end{cases}$$

where A, B, C, E, F are complex numbers such that

(10)
$$|A|^2 + |D|^2 + |E|^2 + 2\Re(\overline{B}C) = 0$$

and $\omega^{i\overline{j}}$ stands for $\omega^i \wedge \overline{\omega}^j$. Moreover, the Lie algebra \mathfrak{g} of G is isomorphic to one of the following:

$$\begin{split} \mathfrak{h}_2 &= (0,0,0,0,e^{12},e^{34})\,,\\ \mathfrak{h}_4 &= (0,0,0,0,0,e^{12},e^{14}+e^{23}),\\ \mathfrak{h}_5 &= (0,0,0,0,e^{13}+e^{42},e^{14}+e^{23})\\ \mathfrak{h}_8 &= (0,0,0,0,0,0,e^{12})\,. \end{split}$$

Remark 3.2. By the previous theorem one has that J_0 is abelian, i.e. the differential of the (1,0)-forms are only of type (1,1), if and only if E = 0.

In the sequel, we will denote by J_0 the strong KT complex structure, i.e. J_0 is the complex structure which gives rise to a strong KT structure, associated to the basis $(\omega^1, \omega^2, \omega^3)$ satisfying the condition (10).

We will use as in [11] the notation

$$\mathbf{Y}_{\omega} = A\omega^{\overline{1}2} + B\omega^{\overline{2}2} + C\omega^{1\overline{1}} + D\omega^{1\overline{2}},$$

where

$$\mathbf{Y} = \left(\begin{array}{cc} A & B \\ C & D \end{array}\right)$$

so that

$$d\omega^3 = \mathbf{Y}_\omega + E\omega^{12} \,.$$

Moreover, we will denote as in [11] by

$$\operatorname{adj}(\mathbf{Y}) = \begin{pmatrix} D & -B \\ -C & A \end{pmatrix}.$$

The 1-forms ω^j , j = 1, 2, 3, associated to the strong KT complex structure J_0 , are left-invariant on G and they define a basis (e^1, \ldots, e^6) of real 1-forms by setting

$$\omega^1 = e^1 + i e^2 \,, \quad \omega^2 = e^3 + i e^4 \,, \quad \omega^3 = e^5 + i e^6 \,.$$

These 1-forms are pull-backs of corresponding 1-forms on M^6 , which we denote by the same symbols.

Since for the dual basis (e_1, \ldots, e_6) we have

$$[e_j, e_k] \subseteq \operatorname{span} \langle e_5, e_6 \rangle,$$

for any j, k = 1, ..., 4, the quotient M^6 is the total space of a principal \mathbb{T}^2 -bundle over \mathbb{T}^4 . The space of invariant 1-forms annihilating the fibres of $\pi : M \to \mathbb{T}^4$ is

$$\mathbb{V} = \operatorname{span} \langle e^1, e^2, e^3, e^4 \rangle \subseteq \ker(d : \mathfrak{g}^* \to \Lambda^2 \mathfrak{g}^*),$$

with equality for the Lie algebras \mathfrak{h}_j , j = 2, 4, 5.

As in [27] we can prove the following

Lemma 3.3. Let J be any invariant complex structure on a strong KT nilmanifold $M^6 = \Gamma \setminus G$. Then the projection π induces a complex structure \tilde{J} on \mathbb{T}^4 such that $\pi : (M^6, J) \to (\mathbb{T}^4, \tilde{J})$ is holomorphic.

Proof. In order to prove the result is sufficient to show that \mathbb{V} is *J*-invariant. By Theorem 3.1 the Lie algebra \mathfrak{g} is isomorphic to $\mathfrak{h}_2, \mathfrak{h}_4, \mathfrak{h}_5$ or \mathfrak{h}_8 .

Any complex structure on \mathfrak{h}_8 is abelian, so the center given by span $\langle e_5, e_6 \rangle$ is preserved by the complex structure and therefore \mathbb{V} is *J*-invariant. By [41, Lemma 11] for any invariant (non bi-invariant) complex structure on the Lie algebras \mathfrak{h}_2 , \mathfrak{h}_4 and \mathfrak{h}_5 there is a basis (η^1, η^2, η^3) of (1, 0)-forms such that

$$\begin{aligned} d\eta^{j} &= 0, \quad j = 1, 2, \\ d\eta^{3} &= \rho \eta^{12} + \eta^{1\overline{1}} + G \eta^{1\overline{2}} + H \eta^{2\overline{2}}, \end{aligned}$$

with $\rho = 0, 1$ and $G, H \in \mathbb{C}$. Then, since the real space associated to the complex space spanned by η^1, η^2 coincides with the kernel of $d : \mathfrak{g}^* \to \Lambda^2 \mathfrak{g}^*$, we have that also in this case \mathbb{V} is *J*-invariant.

If J is bi-invariant, then the Lie algebra \mathfrak{g} has to be isomorphic to \mathfrak{h}_5 and the result follows by [27].

Let $(M^6 = \Gamma \setminus G, J_0)$ be a 6-dimensional nilmanifold with J_0 an invariant strong KT complex structure. By Theorem 3.1 we know that \mathfrak{g} is 2-step nilpotent with dim $\mathfrak{g}^1 \geq 2$ and with center of dimension 1 or 2. Therefore, we may apply Theorem 4.1 and 4.3 by Rollenske in [37] and conclude that any small and large deformation of J_0 is still invariant. Consequently, we may consider invariant deformations and work on the space of complex structures of \mathfrak{g}

$$\mathcal{C}(\mathfrak{g}) = \{ J \in \operatorname{End}(\mathfrak{g}) \, | J^2 = -1, \, N_J = 0 \},\$$

where by N_J we denote the Nijenhuis tensor.

We denote by $\mathcal{C}^+(\mathfrak{g})$ the space of complex structures of \mathfrak{g} inducing the same orientation on \mathbb{V} as J_0 and by $\mathcal{C}_0(\mathfrak{g})$ the open subset of $\mathcal{C}^+(\mathfrak{g})$ whose elements are such that there exists a basis (η^1, η^2, η^3) for which $\eta^{123} \wedge \omega^{\overline{123}} \neq 0$.

We can prove the following

Theorem 3.4. Let (M^6, J_0) be a 6-dimensional strong KT nilmanifold with J_0 defined by the (1,0)-forms $(\omega_1, \omega_2, \omega_3)$. If $J \in C_0(\mathfrak{g})$, then there exists a basis of (1,0)-forms (η^1, η^2, η^3) such that

(11)
$$\begin{cases} \eta^1 = \omega^1 + a\omega^{\overline{1}} + b\omega^{\overline{2}} \\ \eta^2 = \omega^2 + c\omega^{\overline{1}} + f\omega^{\overline{2}} \\ \eta^3 = \omega^3 + x\omega^{\overline{1}} + y\omega^{\overline{2}} + u\omega^{\overline{3}} \end{cases}$$

with $a, b, c, f, x, y, u \in \mathbb{C}$ satisfying

(12)
$$- (\det \mathbf{X})E + (\operatorname{tr}(\mathbf{X}\overline{\mathbf{Y}}) - \overline{E})u + \operatorname{tr}(\mathbf{X}\operatorname{adj}(\mathbf{Y})) = 0,$$

where

(13)
$$\mathbf{X} = \begin{pmatrix} a & b \\ c & f \end{pmatrix}.$$

Therefore, the generic complex structure on M^6 has a space of (1,0)-forms generated by the previous forms η^1 , η^2 and η^3 .

Proof. In view of Lemma 3.3, we have that η^1 and η^2 can be chosen so that their real and imaginary components span \mathbb{V} . The condition

$$\eta^{123} \wedge \omega^{\overline{123}} \neq 0$$

implies that $\omega^1, \omega^2, \omega^3$ appear with non-zero coefficients.

The equation (12) follows from the integrability condition $d\eta^3 \wedge \eta^{12} = 0$ expressing the fact that $d\eta^3$ has no term involving $\eta^{\overline{12}}$.

By a direct computation one has that

$$d\eta^3 \wedge \eta^{12} = -(cEb - cB + cu\overline{B} - fEa + fA + fu\overline{D} - u\overline{E} - bC + bu\overline{C} + aD + au\overline{A})\omega^{12\overline{12}} + bC + bu\overline{C} + bu\overline{C} + aD + au\overline{A})\omega^{12\overline{12}} + bC + bu\overline{C} + bU\overline$$

Therefore the complex structure is integrable if and only if the equation (12) is satisfied. $\hfill \Box$

Consider the space $\Lambda = \langle \eta^1, \eta^2, \eta^3 \rangle$ generated by the modified complex 1-forms (11). If Λ is maximally complex, then it defines an invariant almost complex structure on M^6 that we will denote by $J_{\mathbf{X},x,y}$, where \mathbf{X} is given by (13).

By the previous theorem, the almost complex structure $J_{\mathbf{X},x,y}$ is integrable if and only if the equation (12) holds.

Remark 3.5. In the case of Iwasawa manifold (with the bi-invariant complex structure J_0 which is therefore non strong KT) one has E = 1 and $\mathbf{Y} = 0$ and therefore the integrability condition reduces to the equation

$$u = bc - af = -\det \mathbf{X}$$

already considered in [11].

We are ready to prove the following

Theorem 3.6. Let $(M^6 = \Gamma \setminus G, J_0)$ be a 6-dimensional nilmanifold with an invariant strong KT complex structure J_0 . Then the space of deformations of J_0 for which there exists a strong KT metric is parametrized generically by a real algebraic hypersurface of degree 4 in \mathbb{C}^4 through the origin O = (0, 0, 0, 0). Furthermore, O is non singular (respectively singular) according to the fact that J_0 is non abelian (respectively abelian).

Proof. By Theorem 3.4, the generic complex structure J on M^6 is defined by the (1,0)-forms (η^1, η^2, η^3) given by (11). We have

$$\eta^{12} = \omega^{12} + a\omega^{\overline{12}} + b\omega^{\overline{22}} + c\omega^{1\overline{1}} + f\omega^{1\overline{2}} + (af - bc)\omega^{\overline{12}}$$
$$= \omega^{12} + \mathbf{X}_{\omega} + (\det \mathbf{X})\omega^{\overline{12}}.$$

We write the characteristic polynomial of $\mathbf{X}\overline{\mathbf{X}}$ as $p(x) = x^2 - \gamma x + \delta$, so that

$$\gamma = \operatorname{tr}(\mathbf{X}\overline{\mathbf{X}}) = |a|^2 + |f|^2 + b\overline{c} + \overline{b}c,$$

$$\delta = \det(\mathbf{X}\overline{\mathbf{X}}) = |a|^2|f|^2 + |b|^2|c|^2 - af\overline{b}\overline{c} - bc\overline{a}\overline{f}.$$

The relations

$$\begin{split} \eta^{1\overline{1}2\overline{2}} &= (1 - \gamma + \delta)\omega^{1\overline{1}2\overline{2}} \,, \\ \eta^{1\overline{1}2\overline{2}3\overline{3}} &= (1 - \gamma + \delta)(1 - |u|^2)\omega^{1\overline{1}2\overline{2}3\overline{3}} \end{split}$$

express volume changes associated to a switch of basis from (ω^i) to (η^i) . As a consequence, $\Lambda \cap \overline{\Lambda} = \{0\}$ if and only if

(14)
$$|u| \neq 1$$
 and $p(1) \neq 0$

and these are the conditions that ensure that the complex structure $J_{\mathbf{X},x,y}$ is well defined.

From now we suppose that the conditions (10), (12) and (14) hold.

For simplicity, we will also assume that x = y = 0, and we will denote the complex structure $J_{\mathbf{X},0,0}$ by $J_{\mathbf{X}}$, since the strong KT condition is determined in terms of $d\eta^3$, which does not involve x, y.

As in the proof of Lemma 4.1 of [11] consider the two bases $(\omega^1, \omega^2, \overline{\omega}^1, \overline{\omega}^2)$ and $(\eta^1, \eta^2, \overline{\eta}^1, \overline{\eta}^2)$. The second is related to the first one by the block matrix

$$\mathbf{M} = \left(\begin{array}{cc} I & \mathbf{X} \\ \overline{\mathbf{X}} & I \end{array}\right).$$

Set $\mathbf{Z} = (I - \mathbf{X}\overline{\mathbf{X}})^{-1}$ so that $p(1) = \det(\mathbf{Z})^{-1}$. Then the inverse of **M** is given by

$$\mathbf{M}^{-1} = \left(egin{array}{cc} \mathbf{Z} & -\mathbf{Z}\mathbf{X} \ -\mathbf{\overline{X}}\mathbf{Z} & \mathbf{\overline{Z}} \end{array}
ight)$$

with

$$\mathbf{Z} = \frac{1}{p(1)} \left(\begin{array}{cc} 1 - \overline{b}c - |f|^2 & a\overline{b} + b\overline{f}, \\ c\overline{a} + f\overline{c} & 1 - \overline{c}b - |a|^2 \end{array} \right)$$

and

$$\mathbf{ZX} = \frac{1}{p(1)} \left(\begin{array}{cc} a-a|f|^2 + bc\overline{f} & b-|b|^2c + af\overline{b} \\ c-|c|^2b + af\overline{c} & f-|a|^2f + cb\overline{a} \end{array} \right).$$

Therefore

$$\begin{split} p(1)\,\omega^1 &= (1-\overline{b}c-|f|^2)\eta^1 + \\ &+ (a\overline{b}+b\overline{f})\eta^2 + (-a+a|f|^2-bc\overline{f})\overline{\eta}^1 + (-b+|b|^2c-af\overline{b})\overline{\eta}^2 \\ p(1)\,\omega^2 &= (c\overline{a}+f\overline{c})\eta^1 + (1-\overline{c}b-|a|^2)\eta^2 + \\ &+ (-c+|c|^2b-af\overline{c})\overline{\eta}^1 + (-f+|a|^2f-cb\overline{a})\overline{\eta}^2. \end{split}$$

By [41, Theorem 19] a complex structure J_0 on the Lie algebra \mathfrak{g} of G is strong KT if and only if the complex structure J_0 is equivalent to the one defined by

(15)
$$\begin{cases} d\omega^{j} = 0, \quad j = 1, 2, \\ d\omega^{3} = \rho \omega^{12} + \omega^{1\overline{1}} + G \omega^{1\overline{2}} + H \omega^{2\overline{2}} \end{cases}$$

with $\rho = 0, 1, G, H \in \mathbb{C}$ such that $\rho + |G|^2 = 2\Re \mathfrak{e}(H)$. Therefore, comparing with the structure equations (9), the expression of the matrix **Y** in the new parameters ρ, G, H is the matrix

$$\mathbf{Y} = \left(\begin{array}{cc} 0 & -H \\ \rho & G \end{array}\right).$$

We will examine separately the two cases: $\rho = 1$ (J_0 non abelian) and $\rho = 0$ (J_0 abelian).

For $\rho = 1$, the equation (12) reduces to

$$-(\det \mathbf{X}) + (\operatorname{tr}(\mathbf{X}\overline{\mathbf{Y}}) - 1)u + \operatorname{tr}(\mathbf{X}\operatorname{adj}(\mathbf{Y})) = 0$$

i.e. to

$$(fa - cb - cH + b - aG) - (f\overline{G} - c\overline{H} + b - 1)u = 0.$$

If we denote by

$$\begin{split} \gamma_1 &= -1 + c + |f|^2 - G\overline{f}c + |c|^2H \,, \quad \gamma_2 = b + Ga\overline{b} - |b|^2 + H - H|a|^2 \,, \\ \gamma_3 &= G - f + \overline{b}f + \overline{a}cH - \overline{b}cG \,, \qquad \gamma_4 = -\overline{a} - \overline{b}f + \overline{a}f\overline{G} - \overline{a}c\overline{H} \,, \\ \gamma_5 &= 1 - \overline{b} - G\overline{f} + \overline{c}H \,, \qquad \gamma_6 = af - cb - cH - aG \,, \end{split}$$

then we have that given the complex structure J_0 equivalent to (15) with the complex numbers G, H satisfying the condition

$$1 - 2\mathfrak{Re}H + |G|^2 = 0$$

the new complex structure $J_{\mathbf{X}}$ is integrable and strong KT if and only if the following equations hold

(16)
$$\begin{cases} \overline{\gamma}_5 u + \gamma_6 = 0, \\ (1+|u|^2) \left(|\gamma_3|^2 + |\gamma_4|^2 + 2\Re \mathfrak{e}(\overline{\gamma}_1 \gamma_2) \right) + |\gamma_5|^2 \left(1 - |u|^2 \right)^2 + \\ +4\Re \mathfrak{e} \left(u(\overline{\gamma}_3 \gamma_4 - \overline{\gamma}_1 \overline{\gamma}_2) \right) = 0, \end{cases}$$

since in terms of the γ_j , $j = 1, \ldots, 5$, we have that

$$p(1)d\eta^{3} = \gamma_{5}(1-|u|^{2})\eta^{12} + (-\gamma_{1}+u\overline{\gamma}_{1})\eta^{11} + (\gamma_{3}+u\gamma_{4})\eta^{12} + (\overline{\gamma}_{4}+u\overline{\gamma}_{3})\eta^{\overline{12}} + (-\gamma_{2}+u\overline{\gamma}_{2})\eta^{\overline{22}}.$$

Assuming that $\gamma_5 \neq 0$ and by using the first equation of (16), we may eliminate the complex parameter u. Then, the second equation of (16) becomes

$$\begin{aligned} (|\gamma_5|^2 + |\gamma_6|^2) \left(|\gamma_3|^2 + |\gamma_4|^2 + 2\mathfrak{Re}(\overline{\gamma}_1\gamma_2) \right) + \left(|\gamma_5|^2 - |\gamma_6|^2 \right)^2 + \\ -4\mathfrak{Re} \left(\gamma_5\gamma_6(\overline{\gamma}_3\gamma_4 - \overline{\gamma}_1\overline{\gamma}_2) \right) = 0 \,, \end{aligned}$$

which is a real equation in the complex variables a, b, c and f and thus defines a real hypersurface of degree 4 in \mathbb{C}^4 , non singular at the point O = (a = 0, b = 0, c = 0, f = 0). Therefore, we have that the complex structure $J_{\mathbf{X}}$ (deformation of J_0), defined by the (1, 0)-forms

$$\left\{ \begin{array}{l} \eta^1 = \omega^1 + a\overline{\omega}^1 + b\overline{\omega}^2 \,, \\ \eta^2 = \omega^2 + c\overline{\omega}^1 + f\overline{\omega}^2 \,, \\ \eta^3 = \omega^3 - \frac{\gamma_6}{\overline{\gamma_5}}\overline{\omega}^3 \,, \end{array} \right.$$

has a compatible strong KT metric if and only if (a, b, c, f) belongs to the previous hypersurface. This completes the case $\rho = 1$.

For $\rho = 0$, the equation (12) reduces to

$$\operatorname{tr}(\mathbf{X}\overline{\mathbf{Y}})u + \operatorname{tr}(\mathbf{X}\operatorname{adj}(\mathbf{Y})) = 0,$$

or equivalently to

$$-fu\overline{G} - cH + cu\overline{H} + b - bu - aG = 0.$$

We set

$$\begin{split} \delta_1 &= |b|^2 - \overline{G}\overline{a}b - \overline{H} + \overline{H}|a|^2 \,, \quad \delta_2 &= -1 + |f|^2 - \overline{G}f\overline{c} + \overline{H}|c|^2 \,, \\ \delta_3 &= G + \overline{b}f - G\overline{b}c + c\overline{a}H \,, \quad \delta_4 &= -\overline{b}f + f\overline{a}\overline{G} - c\overline{a}\overline{H} \,, \\ \delta_5 &= -\overline{f}G + \overline{c}H - \overline{b} \,, \quad \delta_6 &= -cH + b - aG \,. \end{split}$$

Then, we have that, given the complex structure J_0 equivalent to (15) with $\rho = 0$ and G, H such that $|G|^2 = 2\Re \mathfrak{e}(H)$, the new almost complex structure $J_{\mathbf{X}}$ is integrable and strong KT if and only if the following equations hold

(17)
$$\begin{cases} \overline{\delta}_5 u + \delta_6 = 0, \\ (1+|u|^2)(|\delta_3|^2 + |\delta_4|^2 - 2\mathfrak{Re}(\delta_1\overline{\delta}_2)) + |\delta_5|^2(1-|u|^2)^2 + \\ +4\mathfrak{Re}(u(\delta_1\delta_2 + \overline{\delta}_3\delta_4 + \delta_4\overline{\delta}_3)) = 0 \end{cases}$$

since

$$p(1)d\eta^3 = (\delta_1 u - \overline{\delta}_1)\eta^{2\overline{2}} + (\delta_2 u - \overline{\delta}_2)\eta^{1\overline{1}} + (\delta_4 u + \delta_3)\eta^{1\overline{2}} + (-\overline{\delta}_3 u - \overline{\delta}_4)\eta^{2\overline{1}} + (\overline{\delta}_6 u + \delta_5)\eta^{12}.$$

Assuming that $\delta_5 \neq 0$ and by using the first equation of (17) we may eliminate the complex parameter u. Then, the second equation of (17) becomes the real equation in the complex variables a, b, c, f

$$\begin{aligned} (|\delta_5|^2 + |\delta_6|^2)(|\delta_3|^2 + |\delta_4|^2 - 2\mathfrak{Re}\left(\delta_1\overline{\delta}_2\right)) + (|\delta_5|^2 - |\delta_6|^2)^2 + \\ -4\mathfrak{Re}\left(\delta_5\delta_6(\delta_1\delta_2 + \overline{\delta}_3\delta_4 + \delta_4\overline{\delta}_3)\right) = 0 \end{aligned}$$

which defines a real hypersurface of degree 4 in \mathbb{C}^4 , singular at the point (a = 0, b = 0, c = 0, f = 0). In this way we prove that $J_{\mathbf{X}}$, deformation of J_0 , defined by the (1,0)-forms

$$\left\{ \begin{array}{l} \eta^1 = \omega^1 + a\overline{\omega}^1 + b\overline{\omega}^2, \\ \eta^2 = \omega^2 + c\overline{\omega}^1 + f\overline{\omega}^2, \\ \eta^3 = \omega^3 - \frac{\delta_6}{\delta_5}\overline{\omega}^3, \end{array} \right.$$

has a compatible strong KT metric if and only if (a, b, c, f) belongs to the previous hypersurface. This completes the case $\rho = 0$. Then the theorem is proved.

4. LOCALLY CONFORMALLY BALANCED STRUCTURES

In general, if a Hermitian manifold (M, J, g) is compact, then by using its fundamental 2-form F, one has two natural linear operators acting on differential forms:

$$L\varphi = F \wedge \varphi,$$

and the adjoint operator L^\ast of L with respect to the global scalar product defined by

$$< \varphi, \psi >= p! \int_{M} (\varphi, \psi) \mathrm{vol}_{g},$$

where (φ, ψ) is the poinwise g-scalar product and vol_g is the volume form.

As in the strong KT case, the astheno-Kähler condition on a compact Hermitian manifold is complementary to the balanced one, since by [33, Theorem 1.1], for $n \geq 3$, one has

(18)
$$L^{*n-1}(2i\partial\overline{\partial}F^{n-2}) = 4^{n-1}(n-1)!(n-2)[2(n-2)d^*\theta + 2||\theta||^2 - ||T||^2],$$

where $\theta = Jd^*F$ is the *Lee form*, $d^*\theta$ its co-differential, $||\theta||$ its g-norm and T is the torsion of the Chern connection ∇^C on (M, g).

Therefore, if (J,g) is balanced, then $\theta = 0$ and, consequently, the astheno-Kähler condition implies that T = 0, i.e. g is Kähler.

By [23, 35], a conformally balanced strong KT structure on a compact manifol of complex dimension n whose Bismut connection has (restricted) holonomy contained in SU(n) is necessarily Kähler. We now prove a similar result for the astheno-Kähler metrics.

Theorem 4.1. A conformally balanced astheno-Kähler structure (J, g) on a compact manifold of complex dimension $n \ge 3$ whose Bismut connection has (restricted) holonomy contained in SU(n) is necessarily Kähler and therefore it is a Calabi-Yau structure.

Proof. Since the Hermitian structure is astheno-Kähler, then by (18), we have

$$2(n-2)d^{*}\theta + 2||\theta||^{2} - ||T||^{2} = 0.$$

Therefore,

(19)
$$d^*\theta = \frac{1}{2(n-2)} [||T||^2 - 2||\theta||^2].$$

By [1, formula (2.11)], the trace 2u of the Ricci form of the Chern connection is related to the trace b of the Ricci form of the Bismut connection by the equation

(20)
$$2u = b + 2(n-1)d^*\theta + 2(n-1)^2||\theta||^2.$$

We recall that the condition that the Bismut connection has (restricted) holonomy contained in SU(n) implies that the Ricci form of the Bismut connection vanishes and, if in addition M is compact, then the first Chern class $c_1(M)$ vanishes.

Therefore, by using (19) and (20), we get

(21)
$$2u = \frac{(n-1)}{(n-2)} [||T||^2 - 2||\theta||^2] + 2(n-1)^2 ||\theta||^2$$

and then, if (J, g) is not Kähler, we must have u > 0.

Since in addition (J, g) is conformally balanced, then it was shown in [39, 35] that there exists a nowhere vanishing holomorphic (n, 0)-form $\tilde{\alpha}$.

Let be $f = -\frac{1}{2} ||\alpha||^2$ and denote by $L^{\mathbb{C}}$ the complex Laplacian defined by

$$L^{\mathbb{C}}(f) = \Delta f + (df, \theta),$$

where Δ is the standard Laplace operator and (,) is the scalar product on the forms induced by g. Then, since $\tilde{\alpha}$ is holomorphic, as in [35] (see also [24, formula (19)]) we have

$$L(f) = 2u ||\tilde{\alpha}||^2 + ||\nabla^C \tilde{\alpha}||^2$$

where ∇^{C} is the Chern connection and u is given by (21).

By the fact that u > 0 and $\tilde{\alpha} \neq 0$, it follows that $L^{\mathbb{C}}f > 0$. From the maximum principle, we have that f is constant which implies that u = 0. The theorem is proved.

Remark 4.2. The previous theorem for n = 3 was already proved in [23, 35].

5. \mathbb{T}^2 -bundles over complex surfaces

By [20] a complex (non-Kähler) surface diffeomorphic to a 4-dimensional compact homogeneous manifold $X = \Theta \backslash L$, where Θ is a uniform discrete subgroup of L, and which does not admit any Kähler structure is one of the following:

a) Hopf surface;

b) Inoue surface of type \mathcal{S}^0 ;

c) Inoue surface of type \mathcal{S}^{\pm} ;

d) primary Kodaira surface;

e) secondary Kodaira surface;

f) properly elliptic surface with first odd Betti number.

A \mathbb{T}^2 -bundle over the Inoue surface of type \mathcal{S}^0 was considered in [12] in order to construct a 6-dimensional compact solvmanifold with a non-trivial generalized Kähler structure. A similar construction can be done for any of the non-Kähler complex homogeneous surfaces, by using the description of L and Θ in [20]. Indeed we can prove the following

Theorem 5.1. On any non-Kähler compact homogeneous complex surface $X = \Theta \setminus L$ there exists a non-trivial compact \mathbb{T}^2 -bundle M carrying a locally conformally balanced strong KT metric.

Proof. For the Inoue surface of type S^0 , we already proved the result. For the surfaces a), c), d) and e) we may consider respectively the 6-dimensional Lie algebras:

$$\begin{aligned} \mathfrak{g}_1 &= (e^{23}, e^{31}, e^{12}, 0, \frac{\pi}{2}e^{64}, \frac{\pi}{2}e^{45}), \\ \mathfrak{g}_2 &= (e^{12}, 0, e^{14}, e^{24}, \frac{\pi}{2}e^{26}, -\frac{\pi}{2}e^{25}), \\ \mathfrak{g}_3 &= (0, 0, e^{12}, 0, \frac{\pi}{2}e^{46}, -\frac{\pi}{2}e^{45}), \\ \mathfrak{g}_4 &= (e^{24}, -e^{14}, e^{12}, 0, \frac{\pi}{2}e^{46}, -\frac{\pi}{2}e^{45}). \end{aligned}$$

endowed with the complex structure J, defined by the (1, 0)-forms

$$\eta^1 = e^1 + ie^4, \quad \eta^2 = e^2 + ie^3, \quad \eta^3 = e^5 + ie^6$$

and the inner product g defined by $g = \sum_{j=1}^{6} e^j \otimes e^j$. Thus g is *J*-Hermitian and, denoting by F the fundamental 2-form associated with the Hermitian structure (J,g), by a direct computation we have that JdF is closed and that the Lee form is closed.

For the surface f) we may take the Lie algebra

$$\mathfrak{g}_5 = \left(2e^{13}, -2e^{23}, -e^{12}, 0, \frac{\pi}{2}e^{46}, -\frac{\pi}{2}e^{45}\right)$$

endowed with the complex structure

$$\begin{aligned} Je_1 &= \frac{1}{2}(e_3 + e_4) \,, \qquad Je_2 &= \frac{1}{2}(e_3 - e_4) \,, \qquad Je_3 &= -(e_1 + e_2) \,, \\ Je_4 &= -e_1 + e_2 \,, \qquad Je_5 &= e_6 \,, \qquad \qquad Je_6 &= -e_5 \,. \end{aligned}$$

and the inner product

$$g=e^1\otimes e^1+e^2\otimes e^2+2(e^3\otimes e^3+e^4\otimes e^4)+e^5\otimes e^5+e^6\otimes e^6.$$

The fundamental form

$$F = -e^{13} - e^{14} - e^{23} + e^{24} + e^{56}$$

is such that the 3-form $JdF = 2e^{123} + 2e^{124}$ is closed. Moreover, the Lee form is closed.

For every Lie algebra \mathfrak{g}_i we have that the span of $\{e^1, e^2, e^3, e^4\}$ can be viewed as the dual of the Lie algebra of the Lie group L.

Let \mathbb{H}_3 be the real 3-dimensional Heisenberg Lie group and $H = \mathbb{R} \ltimes_{\varphi} \mathbb{R}^2$ be the semidirect product of the groups \mathbb{R} and \mathbb{R}^2 , where $\varphi : \mathbb{R} \to GL(2,\mathbb{R})$ is the homomorphism given by

$$\varphi(t) = \begin{pmatrix} \cos(\frac{\pi}{2}t) & \sin(\frac{\pi}{2}t) \\ -\sin(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t) \end{pmatrix}.$$

We have

$$G_1 = SU(2) \times H$$
, $G_3 = \mathbb{H}_3 \times H$, $G_5 = SL(2, \mathbb{R}) \times H$

and a uniform discrete subgroup of H is of the form $\Gamma' = \mathbb{Z} \ltimes_{\varphi} \mathbb{Z}^2$. Therefore the Lie groups G_i , i = 1, 3, 5 admit a uniform discrete subgroup.

Let $(m_{jk}) \in SL(2, \mathbb{Z})$ with two real positive eigenvalues a and b and denote by (a_1, a_2) and (b_1, b_2) the corresponding eigenvectors. The remaining Lie groups G_2 and G_4 are the semidirect products (see [20] for the description of the corresponding Lie group L)

$$G_2 = \mathbb{R} \ltimes_{\nu} (\mathbb{H}_3 \times \mathbb{C}), \quad G_4 = \mathbb{R} \ltimes_{\tilde{\nu}} (\mathbb{H}_3 \times \mathbb{C}),$$

where the automorphisms $\nu(t)$ and $\tilde{\nu}(t)$ are given respectively by

$$\begin{split} \nu(t) &: (x+iy,u,z) \mapsto \left(a^tx+ib^ty,u,e^{i\frac{\pi}{2}t}z\right), \\ \tilde{\nu}(t) &: (x+iy,u,z) \mapsto \left(e^{i\frac{\pi}{2}t}(x+iy),u,e^{i\frac{\pi}{2}t}z\right), \end{split}$$

by identifying the matrix

$$\left(\begin{array}{rrrr}1 & x & u\\0 & 1 & y\\0 & 0 & 1\end{array}\right)$$

in \mathbb{H}_3 with $(x + iy, u) \in \mathbb{C} \times \mathbb{R}$.

The Lie group G_2 admits a compact quotient by a uniform discrete subgroup of the form $\Gamma_2 = \mathbb{Z} \ltimes_{\nu} (\tilde{\Gamma}_n \times \mathbb{Z}^2)$, where \mathbb{Z}^2 is the standard lattice of \mathbb{C} and $\tilde{\Gamma}_n$ is the lattice of \mathbb{H}_3 generated by the elements

$$g_1 = \begin{pmatrix} 1 & a_1 & c_1 \\ 0 & 1 & b_1 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & a_2 & c_2 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 1 & 0 & c_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad c_i \in \mathbb{R}$$

such that

i)
$$[g_1, g_2] = g_3^n$$
,

ii)
$$\nu(1)(g_1) = g_1^{m_{11}} g_2^{m_{12}} g_3^k$$
, $\nu(1)(g_2) = g_1^{m_{21}} g_2^{m_{22}} g_3^l$, with $l, k \in \mathbb{Z}$.

Let Θ_n be the discrete sugroup of \mathbb{H}_3 defined by

$$\Theta_n = \left\{ \left(\begin{array}{ccc} 1 & a & \frac{c}{n} \\ 0 & 1 & b \\ 0 & 0 & 1 \end{array} \right), \quad a, b, c \in \mathbb{Z} \right\},$$

then $\mathbb{Z} \ltimes_{\tilde{\nu}} (\Gamma_n \times \mathbb{Z}^2)$ is a uniform discrete subgroup of the solvable Lie group G_4 .

By construction any quotient $\Gamma_i \setminus G_i$ is a \mathbb{T}^2 -bundle over the complex surface $\Theta \setminus L$.

If X is either a Hopf surface or a primary Kodaira surface or a properly elliptic surface with odd first Betti number, then the \mathbb{T}^2 -bundle M is a product of two 3-dimensional manifolds. Then the interesting cases are the remaining.

The \mathbb{T}^2 -bundle $\Gamma_4 \setminus G_4$ over the secondary Kodaira surface satisfies the equation

(22)
$$0 = 2i\partial\overline{\partial}F = \frac{\alpha'}{4}\mathrm{tr}\,(R^B \wedge R^B),$$

where we denote by R^B the curvature of the Bismut connection and by F the fundamental 2-form. This equation is of interest in the context of superstring theory, since it is a particular case of an equation in the Strominger's system considered in [39] for Hermitian manifolds of complex dimension three:

(23)
$$dH = 2i\partial\overline{\partial}F = \frac{\alpha'}{4} \left[\operatorname{tr} \left(R \wedge R \right) - \operatorname{tr} \left(F_A \wedge F_A \right) \right],$$

where A is a Hermitian-Einstein connection on an auxiliary semi-stable bundle on M, ∇ is a metric connection with skew-symmetric torsion H on M, F_A and R denote respectively the curvature of the two connections A and ∇ .

By [39, 35] the Hermitian manifold has to be conformally balanced with a holomorphic (3, 0)-form.

The first solutions of the complete Strominger's system on non-Kähler manifolds were constructed by J. X. Fu and S. -T. Yau (see [14]).

The locally conformally balanced strong KT manifold $\Gamma_4 \backslash G_4$ gives a solution in dimension 6 of the equation (23) with $F_A = 0$. Indeed, for the Lie algebra \mathfrak{g}_4 we have that $JdF = -e^{123}$. Thus, the non-zero torsion 2-forms and connections 1-forms of the Bismut connection ∇^B are

$$\begin{split} \tau^1 &= e^{23} \,, \quad \tau^2 = -e^{13} \,, \quad \tau^3 = e^{12} \,, \\ \omega_2^1 &= -e^3 + e^4 = -\omega_1^2 \,, \quad \omega_6^5 = -\frac{\pi}{2} e^4 = -\omega_5^6 \,. \end{split}$$

Therefore, we get that the only non-zero curvature forms for ∇^B are given by

$$\Omega_2^1 = -e^{12} = -\Omega_1^2$$

and consequently

$$\operatorname{tr}\left(\Omega\wedge\Omega\right)=\sum_{i,j}\Omega^{i}_{j}\wedge\Omega^{i}_{j}=0$$

We will show now that $\Gamma_4 \setminus G_4$ does not admit any non-vanishing holomorphic (3, 0)form. By a straightforward computation, we have that the non-vanishing curvature forms for the Chern connection ∇^C are

$$\tilde{\Omega}_2^1 = -\tilde{\Omega}_4^3 = \frac{1}{2}e^{12}\,, \qquad \tilde{\Omega}_6^5 = -e^{12}\,.$$

Denote by \hat{R}^i_{jhk} the curvature components defined by

$$\tilde{\Omega}^i_j = \sum_{h,k=1}^6 \tilde{R}^i_{jhk} e^h \wedge e^k,$$

and consider the curvature operator $\tilde{R}(X,Y)$ of the Chern connection defined by

$$\tilde{R}(X,Y)Z = \sum_{i,j,h,k=1}^{o} \tilde{R}^i_{jhk}(e^h \wedge e^k)(X,Y) e^j(Z) e_i.$$

By [28, Lemma 2, p. 151], if $(\Gamma_4 \setminus G_4, J, g)$ admits a non-zero holomorphic (3,0)form, then the traces of the two operators $\tilde{R}(X, Y)$ and $J \circ \tilde{R}(X, Y)$ must vanish, but, by a direct computation, we have that the

$$\operatorname{tr} (J \circ R)(e_1, e_2) = -\pi.$$

Although in physics the most preferred connection for the anomaly cancellation condition is the non-Hermitian connection with skew-symmetric torsion equal to the opposite of the torsion of ∇^B , also the case of a Hermitian connection may be interesting. Indeed, we obtain an example, which can be interpreted as a "locally conformal solution" of the Strominger's system, since locally there is a holomorphic (3,0)-form and conformal change to a balanced metric, plus the anomaly cancellation.

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