# SELF-DUAL PROJECTIVE TORIC VARIETIES 

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#### Abstract

Let $T$ be a torus over an algebraically closed field $\mathbb{k}$ of characteristic 0 , and consider a projective $T$-module $\mathbb{P}(V)$. We determine when a projective toric subvariety $X \subset \mathbb{P}(V)$ is self-dual, in terms of the configuration of weights of $V$.


## 1. Introduction

The notion of duality of projective varieties, which appears in various branches of mathematics, has been a subject of study since the beginnings of algebraic geometry [12, 17. Given an embedded projective variety $X \subset \mathbb{P}(V)$, its dual variety $X^{*}$ is the closure in the dual projective space $\mathbb{P}\left(V^{\vee}\right)$ of the hyperplanes intersecting the regular points of $X$ non transversally.

A projective variety $X$ is self-dual if it is isomorphic to its dual $X^{*}$ as embedded projective varieties. The expected codimension of the dual variety is one. If this is not the case, $X$ is said to be defective. Self-dual varieties other than hypersurfaces are defective varieties with "maximal" defect.

Let $\mathbb{k}$ be an algebraically closed field of characteristic 0 . Let $T$ be an algebraic torus over $\mathbb{k}$ and $V$ a finite dimensional $T$-module. In this paper we characterize self-dual projective toric varieties $X \subset \mathbb{P}(V)$ equivariantly embedded, in terms of the combinatorics of the associated configuration of weights $A$ (cf. Theorems 4.4 and 4.16 ) and in terms of the interaction of the space of relations of these weights with the torus orbits (cf. Theorems 3.2 and 3.8). In particular, we show that $X$ is self-dual if and only if $\operatorname{dim} X=\operatorname{dim} X^{*}$ and the smallest linear subspaces containing $X=X_{A}$ and $X^{*}$ have the same dimension, see Theorems 3.3 and 3.7.

Given a basis of eigenvectors of $V$ and the configuration of weights of the torus action on $V$, it is not difficult to check the equality of the dimensions of $X$ and its dual (for instance, by means of the combinatorial characterization of the tropicalization given in [8]). But the complete classification of defective projective toric varieties in an equivariant embedding is open in full generality and involves a complicated combinatorial problem. For smooth toric varieties this characterization is obtained in [9]; the case of $\mathbb{Q}$-factorial toric varieties is studied in 3]. For non necessarily normal projective toric varieties of codimension two, a characterization is given in (7]. This has been extended for codimensions three and four in 6].

For smooth projective varieties, a full list of self-dual varieties is known 10, 11, 17. This list is indeed short and reduces in the case of toric varieties to hypersurfaces or Segre embeddings of $\mathbb{P}^{1} \times \mathbb{P}^{m-1}$, for any $m \geq 2$, under the assumption

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that $\operatorname{dim} X \leq \frac{2 \operatorname{dim} \mathbb{P}(V)}{3}$. This was expected to be the whole classification under the validity of Hartshorne's conjecture [10]. We prove that this is indeed the whole list of self-dual smooth projective toric varieties in Theorem 5.8 .

There exist some classical examples of self-dual non smooth varieties, as the quartic Kummer surface. Popov and Tevelev gave new families of non smooth self-dual varieties that come from actions of isotropy groups of complex symmetric spaces on the projectivized nilpotent varieties of isotropy modules ([14, [15]). As a consequence of Theorem 4.4, it is easy to construct new families of self-dual projective toric varieties in terms of the Gale dual configuration (see Definition 4.1).

A big class of self-dual toric varieties are the toric varieties associated to Lawrence configurations (see Definition 5.1), which contain the configurations associated to the Segre embeddings. Lawrence constructions are well known in the domain of geometric combinatorics, where they are one of the prominent tools to visualize the geometry of higher dimensional polytopes (see [19, Chapter 6]); the commutative algebraic properties of the associated toric ideals are studied in [2]. We show in Section 5 other non Lawrence concrete examples for any dimension bigger than 2 and any codimension bigger than 1 .

We also introduce the notion of strongly self-dual toric varieties (see Definition 6.1), which is not only related to the geometry of the configuration of weights but also to number theoretic aspects. This concept is useful for the study of the existence of rational multivariate hypergeometric functions [13, 4].

In Section 2 we gather some preliminary results about embedded projective toric varieties and duality of projective varieties. In Section 3 we characterize self-dual projective toric varieties in terms of the geometry of the action of the torus and we give precise assumptions under which self-dual projective varieties are precisely those with maximal defect. In Section 4 we give two (equivalent) combinatorial characterizations of self-duality. In Section 5 we collect several new examples of self-dual (non smooth) projective toric varieties. Finally, in Section 6 we study strongly self-dual toric varieties.

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## 2. Preliminaries

In this section we collect some well known results and useful observations on projective toric varieties and duality of projective varieties.
2.1. Actions of tori. Let $T$ be an algebraic torus over an algebraically closed field $\mathbb{k}$ of characteristic 0 . We denote by $\mathcal{X}(T)$ the lattice of characters of $T$; recall that $\mathbb{k}[T]=\bigoplus_{\lambda \in \mathcal{X}(T)} \mathbb{k} \lambda$. Any finite dimensional rational $T$-module $V, \operatorname{dim} V=n$, decomposes as a direct sum of irreducible representations

$$
\begin{equation*}
V \cong \bigoplus_{i=1}^{n} \mathbb{k} v_{i} \tag{1}
\end{equation*}
$$

with $t \cdot v_{i}=\lambda_{i}(t) v_{i}, \lambda_{i} \in \mathcal{X}(T)$, for all $t \in T$.
The action of $T$ on $V$ canonically induces an action $T \times \mathbb{P}(V) \rightarrow \mathbb{P}(V)$ on the associated projective space, given by $t \cdot[v]=[t \cdot v]$, where $[v] \in \mathbb{P}(V)$ denotes the
class of $v \in V \backslash\{0\}$. Recall that an irreducible $T$-variety $X$ is called toric if there exists $x_{0} \in X$ such that the orbit $\mathcal{O}\left(x_{0}\right)$ is open in $X$.

Let $A=\left\{\lambda_{1}, \ldots, \lambda_{n}\right\}$ (which may contain repeated elements) be the associated set of weights of a finite dimensional $T$-module $V$ - we call $A$ the configuration of weights associated to the $T$-module $V$. To any basis $\mathcal{B}=\left\{v_{1}, \ldots, v_{n}\right\} \subset V$ of eigenvectors we can associate a projective toric variety by

$$
X_{V, \mathcal{B}}=\overline{\mathcal{O}\left(\left[\sum v_{i}\right]\right)} \subset \mathbb{P}(V)
$$

Denote by $\mathbb{T}^{n-1}=\left\{\sum p_{i} v_{i} \in \mathbb{P}(V): \prod p_{i} \neq 0\right\}$. The dense orbit $\mathcal{O}\left(\left[\sum v_{i}\right]\right)$ in $X_{V, \mathcal{B}}$ coincides with the intersection $X_{V, \mathcal{B}} \cap \mathbb{T}^{n-1}$. Observe that since $\operatorname{dim} X_{V, \mathcal{B}}$ is equal to $\operatorname{dim} \mathcal{O}\left(\left[\sum v_{i}\right]\right)$, it follows that $\operatorname{dim} X_{V, \mathcal{B}}$ is maximal among the dimensions of the toric subvarieties of $\mathbb{P}(V)$ - i.e. those of the form $\overline{\mathcal{O}([v])}$ for some $[v] \in \mathbb{P}(V)$.

Based on the decomposition (1), in [12, Proposition II.5.1.5] it is proved that any projective toric variety in an equivariant embedding is of type $X_{V, \mathcal{B}}$ for some $T$-module $V$ and a basis of eigenvectors $\mathcal{B}=\left\{v_{1}, \ldots, v_{n}\right\}$ of $V$, in the following sense. Let $U$ be a $T$-module and $Y \subset \mathbb{P}(U)$ a toric subvariety; then there exists $A=$ $\left\{\lambda_{1}, \ldots, \lambda_{n}\right\} \subset \mathcal{X}(T)$ (with possible repetitions) and a $T$-equivariant linear injection $f: W:=\bigoplus_{i=1}^{n} \mathbb{k} w_{i} \hookrightarrow U, t \cdot w_{i}=\lambda_{i}(t) w_{i}$, such that the induced equivariant morphism $\widehat{f}: \mathbb{P}(W) \hookrightarrow \mathbb{P}(U)$ gives an isomorphism $X_{W, \mathcal{B}} \cong Y$. Moreover, let $W^{\prime}$ be another $T$-module, $\mathcal{B}^{\prime}=\left\{w_{1}^{\prime}, \ldots, w_{n}^{\prime}\right\} \subset W^{\prime}$ a basis of eigenvectors of $W^{\prime}$ such that $t \cdot w_{i}^{\prime}=\lambda_{i}(t) w_{i}^{\prime}$, and consider $f \in \operatorname{Hom}_{T}\left(W, W^{\prime}\right)$, given by $f\left(w_{i}\right)=w_{i}^{\prime}$. Clearly, $f$ is an isomorphism of $T$-modules, and its induced morphism $\widehat{f}: \mathbb{P}(W) \rightarrow \mathbb{P}\left(W^{\prime}\right)$ is an isomorphism such that $\widehat{f}\left(X_{W, \mathcal{B}}\right)=X_{W^{\prime}, \mathcal{B}^{\prime}}$.

In view of the preceding remark, the following notation makes sense.
Definition 2.1. The projective toric variety $X_{A}$ associated to the configuration of weights $A$ is defined as

$$
X_{A}=X_{V, \mathcal{B}}=\overline{\mathcal{O}\left(\left[\sum v_{i}\right]\right)} \subset \mathbb{P}(V)
$$

where $V$ is a $T$-module with $A$ as associated configuration of weights.
We can make a series of reductions on $A$ and $T$, as in 8 . First, the following easy lemma allows to reduce our problem to the case of a faithful representation.

Lemma 2.2. Given a $T$-module $V$ of finite dimension and $A$ the associated configuration of weights, consider the torus $T^{\prime}=\operatorname{Hom}_{\mathbb{Z}}\left(\langle A\rangle_{\mathbb{Z}}, \mathbb{k}^{*}\right)$, where $\langle A\rangle_{\mathbb{Z}} \subset \mathcal{X}(T)$ denotes the $\mathbb{Z}$-submodule generated by $A$. The representation of $T$ in $\mathrm{GL}(V)$ induces a faithful representation $T^{\prime} \rightarrow \mathrm{GL}(V)$ which has the same set theoretical orbits in $V$.

We can then replace $T$ by the torus $T^{\prime}$. It is easy to show that this is equivalent to the fact that $\langle A\rangle_{\mathbb{Z}}=\mathcal{X}(T)$, which we will assume from now on without loss of generality.

Next, we enlarge the torus without affecting the action on $\mathbb{P}(V)$; this will allow us to easily translate affine relations to linear relations on the configuration of weights. If we let the algebraic torus $\mathbb{k}^{*} \times T$ act on $V$ by $\left(t_{0}, t\right) \cdot v=t_{0}(t \cdot v)$, then the actions $T \times \mathbb{P}(V) \rightarrow \mathbb{P}(V)$ and $\left(\mathbb{k}^{*} \times T\right) \times \mathbb{P}(V) \rightarrow \mathbb{P}(V)$ have the same set theoretical orbits. More in general, let $\lambda \in \mathcal{X}(T)$ and $A^{\prime}=\left\{\lambda+\lambda_{1}, \ldots, \lambda+\lambda_{n}\right\}$. Consider the $T$-action on $V$ given by $t \cdot{ }_{\lambda} v_{i}=\left(\lambda+\lambda_{i}\right)(t) v_{i}$. The actions $\cdot$ and $\cdot \lambda$ coincide on $\mathbb{P}(V)$, and the corresponding variety $X_{A^{\prime}}$ coincides with $X_{A}$. Hence, we can assume that there is
a splitting of $T=\mathbb{k}^{*} \times S$ in such a way that $\left(t_{0}, s\right) \cdot v=t_{0}(s \cdot v)$ for all $v \in V$, $t_{0} \in \mathbb{k}^{*}$ and $s \in S$.

In fact, the previous reductions are comprised in the following more general setting:

Lemma 2.3 ([12, Proposition II.5.1.2]). Consider $T, T^{\prime}$ two tori and two finite configurations of $n$ weights $A=\left\{\lambda_{1}, \ldots, \lambda_{n}\right\} \subset \mathcal{X}(T), A^{\prime}=\left\{\lambda_{1}^{\prime}, \ldots, \lambda_{n}^{\prime}\right\} \subset \mathcal{X}\left(T^{\prime}\right)$. Assume that there exists a $\mathbb{Q}$-affine transformation $\psi: \mathcal{X}(T) \otimes \mathbb{Q} \rightarrow \mathcal{X}\left(T^{\prime}\right) \otimes \mathbb{Q}$ such that $\psi\left(\lambda_{i}\right)=\lambda_{i}^{\prime}$ for all $i=1, \ldots, n$. Then $X_{A}=X_{A^{\prime}}$.
Remark 2.4. (1) The dimension of the projective toric variety $X_{A}$ equals the dimension of the affine span of $A$.
(2) Note that if $A=\left\{\lambda_{1}, \ldots, \lambda_{n}\right\}$ is contained in a hyperplane off the origin, then $X_{A}=\mathbb{P}(V)$ precisely when $\operatorname{dim} T=n$ and the elements in $A$ are a basis of $\mathcal{X}(T)$.
(3) If we denote by $d$ the dimension of the affine span of $A$, then $X_{A}$ is a hypersurface if and only if $n=d+2$. In this situation, either $A$ coincides with the set of vertices of its convex hull $\operatorname{Conv}(A) \subset \mathcal{X}(T) \otimes \mathbb{R}$, or $\operatorname{Conv}(A)$ contains only one element $\lambda \in A$ in its relative interior, and $A \backslash\{\lambda\}$ is the set of vertices.

We end this paragraph by recalling some basic facts about the geometric structure of a toric variety $X_{A}$.

Lemma $2.5(\underline{5})$. Let $A=\left\{\lambda_{1}, \ldots, \lambda_{n}\right\} \subset \mathcal{X}(T)$ be a configuration, where $\left\{\lambda_{1}, \ldots, \lambda_{s}\right\}$ is the set of vertices of $\operatorname{Conv}(A)$. Set $X_{i}=\operatorname{Spec}\left(\mathbb{k}\left[\mathbb{Z}^{+}\left\langle\left(\lambda_{j}-\lambda_{i}\right): \lambda_{j} \in A\right\rangle\right]\right)$, $i=1, \ldots, s$. Then $X_{i}$ is an affine toric $T$-variety, and there exist $T$-equivariant open immersions $\varphi_{i}: X_{i} \hookrightarrow X_{A}$, in such a way that

$$
X_{A}=\cup_{i=1}^{s} \varphi_{i}\left(X_{i}\right)=\cup_{i=1}^{s} \operatorname{Spec}\left(\mathbb{k}\left[\mathbb{Z}^{+}\left\langle\lambda_{j}-\lambda_{i}: \lambda_{j} \in A\right\rangle\right]\right)
$$

In particular, $X_{A}$ is a normal variety if and only if $\mathbb{Z}^{+}\left\langle\lambda_{j}-\lambda_{i}: \lambda_{j} \in A\right\rangle=$ $\left(\mathbb{R}^{+}\left\langle\lambda_{j}-\lambda_{i}: \lambda_{j} \in A\right\rangle\right) \cap \mathcal{X}(T)$ for all $i=1, \ldots, s$.

Moreover, $X_{A}$ is a smooth variety if for all $i=1, \ldots, s$, there are exactly $\operatorname{dim} X_{A}$ edges of $\operatorname{Conv}(A)$ from $\lambda_{i}$, and the subset $\left\{\lambda_{j_{h}}-\lambda_{i}: h=1, \ldots, \operatorname{dim} X_{A}\right\}$ is a basis of $\mathcal{X}(T)$, where $\lambda_{j_{h}}$ is the "first" point on an edge from $\lambda_{i}$.

Proof. See for example [5, Appendix to Chapter 3].
2.2. Configurations in lattices, pyramids and projective joins. Let $M^{\prime}$ be a lattice of rank $d-1$. We let $M=\mathbb{Z} \times M^{\prime}$ and consider the $\mathbb{k}$-vector space $M_{\mathbb{k}}=M \otimes_{\mathbb{Z}} \mathbb{k}$. Recall that given a basis $\left\{\mu_{1}, \ldots, \mu_{d}\right\}$ of $M$, we can identify $M$ with $\mathbb{Z}^{d}$ and $M_{\mathrm{k}}$ with $\mathbb{K}^{d}$.

Definition 2.6. A lattice configuration $A=\left\{\lambda_{1}, \ldots, \lambda_{n}\right\} \subset M$ is a finite sequence of lattice points. We say that a configuration $A$ is regular if it is contained in a hyperplane off the origin.

Remark 2.7. Let $T$ be an algebraic torus, and let $A=\left\{\lambda_{1}, \ldots, \lambda_{n}\right\} \subset \mathcal{X}(T)$ be a configuration of weights. Then the following are equivalent:
(1) the configuration $A$ is regular;
(2) up to affine isomorphism, $A$ has the form $\lambda_{i}=\left(1, \lambda_{i}^{\prime}\right)$ for all $i=1, \ldots, n$;
(3) there exists a splitting $T=\mathbb{k}^{*} \times S$, such that under the identification $\mathcal{X}(T)=$ $\mathbb{Z} \times \mathcal{X}(S)$, the weights of $A$ are of the form $\lambda_{i}=\left(1, \lambda_{i}^{\prime}\right), i=1, \ldots, n$. See also the reductions made before Lemma 2.3 .

Definition 2.8. We denote by $\mathcal{R}_{A} \subset \mathbb{Z}^{n}$ the lattice of affine relations among the elements of $A$, that is $\left(a_{1}, \ldots, a_{n}\right)$ belongs to $\mathcal{R}_{A}$ if and only if $\sum_{i} a_{i} \lambda_{i}=0$ and $\sum_{i} a_{i}=0$.

If $A$ is regular, then $\mathcal{R}_{A}$ coincides with the lattice of linear relations among the elements of $A$. Note that these (affine or linear) relations among the elements of $A$ can be identified with the affine relations among the elements of the configuration $\left\{\lambda_{1}^{\prime}, \ldots, \lambda_{n}^{\prime}\right\} \subset M^{\prime}$. Thus, given any configuration $\left\{\lambda_{1}^{\prime}, \ldots, \lambda_{n}^{\prime}\right\} \subset M^{\prime}$, we can embed it in $M=\mathbb{Z} \times M^{\prime}$ via $\lambda^{\prime} \mapsto\left(1, \lambda^{\prime}\right)$ so that affine dependencies are translated to linear dependencies. In fact, the map $\lambda^{\prime} \mapsto\left(1, \lambda^{\prime}\right)$ is an injective affine linear map. More in general, we have the following definition.
Definition 2.9. We say that two configurations $A_{i} \subset \mathcal{X}\left(T_{i}\right), i=1,2$, are affinely equivalent if there exists an affine linear map $\varphi: \mathcal{X}\left(T_{1}\right) \otimes \mathbb{R} \rightarrow \mathcal{X}\left(T_{2}\right) \otimes \mathbb{R}$ (defined over $\mathbb{Q}$ ) such that $\varphi$ sends $A_{1}$ bijectively to $A_{2}$ (in particular, $\varphi$ defines an injective map from the affine span of $A_{1}$ to the affine span of $A_{2}$ ).

So, if $A_{1}$ and $A_{2}$ are affinely equivalent, they have the same cardinal and moreover, $\mathcal{R}_{A_{1}}=\mathcal{R}_{A_{2}}$. Any property of a configuration $A$ shared by all its affinely equivalent configurations is called an affine invariant of $A$. In this terminology, Lemma 2.3 asserts that the projective toric variety $X_{A}$ is an affine invariant of the configuration $A$.
Definition 2.10. We say that $A=\left\{\lambda_{1}, \ldots, \lambda_{n}\right\} \subset M$ is a pyramid (or a pyramidal configuration) if there exists an affine hyperplane $H$ such that $\#\left\{i / \lambda_{i} \notin H\right\}=1$, i.e. if all points in $A$ but one lie in $H$, or equivalently, if there exist an index $i_{0} \in\{1, \ldots, n\}$ and an affine linear function $\ell: M_{\mathbb{k}} \rightarrow \mathbb{k}$ such that $\ell\left(\lambda_{i}\right)=0$ for all $i \neq i_{0}$ and $\ell\left(\lambda_{i_{0}}\right)=1$.

More precisely, we say that $A$ is a $k$-pyramidal configuration if, after reordering, there exists a splitting of the lattice as a direct sum of lattices $M=M_{1} \oplus M_{2}$, with $A_{1}=\left\{\lambda_{1}, \ldots, \lambda_{r}\right\}$ a basis of $M_{1}$ and $A_{2}=\left\{\lambda_{r+1}, \ldots, \lambda_{n}\right\} \subset M_{2}$, with $A_{2}$ not a pyramidal configuration of $M_{2} \otimes_{\mathbb{Z}} \mathbb{k}$. In particular, the 0-pyramidal configurations are the non pyramidal configurations.

Remark 2.11. Being a pyramid is clearly an affine invariant of a configuration. It is straightforward to check that $A$ is a non pyramidal configuration if and only if there exists a relation $\left(p_{1}, \ldots, p_{n}\right) \in \mathcal{R}_{A}$ with $\prod_{i} p_{i} \neq 0$, i.e. if $\mathcal{R}_{A}$ is not contained in a coordinate hyperplane.
Definition 2.12. Let $V_{1}, V_{2}$ two $\mathbb{k}$-vector spaces of respective dimensions $h_{1}+$ $1, h_{2}+1$ and $X \subset \mathbb{P}\left(V_{1}\right), Y \subset \mathbb{P}\left(V_{2}\right)$ two projective varieties. Recall that the join of $X$ and $Y$ is the projective variety

$$
\mathrm{J}_{h_{1}, h_{2}}(X, Y)=\overline{\{[x: y]:[x] \in X,[y] \in Y\}} \subset \mathbb{P}\left(V_{1} \times V_{2}\right)
$$

that is, the cone over the join $\mathrm{J}_{h_{1}, h_{2}}(X, Y)$ is the product of the cones over $X$ and $Y$. We set

$$
\mathrm{J}_{h_{1}, h_{2}}(\emptyset, Y)=\{[\underbrace{0: \cdots: 0}_{h_{1}+1}: y] \in \mathbb{P}\left(V_{1} \times V_{2}\right),[y] \in Y\} \subset \mathbb{P}\left(V_{1} \times V_{2}\right) .
$$

We define analogously $\mathrm{J}_{h_{1}, h_{2}}(X, \emptyset)$.
We will denote $\mathbb{P}^{h}=\mathbb{P}\left(\mathbb{k}^{h+1}\right)$. Observe that for any $Y \subset \mathbb{P}^{h_{2}}, Y \cong \mathrm{~J}_{h_{1}, h_{2}}(\emptyset, Y) \subset$ $\mathbb{P}^{h_{1}+h_{2}+1}$ for any $h_{1} \in \mathbb{N}$. If $X$ and $Y$ are non empty, then $\operatorname{dim} \mathrm{J}_{h_{1}, h_{2}}(X, Y)=$ $\operatorname{dim} X+\operatorname{dim} Y+1$.

Remark that given $X_{i} \subset \mathbb{P}\left(V_{i}\right), \operatorname{dim} V_{i}=h_{i}+1, i=1,2,3$, then
$\mathrm{J}_{h_{1}+h_{2}+1, h_{3}}\left(\mathrm{~J}_{h_{1}, h_{2}}\left(X_{1}, X_{2}\right), X_{3}\right)=\mathrm{J}_{h_{1}, h_{2}+h_{3}+1}\left(X_{1}, \mathrm{~J}_{h_{2}, h_{3}}\left(X_{2}, X_{3}\right)\right) \subset \mathbb{P}\left(V_{1} \times V_{2} \times V_{3}\right)$.
We will denote this variety by $\mathrm{J}_{h_{1}, h_{2}, h_{3}}\left(X_{1}, X_{2}, X_{3}\right)$.
Given two projective toric varieties $X_{A_{1}}$ and $X_{A_{2}}$, then their join is also a toric variety:

Remark 2.13. (1) Let $T=S_{1} \times S_{2}$ be a splitting of $T$ as a product of tori, and $A_{1}=$ $\left\{\lambda_{1}, \ldots, \lambda_{k}\right\} \subset \mathcal{X}\left(S_{1}\right), A_{2}=\left\{\lambda_{k+1}, \ldots, \lambda_{n}\right\} \subset \mathcal{X}\left(S_{2}\right)$ two regular configurations.

Let $V_{1}=\bigoplus_{i=1}^{k} \mathbb{k} v_{i}, s_{1} \cdot v_{i}=\lambda_{i}\left(s_{1}\right) v_{i}$ for all $s_{1} \in S_{1}$, and $V_{2}=\bigoplus_{i=k+1}^{n} \mathbb{k} v_{i}$, $s_{2} \cdot v_{i}=\lambda_{i}\left(s_{2}\right) v_{i}$ for all $s_{2} \in S_{2}$. Then $V=V_{1} \times V_{2}$ is a $T$-module for the product action $\left(s_{1}, s_{2}\right) \cdot\left(w_{1}, w_{2}\right)=\left(s_{1} \cdot w_{1}, s_{2} \cdot w_{2}\right)$. Moreover, $V$ decomposes in simple submodules as $V=\bigoplus_{i=1}^{k} \mathbb{k}\left(v_{i}, 0\right) \oplus \bigoplus_{i=k+1}^{n} \mathbb{k}\left(0, v_{i}\right)$.

Consider the $S_{i}$-toric varieties $X_{A_{i}} \subset \mathbb{P}\left(V_{i}\right)(i=1,2)$ and let $A=A_{1} \times\{0\} \cup$ $\{0\} \times A_{2} \subset \mathcal{X}\left(S_{1}\right) \times \mathcal{X}\left(S_{2}\right)=\mathcal{X}(T)$. The projective toric variety associated to $A$ is then the join $X_{A}=\mathrm{J}_{k-1, n-k-1}\left(X_{A_{1}}, X_{A_{2}}\right)$.
(2) In the particular case when $A \subset M=M_{1} \oplus M_{2}$ is a $k$-pyramidal configuration with $A_{1} \subset M_{1}, A_{2} \subset M_{2}$ as in Definition 2.10, let $S_{1}=\operatorname{Hom}_{\mathbb{Z}}\left(M_{1}, \mathbb{k}^{*}\right)$, $S_{2}=\operatorname{Hom}_{\mathbb{Z}}\left(M_{2}, \mathbb{k}^{*}\right), T=S_{1} \times S_{2}$ and $V$ as above. We then have that $X_{A}=$ $\mathrm{J}_{k-1, n-k-1}\left(\mathbb{P}\left(V_{1}\right), X_{A_{2}}\right)$. That is, $X_{A}$ is the cone over $X_{A_{2}}$ with vertex $\mathbb{P}\left(V_{1}\right)$.

Next, we describe the toric varieties associated to configurations with repeated weights. Recall that a projective variety is called non degenerate if it is not contained in a proper linear subspace.

Lemma 2.14. Let $A=\left\{\lambda_{1}, \ldots, \lambda_{1}, \ldots, \lambda_{h}, \ldots, \lambda_{h}\right\} \subset \mathcal{X}(T)$ be a configuration of $n$ weights, with $\lambda_{i}$ appearing $k_{i}+1$ times and $\lambda_{i} \neq \lambda_{j}$ if $i \neq j$. If we set $k=$ $\sum_{i} k_{i}=n-h$, then the smallest linear subspace that contains $X_{A}$ has codimension $k$.

In particular, $X_{A}$ is a non degenerate variety if and only if the configuration $A$ has no repeated elements.

Proof. Let $\mathcal{B}=\left\{v_{1,1}, \ldots, v_{1, k_{1}+1}, \ldots, v_{h, 1}, \ldots, v_{h, k_{h}+1}\right\}$ be a basis of associated eigenvectors of $V$, with $t \cdot v_{i, j_{i}}=\lambda_{i}(t) v_{i, j_{i}}$ for all $i=1, \ldots, h, j_{i}=1, \ldots, k_{i}+1$. Consider a hyperplane $\Pi \subset \mathbb{P}\left(\bigoplus_{i=1}^{h}\left(\bigoplus_{j_{i}=1}^{k_{i}+1} \mathbb{k} v_{i, j_{i}}\right)\right)$ of equation

$$
\sum_{i, j_{1}, \ldots, j_{h}} c_{i, j_{i}} x_{i, j_{i}}=0
$$

where $x_{i, j_{i}}$ are the coordinates in the basis $\mathcal{B}$. Then $X_{A} \subset \Pi$ if and only if $[t$. $\left.\sum v_{i, j_{i}}\right] \subset \Pi$ for all $t \in T$. As $\left[t \cdot \sum v_{i, j_{i}}\right]=\left[\sum \lambda_{i}(t) v_{i, j_{i}}\right] \in \Pi$, this is equivalent to the equalities

$$
\sum_{i=1}^{h} \sum_{j_{i}=1}^{k_{i}+1} c_{i, j_{i}} \lambda_{i}(t)=0, t \in T
$$

Since $\left\{\lambda_{1}, \ldots, \lambda_{h}\right\}$ are different weights, we deduce that $\sum_{j_{i}=1}^{k_{i}+1} c_{i, j_{i}}=0$ for all $i=1, \ldots, h$. It follows that the maximum codimension of a subspace that contains $X_{A}$ is $\sum_{i=1}^{h} k_{i}=k$.

On the other hand, clearly

$$
X_{A} \subset H=\left\{\sum_{i=1}^{h} x_{i} \sum_{j_{i}=1}^{k_{i}+1} v_{i, j_{i}}: x_{i} \in \mathbb{k}\right\}
$$

where the subspace $H \subset \mathbb{P}(V)$ has codimension $k$.
Lemma 2.15. Let $A=\left\{\lambda_{1}, \ldots, \lambda_{1}, \ldots, \lambda_{h}, \ldots, \lambda_{h}\right\} \subset \mathcal{X}(T)$ be a configuration of $n$ weights, with $\lambda_{i}$ appearing $k_{i}+1$ times and $\lambda_{i} \neq \lambda_{j}$ if $i \neq j$. Set $k=\sum_{i} k_{i}=n-h$ and let

$$
V=\bigoplus_{i=1}^{h}\left(\bigoplus_{j_{i}=1}^{k_{i}+1} \mathbb{k} v_{i, j_{i}}\right)=\left(\bigoplus_{i=1}^{h}\left(\bigoplus_{j_{i}=1}^{k_{i}} \mathbb{k} v_{i, j_{i}}\right)\right) \oplus\left(\bigoplus_{i=1}^{h} \mathbb{k} v_{i, j_{k_{i}+1}}\right)
$$

with $t \cdot v_{i, j_{i}}=\lambda_{i}(t) v_{i, j_{i}}$ for all $t \in T, i=1, \ldots h, j_{i}=1, \ldots, k_{i}+1$.
Let $C=\left\{\lambda_{1}, \ldots, \lambda_{h}\right\}$ and consider $X_{C} \subset \mathbb{P}\left(\bigoplus_{i=1}^{h} \mathbb{k} v_{i, j_{k_{i}+1}}\right)$. Then $X_{A}$ is isomorphic to the cone $\mathrm{J}_{k-1, h-1}\left(\emptyset, X_{C}\right)$ over the non degenerate projective toric variety $X_{C}$.

Proof. Let $f: V \rightarrow V$ the linear isomorphism defined by

$$
\begin{array}{r}
f\left(\left(x_{i, j_{i}}\right)_{i=1, \ldots, h, j_{i}=1, \ldots, k_{i}},\left(x_{i, j_{k_{i}+1}}\right)_{i=1, \ldots, h}\right)= \\
\left(\left(x_{i, j_{i}}-x_{i, j_{k_{i}+1}}\right)_{i=1, \ldots, h, j_{i}=1, \ldots, k_{i}},\left(x_{i, j_{k_{i}+1}}\right)_{i=1, \ldots, h}\right)
\end{array}
$$

The associated projective map clearly sends $X_{A}$ to the join $\mathrm{J}_{k-1, h-1}\left(\emptyset, X_{C}\right)$.
In Proposition 2.17 below we combine Remark 2.13 and Lemmas 2.14 and 2.15 , in order to describe a projective toric variety as a cone over a non degenerate projective toric variety that is not a cone (that is, the associated configuration is non pyramidal).
Remark 2.16. Let $X \subset \mathbb{P}^{n-1}$ be a non linear irreducible projective variety. Let $H \subset \mathbb{P}^{n-1}$ be the minimal linear subspace containing $X$, and let $k$ be the codimension of $H$. Then $H \cong \mathbb{P}^{n-k-1}$ and if $X^{\prime}$ denotes the variety $X$ as a subvariety of $H$, then $X=\mathrm{J}_{k-1, n-k-1}\left(\emptyset, X^{\prime}\right)$. Since $X^{\prime}$ is non degenerate, it follows that there exists $Y \subset \mathbb{P}^{m-1}$ such that $X^{\prime}=\mathrm{J}_{h-1, m-1}\left(\mathbb{P}^{h-1}, Y\right)$, where $n-k-1=h+m-1$. Hence, we have an identification

$$
X=\mathrm{J}_{k-1, h-1, m-1}\left(\emptyset, \mathbb{P}^{h-1}, Y\right)
$$

In particular, $\operatorname{dim} X=h+\operatorname{dim} Y$.
Observe that $Y \subset \mathbb{P}^{m-1}$ is a non degenerate subvariety. Moreover, we can assume that $Y$ is not a cone. In this case, we will denote $X_{\text {nd }}=Y$. If moreover $X$ is an equivariantly embedded toric variety, then we can choose $X_{\mathrm{nd}}$ as $X_{C_{2}}$ in the following proposition.

When $X$ is linear, $X=H, m=1$ and $Y$ is empty.
Proposition 2.17. Let $A=\left\{\lambda_{1}, \ldots, \lambda_{1}, \ldots, \lambda_{h}, \ldots, \lambda_{h}\right\} \subset \mathcal{X}(T)$ be a configuration of $n$ weights, with $\lambda_{i}$ appearing $k_{i}+1$ times and $\lambda_{i} \neq \lambda_{j}$ if $i \neq j$. Set $k=$ $\sum_{i} k_{i}=n-h$ and assume that $C=\left\{\lambda_{1}, \ldots, \lambda_{h}\right\}$ is r-pyramidal. Then there exists a splitting $T=S_{1} \times S_{2}$ such that, after reordering of the elements in $C$, it holds that $C=C_{1} \cup C_{2}$, where $C_{1}=\left\{\lambda_{1}, \ldots, \lambda_{r}\right\}$ is a basis of $\mathcal{X}\left(S_{1}\right)$ and $C_{2}=$ $\left\{\lambda_{r+1}, \ldots, \lambda_{h}\right\} \subset \mathcal{X}\left(S_{2}\right)$ is a non pyramidal configuration, as in Definition 2.10. Moreover, we have that

$$
X_{A}=\mathrm{J}_{k-1, r-1, h-r-1}\left(\emptyset, \mathbb{P}^{r-1}, X_{C_{2}}\right)
$$

In the special case when $X_{A}$ is linear, $C_{2}$ is empty.

Proof. We set $V=\bigoplus_{i=1}^{h}\left(\bigoplus_{j_{i}=1}^{k_{i}+1} \mathbb{k} v_{i, j_{i}}\right)$, with $t \cdot v_{i, j_{i}}=\lambda_{i}(t) v_{i, j_{i}}$ for all $t \in T$, $i=1, \ldots h, j_{i}=1, \ldots, k_{i}+1$, and $w_{i}=v_{i, k_{i}+1}$.

Assume that $C$ is a $r$-pyramidal configuration, and let $X_{C} \subset \mathbb{P}\left(\bigoplus_{i=1}^{h} \mathbb{k} w_{i}\right)$. Then, there exists a splitting $T=S_{1} \times S_{2}$ such that, after reordering of $C$, $C_{1}=\left\{\lambda_{1}, \ldots, \lambda_{r}\right\}$ is a basis of $\mathcal{X}\left(S_{1}\right)$ and $C_{2}=\left\{\lambda_{r+1}, \ldots, \lambda_{h}\right\} \subset \mathcal{X}\left(S_{2}\right)$ is a non pyramidal configuration. Hence,

$$
X_{C}=\mathrm{J}_{r-1, h-r-1}\left(X_{\mathrm{id}_{r}}, X_{C_{2}}\right)=\mathrm{J}_{r-1, h-r-1}\left(\mathbb{P}\left(\oplus_{i=1}^{r} \mathbb{k} v_{i, k_{i}+1}\right), X_{C_{2}}\right)
$$

By Lemma 2.15, we can assume that $X_{A}=\mathrm{J}_{k-1, h-1}\left(\emptyset, X_{C}\right)$, and so

$$
\begin{aligned}
X_{A}= & \mathrm{J}_{k-1, h-1}\left(\emptyset, \mathrm{~J}_{r-1, h-r-1}\left(X_{\mathrm{id}_{r}}, X_{C_{2}}\right)\right)= \\
& \mathrm{J}_{k-1, h-1}\left(\emptyset, \mathrm{~J}_{r-1, h-r-1}\left(\mathbb{P}\left(\oplus_{i=1}^{r} \mathbb{k} v_{i, k_{i}+1}\right), X_{C_{2}}\right)\right)= \\
& \mathrm{J}_{k-1, r-1, h-r-1}\left(\emptyset, \mathbb{P}\left(\oplus_{i=1}^{r} \mathbb{k} v_{i, k_{i}+1}\right), X_{C_{2}}\right),
\end{aligned}
$$

as claimed.
2.3. Dual of a projective toric variety. We recall the classical notion of the dual variety of a projective variety.

Definition 2.18. Let $V$ be a $\mathbb{k}$-vector space of finite dimension and denote by $V^{\vee}$ its dual $\mathbb{k}$-vector space. Let $X \subset \mathbb{P}(V)$ be an irreducible projective variety. The dual variety of $X$ is defined as the closure of the hyperplanes intersecting the regular part $X_{\text {reg }}$ of $X$ non transversally:

$$
X^{*}=\overline{\left\{[f] \in \mathbb{P}\left(V^{\vee}\right): \exists x \in X_{r e g},\left.f\right|_{T_{x} X} \equiv 0\right\}} \subset \mathbb{P}\left(V^{\vee}\right)
$$

As usual, $T_{x} X$ denotes the embedded tangent space of $X$ at $x \in X_{\text {reg }}$.
Note that $\mathbb{P}(V)^{*}=\emptyset$. We set by convention, $\emptyset^{*}=\mathbb{P}\left(V^{\vee}\right)$.
Self-duality is not an intrinsic property, it depends on the projective embedding. It can be proved that $X^{*}$ is an irreducible projective variety and that $\left(X^{*}\right)^{*}=X$ (see for example [12]).

For a generic variety $X \subset \mathbb{P}(V), \operatorname{codim} X^{*}=1$. If $\operatorname{codim} X^{*} \neq 1$, it is said that $X$ has defect codim $X^{*}-1$.

Definition 2.19. An irreducible projective variety $X \subset \mathbb{P}(V)$ is called self-dual if $X$ is isomorphic to $X^{*}$ as embedded projective varieties, that is if there exists a (necessarily linear) isomorphism $\varphi: \mathbb{P}(V) \rightarrow \mathbb{P}\left(V^{\vee}\right)$ such that $\varphi(X)=X^{*}$.

A self-dual projective variety $X \subset \mathbb{P}^{n-1}$ of dimension $d-1<n-1$ (i.e., which is not a hypersurface) has positive defect $n-d-1$. The defect of the whole projective space $\mathbb{P}^{n-1}$ is $n-1$.

Remark 2.20. Recall that given a basis $\mathcal{B}=\left\{v_{1}, \ldots, v_{n}\right\}$ of $V$, we can identify $\mathbb{P}(V)$ with $\mathbb{P}\left(V^{\vee}\right)$ by means of $v_{i} \nVdash v_{i}^{\vee}$, where $\left\{v_{1}^{\vee}, \ldots, v_{n}^{\vee}\right\}$ is the dual basis of $\mathcal{B}$. Then, via the choice of a basis of $V$, we can look at the dual variety inside the same projective space. Self-duality can be reformulated as follows: $X \subset \mathbb{P}(V)$ is self-dual if there exists $\varphi \in \operatorname{Aut}(\mathbb{P}(V))$ such that $\varphi(X)=X^{*}$.

Let $V$ be a $T$-module of finite dimension $n$ over a $d$-dimensional torus $T$ and let $A$ be the associated configuration of weights. In view of the considerations of the preceding subsections, we assume from now on and without loss of generality, that $A=\left\{\lambda_{1}, \ldots, \lambda_{n}\right\} \subset \mathcal{X}(T)$ is a regular configuration, possibly with repeated elements, such that $\langle A\rangle_{\mathbb{Z}}=\mathcal{X}(T)$.

The regularity of $A$ implies in particular the existence of a splitting $T=\mathbb{k}^{*} \times S$ as in Remark 2.7. Then, $X_{A}$ is a $(d-1)$-dimensional subvariety of the $(n-1)$ dimensional projective space $\mathbb{P}(V)$ and the lattice $\mathcal{R}_{A}$ has rank $n-d$.

The dual variety $X_{A}^{*}$ has the following interpretation. For $[\xi] \in \mathbb{P}\left(V^{\vee}\right)$, let $f_{\xi} \in \mathbb{k}[T], f_{\xi}(t)=\xi\left(t \cdot \sum v_{i}\right) \in \mathbb{k}[T]$. Then $X_{A}^{*}$ is obtained as the closure of the set of those $[\xi] \in \mathbb{P}\left(V^{\vee}\right)$ such that there exists $t \in T$ with $f_{\xi}(t)=\frac{\partial f_{\xi}}{\partial t_{i}}(t)=0$ for all $i=1, \ldots, n$.

$$
X_{A}^{*}=\overline{\left\{\xi \in \mathbb{P}\left(V^{\vee}\right): \exists t \in T, f_{A}(t)=\frac{\partial f_{\xi}}{\partial t_{1}}(t)=\frac{\partial f_{\xi}}{\partial t_{2}}(t)=\cdots=\frac{\partial f_{\xi}}{\partial t_{d}}(t)=0\right\}}
$$

In [8] a rational parameterization of the dual variety $X_{A}^{*}$ was obtained. We adapt this result to our notations. As before, $\mathcal{B}=\left\{v_{1}, \ldots, v_{n}\right\}$ is a basis of eigenvectors, $t \cdot v_{i}=\lambda_{i}(t) v_{i}$, and $\mathcal{B}_{A}=\left\{u_{1}, \ldots, u_{n-d}\right\}$ is a basis of $\mathcal{R}_{A}$. We denote by $\mathcal{R}_{A, k}$ the $(n-d)$-dimensional $\mathbb{k}$-vector space $\mathcal{R}_{A} \otimes_{\mathbb{Z}} \mathbb{k}$ and we identify $\mathbb{P}(V)$ with $\mathbb{P}\left(V^{\vee}\right)$ by means of the chosen basis $\mathcal{B}$ of eigenvectors (and its dual basis) as in Remark 2.20

Proposition 2.21 ( 8 , Proposition 4.1]). Let $T=\mathbb{k}^{*} \times S, V, A, \mathcal{B}, \mathcal{B}_{A}$ as before. Then the mapping $\mathbb{P}\left(\mathcal{R}_{A, \mathbb{k}}\right) \times S \rightarrow \mathbb{P}(V)$ defined by

$$
\left(\left[a_{1}: \cdots: a_{n}\right], s\right) \mapsto s \cdot\left[\sum a_{i} v_{i}\right]
$$

has image dense in $X_{A}^{*}$. That is, the morphism

$$
\left(\mathbb{k}^{*}\right)^{n-d} \times T \rightarrow \mathbb{P}(V), \quad(c, t) \mapsto t \cdot\left[\sum c_{i} u_{i}\right]
$$

is a rational parameterization of $X_{A}^{*}$, and

$$
X_{A}^{*}=\overline{\bigcup_{p \in \mathbb{P}\left(\mathcal{R}_{A, k}\right)} \mathcal{O}(p)}=\overline{T \cdot \mathbb{P}\left(\mathcal{R}_{A, k}\right)} .
$$

This last equality, which expresses the dual variety as the closure of the union of the torus orbits of all the classes in the vector space of relations of the configuration $A$, is the starting point of our classification of self-dual projective toric varieties, which we describe in the sequel.

## 3. Characterization of self-duality in terms of orbits

Let $T$ be a torus of dimension $d$ and $V$ a rational $T$-module of dimension $n$ with associated configuration of weights $A=\left\{\lambda_{1}, \ldots, \lambda_{n}\right\}$. We assume that $\langle A\rangle_{\mathbb{Z}}=\mathcal{X}(T)$ and keep the notations of the preceding section. Given $p=\left[\sum p_{i} v_{i}\right] \in \mathbb{T}^{n-1}$, we denote by $m_{p}\left(\left[\sum x_{i} v_{i}\right]\right)=\left[\sum p_{i} x_{i} v_{i}\right]$ the diagonal linear isomorphism defined by $p$.
3.1. Non pyramidal configurations. In this subsection we characterize self-dual projective toric varieties associated to a configuration of weights $A$ which define a non pyramidal configuration, in terms of the orbits of the torus action.

Note that the whole projective space $\mathbb{P}(V)$ can be seen as a toric projective variety associated to a $\operatorname{dim} V$-pyramidal configuration and its dual variety is empty. But we now show that for non pyramidal configurations the dimension of the dual variety $X_{A}^{*}$ cannot be smaller than the dimension of the toric variety $X_{A}$. This result has been proved by Zak [18] for any non degenerate smooth projective variety.

Lemma 3.1. If $A$ is a non pyramidal configuration, then $\operatorname{dim} X_{A}^{*} \geq \operatorname{dim} X_{A}$.
Proof. Indeed, if $A$ is not a pyramidal configuration, then by Remark 2.11 we know that there exists $p=\sum p_{i} v_{i} \in \mathcal{R}_{A, \mathbb{k}}$ such that $p_{i} \neq 0$ for all $i=1, \ldots, n$. Hence, if we identify $\mathbb{P}(V)$ with $\mathbb{P}\left(V^{\vee}\right)$ by means of the dual basis, then

$$
X_{A}^{*}=\overline{T \cdot \mathbb{P}\left(\mathcal{R}_{A, k}\right)} \supset \overline{\mathcal{O}([p])}=m_{p}\left(\overline{\mathcal{O}\left(\left[\sum_{i} v_{i}\right]\right)}\right)=m_{p}\left(X_{A}\right)
$$

Since $p \in \mathbb{T}^{n-1}$, we have that $\operatorname{dim} m_{p}\left(X_{A}\right)=\operatorname{dim} X_{A}$ and the result follows.
We identify $\mathbb{P}(V)$ with $\mathbb{P}\left(V^{\vee}\right)$ by means of the chosen basis $\mathcal{B}$ of eigenvectors (and its dual basis) as in Remark 2.20. The following is the main result of this subsection.

Theorem 3.2. Let $A \subset \mathcal{X}(T)$ be a non pyramidal configuration.
The following assertions are equivalent.
(1) $X_{A}$ is a self-dual projective variety.
(2) There exists $p_{0} \in \mathbb{P}\left(\mathcal{R}_{A, \mathbb{k}}\right) \cap \mathbb{T}^{n-1}$ such that $\mathbb{P}\left(\mathcal{R}_{A, \mathbf{k}}\right) \subset \overline{\mathcal{O}\left(p_{0}\right)}$.
(3) There exists $p_{0} \in \mathbb{P}\left(\mathcal{R}_{A, k}\right) \cap \mathbb{T}^{n-1}$ such that $X_{A}^{*}=m_{p_{0}}\left(X_{A}\right)$.
(4) For all $q \in \mathbb{P}\left(\mathcal{R}_{A, k}\right) \cap \mathbb{T}^{n-1}, \mathbb{P}\left(\mathcal{R}_{A, k}\right) \subset \overline{\mathcal{O}(q)}$.
(5) For all $q \in \mathbb{P}\left(\mathcal{R}_{A, k}\right) \cap \mathbb{T}^{n-1}, X_{A}^{*}=m_{q}\left(X_{A}\right)$.

Proof. We prove $(1) \Rightarrow(5)$ and $(2) \Rightarrow(4)$, the rest of the implications being trivial. $(1) \Rightarrow(5)$ : By Proposition 2.21 .

$$
X_{A}^{*}=\overline{\bigcup_{p \in \mathbb{P}\left(\mathcal{R}_{A, k}\right)} \mathcal{O}(p)} \supset \overline{\bigcup_{p \in \mathbb{P}\left(\mathcal{R}_{A, k}\right) \cap \mathbb{T}^{n-1}} \mathcal{O}(p)} \supset \overline{\mathcal{O}(q)}=m_{q}\left(X_{A}\right)
$$

for all $q \in \mathbb{P}\left(\mathcal{R}_{A, k}\right) \cap \mathbb{T}^{n-1}$. Since $\operatorname{dim} X_{A}=\operatorname{dim} X_{A}^{*}$, equality holds in the last equation.
$(2) \Rightarrow(4)$ : Let $p_{0} \in \mathbb{P}\left(\mathcal{R}_{A, \underline{k}}\right) \cap \mathbb{T}^{n-1}$ be such that $\mathbb{P}\left(\mathcal{R}_{A, k}\right) \subset \overline{\mathcal{O}\left(p_{0}\right)}$. If $q \in$ $\mathbb{P}\left(\mathcal{R}_{A, k}\right) \cap \mathbb{T}^{n-1}$, then $q \in \overline{\mathcal{O}\left(p_{0}\right)} \cap \mathbb{T}^{n-1}=\mathcal{O}\left(p_{0}\right)$. Then, $\mathcal{O}(q)=\mathcal{O}\left(p_{0}\right)$ and the result follows.

The equivalence between (1) and (5) in Theorem 3.2 implies that as soon as the dual of an equivariantly embedded projective toric variety of the form $X_{A}$ has the same dimension of the variety, there exists a linear isomorphism between them.

Theorem 3.3. Let $A \subset \mathcal{X}(T)$ be a configuration of weights which is non pyramidal. Then $X_{A}$ is self-dual if and only if $\operatorname{dim} X_{A}=\operatorname{dim} X_{A}^{*}$.

This result is not true in general for projective toric varieties not equivariantly embedded, even for rational planar curves (for which the dual is again a curve, but not necessarily isomorphic).
3.2. The general case. We now address the complete characterization of selfdual projective toric varieties associated to an arbitrary configuration of weights $A \subset \mathcal{X}(T)$. We keep the notations of the preceding section.

We begin by recalling a well known result about duality of projective varieties:
Lemma 3.4 ([17, Theorem 1.23]). Let $X \subset \mathbb{P}^{n}$ be a non linear irreducible subvariety.
(1) Assume that $X$ is contained in a hyperplane $H=\mathbb{P}^{n-1}$. If $X^{\prime *}$ is the dual variety of $X$, when we consider $X$ as a subvariety of $\mathbb{P}^{n-1}$, then $X^{*}$ is the cone over $X^{\prime *}$ with vertex $p$ corresponding to $H$.
(2) Conversely, if $X^{*}$ is a cone with vertex $p$, then $X$ is contained in the corresponding hyperplane $H$.

When $X$ is linear, $\left(X^{\prime}\right)^{*}$ is empty.
As an immediate application of Lemma 3.4, we have the following characterization of self-dual equivariantly embedded projective toric hypersurfaces. Note that the only linear varieties which are self-dual are the subspaces of dimension $k-1$ in $\mathbb{P}^{2 k-1}$. In particular, the only hyperplanes which are self dual are points in $\mathbb{P}^{1}$.

Corollary 3.5. Let $T$ be an algebraic torus and $A \subset \mathcal{X}(T)$ a configuration such that $X_{A}$ is a non linear hypersurface. Then $X_{A}$ is self-dual if and only if $X_{A}$ is not a cone.

Proof. Assume that $X_{A}$ is a cone. Then by Lemma 3.4, it follows that $X_{A}^{*}$ is contained in a hyperplane, hence $X_{A}$ is not self-dual.

If $X_{A}$ is not a cone, then $A$ is non pyramidal (see Remark 2.13), and it follows from Lemma 3.1 that $\operatorname{dim} X_{A}^{*} \geq \operatorname{dim} X_{A}$. If $\operatorname{dim} X_{A}^{*}>\operatorname{dim} X_{A}$, then $X_{A}^{*}=\mathbb{P}(V)$ and hence $X_{A}=\left(X_{A}^{*}\right)^{*}=\emptyset$, which is a contradiction. It follows that $\operatorname{dim} X_{A}^{*}=$ $\operatorname{dim} X_{A}$ and hence Theorem 3.3 implies that $X_{A}$ is self-dual.

Applying Lemma 3.4, we can reduce the study of duality of projective varieties to the study of non degenerate projective varieties that are not a cone.

Proposition 3.6. Let $X \subset \mathbb{P}^{n-1}$ be an irreducible projective variety. Let $k-1$ be the codimension of the minimal subspace of $\mathbb{P}^{n-1}$ containing $X$. Then, with the notations of Remark 2.16, the following assertions hold:
(1) If $X=\mathrm{J}_{k-1, k-1, m-1}\left(\emptyset, \mathbb{P}^{k-1}, X_{\mathrm{nd}}\right)$, with $X_{\mathrm{nd}} \subset \mathbb{P}^{m-1}$ self-dual, then $X$ is self-dual.
(2) If $X$ is self-dual, then $\operatorname{dim} X_{\mathrm{nd}}=\operatorname{dim}\left(X_{\mathrm{nd}}\right)^{*}$, and $h=k$, that is

$$
\begin{equation*}
X=\mathrm{J}_{k-1, k-1, m-1}\left(\emptyset, \mathbb{P}^{k-1}, X_{\mathrm{nd}}\right) \tag{2}
\end{equation*}
$$

Proof. Let $X=\mathrm{J}_{k-1, h-1, m-1}\left(\emptyset, \mathbb{P}^{h-1}, X_{\mathrm{nd}}\right)$. Applying recursively Lemma 3.4 (see Remark 2.16 we obtain that

$$
\begin{aligned}
X^{*}= & \mathrm{J}_{k-1, h-1, m-1}\left(\emptyset, \mathbb{P}^{h-1}, X_{\mathrm{nd}}\right)^{*}=\mathrm{J}_{k-1, h+m-1}\left(\emptyset, \mathrm{~J}_{h-1, m-1}\left(\mathbb{P}^{h-1}, X_{\mathrm{nd}}\right)\right)^{*}= \\
& \mathrm{J}_{k-1, h+m-1}\left(\mathbb{P}^{k-1}, \mathrm{~J}_{h-1, m-1}\left(\mathbb{P}^{h-1}, X_{\mathrm{nd}}\right)^{*}\right)= \\
& \mathrm{J}_{k-1, h+m-1}\left(\mathbb{P}^{k-1}, \mathrm{~J}_{h-1, m-1}\left(\emptyset, X_{\mathrm{nd}}^{*}\right)\right)= \\
& \mathrm{J}_{k-1, h-1, m-1}\left(\mathbb{P}^{k-1}, \emptyset, X_{\mathrm{nd}}^{*}\right)=\mathrm{J}_{h-1, k-1, m-1}\left(\emptyset, \mathbb{P}^{k-1}, X_{\mathrm{nd}}^{*}\right)= \\
& \mathrm{J}_{h-1, k+m-1}\left(\emptyset, \mathrm{~J}_{k-1, m-1}\left(\mathbb{P}^{k-1}, X_{\mathrm{nd}}^{*}\right)\right) .
\end{aligned}
$$

In particular, $\operatorname{dim} X^{*}=k+\operatorname{dim} X_{\mathrm{nd}}^{*}$, and the maximal subspace that contains $X^{*}$ has codimension $h$.

In order to prove (1), assume that $h=k$ and $X_{\text {nd }}$ is self-dual. Then $X^{*}=$ $\mathrm{J}_{k-1, k-1, m-1}\left(\emptyset, \mathbb{P}^{k-1}, X_{\mathrm{nd}}^{*}\right)$. Since $X_{\mathrm{nd}}$ is self-dual, there exists an isomorphism $\varphi: \mathbb{P}^{m-1} \rightarrow \mathbb{P}^{m-1}$ such that $\varphi\left(X_{\mathrm{nd}}\right)=X_{\mathrm{nd}}^{*}$. It is clear that $\varphi$ extends to an isomorphism $\widetilde{\varphi}: \mathbb{P}^{n-1} \rightarrow \mathbb{P}^{n-1}$ such that $\widetilde{\varphi}(X)=X^{*}$.

In order to prove (2), assuming $X$ is self-dual and writing $X$ as in Remark 2.16 . it follows that $h=k$, and hence $h+\operatorname{dim} X_{\mathrm{nd}}=\operatorname{dim} X=\operatorname{dim} X^{*}=k+\operatorname{dim} X_{\mathrm{nd}}$.

In our toric setting, Proposition 3.6 can be improved, so that we obtain a geometric characterization of self-dual projective toric varieties.

Theorem 3.7. Let $A$ be an arbitrary lattice configuration. Then $X_{A}$ is self-dual if and only if $\operatorname{dim} X_{A}=\operatorname{dim} X_{A}^{*}$ and the smallest linear subspaces containing $X_{A}$ and $X_{A}^{*}$ have the same (co)dimension.
Proof. By Proposition 2.17,

$$
X_{A}=\mathrm{J}_{k-1, h-1, m-1}\left(\emptyset, \mathbb{P}^{h-1}, X_{C_{2}}\right)
$$

where $C_{2} \subset A$ is a non pyramidal configuration without repeated weights. By Theorem 3.3. $X_{C_{2}} \subset \mathbb{P}^{m-1}$ is self dual if and only if $\operatorname{dim} X_{C_{2}}=\operatorname{dim} X_{C_{2}}^{*}$. The result follows now from Proposition 3.6 .

Combining Proposition 2.17 and Theorem 3.7 we obtain the following explicit combinatorial description of self-dual toric varieties.

Theorem 3.8. Let $A=\left\{\lambda_{1}, \ldots, \lambda_{1}, \lambda_{2}, \ldots, \lambda_{2}, \ldots, \lambda_{h}, \ldots, \lambda_{h}\right\} \subset \mathcal{X}(T)$ be a configuration of $n$ weights with each $\lambda_{i}$ appearing $k_{i}+1$ times, $\lambda_{i} \neq \lambda_{j}$ if $i \neq j$. Let $C=\left\{\lambda_{1}, \ldots, \lambda_{h}\right\}$ be the associated configuration without repeated weights. Then $X_{A}$ is self-dual if and only if the following assertions hold.
(1) $C$ is a $k$-pyramidal configuration, where $k=n-h=\sum k_{i}$.
(2) There exists a splitting $T=S_{1} \times S_{2}$ such that, after reordering of the elements in $C$, it holds that $C=C_{1} \cup C_{2}$, where $C_{1}=\left\{\lambda_{1}, \ldots, \lambda_{k}\right\}$ is a basis of $\mathcal{X}\left(S_{1}\right)$ and $C_{2}=\left\{\lambda_{k+1}, \ldots, \lambda_{h}\right\} \subset \mathcal{X}\left(S_{2}\right)$ is a non pyramidal configuration, as in Definition 2.10. Moreover, the $S_{2}$-toric projective variety $X_{C_{2}} \subset \mathbb{P}\left(\bigoplus_{i=k+1}^{h} \mathbb{k} w_{i}\right)$, $t \cdot w_{i}=\lambda_{i}(\bar{t}) w_{i}$, is self-dual.

It follows from Theorem 3.8 that if $X_{A}$ is a self-dual toric variety with $A$ pyramidal, then there are repeated weights in $A$. The converse of this statement does not hold. In the next example we show a family of non-pyramidal configurations $A$ with repetitions such that $X_{A}$ is self dual.

Example 3.9. Let $C=\left\{c_{1}, \ldots, c_{s}\right\} \subset \mathbb{Z}^{n-1}$ be any non pyramidal configuration, such that $X_{C}$ is self-dual. Then, the configuration $A=\left\{e_{1}, e_{1},\left(0, c_{1}\right), \ldots,\left(0, c_{s}\right)\right\} \subset$ $\mathbb{Z}^{n}$ has repeated weights, and $X_{A}$ is self-dual by Theorem 3.8. It is straightfoward to check that $A$ is non-pyramidal. Note that these configurations become pyramidal when we avoid repetitions.

## 4. Characterizations of SElf-DUALITY in combinatorial terms

In this section we will characterize self-duality of projective toric varieties of type $X_{A}$ in combinatorial terms. We make explicit calculations for the algebraic torus $\left(\mathbb{k}^{*}\right)^{d}$ acting on $\mathbb{k}^{n}$, in order to give an interpretation of the conditions of Theorem 3.2 in terms of the configuration $A$ and in terms if its Gale dual configuration, whose definition we recall below.

We refer the reader to [19, Chapter 6] for an account of the basic combinatorial notions we use in what follows.
4.1. Explicit calculations for $\left(\mathbb{k}^{*}\right)^{d}$ acting on $\mathbb{k}^{n}$. Let $T=\left(\mathbb{k}^{*}\right)^{d}$. We identify the lattice of characters $\mathcal{X}(T)$ with $\mathbb{Z}^{d}$. Thus, any character $\lambda \in \mathcal{X}(T)$ is of the form $\lambda(t)=t^{m}$, where $m \in \mathbb{Z}^{d}$ and $t^{m}=t_{1}^{m_{1}} \cdots t_{d}^{m_{d}}$. We take the canonical basis of $\mathbb{k}^{n}$ as the basis of eigenvectors of the action of $T$. That is, if $A=\left\{\lambda_{1}, \ldots, \lambda_{n}\right\} \subset \mathbb{Z}^{d}, T$ acts on $\mathbb{k}^{n}$ by $t \cdot\left(z_{1}, \ldots, z_{n}\right)=\left(t^{\lambda_{1}} z_{1}, \ldots, t^{\lambda_{n}} z_{n}\right)$ for all $t=\left(t_{1}, \ldots, t_{d}\right) \in T$. Then,

$$
X_{A}=\overline{\mathcal{O}([1: \cdots: 1])}=\overline{\left\{\left[t^{\lambda_{1}}: \cdots: t^{\lambda_{n}}\right]: t \in\left(\mathbb{k}^{*}\right)^{d}\right\}} \subset \mathbb{P}^{n-1}
$$

By abuse of notation we also set $A \in \mathcal{M}_{d \times n}(\mathbb{Z})$ the matrix with columns the weights $\lambda_{i}$. In view of the reductions made in Section 2 we assume without loss of generality that the first row of $A$ is $(1, \ldots, 1)$ and that the columns of $A$ span $\mathbb{Z}^{d}$.

The homogeneous ideal $I_{A}$ in $\mathbb{k}\left[x_{1}, \ldots, x_{n}\right]$ of the associated projective toric variety $X_{A}$ is the binomial ideal ([16])

$$
I_{A}=\left\langle x^{a}-x^{b}: a, b \in \mathbb{N}^{n}, \sum_{i=1}^{n} a_{i} \lambda_{i}=\sum_{i=1}^{n} b_{i} \lambda_{i}\right\rangle .
$$

Thus, $X_{A}=\left\{[x] \in \mathbb{P}^{n-1}: x^{a}=x^{b}, \forall a, b \in \mathbb{N}^{n}\right.$ such that $\left.A a=A b\right\}$, and it is easy to see that

$$
X_{A}=\left\{[x] \in \mathbb{P}^{n-1}: x^{v^{+}}-x^{v^{-}}=0, \forall v \in \mathcal{R}_{A}\right\}
$$

where $v_{i}^{+}=\max \left\{v_{i}, 0\right\}, v_{i}^{-}=-\min \left\{v_{i}, 0\right\}\left(\right.$ and so $\left.v=v^{+}-v^{-}\right)$.
For $p \in \mathbb{T}^{n-1}$ we then have

$$
m_{p}\left(X_{A}\right)=\overline{\mathcal{O}(p)}=\left\{[x] \in \mathbb{P}^{n-1}: p^{v^{-}} x^{v^{+}}-p^{v^{+}} x^{v^{-}}=0, \forall v \in \mathcal{R}_{A}\right\}
$$

4.2. Characterization of self-duality in terms of the Gale dual configuration. If $A$ is a non pyramidal configuration, then Theorem 3.2 can be rephrased in terms of a geometric condition on the Gale dual of $A$.

Definition 4.1. Let $A \in \mathcal{M}_{d \times n}(\mathbb{Z})$ with rank $d$. Let $\mathcal{B}_{A}=\left\{u_{1}, \ldots, u_{n-d}\right\} \subset \mathbb{Z}^{n}$ be a basis of $\mathcal{R}_{A}$.

We say that the matrix $B_{A} \in \mathcal{M}_{n \times(n-d)}(\mathbb{Z})$ with columns the vectors $u_{i}$ is a Gale dual matrix of $A$. Let $\mathcal{G}_{A}=\left\{b_{1}, \ldots, b_{n}\right\} \subset \mathbb{Z}^{n-d}$ be the configuration of rows of $B_{A}$, that is $B_{A}=\left(\begin{array}{c}b_{1} \\ \vdots \\ b_{n}\end{array}\right)$ (observe that we allow repeated elements). The configuration $\mathcal{G}_{A}$ is called a Gale dual configuration of $A$. Remark that $\sum_{i=1}^{n} b_{i}=0$.
Remark 4.2. (1) Since $\mathcal{R}_{A}$ is an affine invariant of the configuration $A$, it follows that two affinely equivalent configuration share their Gale dual configurations.
(2) The configuration $A$ is non pyramidal if and only $b_{i} \neq 0$ for all $i=1, \ldots, n$.
(3) When $A$ is regular, $\mathcal{R}_{A}$ is the integer kernel $\operatorname{Ker}_{\mathbb{Z}}(A)$ of the matrix $A$.
(4) For any Gale dual matrix of $A$, the morphism $\gamma: \mathbb{k}^{n-d} \rightarrow \mathbb{k}^{n}, \gamma(s)=\left(\left\langle s, b_{1}\right\rangle, \ldots,\left\langle s, b_{n}\right\rangle\right)$ gives a parameterization of $\mathcal{R}_{A, \mathfrak{k}}$, where we denote $\left\langle s, b_{i}\right\rangle=\sum_{j=1}^{n-d} s_{j} b_{i j}$.
Remark 4.3. By Theorem 3.2, $X_{A}$ is self-dual if and only if there exists $p_{0} \in$ $\mathbb{P}\left(\mathcal{R}_{A, k}\right) \cap \mathbb{T}^{n-1}$ such that $\mathbb{P}\left(\mathcal{R}_{A, \underline{k}}\right) \subset \overline{\mathcal{O}\left(p_{0}\right)}$. By the remarks in subsection 4.1, it follows that $X_{A}$ is self-dual if and only if for some such $p_{0}$ we have that $p_{0}^{v^{-}} w^{v^{+}}-$ $p_{0}^{v^{+}} w^{v^{-}}=0$ for all $w \in \mathcal{R}_{A, \mathrm{k}}$ and $v \in \mathcal{R}_{A}$.

Assume $X_{A}$ is self-dual. Then, given any choice of Gale dual configuration, we deduce that for all $s \in \mathbb{k}^{n-d} \backslash\{0\}$ and $j=1, \ldots, n-d$, we have that

$$
p_{0}^{u_{j}^{-}}\left(\left\langle s, b_{1}\right\rangle,\left\langle s, b_{2}\right\rangle, \ldots,\left\langle s, b_{n}\right\rangle\right)^{u_{j}^{+}}=p_{0}^{u_{j}^{+}}\left(\left\langle s, b_{1}\right\rangle,\left\langle s, b_{2}\right\rangle, \ldots,\left\langle s, b_{n}\right\rangle\right)^{u_{j}^{-}}
$$

for (some, or in fact all) $p_{0} \in \mathbb{P}\left(\mathcal{R}_{A, \mathrm{k}}\right) \cap \mathbb{T}^{n-1}$.
Since this gives an equality in the polynomial ring $\mathbb{k}\left[s_{1}, \ldots, s_{n-d}\right]$, both sides must have the same irreducible factors. But $\left\langle s, b_{i}\right\rangle$ and $\left\langle s, b_{k}\right\rangle$ are associated irreducible factors if and only if $b_{i}$ and $b_{k}$ are collinear vectors. We deduce that for
any line $L$ in $B$-space $\mathbb{Z}^{n-d}$ and for all $j$,

$$
\sum_{b_{i} \in L, b_{i j}>0} b_{i j}=-\sum_{b_{i} \in L, b_{i j}<0} b_{i j} .
$$

Hence, $\sum_{b_{i} \in L} b_{i j}=0$ for all $j=1, \ldots, n-d$, or equivalently, $\sum_{b_{i} \in L} b_{i}=0$.
In fact, this last condition is not only necessary but also sufficient. We give a proof of both implications using results about the tropicalization of the dual variety $X_{A}$ as described in [8].

First we recall that given a dual Gale configuration $\mathcal{G}_{A}=\left\{b_{1}, \ldots, b_{n}\right\}$, and a subset $J \subset\{1, \ldots, n\}$, the flat $S_{J}$ of $\mathcal{G}_{A}$ associated to $J$ is the subset of all the indices $i \in\{1, \ldots, n\}$ such that $b_{i}$ belongs to the subspace generated by $\left\{b_{j}: j \in J\right\}$.
Theorem 4.4. Let $A \in \mathcal{M}_{d \times n}(\mathbb{Z})$ a non pyramidal configuration and $B_{A}$ a Gale dual for $A$ as in (4.1). Then $X_{A}$ is self-dual if and only if for any line $L$ through the origin in $\mathbb{Z}^{n-d}$ we have that $\sum_{b_{i} \in L} b_{i}=0$.

Proof. Since we are dealing with affine invariants, we can assume that $A$ is a regular configuration. By Theorem 3.3. we know that $X_{A}$ is self-dual if and only if $\operatorname{dim} X_{A}$ equals $\operatorname{dim} X_{A}^{*}$. Given a vector $v \in \mathbb{Z}^{n}$, we define a new vector $\sigma(v) \in\{0,1\}^{n}$ by $\sigma(v)_{i}=0$ if $v_{i} \neq 0$ and $\sigma(v)_{i}=1$ if $v_{i}=0$.

If follows from [8, Corollary 4.5] that $\operatorname{dim} X_{A}=\operatorname{dim} X_{A}^{*}$ if and only if for any vector $v \in \mathcal{R}_{A}$, the vector $(1, \ldots, 1)-\sigma(v)$ lies in the row span $F$ of the matrix $A$. But since we are assuming that $(1, \ldots, 1) \in F$, this is equivalent to the condition that $\sigma(v) \in F$. By duality, this is in turn equivalent to the fact that for any $j=1, \ldots, n-d$, the inner product

$$
\left\langle\sigma(v), u_{j}\right\rangle=\sum_{v_{i}=0} b_{i j}=0
$$

That is to say, $X_{A}$ is self dual if and only if for any $v \in \mathcal{R}_{A}$ the sum $\sum_{v_{i}=0} b_{i}=0$. But the sets $S$ of non zero coordinates of the vectors in the space of linear relations $\mathcal{R}_{A}$ coincide with the flats of the Gale configuration $\mathcal{G}_{A}$. So, $X_{A}$ is self-dual if and only if for any flat $S \subset\{1, \ldots, n\}$ the sum $\sum_{i \in S} b_{i}=0$. It is clear that this happens if and only if the same condition holds for all the one-dimensional flats, i.e. if for any line $L$ through the origin the sum $\sum_{b_{i} \in L} b_{i}=0$.

The assumption that $A$ is a non pyramidal configuration in Theorem 4.4 is crucial, as the following example shows.
Example 4.5. Let $A$ be a configuration such that $\mathcal{R}_{A}$ has rank 1. Then $\mathcal{R}_{A}$ is spanned by a single vector, whose coordinates add up to 0 . So, the condition in Theorem 4.4 that the sum of the $b_{i}$ in this line equals 0 is satisfied. But by Corollary 3.5 if $A$ is a pyramid, then $X_{A}$ is not self-dual.
4.3. Geometric characterization of self-dual configurations. In this paragraph we characterize the non pyramidal configurations $A \subset \mathbb{Z}^{d}$ whose Gale dual configurations are as in Theorem 4.4. We keep the assumptions that $\langle A\rangle_{\mathbb{Z}}=\mathbb{Z}^{d}$ and that $A$ is non pyramidal. We begin with some basic definitions about configurations.
Definition 4.6. Given $a=\left(a_{1}, \ldots, a_{n}\right) \in \mathcal{R}_{A, \mathbb{k}}$, we call $\left\{i: a_{i} \neq 0\right\}$ the support of the relation and denote $\operatorname{supp}(a)=\left\{i: a_{i} \neq 0\right\}$. We say that $\lambda_{i}$ belongs to the relation if $i \in \operatorname{supp}(a)$.

Recall that any affine relation $a \in \mathcal{R}_{A, \mathrm{k}}$ satisfies $\sum_{i} a_{i}=0$. It is said that $a$ is a circuit if there is no non trivial affine dependency relation with support strictly contained in $\operatorname{supp}(a)$. In other words, a circuit is a minimal affine dependency relation.

Remark 4.7. Let $C$ be a circuit of a configuration $A$, and let $F$ be the minimal face of $\operatorname{Conv}(\mathrm{A})$ containing $C$. If $d^{\prime}$ denotes the dimension of the affine span of $F$, then $C$ has at most $d^{\prime}+2$ elements.

Definition 4.8. Two elements $b, b^{\prime}$ of a configuration $B$ are parallel if they generate the same straight line through the origin. In particular, $b \neq 0$ and $b^{\prime} \neq 0$. The elements $b, b^{\prime}$ are antiparallel if they are parallel and point into opposite directions.

Two elements $\lambda, \lambda^{\prime}$ of a configuration $A$ are coparallel if they belong exactly to the same circuits.

Remark 4.9. (1) Coparallelism is an equivalence relation. We denote by $\operatorname{cc}(\lambda)$ the coparallelism class of the element $\lambda \in A$.
(2) It is easy to see that $\lambda$ and $\lambda^{\prime}$ are coparallel if and only if they belong to the same affine dependency relations.
(3) The definition of coparallelism can be extended to pyramidal configurations as follows. If $\lambda \in A$ is such that it does not belong to any dependency relation, then $\operatorname{cc}(\lambda)=\{\lambda\}$. Otherwise, $\operatorname{cc}(\lambda)$ consists, as above, of all elements of $A$ belonging to the same circuits as $\lambda$. The condition that $A$ is not a pyramid is then equivalent to the condition that $|\operatorname{cc}(\lambda)| \geq 2$ for all $\lambda \in A$.

Lemma 4.10. Let $\mathcal{G}_{A}=\left\{b_{1}, \ldots, b_{n}\right\}$ be a Gale dual of $A$. Then $\lambda_{i}$ is coparallel to $\lambda_{j}$ if and only if $b_{i}$ and $b_{j}$ are parallel elements of $\mathcal{G}_{A}$.
Proof. Let $B_{A}$ the $(n \times(n-d))$-matrix with rows given by $\mathcal{G}_{A}$ as in Definition 4.1. As $A$ is not a pyramid, no row $b_{i}$ of $B_{A}$ is zero. Any element $a \in \mathcal{R}_{A, k}$ is of the form $B_{A} \cdot m$, for some $m \in \mathbb{k}^{n-d}$. Then $\lambda_{i}$ is coparallel to $\lambda_{j}$ if and only if for any nonzero $m \in \mathbb{k}^{n-d}$ it holds that $\left\langle b_{i}, m\right\rangle \neq 0$ precisely when $\left\langle b_{j}, m\right\rangle \neq 0$. It is clear that this happens if and only if $b_{i}=\alpha b_{j}$ for a non zero constant $\alpha \in \mathbb{k}$, that is, if and only if $b_{i}, b_{j}$ are parallel.
Definition 4.11. Let $A=\left\{\lambda_{1}, \ldots, \lambda_{n}\right\} \subset \mathbb{Z}^{d}$ be a configuration. A subconfiguration $C^{\prime} \subset A$ is called facial if there exists a face $F$ of the convex hull $\operatorname{Conv}(A) \subset \mathbb{R}^{d}$ of $A$ such that $C^{\prime}=A \cap F$.

A subconfiguration $C \subset A$ is a face complement if $A \backslash C$ is a facial subconfiguration of $A$.

Remark 4.12. Let $A=\left\{\lambda_{1}, \ldots, \lambda_{n}\right\} \subset \mathbb{Z}^{d}$ be a configuration. A subconfiguration $C=\left\{\lambda_{i_{1}}, \ldots, \lambda_{i_{h}}\right\} \subset A$ is a face complement if and only if there exists a dependency relation such that

$$
\sum_{j=1}^{h} r_{i_{j}} b_{i_{j}}=0 \quad, \quad r_{i_{j}}>0
$$

Indeed, a dependency relation $\sum_{j=1}^{h} r_{i_{j}} b_{i_{j}}=0$ with all $r_{i_{j}}>0$ can be extended with zero coordinates to a relation $r=\left(r_{1}, \ldots, r_{n}\right)$ among all $b_{i}$ 's. Thus, $r$ lies in the row space of $A$ and so there exists $\ell=\left(\ell_{1}, \ldots, \ell_{d}\right)$ such that $r_{i}=\left\langle\ell, \lambda_{i}\right\rangle$. It follows that the linear form associated to $\ell$ vanishes on the complement of $C$, and all the points of $C$ lie in the same open half space delimited by the kernel of $\ell$.

Lemma 4.13. Let $A=\left\{\lambda_{1}, \ldots, \lambda_{n}\right\} \subset \mathbb{Z}^{d}$ be a configuration. A coparallelism class $C=\left\{\lambda_{i_{1}}, \ldots, \lambda_{i_{h}}\right\} \subset A$ is a face complement if and only if and only if there exist $j, k \in\{1, \ldots, h\}$ such that $b_{i_{j}}$ and $b_{i_{k}}$ are antiparallel.
Proof. If $C$ is a coparallelism class, we know by Lemma 4.10 that all $b_{i_{1}}, \ldots, b_{i_{h}}$ are parallel. It is then clear that a dependency relation $r$ as in Remark 4.12 exists if and only if two of the vectors $b_{i_{j}}, b_{i_{k}}$ are antiparallel.

Definition 4.14. Let $A \subset \mathbb{Z}^{d}$ be a configuration and $C \subset A$ a face complement. We say that $C$ is a parallel face complement if $C$ and $A \backslash C$ lie in parallel hyperplanes.

Note that in this case both $C$ and $A \backslash C$ are facial.
Example 4.15. In Figure 1 below there are three configurations of 6 lattice points in 3 -dimensional space (the 6 vertices in each polytope). The 2 vertices marked with big dots in each of the configurations define a coparallelism class $C$. In the first polytope (1), $C$ is not a face complement; in the second polytope (2), $C$ is a face complement but not a parallel face complement; in the third polytope (3), $C$ is a parallel face complement. The characterization in our next theorem proves that only the toric variety corresponding to this last configuration is self-dual.

(1)

(2)

(3)

Figure 1. Only configuration (3) is self-dual.

It is straightforward to check that if $A_{1}, A_{2}$ are affinely equivalent configurations and $\varphi$ is an affine linear map sending bijectively $A_{1}$ to $A_{2}$, then $\varphi$ preserves coparallelism classes, faces and parallelism relations. Indeed, all these notions can be read in a common Gale dual configuration. Moreover, we can translate Theorem 4.4 as follows.

Theorem 4.16. Let $A \subset \mathbb{Z}^{d}$ be a non pyramidal configuration. The projective toric variety $X_{A}$ is self-dual if and only if any coparallelism class of $A$ is a parallel face complement.

Proof. Let $\mathcal{G}_{A}$ be a Gale dual of $A$ as in Definition 4.1. By Lemma 4.10, coparallelism classes $C=\left\{\lambda_{i_{1}} \ldots, \lambda_{i_{h}}\right\}$ in $A$ are in correspondence with parallel vectors $b_{i_{1}}, \ldots, b_{i_{h}}$ in the dual space (i.e. lines containing vectors of $\mathcal{G}_{A}$ ). But now, $C$ is a parallel face complement if and only if there exists $\ell=\left(\ell_{1}, \ldots, \ell_{d}\right)$ such that
$\left\langle\ell, \lambda_{i}\right\rangle=0$ for all $\lambda_{i} \notin C$ and $\left\langle\ell, \lambda_{i_{j}}\right\rangle=1$ for all $j=1, \ldots, h$. Reciprocally, the sum of the vectors $\sum_{j=1}^{h} b_{i_{j}}=0$ implies the existence of such an $\ell$ as in Remark 4.12 The result now follows from Theorem 4.4.

We have the following easy lemma.
Lemma 4.17. Assume that $A$ is a non pyramidal self-dual lattice configuration. Then, for any $\mu \in A$, the coparallelism class $\operatorname{cc}(\mu)$ has at least two elements and it is a facial subconfiguration of $A$.

Proof. It follows from Definition 4.14 that there exists a linear function $f$ taking value 0 on $A \backslash \operatorname{cc}(\mu)$ and value 1 on $\operatorname{cc}(\mu)$. Then, $\operatorname{cc}(\mu)$ is the facial subconfiguration of $A$ supported by the hyperplane $(f-1)=0$. If cc $(\mu)=\{\mu\}$, then by Theorem $4.16\{\mu\}$ is a vertex, and hence $A$ would be a pyramid. It follows that so $|\operatorname{cc}(\mu)| \geq 2$, for any $\mu \in A$.

We give in Lemma 5.4 (2) an example of a self-dual lattice configuration $A$ which contains an interior point of $\operatorname{Conv}(A)$. However, this cannot happen if $X_{A}$ is not a hypersurface, as the following proposition shows.
Proposition 4.18. Let $A \subset \mathcal{X}(T)$ be a configuration without repetitions such that $X_{A}$ is self-dual, with codim $X_{A}>1$. Then, the interior of the convex hull $\operatorname{Conv}(A)$ does not contain elements of $A$ and for any facial subconfiguration $C^{\prime}$ of $A$, at most one point of $C^{\prime}$ lies in the relative interior of $\operatorname{Conv}\left(C^{\prime}\right)$.

Proof. Since $A=\left\{\lambda_{1}, \ldots, \lambda_{n}\right\} \subset \mathcal{X}(T)$ has no repeated elements, it follows from Theorem 3.8 that $A$ is non pyramidal. Then, as $X_{A}$ is not a hypersurface, if follows from Remark 2.4 that $n \geq d+3$, where $d$ is the dimension of the affine span of $A$.

Assume that there exists $\mu \in A$ belonging to the relative interior of an $s$ dimensional face $F$ of $\operatorname{Conv}(A)$. Therefore, $\mu$ is a convex combination of the vertices of $F$, and thus $\operatorname{cc}(\mu) \subset F$. But by Lemma 4.17, $\operatorname{cc}(\mu)$ is a facial subconfiguration of $A$, and thus a facial subconfiguration of $F \cap A$, which intersects the relative interior of $F$. Then, $\operatorname{cc}(\mu)=F \cap A$. Let $\operatorname{cc}(\mu)=\left\{\mu, \lambda_{1}, \ldots, \lambda_{r}\right\}$. We claim that $\left\{\lambda_{1}, \ldots, \lambda_{r}\right\}$ are affinely independent and thus, $\operatorname{cc}(\mu)$ is a circuit. Indeed, for any $i=1, \ldots, r$, $\operatorname{cc}\left(\lambda_{i}\right)=\operatorname{cc}(\mu)=F \cap A$, and so there cannot be any non-trivial affine dependence relation involving only $\left\{\lambda_{1}, \ldots, \lambda_{r}\right\}$. In particular, $r=s+1,\left\{\lambda_{1}, \ldots, \lambda_{s+1}\right\}$ are the vertices of $F$ and $\mu$ is the only point in $F \cap A$ belonging to the relative interior of $\operatorname{Conv}(F)$.

Therefore, if the relative interior of $\operatorname{Conv}(A)$ contains one element $\mu \in A$, it follows that $A$ is a circuit, and hence $n=d+2$, see Remark 4.7. That is, $X_{A}$ is a hypersurface.

Example 4.19. Consider the self-dual configuration $A$ given by the columns of the matrix

$$
A=\left(\begin{array}{llllll}
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 2
\end{array}\right)
$$

The associated toric variety has dimension 3 in $\mathbb{P}^{5}$, so it is not a hypersurface. No point of $A=\operatorname{Conv}(A) \cap \mathbb{Z}^{4}$ lies in the interior, but there are two facial subconfigurations of $A$ (namely, the segments with vertices $\{(1,0,0,0),(1,0,2,0)\}$ and
$\{(0,1,0,0),(0,1,0,2)\}$, respectively) which do have a point of $A$ in their relative interior. Note that

$$
X_{A}=\left\{\left(x_{1}, \ldots, x_{6}\right) \in \mathbb{P}^{5} / x_{2}^{2}-x_{1} x_{3}=x_{4}^{2}-x_{5} x_{6}\right\}
$$

is not smooth. It is a complete intersection but the four fixed points
$(0,0,1,0,0,0),(1,0,0,0,0,0),(0,0,0,1,0,0),(0,0,0,0,0,1)$ are not regular, as can be checked by the drop in rank of the Jacobian matrix. This could be seen directly in the geometry of the configuration. The convex hull of $A$ is a simple polytope (in fact, it is a simplex) of dimension 3 lying in the hyperplane $H=\left\{\left(y_{1}, \ldots, y_{4}\right) \in\right.$ $\left.\mathbb{R}^{4} / y_{1}+y_{2}=1\right\}$, but fixing the origin at any of the four vertices, the first lattice points in the 3 rays from that vertex do not form a basis of the lattice $H \cap \mathbb{Z}^{4}$. Note that there is a splitting of the 4 -torus $T$ as a product of tori of dimension 2 corresponding respectively to the first three and last three weights in $A$.

We end this paragraph by showing another interesting combinatorial property of configurations associated to self-dual toric varieties.

Proposition 4.20. Let $A=\left\{\lambda_{1}, \ldots, \lambda_{n}\right\} \subset \mathcal{X}(T)$ be a non pyramidal configuration such that $X_{A}$ is self dual and let $D$ be an arbitrary non empty subset of $A$. Then, either $D$ is a pyramidal configuration or $X_{D}$ is self-dual and, moreover, $D$ is a facial subconfiguration of $A$.

Proof. Assume that $D=\left\{\lambda_{1}, \ldots, \lambda_{s}\right\} \subset A$ is non pyramidal, and consider $\mathcal{R}_{D} \subset \mathbb{Z}^{s}$ . It is clear that $\mathcal{R}_{D} \times\{0\} \subset \mathcal{R}_{A}$. Hence, if $\mathcal{B}_{D}$ is a basis of $\mathcal{R}_{D}$, then there exists a $\mathbb{Q}$-basis of $\mathcal{R}_{A} \otimes \mathbb{Q}$ of the form $\mathcal{G}_{D} \times\{0\} \cup \mathcal{C}$. Let $\mathcal{B}_{A}$ be a $\mathbb{Z}$-basis of $\mathcal{R}_{A}$, and $\mathcal{G}_{A}=\left\{b_{1}, \ldots, b_{n}\right\}$ its associated Gale dual configuration. Then there exists an invertible $\mathbb{Q}$-matrix $M$ such that

$$
B^{\prime}=\left(\begin{array}{c|c}
B_{D} & C_{1} \\
\hline 0 & C_{2}
\end{array}\right)=\left(\begin{array}{c}
b_{1} \\
\vdots \\
b_{s} \\
b_{s-1} \\
\vdots \\
b_{n}
\end{array}\right) M
$$

Since $A$ is self-dual, it follows from Theorem 4.4 that the rows $b_{i}$ are such that the sum of vectors $b_{i}$ in the same line through the origin is zero. Hence, the matrix $B^{\prime}$ satisfies the same property. As $D$ is non pyramidal, no row of $B_{D}$ is zero. Therefore, $\left(B_{D}, C_{1}\right)$, and hence $B_{D}$, also satisfy the property that the sum of all its row vectors in a line through the origin is equal to zero. Hence, $X_{D}$ is self-dual. Moreover, the sum of the row vectors of $C_{2}$ is zero, and it follows from Remark 4.12 that $D$ is facial.

## 5. Families of self-dual projective varieties.

In this section we use our previous results in order to obtain new families of projective toric varieties that are self-dual. In particular, we obtain many new examples of non smooth self-dual projective varieties. We also identify all the smooth self-dual projective varieties of the form $X_{A}$. We retrieve in this (toric) case Ein's result, without needing to rely on Hartshorne's conjecture.

### 5.1. Projective toric varieties associated to Lawrence configurations.

Definition 5.1. We say that a configuration $A$ of $2 n$ lattice points is Lawrence if it is affinely equivalent to a configuration whose associated matrix has the form

$$
\left(\begin{array}{cc}
I d_{n} & I d_{n}  \tag{3}\\
0 & M
\end{array}\right)
$$

where $I d_{n}$ denotes the $n \times n$ identity matrix. Equivalently, $A$ is a Lawrence configuration if it is affinely equivalent to a Cayley sum of $n$ subsets, each one containing the vector 0 and one of the column vectors of $M$.

Lawrence configurations are a special case of Cayley configurations (see [4). The Lawrence configuration associated to the matrix (3) is the Cayley configuration of the two-point configurations consisting of the origin and one column vector of $M$. In the smooth case, Cayley configuration of strictly equivalent polytopes correspond to toric fibrations (see [9]).

It is straightforward to verify that if $A$ is Lawrence, then
(i) $\mathcal{R}_{A}=\left\{\binom{-v}{v}: v \in \operatorname{Ker}_{\mathbb{Z}}(M)\right\}$.
(ii) $A$ is pyramidal if and only if $M$ is pyramidal.

We immediately deduce from Theorem 4.4 the following result.
Corollary 5.2. If $A$ is a non pyramidal Lawrence matrix then $X_{A}$ is self-dual.
Example 5.3. The well known fact that the Segre embedding of $\mathbb{P}^{1} \times \mathbb{P}^{m-1}$ in $\mathbb{P}^{2 m-1}$ is self-dual follows directly from Corollary 5.2, the image of the Segre morphism

$$
\varphi(x, y)=\left[y_{0} x_{0}: y_{1} x_{0}: \cdots: y_{m} x_{0}: y_{0} x_{1}: y_{1} x_{1}: \cdots: y_{m} x_{1}\right]
$$

where $x=\left[x_{0}, x_{1}\right], y=\left[y_{0}: y_{1}: \cdots: y_{m}\right]$, is a projective toric variety with associated matrix

$$
A=\left(\begin{array}{cccccccccc}
1 & \cdots & \cdots & \cdots & 1 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & 0 & 1 & \cdots & \cdots \cdots & 1 \\
1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\
0 & \ddots & \ddots & & \vdots & 0 & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1
\end{array}\right)
$$

The sum of the first two rows equals the sum of the last $m$ rows. It is easy to see that $A$ is affinely equivalent to the configuration $A^{\prime}$ with associated matrix

$$
A^{\prime}=\left(\begin{array}{ccc}
I d_{n} & I d_{n}  \tag{4}\\
0 \cdots & 0 & 1 \cdots
\end{array}\right)
$$

The matrix $A^{\prime}$ is a non pyramidal Lawrence matrix, hence $X_{A^{\prime}}=X_{A}$ is self-dual.
We finish this paragraph by proving that Segre embeddings of $\mathbb{P}^{1} \times \mathbb{P}^{m-1}, m \geq 2$ are the unique smooth self-dual projective toric varieties that are not a hypersurface. We begin with an easy lemma which classifies all smooth hypersurfaces of the form $X_{A}$
Lemma 5.4. Let $A$ be a lattice configuration such that $X_{A}$ is a smooth hypersurface. Then, $A$ is of one of the following forms:
(1) A consists of two equal points, and so $X_{A}=\{(1: 1)\}=\left\{\left(x_{0}: x_{1}\right) \in\right.$ $\left.\mathbb{P}^{1} / x_{0}-x_{1}=0\right\}$.
(2) A consists of three collinear points with one of them the mid point of the others, and so $X_{A}=\left\{\left(x_{0}: x_{1}: x_{2}\right) \in \mathbb{P}^{2} / x_{1}^{2}-x_{0} x_{2}=0\right\}$.
(3) A consists of four points $a, b, c, d$ with $a+c=b+d$, and so $X_{A}=$ $\left\{\left(x_{0}, x_{1}, x_{2}, x_{3}\right) \in \mathbb{P}^{3} / x_{0} x_{3}-x_{1} x_{2}=0\right\}$ is the Segre embedding of $\mathbb{P}^{1} \times \mathbb{P}^{1}$ in $\mathbb{P}^{3}$.

Proof. When $X_{A}$ is a hypersurface, an equation for $X_{A}$ is given by $b_{A}(x)=$ $\prod_{b_{i}>0} x_{i}^{b_{i}}-\prod_{b_{i}<0} x_{i}^{-b_{i}}$, where the transpose of the row vector $\left(b_{1}, \ldots, b_{n}\right)$ is a choice of Gale dual of $A$. The cases (1), (2) and (3) in the statement correspond to the row vectors $(1,1),(1,-2,1)$ and $1,-1,-1,1$ ), respectively (or any permutation of the coordinates), and it is straightforward to check that $X_{A}$ is smooth. It is easy to verify that in any other case, there exists a point $x \in X_{A}$ where $b_{A}$ and all its partial derivatives vanish at $x$.

We saw in Example 4.19 that a non-pyramidal self-dual lattice configuration $A$ with $\operatorname{codim}\left(X_{A}\right)>1$ can have a point in the interior of a proper face. Moreover, more complicated situations can happen:

Example 5.5. Consider the following dimension 3 configuration $A \subset \mathbb{Z}^{4}, A=$ $\{(1,0,0,2),(1,0,0,0),(0,1,0,0),(0,1,0,2),(0,0,1,0),(0,0,1,1)\}$. Then, $\mathbb{Z} A=\mathbb{Z}^{4}$ and $X_{A}$ is self-dual because the following is a choice of Gale dual $B \in \mathbb{Z}^{6 \times 2}$ :

$$
B=\left(\begin{array}{rr}
1 & 0 \\
-1 & 0 \\
0 & 1 \\
0 & -1 \\
2 & -2 \\
-2 & 2
\end{array}\right)
$$

All the points in $A$ are vertices of the polytope $P:=\operatorname{Conv}(\mathrm{A})$, but $A \neq P \cap \mathbb{Z}^{4}$. Indeed, there is a lattice point in the middle of each of the segments $[(1,0,0,2),(1,0,0,0)]$, $[0,1,0,0),(0,1,0,2)]$, which are faces of $P$. It is clear that $X_{A}$ is not smooth (for instance looking at the first lattice points in all the edges emanating from $(1,0,0,0)$ ), nor embedded by a complete linear system.

However, the following result shows that when $X_{A}$ is smooth and self-dual, the situation is nicer.

Lemma 5.6. Let $A$ be a lattice configuration without repeated points such that $X_{A}$ is self-dual and smooth. Then, unless $X_{A}$ is the quadratic rational normal curve in (2) of Lemma5.4, no facial subconfiguration $C \subseteq A$ contains a point of $A$ in the relative interior of $\operatorname{Conv}(C)$.
Proof. Assume $A=\left\{\lambda_{1}, \ldots, \lambda_{n}\right\}$ has no repeated points and there exists $\mu \in A$ and a proper face $F$ of $\operatorname{Conv}(A)$ containing $\mu$ in its relative interior. Then, $F \cap A$ is not a pyramid, and it follows from Proposition 4.20 that $X_{F \cap A}$ is self-dual. Since $X_{F \cap A}$ is also smooth, Proposition 4.18 implies that $X_{F \cap A}$ is a hypersurface. We deduce from Lemma 5.4 that $F \cap A$ has dimension one and consists (up to reordering) of 3 points $\left\{\lambda_{1}, \lambda_{2}, \lambda_{3}\right\}$ with $\lambda_{1}+\lambda_{3}=2 \lambda_{2}$. We can choose a Gale dual $B$ of $A$ of the form:

$$
B=\left(\begin{array}{c|c}
B_{1} & C_{1} \\
\hline 0 & C_{2}
\end{array}\right),
$$

with $B_{1}$ the $3 \times 1$ column vector with transpose $(1,-2,1)$. We see that the coparallelism class of each $\lambda_{i}$ is contained in $F \cap A$ and no class can consist of a single element because $A$ is not a pyramid. Therefore, $\operatorname{cc}\left(\lambda_{i}\right)=F \cap A, i=1,2,3$; that is, any two of the first 3 rows of $B$ are linearly dependent. We can thus find another choice of Gale dual $B^{\prime}$ of $A$ of the form:

$$
B=\left(\begin{array}{c|c}
B_{1} & 0 \\
\hline 0 & C_{2}
\end{array}\right) .
$$

Then, there is a splitting of the torus and $X_{A}$ cannot be smooth, with arguments similar to those in Example 4.19, because $A$ has no repeated points and so there is no linear equation in the ideal $I_{A}$.

We now characterize the Segre embeddings $\mathbb{P}^{1} \times \mathbb{P}^{m-1}$ in $\mathbb{P}^{2 m-1}$ from Example 5.3 in terms of the Gale dual configuration.

Lemma 5.7. A toric variety $X_{A} \subset \mathbb{P}^{2 m-1}$ is the Segre embedding of $\mathbb{P}^{1} \times \mathbb{P}^{m-1}$ if and only if any Gale dual $B \in \mathbb{Z}^{2 m \times r}$ of $A$ has the following form: $r=m-1$ and, up to reordering, the rows of $b_{1}, \ldots, b_{2 m}$ of $B$ satisfy $\operatorname{det}\left(b_{1}, \ldots, b_{m-1}\right)=$ $1, b_{1}+\cdots+b_{m}=0$ and $b_{m+j}+b_{j}=0$, for all $j=1, \ldots, m$.
Proof. It is clear that any Gale dual to the matrix $A^{\prime}$ in (4) is of this form. And it is also straightforward to check that any matrix $B$ as in the statement is a Gale dual of this $A^{\prime}$.

We can now prove the complete characterization of smooth self-dual varieties $X_{A}$.

Theorem 5.8. The only self-dual smooth non linear projective toric varieties equivariantly embedded are the toric hypersurfaces described in (2) and (3) of Lemma 5.4 and the Segre embeddings $\mathbb{P}^{1} \times \mathbb{P}^{m-1}$ in $\mathbb{P}^{2 m-1}$ for $m \geq 3$.

Proof. We proceed by induction in the codimension of $A$. By Lemma 5.4 , the result is true when $X_{A}$ is a hypersurface. Assume then that $\operatorname{codim}\left(X_{A}\right)>1$. Now, by Lemma 5.6, we know that all the points in $A$ are vertices of $\operatorname{Conv}(A)$. Let $C$ be a coparallelism class and let $D:=A \backslash C$. Then, $X_{D}$ is smooth and it is non pyramidal. Indeed, we can choose a Gale dual $B$ of $A$ of the form:

$$
B=\left(\begin{array}{c|c}
b_{11} & \\
\vdots & 0 \\
\frac{b_{r 1}}{b_{r+1,1}} & \\
\vdots & D_{2} \\
b_{n 1} &
\end{array}\right)
$$

where $\left(b_{11}, 0\right), \ldots,\left(b_{r 1}, 0\right)$ correspond to the elements of $C$. If $D$ is a pyramid, is is easy to show that at least one row of $D_{2}$ must be zero, and it follows that the corresponding point of the configuration belongs also to $C$, and thus is a contradiction.

Hence, it follows from Proposition 4.20 that $X_{D}$ is self-dual with $\operatorname{codim}\left(X_{D}\right)=$ $\operatorname{codim}\left(X_{A}\right)-1<\operatorname{codim}\left(X_{A}\right)$ and no point of $D$ belongs to the relative interior of $\operatorname{Conv}(D)$. Therefore, by induction, $X_{D}$ is the Segre embedding of $\mathbb{P}^{1} \times \mathbb{P}^{m^{\prime}-1}$
in $\mathbb{P}^{2 m^{\prime}-1}$ for $m^{\prime} \geq 2$ (including the hypersurface case $\mathbb{P}^{1} \times \mathbb{P}^{1}$ ). In particular, $|D|=2 m^{\prime}$ is even.

Assume $C=\left\{\mu_{1}, \ldots, \mu_{r}\right\}$. Let $B_{D} \in \mathbb{Z}^{2 m^{\prime} \times\left(m^{\prime}-1\right)}$ be a choice of Gale dual of $D$ as in Lemma 5.7. with rows $e_{1}^{\prime}, \ldots, e_{m^{\prime}}^{\prime},-e_{1}^{\prime}, \ldots,-e_{m^{\prime}}^{\prime}$ with $\left\{e_{1}^{\prime}, \ldots, e_{m^{\prime}-1}^{\prime}\right\}$ a basis of $\mathbb{Z}^{m^{\prime}-1}$ and $e_{1}^{\prime}+\cdots+e_{m}^{\prime}=0$. Add another integer affine relation with coprime entries as the first column, to form a matrix $B^{\prime}$ whose columns are a $\mathbb{Q}$-basis of relations of $A$ of the form:

$$
B^{\prime}=\left(\begin{array}{c|c}
B_{1} & 0 \\
\hline B_{2} & B_{D}
\end{array}\right)
$$

Now, each coparallelism class of any $\mu \in D$ (with respect to $D$ ) has two elements when $m^{\prime}>2$, and so it cannot be "broken" when considering coparallelism classes in $A$, since it is not a pyramid. Then, via column operations we can assume that $B_{2}$ is of the form $B_{2}^{t}=(0, \ldots, 0, a, 0, \ldots, 0,-a),\left(a \in \mathbb{Z}_{\geq 0}\right)$. In case $m^{\prime}=2$, then $B_{D}^{t}=(1,-1,-1,1)$ and the unique coparallelism class could be broken, but at most in two pieces with two elements each, and again we have the same formulation for $B_{2}$. In both cases, if $a=0$, then we have a splitting, which implies that either there is a repeated point (if $B_{1}^{t}=(1,-1)$ ) or $X_{A}$ is not smooth. Then $a \geq 1$. Consider the subconfiguration $E$ of $A$ obtained by forgetting the two columns corresponding to the rows $m^{\prime}$ and $2 m^{\prime}$ of $B_{D}$. Since the vectors $b_{i}$ with complementary indices add up to zero, it follows that $E$ is facial and again, $X_{E}$ is smooth. We deduce that $a=1$ and $B_{1}^{t}= \pm(1,-1)$, which implies that $X_{A}$ is the Segre embedding of $\mathbb{P}^{1} \times \mathbb{P}^{m^{\prime}+1}$ in $\mathbb{P}^{2 m^{\prime}+1}$.
5.2. Non Lawrence families of examples. We have the following obvious corollaries of Theorem 5.8.

Corollary 5.9. Let $A \in \mathcal{M}_{d \times n}(\mathbb{Z})$ with maximal rank $d$ associated to a regular configuration of weights and let $X_{A} \subset \mathbb{P}^{n-1}$ be the projective toric variety associated to $A$. Assume $X_{A}$ is not a hypersurface, non linear, smooth and self-dual. Then, $n$ is even.

As the defect of the Segre embedding $X_{m}=\mathbb{P}^{1} \times \mathbb{P}^{m-1}$ in $\mathbb{P}^{2 m-1}$ for any $m \geq 2$ equals $2 m-2-m=m-2=\operatorname{dim} X_{m}-2$, we recover for smooth varieties $X_{A}$ the following result, due to Landman ([10]) for any projective smooth variety.

Corollary 5.10. [Landman] If $X_{A} \subset \mathbb{P}^{n-1}$ is a non linear smooth projective variety such that $\operatorname{dim} X<n-2$ with defect $k>0$, then $\operatorname{dim} X \equiv k(2)$.

We use the previous corollaries together with Theorem 4.4 to construct families of non regular self-dual varieties.

Example 5.11. Consider the families of matrices $\left\{A_{\alpha}\right\},\left\{B_{\alpha}\right\}$ for $\alpha \in \mathbb{Z}, \alpha \neq 0$, defined by:

$$
A_{\alpha}=\left(\begin{array}{rrrrrrr}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & \alpha & 0 & -\alpha & 0 \\
0 & 0 & 1 & 0 & -\alpha & 0 & \alpha
\end{array}\right), \quad B_{\alpha}=\left(\begin{array}{rrr}
2 \alpha & 0 \\
-\alpha & 0 \\
-\alpha & 0 \\
1 & 1 \\
-1 & -1 \\
0 & 1 \\
0 & -1
\end{array}\right) .
$$

Clearly, $B_{\alpha}$ is a choice of a Gale dual matrix of $A_{\alpha}$.

Observe that as $\alpha \neq 0$, the configuration $A_{\alpha}$ is not a pyramid and $\operatorname{dim}\left(X_{A_{\alpha}}\right)=4$. Moreover, it is easy to show that if $\alpha \neq \alpha^{\prime}$, then $X_{A_{\alpha}}$ and $X_{A_{\alpha^{\prime}}}$ are not isomorphic as embedded varieties because they have different degrees. The degree of $X_{A_{\alpha}}$ is the normalized volume of the convex hull of the points in the configuration $A_{\alpha}$ ([16]) and it can be computed easily in terms of the Gale dual configuration.

Since the conditions of Theorem 4.4 hold, it follows that $X_{A_{\alpha}}$ is self-dual for all $\alpha \in \mathbb{Z}, \alpha \neq 0$. Moreover, $n=7$ is odd and so we deduce from Corollary 5.9 that $X_{A_{\alpha}}$ is a singular variety. The difference between its dimension and its defect is $4-1=3 \not \equiv 0(2)$.

We can generalize Example 5.11 in order to construct families of non degenerate projective toric self-dual varieties of arbitrary dimension greater than or equal to 3 and of arbitrary codimension greater than or equal to 2 .

Example 5.12. Families of self-dual varieties of any dimension $\geq 3$. Let any $r \geq 2$ and $\alpha_{1}, \ldots, \alpha_{r}$ non zero integer numbers satisfying $\sum_{i=1}^{r} \alpha_{i}=0$. Consider the planar lattice configuration

$$
\mathcal{G}_{\alpha}=\left\{\left(\alpha_{1}, 0\right), \ldots,\left(\alpha_{r}, 0\right),(0,1),(0,-1),(1,1),(-1,-1)\right\} .
$$

Let $A$ be any lattice configuration with Gale dual $\mathcal{G}_{\alpha}$. Then, $A$ is not a pyramid and the associated projective toric variety $X_{A} \subset \mathbb{P}^{r+3}$ is self-dual by Theorem 4.4, with dimension $\operatorname{dim} X_{A}=(r+4)-2-1=r+1$.

When $r=2$, the dimension of $X_{A_{\alpha}}$ is 3 . The case $\alpha_{1}, \alpha_{2}= \pm 1$ corresponds to the Segre embedding of $\mathbb{P}^{1} \times \mathbb{P}^{2}$ in $\mathbb{P}^{5}$. Already for $\alpha_{1}, \alpha_{2}= \pm 2$, the configuration $A_{\alpha}$ does not contain all the lattice points in its convex hull. If we add those "remaining" points to the configuration, the associated toric variety is no longer self-dual.

Example 5.13. Families of self-dual varieties of any codimension $\geq 2$. Using the same ideas of the previous example, we can construct pairs $(A, B)$ with $A$ a non pyramidal configuration and $B$ its Gale dual satisfying the hypothesis of Theorem 4.4, so that $X_{A}$ is self-dual, with arbitrary codimension $m \geq 2$.

For any $r \geq 2$ set $n=2 m+r$. As usual, $e_{1}, \ldots, e_{m}$ denotes the canonical basis in $\mathbb{Z}^{m}$. For any choice of non zero integers $\alpha_{1}, \ldots, \alpha_{r}$ with $\sum_{i=1}^{r} \alpha_{i}=0$ consider the following lattice configuration in $\mathbb{Z}^{m}$ :

$$
\mathcal{G}_{\alpha}:=\left\{\alpha_{1} e_{1}, \ldots, \alpha_{r} e_{1}, e_{2},-e_{2}, \ldots, e_{m},-e_{m}, e_{1}+\cdots+e_{m},-\left(e_{1}+\cdots+e_{m}\right)\right\} .
$$

For any lattice configuration $A_{\alpha} \subset \mathbb{Z}^{n}$ with this Gale dual, $A_{\alpha}$ is not a pyramid and its associated self-dual toric variety $X_{A_{\alpha}} \subset \mathbb{P}^{n}$ has dimension $m+r-1$ and codimension $m$.

## 6. Strongly self-dual varieties

We are interested now in characterizing a particular interesting case of self-dual projective toric varieties.

Definition 6.1. Let $A$ be a regular lattice configuration without repetitions. We say that the projective variety $X_{A} \subset \mathbb{P}^{n-1}$ is strongly self-dual if $X_{A}$ coincides with $X_{A}^{*}$ under the canonical identification between $\mathbb{P}^{n-1}$ and its dual projective space as in Remark 2.20

We deduce from Theorem 3.2 the following characterization of strongly self-dual projective toric varieties of the form $X_{A}$.

Proposition 6.2. Let $A$ be a regular lattice configuration without repetitions. Then $X_{A}$ is strongly self-dual if and only if $\mathbb{P}\left(\mathcal{R}_{A, k}\right) \subset X_{A}$.

Proof. If $X_{A}$ is strongly self-dual, the containment $\mathbb{P}\left(\mathcal{R}_{A, \mathrm{k}}\right) \subset X_{A}^{*}$ implies that the condition $\mathbb{P}\left(\mathcal{R}_{A, \mathbb{k}}\right) \subset X_{A}$ is necessary.

Assume that this condition holds and $A$ has no repetitions. As we already observed, Theorem 3.8 implies that $A$ is not pyramidal. Then, it follows from Theorem 3.2 that for any $q \in \mathbb{P}\left(\mathcal{R}_{A, \mathbb{k}}\right) \cap \mathbb{T}^{n-1} \subset X_{A} \cap \mathbb{T}^{n-1}, m_{q}\left(X_{A}\right)=X_{A}^{*}$. But since $q \in \mathcal{O}([1: \cdots: 1])$, we deduce that $m_{q}\left(X_{A}\right)=\overline{\mathcal{O}(q)}=\overline{\mathcal{O}([1: \cdots: 1])}=X_{A}$, that is $X_{A}^{*}=X_{A}$.

Using the same notation of Theorem 4.4, we have:
Theorem 6.3. Let $A$ be a non pyramidal regular lattice configuration $A$ of $n$ weights spanning $\mathbb{Z}^{d}$ and let $B_{A}$ be a Gale dual of $A$. Then:

$$
X_{A} \text { is strongly self-dual } \Leftrightarrow\left\{\begin{array}{l}
\text { (a) For any line } L \text { through the origin } \\
\text { we have } \sum_{\substack{ }} b_{i}=0 . \\
\text { (b) } \prod_{\substack{j=1 \\
b_{j i}>0}}^{n} b_{j i}^{b_{j i}}=\prod_{\substack{j=1 \\
b_{j i}<0}}^{n} b_{j i}^{-b_{j i}}, i=1, \ldots, n-d .
\end{array}\right.
$$

In the above statement, we use the convention that $0^{0}=1$.
Proof. Assume that $X_{A}$ is strongly self-dual. Then (a) holds by Theorem 4.4. By Proposition 6.2 , we know that $\mathbb{P}\left(\mathcal{R}_{A, k}\right) \cap \mathbb{T}^{n-1} \subset X_{A} \cap \mathbb{T}^{n-1}$, and this last variety is cut out by the $(n-d)$ binomials

$$
\prod_{\substack{j=1 \\ b_{j i}>0}}^{n} x_{j}^{b_{j i}}=\prod_{\substack{j=1 \\ b_{j i}<0}}^{n} x_{j}^{-b_{j i}}, \quad \forall i=1, \ldots, n-d
$$

Then, we have the following equalities, for all $s \in \mathbb{k}^{n-d}$ :

$$
\begin{equation*}
\prod_{\substack{j=1 \\ b_{j i}>0}}^{n}\left\langle s, b_{j}\right\rangle^{b_{j i}}=\prod_{\substack{j=1 \\ b_{j i}<0}}^{n}\left\langle s, b_{j}\right\rangle^{-b_{j i}}, \forall i=1, \ldots, n-d . \tag{5}
\end{equation*}
$$

We get the conditions (b) by evaluating respectively at $s=e_{1}, \ldots, e_{n-d}$.
Conversely, condition (a) implies the equalities (5) of the polynomials in $s$ on both sides up to constant, as in Remark 4.3. Then, condition (b) ensures that this constant is 1 . Therefore, $\mathbb{P}\left(\mathcal{R}_{A, \mathrm{k}}\right) \cap \mathbb{T}^{n-1} \subset X_{A} \cap \mathbb{T}^{n-1}$, and so $X_{A}$ is strongly self-dual by Proposition 6.2 .
Example 6.4. Consider the matrix $A=\left(\begin{array}{rrrrrrrrr}1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1\end{array}\right)$. Observe that $A$ is non pyramidal. A Gale dual matrix $B_{A}$ for $A$ is given by the transpose of the matrix $\left(\begin{array}{rrrrrrrrr}-2 & -2 & -2 & -2 & 4 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 & -2 & -1 & 0 & -1 & 0\end{array}\right)$.

Clearly, $B_{A}$ satisfies the conditions of Theorem 6.3 and hence $X_{A}$ is strongly self-dual. But note that $A$ is not a Lawrence configuration.

We conclude this section with the complete characterization of strongly self-dual varieties of type $X_{A}$, with $A$ a non pyramidal Lawrence matrix.

Theorem 6.5. Let $A$ be a non pyramidal Lawrence configuration consisting of $2 n$ points in $\mathbb{Z}^{n+d}$, as in (3). Then $X_{A}$ is strongly self-dual if and only if there exists a subset $I$ of rows of the lower matrix $M=\left(m_{j k}\right)$ such that $\sum_{j \in I} m_{j k}$ is an odd number for all $k=1, \ldots, n$.

Proof. By Corollary 5.2, $X_{A}$ is self-dual for any non pyramidal Lawrence configuration $A$. Thus, $X_{A}$ is strongly self-dual if and only if conditions (b) in Theorem 6.3 are satisfied. If $\mathcal{G}_{M}=\left\{c_{1}, \ldots, c_{n}\right\} \subset \mathbb{Z}^{n-d}$ is a Gale dual configuration for $M$, then $\left\{-c_{1}, \ldots,-c_{n}, c_{1}, \ldots, c_{n}\right\}$ defines a Gale dual configuration for $A$. Conditions (b) are then equivalent in this case to the equalities

$$
(-1)^{\sum_{j=1}^{n} c_{j i}}=0, \quad i=1, \ldots, n-d
$$

This is in turn equivalent to the condition that for all $v \in \mathcal{R}_{M}$, the sum $\sum_{j=1}^{n} v_{j} \equiv$ 0 (2). But this is equivalent to the fact that the vector $(1, \ldots, 1)$ lies in the row span of $M$ when we reduce all its entries modulo 2 . Denoting classes in $\mathbb{Z}_{2}$ with an over-line, this condition means that there exist $\alpha_{1}, \ldots, \alpha_{d} \in \mathbb{Z}_{2}=\{0,1\}$ such that

$$
(1, \ldots, 1)=\sum_{i=1}^{d} \alpha_{i}\left(\overline{m_{i 1}}, \ldots, \overline{m_{i n}}\right)=\sum_{\alpha_{i}=1}\left(\overline{m_{i 1}}, \ldots, \overline{m_{i n}}\right)
$$

It suffices to call $I=\left\{i \in\{1, \ldots, d\}: \alpha_{i}=1\right\}$.
Example 6.6. The Segre embeddings in Example 5.3 have associated Lawrence matrices as in (4), where $M$ is a matrix with a single row of with all entries equal to 1 . They clearly satisfy the hypotheses of Theorem 6.5. Then, for any $m>1$, the Segre embedding of $\mathbb{P}^{1} \times \mathbb{P}^{m-1}$ is a strongly-self dual projective toric variety.

## References

[1] A. Borel, Linear Algebraic Groups, second edition. Graduate Texts in Mathematics, 126. New York, Springer, 1991.
[2] D. Bayer, S. Popescu and B. Sturmfels, Syzygies of Unimodular Lawrence Ideals, J. Reine Angew. Math. 534 (2001), 169-186.
[3] C. Casagrande, S. Di Rocco, Projective $Q$-factorial toric varieties covered by lines, Commun. Contemp. Math. 10 (2008), no. 3, 363-389. arXiv:math.AG/0512385
[4] E. Cattani, A. Dickenstein, B. Sturmfels, Rational Hypergeometric Functions, Compositio Math. 128 (2001), no. 2, 217-239.
[5] D. Cox, J. Little, H. Schenk, Toric Varieties, Graduate Studies in Mathematics, Amer. Math. Soc., Providence, RI, to appear.
[6] R. Curran, E. Cattani, Restriction of A- Discriminants and dual defect toric varieties, J. Symbolic Comput. 42 (2007), no. 1-2, 115-135.
[7] A. Dickenstein, B. Sturmfels, Elimination theory in codimension two, Journal of Symbolic Computation, 34 (2002), 119-135.
[8] A. Dickenstein, E.M. Feichtner, B. Sturmfels, Tropical Discriminants, J. Amer. Math. Soc. 20 (2007), no. 4, 1111-1133.
[9] S. Di Rocco, Projective duality of toric manifolds and defect polytopes. Proc. London Math. Soc. (3) 93 (2006), no. 1, 85-104.
[10] L. Ein, Varieties with small dual varieties, I, Invent. Math. 86 (1986), no. 1, 63-74.
[11] L. Ein, Varieties with small dual varieties, II, Duke. Math. J. 52 (1985), no. 4, 895-907.
[12] I.M. Gelfand; M.M. Kapranov; A.V. Zelevinsky, Discriminants, resultants, and multidimensional determinants. Boston, Birkhäuser, 1994.
[13] I.M. Gelfand; M.M. Kapranov y A.V. Zelevinsky, Hypergeometric functions and toral manifolds. Funct. Anal. Appl. 23 (1989), no. 2, 94-106.
[14] V.L. Popov, Self-dual algebraic varieties and nilpotents orbits. Proceedings of the International Colloquium on Algebra, Arithmetic and Geometry, Mumbai, 2000, Tata Inst. Fund. Research, Narosa Publ. House, 2002, 509-533.
[15] V.L. Popov; E.A. Tevelev, Self-dual projective algebraic varieties associated with symmetric spaces. In: Popov, Vladimir L. (ed.), Algebraic groups and algebraic varieties, Encyclopaedia of Mathematical Sciences, Vol. 132, Invariant Theory and Algebraic Transformation Groups Vol. III, Springer Verlag, 131-167 (2004).
[16] B. Sturmfels, Gröbner bases and convex polytopes, University Lecture Series, 8. American Mathematical Society, Providence, RI, 1996. xii+162 pp..
[17] E. Tevelev, Projective duality and homogeneous spaces, Encyclopaedia of Mathematical Sciences, 133. Invariant Theory and Algebraic Transformation Groups 4. New York, Springer, 2005.
[18] F. Zak, Projection of algebraic varieties, (Russian) Mat. Sb. (N.S.) 116(158) (1981), no. 4, 593-602, 608. English translation: Math. Sbornik 44 (1983), 535-544.
[19] G. Ziegler, Lectures on polytopes, GTM, 152. Springer-Verlag, New York, 1995. x+370 pp.

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