

INVERSE SYSTEMS, GELFAND-TSETLIN PATTERNS AND THE WEAK LEFSCHETZ PROPERTY

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ABSTRACT. In [19], Migliore–Miró-Roig–Nagel show that the Weak Lefschetz property can fail for an ideal $I \subseteq \mathbb{K}[x_1, \dots, x_4]$ generated by powers of linear forms. This is in contrast to the analogous situation in $\mathbb{K}[x_1, x_2, x_3]$, where WLP always holds [24]. We use the inverse system dictionary to connect I to an ideal of fat points, and show that failure of WLP for powers of linear forms is connected to the geometry of the associated fat point scheme. Recent results of Sturmfels–Xu in [26] allow us to relate WLP to Gelfand–Tsetlin patterns.

1. INTRODUCTION

Let $I \subseteq S = \mathbb{K}[x_1, \dots, x_r]$ be an ideal such that $A = S/I$ is Artinian. Then A has the *Weak Lefschetz Property* (WLP) if there is an $\ell \in S_1$ such that for all m , the map $\mu_\ell : A_m \xrightarrow{-\ell} A_{m+1}$ is either injective or surjective. We assume $\text{char}(\mathbb{K}) = 0$; this simplifies our use of inverse systems. The case $r = 1$ is trivial, and WLP always holds for $r = 2$ [15]. For $r = 3$, WLP holds for ideals of generic forms [2], complete intersections [15], ideals with semistable syzygy bundle and certain splitting type, and almost complete intersections with unstable syzygy bundle [3], certain monomial ideals [19] and ideals generated by powers of linear forms [24]. The following example of Migliore–Miró-Roig–Nagel [19] shows that the result of [24] can fail for $r \geq 4$, and motivates this paper.

Example 1.1. $\mathbb{K}[x_1, x_2, x_3, x_4]/\langle x_1^3, x_2^3, x_3^3, x_4^3, (x_1 + x_2 + x_3 + x_4)^3 \rangle$ does not have WLP. The Hilbert function of A is $(1, 4, 10, 15, 15, 6)$, and $A_3 \rightarrow A_4$ is not full rank.

This example is explained by the following result, proved in §3.

Proposition 1.2. *For generic forms $l_i \in S_1$ with $A = S/\langle l_1^t, \dots, l_n^t \rangle$ Artinian, the map $A_t \rightarrow A_{t+1}$ has full rank iff $(r, t, n) \notin \{(4, 3, 5), (5, 3, 9), (6, 3, 14), (6, 2, 7)\}$.*

The failure of WLP in Example 1.1 stems from the fact that the space of quartics in \mathbb{P}^2 passing through five double points is nonempty: WLP fails for geometric reasons. We use inverse systems to translate questions about powers of linear forms to questions about ideals of fatpoints. Then results of Alexander–Hirschowitz [1], Nagata [23] and De Volder–Laface [6] can be applied to the syzygy bundle [15], which allows us to analyze WLP for $r = 4$ when $n = 5, 6, 7, 8$ (see §4):

Theorem 1.3. *Let $I = \langle l_1^t, \dots, l_n^t \rangle \subseteq \mathbb{K}[x_1, x_2, x_3, x_4]$ with $l_i \in S_1$ generic. If $n \in \{5, 6, 7, 8\}$, then WLP fails, respectively, for $t \geq \{3, 27, 140, 704\}$.*

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This is surprising: for $I \subseteq \mathbb{K}[x_1, x_2, x_3, x_4]$ generated by general forms, Migliore and Miro-Roig show in [21] that the quotient ring always has WLP. It also contrasts to most known cases of powers of linear forms: WLP always holds in the three variable case [24] and for complete intersections (i.e., $r = n$). The result on complete intersections is due to Stanley [25], who showed that if $I = \langle l_1^{t_1}, \dots, l_n^{t_n} \rangle$ is a complete intersection, then S/I has the strong Lefschetz property. In §5 we use this and results of D'Cruz-Iarrobino [5] to prove

Theorem 1.4. *For $I = \langle l_1^t, \dots, l_{r+1}^t \rangle \subseteq \mathbb{K}[x_1, \dots, x_r]$ with $l_i \in S_1$ generic, $r = 2k$, $k \geq 2$ and $t \gg 0$, WLP fails in degree $\frac{r}{2}(t-1) - 1$.*

Migliore-Miro-Roig-Nagel [20] have recently strengthened this result to hold for all t . They also obtain very precise results on WLP for almost complete intersections for $r = 4, 5$, when the powers of the linear forms are not uniform. Using a result of Sturmfels-Xu on Gelfand-Tsetlin patterns, we obtain partial results for r odd. Based on our results and computational evidence, we believe

Conjecture 1.5. *For $I = \langle l_1^t, \dots, l_n^t \rangle \subseteq \mathbb{K}[x_1, \dots, x_r]$ with $l_i \in S_1$ generic and $n \geq r+1 \geq 5$, WLP fails for all $t \gg 0$.*

2. BACKGROUND

2.1. Inverse systems. In [7], Emsalem and Iarrobino proved that there is a close connection between ideals generated by powers of linear forms, and ideals of fatpoints. Let $p_i = [p_{i1} : \dots : p_{ir}] \in \mathbb{P}^{r-1}$, $I(p_i) = \wp_i \subseteq R = \mathbb{K}[y_1, \dots, y_r]$, and $\{p_1, \dots, p_n\} \subseteq \mathbb{P}^{r-1}$ be a set of distinct points. A fat point ideal is an ideal of the form

$$F = \bigcap_{i=1}^n \wp_i^{\alpha_i+1} \subset R.$$

Recall $S = \mathbb{K}[x_1, \dots, x_r]$ and let $L_{p_i} = \sum_{j=1}^r p_{ij} x_j$. Define an action of R on S by partial differentiation: $y_j \cdot x_i = \partial x_i / \partial x_j$. Since F is a submodule of R , it acts on S . The set of elements annihilated by the action of F is denoted F^{-1} . Emsalem and Iarrobino show that for $j \geq \max\{\alpha_i + 1\}$, $(F^{-1})_j = \langle L_{p_1}^{j-\alpha_1}, \dots, L_{p_m}^{j-\alpha_m} \rangle_j$, and that $\dim_{\mathbb{K}}(F^{-1})_j = \dim_{\mathbb{K}}(R/F)_j$. This generalizes Terracini's lemma, where the α_i are all two. For more on inverse systems, see [8].

Theorem 2.1.1 (Emsalem and Iarrobino, [7]). *Let F be an ideal of fatpoints:*

$$F = \wp_1^{\alpha_1+1} \cap \dots \cap \wp_n^{\alpha_n+1} \subset R.$$

Then

$$(F^{-1})_j = \begin{cases} S_j & \text{for } j \leq \max\{\alpha_i\} \\ L_{p_1}^{j-\alpha_1} S_{\alpha_1} + \dots + L_{p_n}^{j-\alpha_n} S_{\alpha_n} & \text{for } j \geq \max\{\alpha_i + 1\} \end{cases}$$

and

$$\dim_{\mathbb{K}}(F^{-1})_j = \dim_{\mathbb{K}}(R/F)_j.$$

The following corollary is just a special case version of Theorem 2.1.1, but one that we will use repeatedly.

Corollary 2.1.2. *Let $t \geq 1$ be an integer, let*

$$J = \wp_1^{j-t+1} \cap \dots \cap \wp_n^{j-t+1} \subset R$$

be an ideal of fatpoints and consider the ideal $I = \langle L_{p_1}^t, \dots, L_{p_n}^t \rangle \subset S$. Then

$$\dim_{\mathbb{K}} I_j = \begin{cases} \dim_{\mathbb{K}}(R/J)_j & \text{for } j \geq t \\ 0 & \text{for } 0 \leq j < t \end{cases}$$

and hence

$$\dim_{\mathbb{K}}(S/I)_j = \begin{cases} \dim_{\mathbb{K}} J_j & \text{for } j \geq t \\ \binom{r-1+j}{r-1} & \text{for } 0 \leq j < t \end{cases}$$

Note that to obtain the Hilbert function of a fixed ideal of linear forms, it is necessary to consider an infinite family of ideals of fat points.

Example 2.1.3. Here we apply Corollary 2.1.2 to obtain the Hilbert function for A from Example 1.1:

j	0	1	2	3	4	5	6	...
$\dim_{\mathbb{K}} A_j$	1	4	10	15	15	6	0	...
$HF(\cap_{i=1}^5 \wp_i^{j-2}, j)$	0	0	0	15	15	6	0	...

We consider the restriction of this example to \mathbb{P}^2 in Example 2.2.1.

2.2. Blowups of points in projective space. There is a well-known correspondence between the graded pieces of an ideal of fat points $F \subseteq \mathbb{K}[x_1, \dots, x_r]$ and the global sections of a line bundle on the variety X which is the blow up of \mathbb{P}^{r-1} at the points. We briefly review this. Let E_i be the class of the exceptional divisor over the point p_i , and E_0 the pullback of a hyperplane on \mathbb{P}^{r-1} . Given non-negative integers m_i , consider the fatpoints ideal $J = \wp_1^{m_1} \cap \dots \cap \wp_n^{m_n} \subset R$ and let

$$D = jE_0 - \sum_{i=1}^n m_i E_i.$$

Of course, $\dim_{\mathbb{K}} J_j = h^0(\mathbb{P}^{r-1}, \mathcal{I}_Z(j))$, where $\mathcal{I}_Z(j)$ is the ideal sheaf of the fat-points subscheme Z defined by F . Moreover, by [11, Proposition 4.1.1], $h^i(X, D) = h^i(\mathbb{P}^{r-1}, \mathcal{I}_Z(j))$ for all $i \geq 0$. Taking cohomology of the exact sequence

$$0 \longrightarrow \mathcal{I}_Z(j) \longrightarrow \mathcal{O}_{\mathbb{P}^{r-1}}(j) \longrightarrow \mathcal{O}_Z(j) \longrightarrow 0$$

and using the fact that $\mathcal{O}_Z(j) \cong \mathcal{O}_Z$ and thus $h^0(Z, \mathcal{O}_Z(j)) = h^0(Z, \mathcal{O}_Z) = \sum_i \binom{r-2+m_i}{r-1}$, shows that

$$(1) \quad h^0(X, D) = h^0(\mathcal{I}_Z(j)) = \binom{r-1+j}{r-1} - \sum_i \binom{r-2+m_i}{r-1} + h^1(\mathcal{I}_Z(j)).$$

In the context of Corollary 2.1.2, taking $m_i = j - t + 1$ for all i and defining D_j to be $D_j = jE_0 - (j - t + 1)(E_1 + \dots + E_n)$, we thus have:

$$(2) \quad \dim_{\mathbb{K}} I_j = \begin{cases} n \binom{r+j-t-1}{r-1} - h^1(\mathcal{I}_Z(j)) = n \binom{r+j-t-1}{r-1} - h^1(D_j) & \text{for } j \geq t \\ 0 & \text{for } 0 \leq j < t \end{cases}$$

Alternatively, this can be stated for the quotient $S/I = A$ as:

$$(3) \quad \dim_{\mathbb{K}} A_j = \begin{cases} h^0(D_j) & \text{for } j \geq t \\ \binom{r-1+j}{r-1} & \text{for } 0 \leq j < t \end{cases}$$

We will say that I has *expected dimension* in degree j if either $I_j = 0$ or $h^1(D_j) = 0$. We say D_j is *irregular* if $h^1(D_j) > 0$ and *regular* otherwise. We say D_j is *special* if $h^0(D_j)$ and $h^1(D_j)$ are both positive.

Example 2.2.1. Let A be the quotient of $\mathbb{K}[x_1, x_2, x_3]$ by the cubes of five general linear forms. The corresponding five points in \mathbb{P}^2 are general, and the first interesting computation involves $D_4 = 4E_0 - \sum_{i=1}^5 2E_i$, for which we have

$$\dim_{\mathbb{K}} A_4 = h^0(D_4) = \binom{6}{2} - 15 + h^1(D_4).$$

Since $H^0(D_4)$ contains the double of a conic through the five points, D_4 is special, and in fact we have $h^0(D_4) = 1 = h^1(D_4)$.

2.3. WLP and the syzygy bundle. In [15], Harima-Migliore-Nagel-Watanabe study WLP using the syzygy bundle:

Definition 2.3.1. If $I = \langle f_1, \dots, f_n \rangle$ is $\langle x_1, \dots, x_r \rangle$ -primary, and $\deg(f_i) = d_i$, then the syzygy bundle $\mathcal{S}(I) = \widetilde{\text{Syz}}(I)$ is a rank $n-1$ bundle defined via

$$(4) \quad 0 \longrightarrow \text{Syz}(I) \longrightarrow \bigoplus_{i=1}^n \mathcal{S}(-d_i) \xrightarrow{[f_1, \dots, f_n]} \mathcal{S} \longrightarrow \mathcal{S}/I \longrightarrow 0.$$

or, equivalently, by

$$(5) \quad 0 \longrightarrow \text{Syz}(I) \longrightarrow \bigoplus_{i=1}^n \mathcal{S}(-d_i) \longrightarrow I \longrightarrow 0$$

Let ℓ be a generic form in S_1 with $L = V(\ell)$, and I an ideal such that $A = S/I$ is Artinian. Sheafifying Equation (4) and twisting gives

$$0 \longrightarrow \mathcal{S}(I)(m) \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^{r-1}}(m - d_i) \longrightarrow \mathcal{O}_{\mathbb{P}^{r-1}}(m) \longrightarrow 0.$$

Taking cohomology shows that

$$(6) \quad A = \bigoplus_{m \in \mathbb{Z}} H^1(\mathcal{S}(I)(m)),$$

since A and $\bigoplus_{m \in \mathbb{Z}} H^1(\mathcal{S}(I)(m))$ both are direct sums of cokernels of the same maps on global sections. Similarly,

$$(7) \quad \text{Syz}(I) \simeq \bigoplus_t H^0(\mathcal{S}(I)(t)),$$

since $\text{Syz}(I)$ and $\bigoplus_t H^0(\mathcal{S}(I)(t))$ both are direct sums of kernels of the same maps on global sections. From Equation (5) we also see that

$$(8) \quad \dim_{\mathbb{K}} I_j = \sum_i \binom{j - d_i + r - 1}{r - 1} - \dim_{\mathbb{K}} \text{Syz}(I)_j.$$

In case $f_i = L_{P_i}^t$ for a set of distinct points P_i , setting $D_j = jE_0 - (j - t + 1)(E_1 + \dots + E_n)$ and comparing with Equation (2) shows that

$$(9) \quad h^0(\mathcal{S}(I)(j)) = \dim_{\mathbb{K}} \text{Syz}(I)_j = h^1(D_j)$$

for $j \geq t$.

Since $\mathcal{S}(I)$ is a bundle, tensoring the sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{r-1}}(m) \longrightarrow \mathcal{O}_{\mathbb{P}^{r-1}}(m+1) \longrightarrow \mathcal{O}_L(m+1) \longrightarrow 0$$

with $\mathcal{S}(I)$ gives the exact sequence

$$0 \longrightarrow \mathcal{S}(I)(m) \longrightarrow \mathcal{S}(I)(m+1) \longrightarrow \mathcal{S}(I)|_L(m+1) \longrightarrow 0.$$

The long exact sequence in cohomology yields a sequence

$$(10) \quad \begin{array}{ccccccc} 0 & \longrightarrow & H^0(\mathcal{S}(I)(m)) & \longrightarrow & H^0(\mathcal{S}(I)(m+1)) & \xrightarrow{\phi_m} & H^0(\mathcal{S}(I)|_L(m+1)) \\ & & & & & \nearrow & \\ & & H^1(\mathcal{S}(I)(m)) & \xrightarrow{\mu_\ell} & H^1(\mathcal{S}(I)(m+1)) & \longrightarrow & H^1(\mathcal{S}(I)|_L(m+1)) \\ & & & & \searrow \psi_m & & \\ & & H^2(\mathcal{S}(I)(m)) & \xrightarrow{\quad} & H^2(\mathcal{S}(I)(m+1)) & \longrightarrow & \cdots \end{array}$$

Surjectivity of μ_ℓ in degree m follows from injectivity of ψ_m , and injectivity of μ_ℓ from surjectivity of ϕ_m . In particular, μ_ℓ is injective in degree m if $h^0(\mathcal{S}(I)|_L(m+1)) = 0$.

Remark 2.3.2. In the situation that f_1, \dots, f_n are t^{th} powers of linear forms L_{P_i} , we can understand $\mathcal{S}(I)|_L$ recursively. Without loss of generality, we may assume $\ell = x_r$. Quotienting by the ideal $(\ell) \subset S$ gives an image ideal $I' = I \otimes S' \subset S' = S/(\ell)$ that is itself generated by t^{th} powers of linear forms (distinct since ℓ is generic), these being the images under the quotient of the generators of I . We let A' denote S'/I' . If $D_j = jE_0 - (j-t+1)(E_1 + \dots + E_n)$ is the divisor on the blow up of \mathbb{P}^{r-1} for the inverse system associated to I_j , we will denote by $D'_j = jE'_0 - (j-t+1)(E'_1 + \dots + E'_n)$ the divisor on the blow up of \mathbb{P}^{r-2} for the inverse system associated to I'_j . We also have $Syz(I') = Syz(I) \otimes S'$ and thus $\mathcal{S}(I') = \mathcal{S}(I)|_L = \mathcal{S}(I) \otimes S'$. Indeed, tensoring Equation (5) by S' yields the sequence

$$(11) \quad 0 \longrightarrow Tor_1^S(I, S') \longrightarrow Syz(I) \otimes S' \longrightarrow \bigoplus_{i=1}^n S'(-t) \longrightarrow I \otimes S' \longrightarrow 0.$$

But $Tor_1^S(I, S') = 0$ since it is the kernel of the injective map $I \xrightarrow{\mu_\ell} I(1)$, so

$$(12) \quad 0 \longrightarrow Syz(I') \longrightarrow \bigoplus_{i=1}^n S'(-d_i) \longrightarrow I' \longrightarrow 0$$

is exact, analogous to Equation (5). Thus we also have

$$(13) \quad A' = \bigoplus_{m \in \mathbb{Z}} H^1(\mathcal{S}(I')(m)) = \bigoplus_{m \in \mathbb{Z}} H^1(\mathcal{S}(I)|_L(m)),$$

$$(14) \quad Syz(I') = Syz(I) \otimes_S S' \simeq \bigoplus_t H^0(\mathcal{S}(I')(t)),$$

$$(15) \quad \dim_{\mathbb{K}} I'_j = \sum_i \binom{j-d_i+r-2}{r-2} - \dim_{\mathbb{K}} Syz(I')_j,$$

and, for $j \geq t$,

$$(16) \quad h^0(\mathcal{S}(I')(j)) = \dim_{\mathbb{K}} Syz(I')_j = h^1(D'_j).$$

Thus μ_ℓ is injective in degree m if $m+1 \geq t$ and $h^1(D'_{m+1}) = 0$, since by Equation (2) applied to I'_{m+1} and D'_{m+1} for \mathbb{P}^{r-2} we have $h^0(\mathcal{S}(I)|_L(m+1)) = h^1(D'_{m+1})$.

3. THE ALEXANDER-HIRSCHOWITZ THEOREM AND GENERIC FORMS

A landmark result on the dimension of linear systems is:

Theorem 3.1 (Alexander–Hirschowitz [1]). *Fix $m, r-1 \geq 2$, and consider the linear system of hypersurfaces of degree m in \mathbb{P}^{r-1} passing through n general points with multiplicity two. Then*

- (1) *For $m = 2$, the system is special iff $2 \leq n \leq r-1$.*
- (2) *For m greater than two, the only special systems are $(r-1, m, n) \in \{(2, 4, 5), (3, 4, 9), (4, 4, 14), (4, 3, 7)\}$. In each of these four cases, the linear system is expected to be empty but in fact has projective dimension 0.*

As a consequence of Theorem 3.1 and the developments from section 2, we have fairly complete information on WLP for quotients A by ideals of powers of n generic linear forms when n is not too small; specifically, we have:

Proposition 3.2. *Given generic linear forms l_i such that $I = \langle l_1^t, \dots, l_n^t \rangle$ and $A = \mathbb{K}[x_1, \dots, x_r]/I = S/I$ is Artinian, consider the maps $\mu_\ell : A_j \rightarrow A_{j+1}$ where $\ell = x_r$ and L is the hyperplane defined by ℓ .*

- (a) *For $0 \leq j \leq t-2$, $\mu_\ell : A_j \rightarrow A_{j+1}$ is injective but not surjective.*
- (b) *If $n \geq \binom{r-2+t}{r-2}$, then $\mu_\ell : A_j \rightarrow A_{j+1}$ is surjective for $j \geq t-1$.*
- (c) *The map $\mu_\ell : A_t \rightarrow A_{t+1}$ is injective if and only if $(r, t, n) \notin \{(4, 3, 5), (5, 3, 9), (6, 3, 14), (6, 2, 7)\}$.*
- (d) *If $n > \binom{r-2+t}{r-2}$, $A_{t-1} \xrightarrow{\mu_\ell} A_t$ is not injective, while if $n \geq \binom{r-2+t}{r-2}$, $A_t \xrightarrow{\mu_\ell} A_{t+1}$ is an isomorphism.*

Proof. (a) If $j \leq t-2$, then $I_{j+1} = I_j = 0$ and hence $A_j = S_j$ and $A_{j+1} = S_{j+1}$, but S is a domain with $\dim_{\mathbb{K}} S_j < \dim_{\mathbb{K}} S_{j+1}$.

(b) Let $S' = S/(\ell)$, $I' = I|_L$ and $A' = S'/I'$. Note that since $\text{char}(\mathbb{K}) = 0$, the locus of t^{th} powers of all linear forms in S' satisfies no non-trivial linear relation (this would be false if $\text{char}(\mathbb{K}) > 0$ and t were a power of the characteristic). Thus the span of the t^{th} powers of n generic linear forms has maximal dimension; i.e., its dimension is the minimum of n and the dimension $\binom{r-2+t}{r-2}$ of the space of all forms of degree t in $r-1$ variables. Since $n \geq \binom{r-2+t}{r-2}$, we see that $I'_{j+1} = S'_{j+1}$ for $j = t-1$ (and hence for $j \geq t-1$), hence $A'_{j+1} = 0$ for $j \geq t-1$.

Now by Equation (13) we have $H^1(\mathcal{S}(I)|_L(j+1)) = A'_{j+1} = 0$ for $j \geq t-1$, so by Equation (10) the map $A_j \xrightarrow{\mu_\ell} A_{j+1}$ is surjective.

(c) We now consider injectivity of $A_t \xrightarrow{\mu_\ell} A_{t+1}$. From the long exact sequence of Equation (10), we have

$$0 \rightarrow H^0(\mathcal{S}(I)(t)) \rightarrow H^0(\mathcal{S}(I)(t+1)) \rightarrow H^0(\mathcal{S}(I)|_L(t+1)) \rightarrow A_t \xrightarrow{\mu_\ell} A_{t+1}$$

Since I is generated in degree t , $h^0(\mathcal{S}(I)(t)) = 0$. Whenever $h^0(\mathcal{S}(I)(t+1)) < h^0(\mathcal{S}(I')(t+1))$ we thus see that μ_ℓ fails to be injective. This is precisely what occurs if $(r, t, n) \in \{(4, 3, 5), (5, 3, 9), (6, 3, 14), (6, 2, 7)\}$. For example, let $(r, t, n) = (4, 3, 5)$ and consider the divisor $D_{t+1} = (t+1)E_0 - 2(E_1 + \dots + E_n)$ on \mathbb{P}^{r-1} and $D'_{t+1} = (t+1)E'_0 - 2(E'_1 + \dots + E'_n)$ on \mathbb{P}^{r-2} . By Equation (1) we know

$h^0(D_{t+1}) \geq \binom{r-1+t+1}{r-1} - n\binom{r}{r-1} = 15 > 0$, so $h^1(D_{t+1}) = 0$ by Theorem 3.1 and $h^0(\mathcal{S}(I)(t+1)) = 0$ by Equation (9), but $h^0(\mathcal{S}(I)|_L(t+1)) = h^1(D'_{t+1}) > 0$ by Equation (16) and Theorem 3.1. The cases $(5, 3, 9), (6, 3, 14), (6, 2, 7)$ work the same way.

Now assume $(r, t, n) \notin \{(4, 3, 5), (5, 3, 9), (6, 3, 14), (6, 2, 7)\}$. The map $A_t \rightarrow A_{t+1}$ will be injective by Equation 10 if

$$h^0(\mathcal{S}(I)|_L(t+1)) = 0.$$

But $h^0(\mathcal{S}(I)|_L(t+1)) = h^1(D'_{t+1})$ by Equation (16). Since the restrictions of generic linear forms to L remain generic, by Theorem 3.1 we have $h^1(D'_{t+1}) = 0$.

(d) Assume $n > \binom{r-2+t}{r-2}$. As shown in (b), $I'_t = S'_t$, hence $h^0(I'(t)) = h^0(S'(t)) = \binom{r-2+t}{r-2}$. Now by Equation (12), using Equation (15),

$$h^0(\mathcal{S}(I)|_L(t)) = h^0(S'(0)^n) - \dim_{\mathbb{K}} I'_t = n - \binom{r-2+t}{r-2} > 0.$$

But we noted in (c) that $h^0(\mathcal{S}(I)(t)) = 0$. Thus by Equation (10), $A_{t-1} \xrightarrow{\mu_\ell} A_t$ is not injective.

If however $n \geq \binom{r-2+t}{r-2}$, applying the statement of parts (b, c) shows that $A_t \xrightarrow{\mu_\ell} A_{t+1}$ is an isomorphism.

□

As pointed out to us by Iarrobino, this proposition is related to a result of Hochster-Laksov [16]. In the situation of the proposition with $n = \binom{r-2+t}{r-2}$, WLP holds at “twin peaks”.

Corollary 3.3. *For generic linear forms l_i and $I = \langle l_1^t, \dots, l_n^t \rangle$ with $A = \mathbb{K}[x_1, \dots, x_r]/I$ Artinian, the map $A_t \rightarrow A_{t+1}$ has full rank if and only if $(r, t, n) \notin \{(4, 3, 5), (5, 3, 9), (6, 3, 14), (6, 2, 7)\}$.*

Proof. By Proposition 3.2(c), it suffices to show in the four exceptional cases that μ_ℓ is not surjective.

- (1) For $I = \langle l_1^3, \dots, l_5^3 \rangle \subseteq \mathbb{K}[x_1, \dots, x_4]$, the Hilbert series for S/I is $(1, 4, 10, 15, 15, 6)$, as in Example 2.1.3. But as in the proof of Proposition 3.2(c), the kernel of $A_3 \rightarrow A_4$ has dimension $h^1(D'_4)$, hence the cokernel has dimension $h^1(D'_4)$, so μ_ℓ fails to have full rank, since $h^1(D'_4) = 1$ by Theorem 3.1.
- (2) Similarly, for $I = \langle l_1^3, \dots, l_9^3 \rangle \subseteq \mathbb{K}[x_1, \dots, x_5]$, the Hilbert series for S/I is $(1, 5, 15, 26, 25)$, and the kernel of $A_3 \rightarrow A_4$ has dimension $h^1(D'_4) = 2$, so the cokernel has dimension 1.
- (3) For $I = \langle l_1^3, \dots, l_{14}^3 \rangle \subseteq \mathbb{K}[x_1, \dots, x_6]$, the Hilbert series for S/I is $(1, 6, 21, 42, 42)$ but the kernel (and hence the cokernel) of $A_3 \rightarrow A_4$ has dimension $h^1(D'_4) = 1$.
- (4) For $I = \langle l_1^2, \dots, l_7^2 \rangle \subseteq \mathbb{K}[x_1, \dots, x_6]$, the Hilbert series for S/I is $(1, 6, 14, 14, 5)$ but the kernel (and hence the cokernel) of $A_2 \rightarrow A_3$ has dimension $h^1(D'_3) = 1$.

Note that all but (2) are instances of failure of WLP at “twin peaks”.

□

4. POWERS OF LINEAR FORMS IN $\mathbb{K}[x_1, x_2, x_3, x_4]$

For powers of linear forms in $\mathbb{K}[x_1, x_2, x_3]$, restriction to ℓ yields powers of linear forms in two variables, and as shown in [9], behaviour of these ideals depends only on the degrees of the generators. This is in contrast to the case of four variables, where restriction to $L = V(\ell) \simeq \mathbb{P}^2$ yields powers of linear forms in $\mathbb{K}[x_1, x_2, x_3]$. In this section, we focus on powers of linear forms in $S = \mathbb{K}[x_1, \dots, x_4]$ for which the Hilbert function of the associated (restricted) fatpoint subscheme is known.

A famous open conjecture on the Hilbert function of fat points in \mathbb{P}^2 is expressed in terms of (-1) -curves (i.e., smooth rational curves E with $E^2 = -1$):

Conjecture 4.1 (Segre-Harbourne-Gimigliano-Hirschowitz [22]). *Suppose that $\{p_1, \dots, p_n\} \subseteq \mathbb{P}^2$ is a collection of points in general position, X is the blowup of \mathbb{P}^2 at the points, and E_i the exceptional divisor over p_i . If $F_j = jE_0 - \sum_{i=1}^n a_i E_i$ is special, then there exists a (-1) -curve E with $E \cdot F_j \leq -2$.*

Example 4.2. Let $C = 2(2E_0 - \sum_{i=1}^5 E_i) + (E_0 - E_1 - E_2)$. Then $h^0(C) = 1$ and $h^1(C) = 1$, so C is special, but $E = 2E_0 - \sum_{i=1}^5 E_i$ is rational by adjunction with $E^2 = -1$ and $E \cdot C = -2$.

Lemma 4.3. *Suppose $l_i \in S_1$ are generic and $I = \langle l_1^t, \dots, l_n^t \rangle$, with $A = S/I$ Artinian. If Conjecture 4.1 holds and the divisor D'_m corresponding to the inverse system of $(I \otimes S/\ell)_m$ is effective but $F \cdot E \geq -1$ for all (-1) -curves E , then the map $A_{m-1} \rightarrow A_m$ is injective.*

Proof. By Equation (16), if D'_m is nonspecial, then $H^0(\mathcal{S}(I)|_L(m)) = 0$. Since by Equation (10) $H^0(\mathcal{S}(I)|_L(m))$ maps onto the kernel of $A_{m-1} \rightarrow A_m$, the result follows. \square

Example 4.4. The failure of WLP for Example 1.1 can be related to the occurrence of an SHGH curve E which as we saw in the proof of Corollary 3.3 causes $A_3 \rightarrow A_4$ not to be injective, in this case $E = 2E_0 - E_1 - \dots - E_5$ (see Example 2.2.1 where we have $D'_4 = 2E$). The hypothesis that A is a quotient by powers of generic linear forms is necessary for $A_3 \rightarrow A_4$ to fail to be injective. For example, if instead $I = \langle x^3, y^3, z^3, w^3, (x+y)^3 \rangle$, then $h^0(\mathcal{S}(I)|_L(4)) = 1 = h^0(\mathcal{S}(I)(4))$, and $h^0(\mathcal{S}(I)(3)) = 0$, so now $A_3 \rightarrow A_4$ is injective. On the other hand, injectivity can fail even when no SHGH curve occurs; for example, let $I = \langle l_1^5, \dots, l_{22}^5 \rangle$ where the l_i are generic linear forms in 4 variables. Then $A_4 \rightarrow A_5$ is not injective by Equation (10), since $D'_5 = 5E'_0 - E'_1 - \dots - E'_{22}$ so $h^0(\mathcal{S}(I)|_L(5)) = h^1(D'_5) = 1$ and $h^0(\mathcal{S}(I)(5)) = h^1(D_5) = 0$ (because 22 general points impose independent conditions on quintics on \mathbb{P}^3 but not on \mathbb{P}^2).

The preceding example involving 22 generic linear forms shows that the putative test $E \cdot F_j \leq -2$ for irregularity for linear systems in Conjecture 4.1 requires in general that F_j be effective. When the number n of general points is at most 8 but not a square a stronger statement can be made; this is Lemma 4.6. But first we find all (-1) -curves E on X when $n \leq 8$.

Lemma 4.5. *If $X \rightarrow \mathbb{P}^2$ is the blow up of distinct points $p_1, \dots, p_8 \in \mathbb{P}^2$ and $E = dE_0 - \sum_{i=1}^8 b_i E_i$ is the divisor of a (-1) -curve on X , then $d \leq 6$ and the*

b_i are a permutation of one of the following: $(-1, 0, 0, 0, 0, 0, 0, 0)$ for $d = 0$, $(0, 0, 0, 0, 0, 0, 1, 1)$ for $d = 1$, $(0, 0, 0, 1, 1, 1, 1, 1)$ for $d = 2$, $(0, 1, 1, 1, 1, 1, 1, 2)$ for $d = 3$, $(1, 1, 1, 1, 1, 2, 2, 2)$ for $d = 4$, $(1, 1, 2, 2, 2, 2, 2, 2)$ for $d = 5$ and $(2, 2, 2, 2, 2, 2, 2, 3)$ for $d = 6$. Moreover, if the points p_i are general, each case does in fact give a smooth rational curve E with $E^2 = -1$.

Proof. It is easy to check that $E^2 = -1$ in each of the cases listed in the statement of the lemma. We also have $h^0(E) > 0$ in each case since a naive dimension count shows the number of conditions imposed by the points is always less than the dimension of the space of all forms of degree d . Moreover, if the points are general, each divisor $dE_0 - \sum_{i=1}^8 b_i E_i$ reduces by Cremona transformations to either $E_0 - E_1 - E_2$ or E_1 , and hence E is always (linearly equivalent to) a prime divisor (see [23]). Adjunction now shows that E is smooth and rational.

Now we show that the list is complete. Since $E^2 = -1$, $d^2 = \sum_{i=1}^8 b_i^2 - 1$, and $KE = \sum b_i - 3d$, adjunction implies $3d = \sum_{i=1}^8 b_i + 1$. By Cauchy-Schwartz,

$$d^2 \sum_{i=1}^8 b_i^2 - 1 \geq \frac{1}{8} \left(\sum_{i=1}^8 b_i \right)^2 - 1 = \frac{1}{8} (3d - 1)^2 - 1 = \frac{9d^2 - 6d - 7}{8}.$$

Thus, $d^2 - 6d - 7 \leq 0$ so $d \in [1, 7]$. However, $d = 7$ forces equality of the b_i , and it is easy to see there are no solutions. Hence $d \in \{1, 2, 3, 4, 5, 6\}$, and a check shows only the b_i above can occur. For a different proof, see [14]. \square

Lemma 4.6. *Let X be the blow up of \mathbb{P}^2 at $1 < n \leq 8$ general points, $n \neq 4$. Let F be of the form $dE_0 - m(E_1 + \cdots + E_n)$ with $d \geq 0$ and $m \geq 0$. Then F is irregular if and only if there is a (-1) -curve E such that $E \cdot F < -1$.*

Proof. Conjecture 4.1 is known to be true for $n \leq 8$ general points (this follows from [23, Theorem 9]). Thus if F is special (i.e., effective and irregular), then there is a smooth rational curve E with $E^2 = -1$ such that $F \cdot E < -1$. Conversely, if F is effective, it is easy to see that there being a smooth rational curve E with $E^2 = -1$ such that $E \cdot F < -1$ implies that F is irregular. In particular, if F is effective but $F \cdot E < 0$, then $F - E$ is effective. But $-K_X = 3E_0 - E - \cdots - E_n$ is effective since $n \leq 8$, hence $K_X - (F - E)$ is not effective, and by duality we have $h^2(F - E) = h^0(K_X - (F - E)) = 0$. However, E is rational, so $E \cdot F < -1$ implies $h^1(E, F|_E) > 0$, and the long exact sequence in cohomology coming from

$$0 \rightarrow \mathcal{O}_X(F - E) \rightarrow \mathcal{O}_X(F) \rightarrow \mathcal{O}_E(F) \rightarrow 0$$

now shows that $h^1(F) > 0$.

In checking individual examples which we will need to do to handle the case that F is not effective, it can be useful to note that the same argument shows $h^1(F) > 0$ when $E \cdot F < -1$ whether or not F is effective if $h^2(F - E) = 0$. (We have $h^2(F - E) = 0$ for example if $(F - E) \cdot E_0 > -3$, by duality since $h^2(F - E) = h^0(K_X - F + E)$ but $(K_X - F + E) \cdot E_0 = -3 + 2 < 0$ hence $K_X - F + E$ is not effective.)

Two additional observations will be helpful. If E is a (-1) -curve, note that $E_0 \cdot E \geq 0$ and hence $F \cdot E < -1$ implies $(F - E_0) \cdot E < -1$. Also, if $h^1(F) > 0$, then $h^1(F - E_0) > 0$. This is because $F \cdot E_0 \geq 0$ by hypothesis, and so $h^1(E_0, \mathcal{O}_{E_0}(F)) = 0$. Taking cohomology of

$$0 \rightarrow \mathcal{O}_X(F - E_0) \rightarrow \mathcal{O}_X(F) \rightarrow \mathcal{O}_{E_0}(F) \rightarrow 0$$

and using $h^1(F) > 0$ shows that $h^1(F - E_0) > 0$.

Now assume F is not effective (and hence $m > 0$); we consider each n individually.

- $n = 1$. We must skip this case, since $F = -2E_1$ is irregular but $E = E_1$ is the only (-1) -curve when $n = 1$, and $E \cdot F > 0$.
- $n = 2$. It is easy to see that $tE_0 - m(E_1 + E_2)$ is effective if and only if $t \geq m$. Thus $d < m$. But $E = E_0 - E_1 - E_2$ is a (-1) -curve with $E \cdot (mE_0 - m(E_1 + E_2)) = -m$ and $F \cdot E = d - 2m < -m$. Thus, if $m > 1$, $h^1(mE_0 - m(E_1 + E_2)) > 0$ (since $mE_0 - m(E_1 + E_2)$ is effective and has intersection with E less than -1), and since F is obtained by subtracting off copies of E_0 , our observations above imply $F \cdot E < -1$ and $h^1(F) > 0$. If $m = 1$ then $F = -E_1 - E_2$ which has $h^1 = 1$ (because two points fail to impose independent conditions on forms of degree zero) and $F \cdot E < -1$.
- $n = 3$. Since $N = 2E_0 - E_1 - E_2 - E_3$ is nef, $G = tE_0 - m(E_1 + E_2 + E_3)$ is not effective if $2t < 3m$ (i.e., if $G \cdot N < 0$). On the other hand, the least t such that $2t \geq 3m$ is $t = 3m/2$ if m is even and $(3m + 1)/2$ if m is odd. Taking G_0 to be G in the case that m is even and $t = 3m/2$, we have $G_0 = (m/2)((E_0 - E_1 - E_2) + (E_0 - E_1 - E_3) + (E_0 - E_2 - E_3))$, which is effective, while taking G_1 to be G when m is odd and $t = (3m + 1)/2$, we have $G_1 = (2E_0 - E_1 - E_2 - E_3) + ((m - 1)/2)((E_0 - E_1 - E_2) + (E_0 - E_1 - E_3) + (E_0 - E_2 - E_3))$, which also is effective. Thus G is not effective if and only if $2t < 3m$. Let $E = E_0 - E_1 - E_2$. Then $G_0 \cdot E = -m/2$, so if m is even $F = G_0 - iE_0$ for some $i > 0$, and we have both $F \cdot E < -1$ and $h^1(F) > 0$ if $m > 2$. If $m = 2$, then F either has $d = 0, 1$ or 2 . In each case one checks directly that $F \cdot E < -1$ and $h^1(F) > 0$ both hold. Similarly, $G_1 \cdot E = -(m - 1)/2$, so if m is odd then $F \cdot E < -1$ and $h^1(F) > 0$ if $m > 3$. If $m = 3$, then $0 \leq d \leq 4$, and in each case one can check that $F \cdot E < -1$ and $h^1(F) > 0$. If $m = 1$, then $F \cdot E < -1$ and $h^1(F) > 0$ for $d = 0$ but if $d = 1$, then $F \cdot E \geq -1$ for every exceptional curve E and $h^1(F) = 0$.
- $n = 4$. We must also skip this case, since $F = E_0 - E_1 - E_2 - E_3 - E_4$ is not effective, but $F \cdot E \geq -1$ for every (-1) -curve E , yet $h^1(F) = 1$.
- $n = 5$. Let $G = tE_0 - m(E_1 + \cdots + E_5)$. Note that $E = 2E_0 - (E_1 + \cdots + E_5)$ is a (-1) -curve and $N = 2E_0 - (E_1 + \cdots + E_4)$ is nef and effective. Thus $t \geq 2m$ implies $G = mE + iE_0$ for some $i \geq 0$, and hence G is effective, while $t < 2m$ implies $G \cdot N < 0$, so G is not effective. Thus $d < 2m$, and we have $F \cdot E < E \cdot mE = -m$. If $m > 1$, using the fact that $h^1(mE) > 0$ when $m > 1$ (i.e., the effective case done above), we thus have both $h^1(F) > 0$ and $F \cdot E < -1$. If $m = 1$, then we have $d = 0$ or 1 , and in both cases we have $h^1(F) > 0$ and $F \cdot E < -1$.
- $n = 6$. Let $G = tE_0 - m(E_1 + \cdots + E_6)$, $E = 2E_0 - (E_1 + \cdots + E_5)$ and $N = 5E_0 - 2(E_1 + \cdots + E_6)$. Let $Q = 12E_0 - 5(E_1 + \cdots + E_6)$; note that $Q = (2E_0 - (E_1 + \cdots + E_6) + E_1) + \cdots + (2E_0 - (E_1 + \cdots + E_6) + E_6)$ is effective, being the sum of six (-1) -curves. Note that N is effective and nef: effective since 6 double points impose at most 18 conditions on the 21 dimensional space of all quintics, and nef since $5N = 2Q + E_0$, and we check that $5N$ meets each of the irreducible components in this sum

non-negatively. Since $G \cdot N = 5t - 12m$, we see if $5t < 12m$, then G is not effective. On the other hand, if $5t \geq 12m$, then G is effective. To see this, work mod 5; i.e., let $m = 5a + i$ for $0 \leq i \leq 4$. The least t such that $5t \geq 12m$ is, respectively, $12a$, $12a + 3$, $12a + 5$, $12a + 8$ and $12a + 10$, and G is, in turn, aQ , $aQ + (3E_0 - E_1 - \cdots - E_6)$, $aQ + N$, $aQ + N + (3E_0 - E_1 - \cdots - E_6)$ and $aQ + 2N$. Each of these is effective (so if $5t \geq 12m$, then G is effective) and, respectively, $G \cdot E$ is $-a$, $-a + 1$, $-a$, $-a + 1$ and $-a$, hence $G \cdot E < -1$ and $h^1(G) > 0$ (and hence $F \cdot E < -1$ and $h^1(F) > 0$, since $F = G - iE_0$ for some $i > 0$) except when $a \leq 1$ or $G = 2Q + (3E_0 - E_1 - \cdots - E_6)$ or $G = 2Q + N + (3E_0 - E_1 - \cdots - E_6)$. A direct check of the exceptional cases shows that $F \cdot E < -1$ and $h^1(F) > 0$ in each case except $F = 2E_0 - (E_1 + \cdots + E_6)$ and $F = 7E_0 - 3(E_1 + \cdots + E_6)$, and in both of these cases we have $F \cdot E = -1$ for every exceptional curve E and $h^1(F) = 0$.

- $n = 7$. Let $G = tE_0 - m(E_1 + \cdots + E_7)$, $E = 3E_0 - (2E_1 + E_2 + \cdots + E_7)$ and let $N = 8E_0 - 3(E_1 + \cdots + E_7)$. Let $Q = 21E_0 - 8(E_1 + \cdots + E_7)$; note that $Q = (3E_0 - (E_1 + \cdots + E_7) - E_1) + \cdots + (3E_0 - (E_1 + \cdots + E_7) - E_7)$ is effective, being the sum of seven (-1) -curves. Note that N is effective and nef: effective since 7 triple points impose at most 42 conditions on the 45 dimensional space of all octics, and nef since $8N = 3Q + E_0$, and we check that $8N$ meets each of the irreducible components in this sum non-negatively. Since $G \cdot N = 8t - 21m$, we see if $8t < 21m$, then G is not effective. On the other hand, if $8t \geq 21m$, then G is effective. To see this, work mod 8; i.e., let $m = 8a + i$ for $0 \leq i \leq 7$. The least t such that $8t \geq 21m$ is, respectively, $21a$, $21a + 3$, $21a + 6$, $21a + 8$, $21a + 11$, $21a + 14$, $21a + 16$ and $21a + 19$, and G is, in turn, aQ , $aQ + (3E_0 - E_1 - \cdots - E_7)$, $aQ + 2(3E_0 - E_1 - \cdots - E_7)$, $aQ + N$ and $aQ + N + (3E_0 - E_1 - \cdots - E_7)$, $aQ + N + 2(3E_0 - E_1 - \cdots - E_7)$, $aQ + 2N$ and $aQ + 2N + (3E_0 - E_1 - \cdots - E_7)$. Each of these is effective (so if $8t \geq 21m$, then G is effective). But $G \cdot E$ is, respectively, $-a$, $-a + 1$, $-a + 2$, $-a$, $-a + 1$, $-a + 2$, $-a$ and $-a + 1$. Thus $h^1(F) > 0$ and $F \cdot E < -1$ unless $a \leq 1$, or G is either $2Q + (3E_0 - E_1 - \cdots - E_7)$, $2Q + 2(3E_0 - E_1 - \cdots - E_7)$, $3Q + 2(3E_0 - E_1 - \cdots - E_7)$, $2Q + N + (3E_0 - E_1 - \cdots - E_7)$, $2Q + N + 2(3E_0 - E_1 - \cdots - E_7)$, $3Q + N + 2(3E_0 - E_1 - \cdots - E_7)$ or $2Q + 2N + (3E_0 - E_1 - \cdots - E_7)$. But in each of these exceptions (except $G = aQ$ with $a = 0$, which does not give rise to any cases of F), we have $(G - E_0 - E) \cdot E_0 > -3$ so $h^2(G - E_0 - E) = 0$ and, unless G is either $0Q + 2(3E_0 - E_1 - \cdots - E_7)$ or $0Q + N + 2(3E_0 - E_1 - \cdots - E_7)$, we have $(G - E_0) \cdot E < -1$, so $h^1(G - E_0) > 0$ and hence $F \cdot E < -1$ and $h^1(F) > 0$. If $G = 2(3E_0 - E_1 - \cdots - E_7)$, then $G - E_0$ has $h^1 = 0$ and $(G - E_0) \cdot E \geq -1$ for every exceptional curve E , while $G - 2E_0$ has $h^1 > 0$ and $G \cdot E < -1$. If $G = N + 2(3E_0 - E_1 - \cdots - E_7)$, then $G - E_0$ has $h^1 = 0$ and $(G - E_0) \cdot E \geq -1$ for every exceptional curve E , while $G - 2E_0$ has $h^1 > 0$ and $G \cdot E < -1$.
- $n = 8$. Let $G = tE_0 - m(E_1 + \cdots + E_8)$, $E = 6E_0 - (3E_1 + 2E_2 + \cdots + 2E_8)$ and let $N = 17E_0 - 6(E_1 + \cdots + E_8)$. Let $Q = 48E_0 - 17(E_1 + \cdots + E_8)$; note that $Q = (6E_0 - 2(E_1 + \cdots + E_8) - E_1) + \cdots + (6E_0 - 2(E_1 + \cdots + E_8) - E_8)$ is effective, being the sum of eight (-1) -curves. Note that N is effective and nef: effective since 8 sextuple points impose at most 168 conditions on

the 171 dimensional space of all 17-ics, and nef since $17N = 6Q + E_0$, and we check that $17N$ meets each of the irreducible components in this sum non-negatively. Since $G \cdot N = 17t - 48m$, we see if $17t < 48m$, then G is not effective. On the other hand, if $17t \geq 48m$, then G is effective. To see this, work mod 17; i.e., let $m = 17a + i$ for $0 \leq i \leq 16$. The least t such that $17t \geq 48m$ is, respectively, $48a + 3i$ for $0 \leq i \leq 5$, $48a + 17 + 3(i - 6)$ for $6 \leq i \leq 11$, and $48a + 34 + 3(i - 12)$ for $12 \leq i \leq 16$, and G is: $G = aQ + i(3E_0 - E_1 - \dots - E_8)$ for $0 \leq i \leq 5$; $G = aQ + N + (i - 6)(3E_0 - E_1 - \dots - E_8)$ for $6 \leq i \leq 11$; and $G = aQ + 2N + (i - 12)(3E_0 - E_1 - \dots - E_8)$ for $12 \leq i \leq 16$. Each of these is effective (so if $17t \geq 48m$, then G is effective).

But F is of the form $G - jE_0$ for some $j \geq 1$ and some G on this list. Taking $j = 1$ for each G (so $F = G - E_0$), we have $(F - E) \cdot E_0 > -3$ and thus $h^2(F - E) = 0$ (except for the cases $G = 0$ and $G = 3E_0 - E_1 - \dots - E_8$, but if $G = 0$ then $F \cdot E_0 < 0$ which we excluded by hypothesis, and if $G = 3E_0 - E_1 - \dots - E_8$, then $F = dE_0 - E_1 - \dots - E_8$ for $0 \leq d \leq 2$, and in these cases we have both $F \cdot E < -1$ and $h^1(F) > 0$). Thus whenever we have $F \cdot E < -1$ we have $h^1(F) > 0$. The only remaining cases $F = G - jE_0$ for which we do not have $F \cdot E < -1$ are: $F = 5(3E_0 - E_1 - \dots - E_8) - E_0$ and $F = N + 5(3E_0 - E_1 - \dots - E_8) - E_0$. A direct check of these exceptional cases shows that $F \cdot E = -1$ for every exceptional curve E and $h^1(F) = 0$. \square

Lemma 4.7. *For $I = \langle l_1^t, \dots, l_n^t \rangle \subseteq S$ with $n \leq 8$ and $l_i \in S_1$ generic, the map $A_{m-1} \xrightarrow{\mu_I} A_m$ is injective for*

- $m < \lceil \frac{17(t-1)+2}{11} \rceil$ if $n = 8$.
- $m < \lceil \frac{8(t-1)+2}{5} \rceil$ if $n = 7$.
- $m < \lceil \frac{5(t-1)+2}{3} \rceil$ if $n = 5, 6$.

Proof. If $m \leq t$, then $A_{m-1} \rightarrow A_m$ is injective since $A_{m-1} = 0$ by Corollary 2.1.2. So suppose $m \geq t$. By Equations (10), (14) and (16), $A_{m-1} \rightarrow A_m$ is injective if $h^1(D'_m) = 0$, where D'_m is the line bundle $mE'_0 - (m - t + 1)(\sum_{i=1}^n E'_i)$ on \mathbb{P}^2 . By Lemma 4.6, $h^1(D'_m) = 0$ if $D'_m \cdot E \geq -1$ for every (-1) -curve $E = dE'_0 - \sum_i b_i E'_i$. Since $m \geq t$, we have $D'_m \cdot E'_i \geq 0$ for all i , so we now need to check the remaining (-1) -curves listed in Lemma 4.5. I.e., we may assume $d \geq 1$. It suffices to show that

$$md - (m - t + 1)(\sum_{i=1}^8 b_i) > -2,$$

but $\sum_{i=1}^8 b_i = 3d + 1$ for any (-1) -curve, so this simplifies to $md - (m - t + 1)(3d - 1) > -2$ or $m < \frac{(3d-1)(t-1)+2}{(2d-1)}$. The right hand side is decreasing as a function of d . Thus for each n we use the largest d available; i.e., $d = 6$ for $n = 8$, $d = 3$ for $n = 7$ and $d = 2$ for $n = 5, 6$. Plugging in these values of d gives the result. \square

Lemma 4.8. *For $I = \langle l_1^t, \dots, l_5^t \rangle \subseteq S$ with $l_i \in S_1$ generic, S/I fails to have WLP for all $t \geq 3$.*

Proof. For $t = 3$, the result follows from Corollary 3.3. For larger t , we will apply the main result of De Volder-Laface [6] on fatpoints in \mathbb{P}^3 . Assume $2a \geq 4b \geq 0$; then the divisor $aE_0 - b \sum_{i=1}^n E_i$ obtained by blowing up $n \leq 8$ general points on

\mathbb{P}^3 is effective since $\binom{a+3}{3} > 5\binom{b+2}{3}$ and by [6] it is non-special since $a > 2b - 2$. So for $D_m = mE_0 - \sum_{i=1}^5 (m-t+1)E_i$ we have $h^1(D_m) = 0$ if $2m \geq 4(m-t+1)$ and $m \geq t$, or equivalently if $2t-2 \geq m \geq t$. So if $2t-2 \geq m \geq t$ we have

$$(17) \quad \dim_{\mathbb{K}} A_m = h^0(D_m) = \binom{m+3}{3} - 5\binom{m-t+3}{3}$$

by Equation (3) and $h^0(\mathcal{S}(I)(m)) = h^1(D_m) = 0$ by Equation (9). Now by Equation (10) we have an exact sequence

$$0 \longrightarrow H^1(D'_m) \longrightarrow A_{m-1} \longrightarrow A_m$$

as long as $2t-2 \geq m \geq t$.

For $m = \lceil \frac{5t}{3} \rceil - 1$, we have $D'_m \cdot (2E_0 - E_1 - \dots - E_5) \leq -2$, so $h^1(D'_m) > 0$ by Lemma 4.6. (Note for this value of m we have $t \leq m-1 < m \leq 2t-2$ for $t \geq 3$.) Thus, to prove failure of WLP, it suffices to show

$$\dim_{\mathbb{K}} A_{\lceil \frac{5t}{3} \rceil - 1} \geq \dim_{\mathbb{K}} A_{\lceil \frac{5t}{3} \rceil - 2}.$$

We obtain these dimensions from Equation (17). So the result will follow if

$$\left(\binom{\lceil \frac{5t}{3} \rceil - 1 + 3}{3} - 5 \binom{\lceil \frac{5t}{3} \rceil - 1 - t + 3}{3} \right) \geq \left(\binom{\lceil \frac{5t}{3} \rceil - 2 + 3}{3} - 5 \binom{\lceil \frac{5t}{3} \rceil - 2 - t + 3}{3} \right).$$

A calculation shows this holds for all $t \geq 6$. For the case $t = 4$, the Hilbert function of A is $(1, 4, 10, 20, 30, 36, 34, 20)$, and we have the sequence

$$0 \longrightarrow H^1(D_6) \longrightarrow A_5 \longrightarrow A_6.$$

Since $\dim_{\mathbb{K}} A_5 = 36$ and $\dim_{\mathbb{K}} A_6 = 34$ and by [23] $h^1(D_6) = 3$, so $A_5 \rightarrow A_6$ has rank 33, and WLP fails.

For $t = 5$, the Hilbert function of A is $(1, 4, 10, 20, 35, 51, 64, 70, 65, 45, 16)$, and we have the sequence

$$0 \longrightarrow H^1(D_8) \longrightarrow A_7 \longrightarrow A_8.$$

Since $\dim_{\mathbb{K}} A_7 = 70$ and $\dim_{\mathbb{K}} A_8 = 65$ and by [23] $h^1(D_8) = 6$, so $A_7 \rightarrow A_8$ has rank 64, and WLP fails. \square

Lemma 4.9. *For $I = \langle l_1^t, \dots, l_6^t \rangle \subseteq S$ with $l_i \in S_1$ generic, S/I has WLP for all $t \leq 14$, and fails to have WLP for all $t \geq 27$.*

Proof. Let $m = \lceil \frac{5t}{3} \rceil - 1$, as in the proof of Lemma 4.8. Mimicking the argument there, as long as $t \geq 3$ we have

$$(18) \quad \dim_{\mathbb{K}} A_m - \dim_{\mathbb{K}} A_{m-1} = \binom{m+3}{3} - 6\binom{m-t+3}{3} - \binom{m+2}{3} + 6\binom{m-t+2}{3}.$$

and an exact sequence

$$0 \longrightarrow H^1(D'_m) \longrightarrow A_{m-1} \longrightarrow A_m.$$

But by Lemma 4.6,

$$C \cdot D'_m = \begin{cases} -2 & \text{if } t \bmod 3 = 0 \\ -3 & \text{if } t \bmod 3 = 1 \\ -4 & \text{if } t \bmod 3 = 2 \end{cases}$$

where $C = 2E'_0 - E'_1 - \dots - E'_5$, hence $h^1(D'_m) > 0$. Since Equation (18) is positive for $t \geq 48$, we see WLP fails for $t \geq 48$. Using Lemma 4.7 and Proposition 2.1 of [19] and analyzing individual cases shows that WLP holds for all $t \leq 14$, and fails

for all $27 \leq t \leq 47$. Finally, for $t = 15$ WLP fails: $h^1(D_m) = 6$, $\dim_{\mathbb{K}} A_{m-1} = 1610$ and $\dim_{\mathbb{K}} A_m = 1605$, and for $t = 26$ WLP holds: $h^1(D_m) = 36 = \dim_{\mathbb{K}} A_{m-1} - \dim_{\mathbb{K}} A_m$. \square

Theorem 4.10. *Let $I = \langle l_1^t, \dots, l_n^t \rangle \subseteq \mathbb{K}[x_1, x_2, x_3, x_4]$ with $l_i \in S_1$ generic. If $n \in \{5, 6, 7, 8\}$, then WLP fails, respectively, for $t \geq \{3, 27, 140, 704\}$.*

Proof. Lemma 4.8 and Lemma 4.9 take care of the cases $n = 5, 6$. For $n = 7$ or 8 , the same argument as used in Lemmas 4.8 and 4.9 shows that as long as $t \geq 3$ we have

$$(19) \quad \dim_{\mathbb{K}} A_m - \dim_{\mathbb{K}} A_{m-1} = \binom{m+3}{3} - 6 \binom{m-t+3}{3} - \binom{m+2}{3} + n \binom{m-t+2}{3}.$$

and an exact sequence

$$0 \longrightarrow H^1(D'_m) \longrightarrow A_{m-1} \longrightarrow A_m.$$

But by Lemma 4.6, for $n = 7$ we have

$$C \cdot D'_m = \begin{cases} -5 & \text{if } t \bmod 5 = 0 \\ -2 & \text{if } t \bmod 5 = 1 \\ -4 & \text{if } t \bmod 5 = 2 \\ -6 & \text{if } t \bmod 5 = 3 \\ -3 & \text{if } t \bmod 5 = 4 \end{cases}$$

where $C = 3E'_0 - 2E'_1 - E'_2 - \dots - E'_7$, hence $h^1(D'_m) > 0$, and for $n = 8$ we have

$$C \cdot D'_m = \begin{cases} -6 & \text{if } t \bmod 11 = 0 \\ -11 & \text{if } t \bmod 11 = 1 \\ -5 & \text{if } t \bmod 11 = 2 \\ -10 & \text{if } t \bmod 11 = 3 \\ -4 & \text{if } t \bmod 11 = 4 \\ -9 & \text{if } t \bmod 11 = 5 \\ -3 & \text{if } t \bmod 11 = 6 \\ -8 & \text{if } t \bmod 11 = 7 \\ -2 & \text{if } t \bmod 11 = 8 \\ -7 & \text{if } t \bmod 11 = 9 \\ -12 & \text{if } t \bmod 11 = 10 \end{cases}$$

where $C = 6E'_0 - 3E'_1 - 2E'_2 - \dots - 2E'_8$, hence again $h^1(D'_m) > 0$. Since Equation 19 is non-negative for $t \geq 140$ when $n = 7$ and for $t \geq 704$ when $n = 8$, the result follows. \square

WLP can hold for small values of t , and individual examples are easy to check:

Example 4.11. Consider $I = \langle l_1^8, \dots, l_8^8 \rangle \subseteq \mathbb{K}[x_1, \dots, x_4] = S$ with $l_i \in S_1$ generic, and $\ell \in S_1$ such that $I \otimes S/(\ell)$ is minimally generated by powers of eight generic linear forms. Let $A = S/I$, $S' = S/(\ell)$, $I|_L = I \otimes_S S'$ and let the divisor associated

via the inverse system corresponding to $(I|_L)_m$ be

$$D'_m = mE'_0 - (m - t + 1) \sum_{i=1}^8 E'_i.$$

For degrees ≥ 8 , the Hilbert function of A is:

i	8	9	10	11	12	13	14	15
$HF(A, i)$	157	188	206	204	175	112	8	0

By Lemma 4.7 the maps $A_{m-1} \xrightarrow{\mu_\ell} A_m$ are injective for $1 \leq m \leq 10$. For $m = 11$, $D'_{11} \cdot (6E'_0 - \sum_{i=1}^7 2E'_i - 3E'_8) < -1$ hence $h^1(D'_{11}) > 0$; in fact $h^1(D'_{11}) = 2$ [13], giving $A_{10} \twoheadrightarrow A_{11}$. By Proposition 2.1 of [19], this gives surjectivity for $m \geq 11$, so A has WLP.

Since Conjecture 4.1 holds for eight or fewer points in general position in \mathbb{P}^2 , the analysis in this section can be carried out for powers of eight or fewer general forms in $\mathbb{K}[x_1, \dots, x_4]$ where the powers differ. In [4], Ciliberto-Miranda show that Conjecture 4.1 holds for points with uniform multiplicity ≤ 12 . However, there is no version of the De Volder-Laface result, so even in the special case of powers of linear forms in four variables, the study of WLP is closely linked to a difficult open problem on fatpoints in \mathbb{P}^2 .

5. POWERS OF $r + 1$ LINEAR FORMS IN $\mathbb{K}[x_1, \dots, x_r]$

We close by tackling the case of an almost complete intersection of powers of linear forms (so $n = r + 1$). For brevity, in this section we denote

$$\begin{aligned} A_{r,t} &= \mathbb{K}[x_1, \dots, x_r] / \langle l_1^t, \dots, l_{r+1}^t \rangle \\ B_{r,t} &= \mathbb{K}[x_1, \dots, x_r] / \langle l_1^t, \dots, l_{r+2}^t \rangle \\ C_{r,t} &= \mathbb{K}[x_1, \dots, x_r] / \langle l_1^t, \dots, l_r^t \rangle, \end{aligned}$$

where all forms are generic. The algebras A, B, C are related by the long exact sequence

$$(20) \quad 0 \longrightarrow (I : \ell) / I \longrightarrow S / I \xrightarrow{\cdot \ell} S(1) / I \longrightarrow S(1) / I + \langle \ell \rangle \longrightarrow 0.$$

5.1. A key tool. We now recall a key tool in analyzing WLP for $A_{r,t}$. Following a suggestion of Iarrobino, Stanley interprets $C_{r,t}$ as the cohomology ring of a product of projective spaces and applies the Lefschetz hyperplane theorem to show that

Lemma 5.1.1. [Lemma C of [18]] *Let $m = \min\{\lfloor \frac{r}{t} \rfloor, r\}$. Then the Hilbert function of $A_{r,t}$ in degree i is*

$$\dim_{\mathbb{K}}(A_{r,t})_i = \begin{cases} \binom{r-1+i}{r-1} + \sum_{j=1}^m (-1)^j \binom{r-1+i-tj}{r-1} \cdot \binom{r+1}{j} & \text{if this quantity is positive,} \\ 0 & \text{otherwise.} \end{cases}$$

5.2. The case of r even. We recall that the socle degree of $B_{r-1,t}$ is the largest degree i such that $\dim_{\mathbb{K}}(B_{r-1,t})_i > 0$. In Lemma 2 of [5], D'Cruz and Iarrobino prove

Lemma 5.2.1. *For $r - 1$ odd, the socle degree of $B_{r-1,t}$ is $(t - 1)\frac{r}{2}$.*

Theorem 5.2.2. *Let $k \geq 2$. Then $A_{2k,t}$ fails to have WLP in degree $c = k(t - 1) - 1$ for all $t \gg 0$.*

Proof. By Lemma 5.2.1 we know $(B_{2k-1,t})_{c+1} \neq 0$ and from (20) we have the exact sequence

$$(A_{2k,t})_c \xrightarrow{\cdot \ell} (A_{2k,t})_{c+1} \longrightarrow (B_{2k-1,t})_{c+1} \longrightarrow 0,$$

so WLP fails if

$$\dim_{\mathbb{K}}(A_{2k,t})_c \geq \dim_{\mathbb{K}}(A_{2k,t})_{c+1}.$$

For the relevant degrees c and $c+1$, the upper limit m in Lemma 5.1.1 is

$$\begin{aligned} \text{For } c : m &= \min\{\lfloor \frac{k(t-1)-1}{t} \rfloor, 2k\} \\ \text{For } c+1 : m &= \min\{\lfloor \frac{k(t-1)}{t} \rfloor, 2k\} \end{aligned}$$

If $t \geq k+1$, both m values equal $k-1$, so by Lemma 5.1.1 it suffices to show

$$\binom{2k-1+c}{r-1} + \sum_{1 \leq j \leq m} (-1)^j \binom{2k-1+c-tj}{2k-1} \cdot \binom{2k+1}{j} \geq \binom{2k+c}{2k-1} + \sum_{1 \leq j \leq m} (-1)^j \binom{2k+c-tj}{2k-1} \cdot \binom{2k+1}{j}$$

Rearranging shows this inequality is equivalent to

$$\sum_{j=0}^{k-1} (-1)^{j+1} \binom{2k-2-k+(k-j)t}{2k-2} \cdot \binom{2k+1}{j} \geq 0.$$

Expanding yields a polynomial of degree $2k-2$ in t , with lead coefficient $\frac{\alpha}{(2k-2)!}$, where

$$\alpha = \sum_{j=0}^{k-1} (-1)^{j+1} (k-j)^{2k-2} \cdot \binom{2k+1}{j}.$$

But α is the difference of two central Eulerian numbers

$$(21) \quad \alpha = \left\langle \begin{matrix} 2k-2 \\ k-2 \end{matrix} \right\rangle - \left\langle \begin{matrix} 2k-2 \\ k-3 \end{matrix} \right\rangle,$$

so the positivity of α now follows, since the Eulerian numbers $\langle \begin{smallmatrix} n \\ j \end{smallmatrix} \rangle$ are increasing for $1 \leq j \leq n/2$. \square

Example 5.2.3. Theorem 5.2.2 does not detect all obstructions to WLP. The Hilbert function of $A_{4,6}$ is

i	0	1	2	3	4	5	6	7	8	9	10	11	12
$HF(A, i)$	1	4	10	20	35	56	79	100	115	120	111	84	45

WLP fails for both $A_8 \rightarrow A_9$ and $A_9 \rightarrow A_{10}$ but only the latter failure is predicted by the theorem.

5.3. Gelfand-Tsetlin patterns. There is an interesting connection to combinatorics which we will apply in the next section.

Definition 5.3.1. A two-row Gelfand-Tsetlin pattern is a non-negative integer $2 \times n$ -matrix (λ_{ij}) that satisfies $\lambda_{2n} = 0$, $\lambda_{1,j+1} \geq \lambda_{2,j}$ and $\lambda_{i,j} \geq \lambda_{i,j+1}$ for $i = 1, 2$ and $j = 1, \dots, n-1$.

In Proposition 3.6 of [26], Sturmfels-Xu show that for generic forms l_i , the Hilbert function of $\mathbb{K}[x_1, \dots, x_r] / \langle l_1^{u_1}, \dots, l_{r+1}^{u_{r+1}} \rangle$ in degree i is the number of two-rowed Gelfand-Tsetlin patterns with $\lambda_{21} = i$ and $\lambda_{1j} + \lambda_{2j} = u_j + \dots + u_{r+1}$ for $j = 1, \dots, r+1$.

Corollary 5.3.2. *Let $m = \min\{\lfloor \frac{i}{t} \rfloor, r\}$. The number of two-rowed Gelfand-Tsetlin patterns with $\lambda_{21} = i$ and $\lambda_{1j} + \lambda_{2j} = (r + 2 - j)t$ for $j = 1, \dots, r + 1$ is*

$$\begin{cases} \binom{r-1+i}{r-1} + \sum_{1 \leq j \leq m} (-1)^j \binom{r-1+i-tj}{r-1} \cdot \binom{r+1}{j} & \text{if this quantity is positive.} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This follows from the result of Sturmfels-Xu and Lemma 5.1.1. \square

5.4. The case of $r = 2k + 1$ odd. Let $SD(A)$ denote the socle degree of an Artinian algebra A . No formula for $SD(B_{2k,t})$ analogous to that of Lemma 5.2.1 is known. However, we can still obtain some partial results on WLP for $A_{2k+1,t}$ by applying results from [26].

Remark 5.4.1. The socle degree of $B_{4,t}$ up to $t = 14$ is:

t	2	3	4	5	6	7	8	9	10	11	12	13	14
$SD(B_{4,t})$	2	4	7	9	12	14	16	19	21	24	26	28	31

Lemma 3 of [5] asserts that $SD(B_{2k,t}) = (t - 1)k$ but the proof shows only that

$$(22) \quad (t - 1)k \leq SD(B_{2k,t}) \leq (t - 1)(k + 1);$$

the table above shows the assertion of the lemma is incorrect for $4 \leq t \leq 14$.

Lemma 5.4.2. *If $c = (t - 1)(k + 1) - 1$ and $t > 2k + 2$, then*

$$\dim_{\mathbb{K}}(A_{2k+1,t})_c \geq \dim_{\mathbb{K}}(A_{2k+1,t})_{c+1}.$$

Proof. Let G_i denote the set of Gelfand-Tsetlin patterns with $\lambda_{21} = i$ and $\lambda_{1j} + \lambda_{2j} = (2k + 1 + 2 - j)t$ for $j = 1, \dots, 2k + 1 + 1$. We will exhibit an injective map $G_{c+1} \rightarrow G_c$. To do this, note that there is no pattern in G_{c+1} with $\lambda_{22} = c + 1$, as this would imply $\lambda_{12} = (2k + 1)t - c - 1$. Since $\lambda_{12} \geq \lambda_{22}$ this yields $(2k + 1)t - c - 1 \geq c + 1$, so $(2k + 1)t - 2 \geq 2c = 2[(t - 1)(k + 1) - 1]$ and so $2k + 2 \geq t$, a contradiction.

Define a map $G_{c+1} \rightarrow G_c$ by sending $\Lambda \in G_{c+1}$ to the pattern obtained by replacing the first column of Λ (given by $\lambda_{11} = (2k + 3)t - c - 1, \lambda_{12} = c + 1$) with $\lambda'_{11} = (2k + 3)t - c, \lambda'_{12} = c$. This new filling is still a Gelfand-Tsetlin pattern since we have shown that $\lambda_{22} \leq c$, therefore the map is an injection of G_{c+1} into G_c . \square

We now have:

Proposition 5.4.3. *For $A_{2k+1,2l+1}$, with l possibly a half integer, then WLP fails for the map $(A_{2k+1,2l+1})_c \rightarrow (A_{2k+1,2l+1})_{c+1}$ if*

(a) *for $c + 1 = 2kl$ we have*

$$\dim_{\mathbb{K}}(A_{2k+1,2l+1})_c + \dim_{\mathbb{K}}(B_{2k,2l+1})_{c+1} > \dim_{\mathbb{K}}(A_{2k+1,2l+1})_{c+1},$$

or if

(b) *$2l + 1 > 2k + 2$ and $SD(B_{2k,2l+1}) = c + 1$ for $c + 1 = 2l(k + 1)$.*

Proof. By (22), the socle degree of $B_{2k,2l+1}$ is at least $c + 1 = 2kl = k(t - 1)$ where $t = 2l + 1$, so by (20) the map is not surjective, while if the stated inequality holds, then the map cannot by dimension considerations be injective, which proves (a). Similarly, if the socle degree of $B_{2k,2l+1}$ is $2l(k + 1) = (k + 1)(t - 1)$ for $t = 2l + 1$, then surjectivity fails so Lemma 5.4.2 implies injectivity fails too, which proves (b). \square

In order to apply Proposition 5.4.3(a), we will need to be able to compute the dimension of $B_{2k,2l+1}$ in degree $c + 1 = 2kl$. In Theorem 7.2 of [26], Sturmfels-Xu use the Verlinde formula to show that for generic linear forms l_i , the Hilbert function of $B_{s,2l+1}$ in degree $i = sl$ is

$$(23) \quad \dim_{\mathbb{K}}(B_{s,2l+1})_i = \frac{1}{2l+1} \sum_{j=0}^{2l} (-1)^{sj} \left(\sin \frac{2j+1}{4l+2} \pi \right)^{-s}.$$

Here l can be a half-integer if s is even but must be an integer if s is odd. In particular, the Verlinde formula gives the Hilbert function of $B_{2k,2l+1}$ in degree $sl = 2k \cdot \frac{t-1}{2} = k(t-1)$.

When $l = 1/2$ and $i = \lceil s/2 \rceil$, the dimension of $(B_{s,2l+1})_i$ takes a particularly simple form:

$$(24) \quad \dim_{\mathbb{K}}(B_{s,2})_i = \begin{cases} 2^i & \text{if } s \text{ is even and } i = \frac{s}{2}, \\ 1 & \text{if } s \text{ is odd and } i = \frac{s+1}{2}. \end{cases}$$

This was conjectured by D'Cruz and Iarrobino in [5], and proved by Sturmfels and Xu in [26, Corollaries 7.3, 7.4].

5.5. Almost complete intersections with $t = 2$. We close by studying almost complete intersections of squares of linear forms. For example, by applying the results above we have:

Example 5.5.1. For $B_{7,2}$, $SD(B_{7,2}) = 4$ by Lemma 5.2.1 (and the socle dimension is 1, but we don't need the specific dimension in this case), while $\dim_{\mathbb{K}}(A_{8,2})_3 = 48$ and $\dim_{\mathbb{K}}(A_{8,2})_4 = 42$ by Lemma 5.1.1 so WLP fails by Theorem 5.2.2. For $B_{8,2}$, $\dim_{\mathbb{K}}(B_{8,2})_4 = 16$ from (23) (and $SD(B_{8,2}) = 4$ but we don't need the specific socle degree in this case), while $\dim_{\mathbb{K}}(A_{9,2})_3 = 75$ and $\dim_{\mathbb{K}}(A_{9,2})_4 = 90$ by Lemma 5.1.1 so WLP fails by Proposition 5.4.3.

More generally, consider the map $(A_{r,2})_{k-1} \rightarrow (A_{r,2})_k$ where r is either $2k$ or $2k+1$. By (22), the socle degree of $B_{r-1,2}$ is at least k , so from Theorem 5.2.2 (if $r = 2k$ is even) or from Proposition 5.4.3(a) (if $r = 2k+1$ is odd), we see WLP fails if $\dim_{\mathbb{K}}(A_{r,2})_{k-1} \geq \dim_{\mathbb{K}}(A_{r,2})_k$. Using Lemma 5.1.1 we can check this for any specific value of k ; numerical experiments suggest this holds for $r \geq 15$ if r is odd and for $r \geq 6$ if r is even. If in fact $r = 2k+1$ is odd, then by Proposition 5.4.3(a) and (24) it is enough to show $\dim_{\mathbb{K}}(A_{r,2})_{k-1} + 2^k > \dim_{\mathbb{K}}(A_{r,2})_k$. Numerical experiments suggest this holds for odd $r \geq 9$. This leads us to make the following conjecture.

Conjecture 5.5.2. *For $A_{r,2}$, WLP fails for $r = 6$ and all $r \geq 8$.*

In [20], Migliore-Miro-Roig-Nagel prove the conjecture is true for an even number of variables. **Acknowledgements** Computations were performed using Macaulay2, by Grayson and Stillman, available at: <http://www.math.uiuc.edu/Macaulay2/>. Scripts to analyze WLP are available at: <http://www.math.uiuc.edu/~asecele2>. Special thanks go to Pietro Majer for explaining Equation (21) to us.

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