# A NEW SOLVABILITY CRITERION FOR FINITE GROUPS 

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#### Abstract

In 1968, John Thompson proved that a finite group $G$ is solvable if and only if every 2 -generator subgroup of $G$ is solvable. In this paper, we prove that solvability of a finite group $G$ is guaranteed by a seemingly weaker condition: $G$ is solvable if for all conjugacy classes $C$ and $D$ of $G$, there exist $x \in C$ and $y \in D$ for which $\langle x, y\rangle$ is solvable. We also prove the following property of finite nonabelian simple groups, which is the key tool for our proof of the solvability criterion: if $G$ is a finite nonabelian simple group, then there exist two integers $a$ and $b$ which represent orders of elements in $G$ and for all elements $x, y \in G$ with $|x|=a$ and $|y|=b$, the subgroup $\langle x, y\rangle$ is nonsolvable.


## 1. Introduction

John G. Thompson's famous 'N-group paper' [T] of 1968 included the following important solvability criterion for finite groups:

A finite group is solvable if and only if every pair of its elements generates a solvable group.
P. Flavell [F] gave a relatively simple proof of Thompson's result in 1995. We prove that solvability of finite groups is guaranteed by a seemingly weaker condition than the solvability of all its 2-generator subgroups.

Theorem A. Let $G$ be a finite group such that, for all $x, y \in G$, there exists an element $g \in G$ for which $\left\langle x, y^{g}\right\rangle$ is solvable. Then $G$ is solvable.

Theorem A can be rephrased as the following equivalent result.

[^0]Theorem A'. Let $G$ be a finite group such that, for all conjugacy classes $C$ and $D$ of $G$ (possibly $C=D$ ), there exist $x \in C$ and $y \in D$ for which $\langle x, y\rangle$ is solvable. Then $G$ is solvable.

Our second main result, which is the key tool for proving Theorem A, deals with the nonsolvability of certain 2-generator subgroups of finite nonabelian simple groups. For a finite group $G$ let

$$
o e(G)=\{m \mid \exists g \in G \text { with }|g|=m\}
$$

denote the set of element orders of $G$. Using the classification of finite simple groups, we prove the following theorem.

Theorem B. Let $G$ be a finite nonabelian simple group. Then there exist $a, b \in$ oe $(G)$, such that, for all $x, y \in G$ with $|x|=a,|y|=b$, the subgroup $\langle x, y\rangle$ is nonsolvable.

Theorem B was proved separately for alternating groups, sporadic groups, classical groups of Lie type and exceptional groups of Lie type in Propositions 2.1, 2.2, 4.2, 4.5, respectively. In view of Propositions 2.1, 2.2, we state the following conjecture.

Conjecture. If $G$ is a finite nonabelian simple group, then there exist two distinct primes $p, q \in$ oe $(G)$, such that, for all $x, y \in G$ with $|x|=p$, $|y|=q$, the subgroup $\langle x, y\rangle$ is nonsolvable (or, maybe, even nonabelian simple).

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1.1. Other generalisations of Thompson's theorem. Several other 'Thompson-like' results have appeared in the literature recently. We mention here four such theorems. In the first three results, solvability of all 2-generator subgroups is replaced by a weaker condition restricting the required set of solvable 2 -generator subgroups, in different ways from our generalisation.

In 2000, Guralnick and Wilson GW obtained a solvability criterion by restricting the proportion of 2 -generator subgroups required to be solvable.

Theorem 1.1. A finite group is solvable if and only if more than $\frac{11}{30}$ of the pairs of elements of $G$ generate a solvable subgroup.

In addition they proved similar results showing that the properties of nilpotency and having odd order are also guaranteed if a sufficient proportion of element pairs generate subgroups with these properties, namely more than $\frac{1}{2}$ for nilpotency, and more than $\frac{11}{30}$ for having odd order.

In contrast to this, in a paper published in 2009, Gordeev, Grunewald, Kunyavskii and Plotkin [GGKP] proved a solvability criterion which involved 2-generation within each conjugacy class. This result was also proved independently by Guest in [G, Corollary 1].

Theorem 1.2. A finite group $G$ is solvable if and only if, for each conjugacy class $C$ of $G$, each pair of elements of $C$ generates a solvable subgroup.

A stronger result of this type was obtained recently by Kaplan and Levy in [KL, Theorem 4]. Their criterion involves only a limited 2generation within the conjugacy classes of elements of odd prime-power order.

Theorem 1.3. A finite group $G$ is solvable if and only if for every $x, y \in G$, with $x$ a p-element for some odd prime $p$ and $y$ a 2-element, the group $\left\langle x, x^{y}\right\rangle$ is solvable.

Our requirement, while ranging over all conjugacy classes, requires only existence of a solvable 2-generator subgroup with one generator from each of two (possibly equal) classes. We know of no similar criteria in this respect.

The forth result we draw attention to is in a 2006 paper of Guralnick, Kunyavski, Plotkin and Shalev [GKPS]. They proved that membership of the solvable radical of a finite group is characterised by solvability of certain 2-generator subgroups. (The solvable radical $R(G)$ of a finite group $G$ is the largest solvable normal subgroup of $G$.)

Theorem 1.4. For a finite group $G$, the solvable radical $R(G)$ coincides with the set of all elements $x \in G$ with the property: " for any $y \in G$, the subgroup $\langle x, y\rangle$ is solvable".

## 2. Alternating and sporadic simple groups

Theorem B for the alternating groups follows from the following proposition.
Proposition 2.1. If $m \geq 5$, then there exist distinct primes $p$ and $q$ satisfying $m / 2<p<q \leq m$ such that, for all $x, y \in A_{m}$ with $|x|=p$ and $|y|=b$, the subgroup $\langle x, y\rangle \cong A_{d}$ for some $d \geq 5$. In particular, $\langle x, y\rangle$ is nonsolvable.

Proof. First we remark that if $n$ is a positive integer and $\pi(n)$ denotes the number of primes at most $n$, then the following is known (see, for example [R, p.188]):

$$
\pi(2 n)-\pi(n)>n /(3 \ln n) \quad \text { for } n \geq 5
$$

In particular, if $n \geq 17$, then

$$
\pi(2 n)-\pi(n)>17 /(3 \ln 17)>2
$$

which implies that $\pi(2 n)-\pi(n) \geq 3$. Thus, if $n \geq 17$, then there are at least 3 distinct primes $p, r$ and $q$ satisfying $n<p<r<q \leq 2 n$. In particular, $n<p<q-3<q \leq 2 n$. Hence, if $m \geq 34$, then there exist primes $p$ and $q$ such that $m / 2<p<q-3<q \leq m$, and elements $x, y \in A_{m}$ of order $p$ and $q$, respectively. Let $x, y$ be any such elements and let $H=\langle x, y\rangle$. Let $\Delta$ be the support of $H$, that is, the subset of $\{1,2, \ldots, m\}$ consisting of all the points moved by $H$, and let $d:=|\Delta|$. Since $q>p>m / 2$, it follows that $H$ is transitive on $\Delta$, and that $q \leq d \leq m<2 p$. By [Wi, Theorem 8.4], $H$ is primitive on $\Delta$. Moreover, since $d \geq q>p+3>5$ and $H$ contains only even permutations, it follows from a theorem of C. Jordan dating from 1873, see [Wi, Theorem 13.9], that $H \cong A_{d}$ for some $d \geq 5$, as claimed.

It remains to deal with $A_{m}$, for $5 \leq m \leq 33$. In each case we will choose primes $p$ and $q$ such that $m / 2 \leq p<q \leq m$, and consider the subgroup $H=\langle x, y\rangle$ generated by elements $x$ and $y$ of $A_{m}$ of order $p$ and $q$, respectively. Denote by $\Delta$ the support of $H$, and let $d=|\Delta|$. In all cases $d \geq 5$ since $d \geq q>p \geq 3$, and hence $A_{d}$ is nonsolvable.

For $17 \leq m \leq 33$, and for $11 \leq m \leq 13$, let $q$ be the largest prime such that $q \leq m$, and let $p$ be the smallest prime such that $p>m / 2$. Then $p \leq q-3$, and the argument above shows that $H \cong A_{d}$.

If $14 \leq m \leq 16$, let $q=13, p=11$. If $d=13$, then $H \leq A_{13}$ and since by the [ATLAS, p. 104] no maximal subgroup of $A_{13}$ has order divisible by $11 \cdot 13$, it follows that $H=A_{13}$. If $d>13$, then as before, $H$ is a primitive group on $\Delta$, and since $d=p+k$ with $k \geq 3$, it follows by [Wi, Theorem 13.9] that $H \cong A_{d}$.

If $7 \leq m \leq 10$, let $q=7, p=5$. It follows from the lists of maximal subgroups of $A_{d}$ in ATLAS, pp. 10, 22, 37, 48] that $H \cong A_{d}$. If $m=5,6$, let $q=5, p=3$. It follows from the lists of maximal subgroups of these groups in [ATLAS, pp. 2, 4] that $H=A_{5}$ if $m=5$, and that $H \cong A_{5}$ or $A_{6}$ if $m=6$. The proof of Proposition 2.1 is complete.

Theorem B for the sporadic simple groups follows from the following proposition. For compactness of notation, we write $L_{2}(q)$ instead of $\operatorname{PSL}(2, q)$ in Table 1.

| $S$ | $p$ | $q$ | $\langle x, y\rangle$ | $S$ | $p$ | $\langle x, y\rangle$ |  |
| ---: | ---: | ---: | :--- | ---: | ---: | :--- | :--- |
| $M_{11}$ | 2 | 11 | $M_{11}, L_{2}(11)$ | $M_{12}$ | 2 | 11 | $M_{12}, M_{11}, L_{2}(11)$ |
| $M_{22}$ | 2 | 11 | $M_{22}, L_{2}(11)$ | $M_{23}$ | 2 | 23 | $M_{23}$ |
| $M_{24}$ | 2 | 23 | $M_{24}, L_{2}(23)$ | $J_{1}$ | 5 | 19 | $J_{1}$ |
| $J_{2}$ | 2 | 7 | $J_{2}, L_{2}(7)$ | $J_{3}$ | 2 | 19 | $J_{3}, L_{2}(19)$ |
| $J_{4}$ | 3 | 43 | $J_{4}$ | $H S$ | 2 | 11 | $H S, M_{11}, L_{2}(11), M_{22}$ |
| $H e$ | 7 | 17 | $H e$ | $M c L$ | 2 | 11 | $M c L, M_{11}, M_{12}, L_{2}(11)$ |
| $S u z$ | 11 | 13 | $S u z$ | $L y$ | 3 | 67 | $L y$ |
| $R u$ | 3 | 29 | $R u$ | $O^{\prime} N$ | 2 | 31 | $O^{\prime} N$ |
| $C o_{1}$ | 13 | 23 | $C o_{1}$ | $C o_{2}$ | 2 | 23 | $C o_{2}, M_{23}$ |
| $C o_{3}$ | 2 | 23 | $C o_{3}, M_{23}$ | $F i_{22}$ | 11 | 13 | $F i_{22}$ |
| $F i_{23}$ | 2 | 23 | $F i_{23}, M_{23}, L_{2}(23)$ | $F i_{24}^{\prime}$ | 3 | 29 | $F i_{24}^{\prime}$ |
| $H N$ | 5 | 19 | $H N$ | $T h$ | 19 | 31 | $T h$ |
| $B$ | 2 | 47 | $B$ | $M$ | 2 | 59 | $M, L_{2}(59)$ |

Table 1. Results table for Proposition 2.2

Proposition 2.2. Let $S$ be a sporadic simple group as in one of the rows of Table 1. Then for the primes $p, q$ in the corresponding row of Table 1, $p, q \in o e(S)$ and, for all $x, y \in S$ with $|x|=p$ and $|y|=q$, the subgroup $\langle x, y\rangle$ is one of the nonabelian simple groups in the row for $S$ and column labeled $\langle x, y\rangle$ of Table 1 .

Proof. The proof uses heavily the lists of maximal subgroups of the sporadic simple groups and of other simple groups, which appear in ATLAS and in ATLAS3. For all sporadic simple groups our results follow from a close examination of these lists. In particular the groups listed in the column labeled $\langle x, y\rangle$ are the only subgroups which could possibly be generated by two elements, the first of order $p$ and the second of order $q$.

As an example, we describe in detail our treatment of the HigmanSims sporadic group $H S$ of order $2^{9} .3^{2} .5^{3} .7 .11$. Having checked the maximal subgroups of $H S$, we choose the primes $p=2$ and $q=11$. If the subgroup $X=\langle x, y\rangle$ is not equal to $H S$, then $X$ is contained in a maximal subgroup of $H S$ of order divisible by 11 . We see from [ATLAS, p. 80] that the only such maximal subgroups of $H S$ are the simple groups $M_{22}$ and $M_{11}$. Suppose first that $X$ is a subgroup of $M_{11}$. If $X \neq M_{11}$, then it is contained in a maximal subgroup of $M_{11}$ of order divisible by 11. By [ATLAS, p.18], each such maximal subgroup of $M_{11}$ is isomorphic to the simple group $L_{2}(11)$. If $X \neq L_{2}(11)$, then $X$ is contained in a maximal subgroup of $L_{2}(11)$ of order divisible by 11. However, such maximal subgroups have order $5 \cdot 11$, which is
not divisible by 2 . Thus we are left with the possibilities: $X=H S$, $X=M_{11}, X=L_{2}(11)$, or $X$ is a subgroup of $M_{22}$. If $X$ is a proper subgroup of $M_{22}$, then $X$ is contained in a maximal subgroup of $M_{22}$ of order divisible by 11. By ATLAS, p.30], the only such maximal subgroup of $M_{22}$ is the simple group $L_{2}(11)$, which we have already examined. Thus, finally, $X$ is one of the simple groups $H S, M_{11}$, $L_{2}(11)$ or $M_{22}$, as in Table 1 .

Thus Proposition 2.2 is proved.

## 3. Primitive prime divisors

In the following, $q=p^{k}$ is a power of a prime $p$. For any positive integer $e$, we say that a prime $r$ is a primitive prime divisor of $q^{e}-1$ if $r$ divides $q^{e}-1$ and $r$ does not divide $q^{i}-1$ for any positive integer $i<e$. Observe that then $e$ is the order of $q$ modulo the prime $r$; so $e$ divides $r-1$ and, in particular, $r \geq e+1$. The set of primitive prime divisors of $q^{e}-1$ will be denoted by $\operatorname{ppd}(q, e)$.

We say that a prime $r$ is a basic primitive prime divisor of $q^{e}-1$, if $r$ is a primitive prime divisor of $p^{k e}-1$, that is to say, if $r \in \operatorname{ppd}(p, e k)$. We denote by $\operatorname{bppd}(q, e)$ the set of basic primitive prime divisors of $q^{e}-1$. Note that $\operatorname{bppd}(q, e) \subseteq \operatorname{ppd}(q, e)$, and that the inclusion can be strict; for example, $\operatorname{ppd}\left(2^{2}, 3\right)=\{7\}$, but $\operatorname{bppd}\left(2^{2}, 3\right)=\operatorname{ppd}(2,6)=\emptyset$. The following result of Zsigmondy [Z] will be used frequently.

Theorem 3.1. Let $q \geq 2$ and $e \geq 2$. Then $\operatorname{bppd}(q, e)=\emptyset$ if and only if one of the following holds.
(i): $q$ is a Mersenne prime, $e=2$, and here $\operatorname{ppd}(q, 2)=\emptyset$;
(ii): $(q, e)=(2,6)$, and here $\operatorname{ppd}(2,6)=\emptyset$;
(iii): $(q, e) \in\{(4,3),(8,2)\}$, and here $\operatorname{ppd}(4,3)=\{7\}$, and $\operatorname{ppd}(8,2)=\{3\}$.
Next, we define the set $\operatorname{Lpd}(q, e)$ of large primitive divisors of $q^{e}-1$ to be the set consisting of primes $r \in \operatorname{ppd}(q, e)$ such that $r>e+1$ together with the integer $(e+1)^{2}$ if $e+1 \in \operatorname{ppd}(q, e)$ and $(e+1)^{2}$ divides $q^{e}-1$.

Finally, we define the set $\operatorname{Lbpd}(q, e)$ of large basic primitive divisors of $q^{e}-1$ to be the set consisting of primes $r \in \operatorname{bppd}(q, e)$ such that $r>e+1$, together with the integer $(e+1)^{2}$ if $e+1 \in \operatorname{bppd}(q, e)$ and $(e+1)^{2}$ divides $q^{e}-1$.

The following observation will be very useful in the sequel.
Proposition 3.2. Let $q \geq 2$ and $e \geq 3$. Then $\operatorname{Lbpd}(q, e)=\emptyset$ if and only if $(q, e)$ is one of the following:

$$
(2,4),(2,6),(2,10),(2,12),(2,18),(3,4),(3,6),(4,3),(5,6) .
$$

Proof. Let Lpd, Lbpd, bppd, ppd denote the sets $\operatorname{Lpd}(q, e), \operatorname{Lbpd}(q, e)$, $\operatorname{bppd}(q, e), \operatorname{ppd}(q, e)$, respectively. We prove first that $\operatorname{Lbpd} \neq \emptyset$ if and only if both bppd $\neq \emptyset$ and Lpd $\neq \emptyset$. If Lbpd $\neq \emptyset$, then clearly $\operatorname{bppd} \neq \emptyset$ and Lpd $\neq \emptyset$. Conversely, suppose that $\operatorname{bppd} \neq \emptyset$ and $\operatorname{Lpd} \neq \emptyset$. Let $r$ be the largest element of $\operatorname{bppd}$. Then $r \in \operatorname{ppd}(p, k e)$, so $r \geq k e+1 \geq e+1$. If $r>e+1$, then $r \in \operatorname{Lbpd}$, so we may assume that $r=e+1$. Then $k=1$ and $\operatorname{bppd}=\operatorname{ppd}=\{e+1\}$. Let $s \in \operatorname{Lpd}$. Since ppd $=\{e+1\}$, it follows that $s=(e+1)^{2}$ and $s$ divides $q^{e}-1$, whence $s \in \operatorname{Lbpd}$. Thus in both cases Lbpd is nonempty, as required.

Hence Lbpd is empty if and only if either Lpd is empty or bppd is empty. Now assume that $q \geq 2$ and $e \geq 3$. By [NP1, Theorem 2.2], the set Lpd is empty only for $q=2$ and $e \in\{4,6,10,12,18\}$, for $q=3$ and $e \in\{4,6\}$, and for $(q, e)=(5,6)$. Moreover, by Theorem 3.1, the set bppd is empty only if $(q, e)=\{(2,6),(4,3)\}$. So Lbpd is empty precisely for the values of $(q, e)$ listed in the proposition.

Remark 3.3. Let $G$ be a subgroup of $\operatorname{GL}(d, q)$ and let $m \in \operatorname{Lpd}(q, e)$ (or $m \in \operatorname{Lbpd}(q, e)$ ) with $d \geq e>d / 2$. If $m$ divides $|G|$, then $m \in$ $o e(G)$. In fact, either $m=r$ or $m=r^{2}=(e+1)^{2}$, where $r$ is a (basic) primitive prime divisor of $q^{e}-1$. Since $e>d / 2$, a Sylow $r$-subgroup of $\mathrm{GL}(d, q)$ is cyclic and so $G$ has elements of order $m$.

We say that an element $g \in G$ is a $\operatorname{bppd}(q, e)$-element if the order of $g$ is divisible by some element of $\operatorname{bppd}(q, e)$. Similarly, we say that $g \in G$ is an $\operatorname{Lbpd}(q, e)$-element if the order of $g$ is divisible by some element of $\operatorname{Lbpd}(q, e)$.

We use results from [NP1] and [NP2] to deal with subgroups of linear groups containing one or two "big" ppd-elements. We observe here that basic primitive prime divisors will be relevant in order to exclude examples of "subfield type" (see case (d) in the proof of Lemma 3.4), while large primitive divisors will be relevant in the proof of Lemma 4.1, Our first lemma deals with irreducible subgroups of $\operatorname{GL}(d, q)$ for $d \geq 3$.

Lemma 3.4. Let $G$ be an irreducible subgroup of $\mathrm{GL}(d, q)$, with $d \geq 3$. Assume that $x, y \in G$ are such that $x$ is a $\operatorname{bppd}(q, e)$-element and $y$ is a $\operatorname{bppd}(q, f)$-element, with $d \geq e>f>d / 2$. Then either $G$ is nonsolvable or one of the following holds.
(1): $d=e=f+1$ is prime, $G$ is conjugate to a subgroup of $\operatorname{GL}\left(1, q^{d}\right) . d$, and $G$ contains no $\operatorname{Lbpd}(q, f)$-elements;
(2): There exists a prime $c$ dividing $\operatorname{gcd}(d, e, f)$ such that $G$ is conjugate to a subgroup of $\mathrm{GL}\left(\frac{d}{c}, q^{c}\right) . c$ and the elements $x^{c}$ and $y^{c}$ lie in $\operatorname{GL}\left(\frac{d}{c}, q^{c}\right)$ as a $\operatorname{bppd}\left(q^{c}, \frac{e}{c}\right)$-element and a $\operatorname{bppd}\left(q^{c}, \frac{f}{c}\right)$ element, respectively.

Proof. By assumption $|x|$ is a multiple of an element $r \in \operatorname{bppd}(q, e)$ and $|y|$ is a multiple of an element $s \in \operatorname{bppd}(q, f)$. Suppose first that $d=3$ and $q \leq 4$. Then $3 \geq e>f>3 / 2$ and so $e=3, f=2$ and $\operatorname{bppd}(q, 3) \neq \emptyset, \operatorname{bppd}(q, 2) \neq \emptyset$. By Theorem 3.1, $q \neq 3,4$. Hence, $q=2$ and so $r=7, s=3, G \leq \operatorname{GL}(3,2)$ and $\operatorname{Lbpd}(q, f)=\emptyset$. By ATLAS, p.3], either $G=\mathrm{GL}(3,2)$ is nonsolvable or $G=\langle x, y\rangle \cong \mathrm{GL}(1,8) .3$ and part (1) holds. Thus if $d=3$ we may assume that $q \geq 5$.

Next we claim that, if $(d, q)=(4,2)$ or $(4,3)$, then $G$ is nonsolvable. In these cases $d=4 \geq e>f>2$ and so $e=4, f=3$ and $\operatorname{bppd}(q, 4) \neq$ $\emptyset, \operatorname{bppd}(q, 3) \neq \emptyset$. If $(d, q)=(4,3)$, then $G \leq \operatorname{GL}(4,3) \cong 2 . \operatorname{PSL}(4,3) .2$ (see [ATLAS, pp. 68 and 69]) and $r=5, s=13$. Since no proper subgroup of $\operatorname{PSL}(4,3)$ is divisible by $5 \cdot 13, \operatorname{PSL}(4,3)$ is a section of $G$ and hence $G$ is nonsolvable, as claimed. If $(d, q)=(4,2)$, then $G \leq \mathrm{GL}(4,2) \cong \mathrm{PSL}(4,2)$ and $r=5, s=7$. It follows from ATLAS, pp. 10 and 22] that $G \cong A_{7}$ or $A_{8}$ and in particular $G$ is nonsolvable. Thus we may assume further that $(d, q) \neq(4,2)$ or $(4,3)$.

Note that $q=p^{k}$ for some prime $p$ and $k \geq 1$. Since $r \in \operatorname{ppd}(p, e k)$, $s \in \operatorname{ppd}(p, f k)$ and $e>f \geq 2$, neither $r$ nor $s$ divides $q-1$. Moreover, if $i \leq d$ and $h \leq \frac{k}{2}$, then neither $r$ nor $s$ divides $p^{i h}-1$ (since $i h \leq$ $\left.\frac{d k}{2}<f k<e k\right)$.

We apply NP1, Theorem 4.7]. First we show that the cases (a), (c), (d) and (e) of [NP1, Theorem 4.7], if they occur, imply that $G$ is nonsolvable.
(a): (Classical type) $G$ has a normal subgroup $\Omega$, where $\Omega$ is one of the following groups: $\operatorname{SL}(d, q), \operatorname{Sp}(d, q)(d$ even $), \operatorname{SU}\left(d, q_{0}\right)(q=$ $\left.q_{0}^{2}\right), \Omega^{ \pm}(d, q)(d$ even $)$, and $\Omega^{\circ}(d, q)$ (d odd). Because of our assumption that $(d, q) \neq(4,2),(4,3)$ or $(3, q)$ with $q \leq 4$, each of these classical groups is nonsolvable and so $G$ is nonsolvable in case (a).
(c): (Nearly Simple Groups) Here $G$ is nearly simple, and hence nonsolvable.
(d): (Subfield type) Denote by $Z$ the subgroup of scalar matrices of $\mathrm{GL}(d, q)$ and let $Z \circ \mathrm{GL}\left(d, q_{0}\right)$ denote a group which can be defined over a subfield modulo scalars. In case (d), $G$ is conjugate to a subgroup of $Z \circ \mathrm{GL}\left(d, q_{0}\right)$ for some proper subfield $\mathbb{F}_{q_{0}}$ of $\mathbb{F}_{q}$, say $q=q_{0}^{a}$ and $q_{0}=p^{h}$, with $k=a h$ and $a \geq 2$. Since $x \in G$ and $r$ does not divide $q-1$ (as noted above), it follows that $r$ divides $\left|\operatorname{GL}\left(d, q_{0}\right)\right|$. Hence $r$ divides $p^{h t}-1$ for some $t \leq d$. Since $h \leq \frac{k}{2}$, this contradicts our observation above, so case (d) does not arise.
(e): (Imprimitive type) Here $G$ preserves a direct-sum decomposition $V=U_{1} \oplus \cdots \oplus U_{d}$ with $\operatorname{dim}\left(U_{i}\right)=1$ for $i=1, \ldots, d$ and $G$ induces $A_{d}$ or $S_{d}$ on the set $\left\{U_{1}, \ldots, U_{d}\right\}$. Since $A_{d}$ is nonsolvable for $d \geq 5$,
we may assume that $d \leq 4$. In particular, $|G|$ divides $(q-1)^{d} d$ ! and since the primes $r, s$ do not divide $q-1$, each of them divides $d!$. In particular, $r, s \leq d \leq 4$. However, $e>f \geq 2$, so $r \geq e+1 \geq 4$, which implies that $r=4$, a contradiction.

This leaves case (b) of [NP1, Theorem 4.7], and in that case $G$ is conjugate to a subgroup of $\operatorname{GL}\left(\frac{d}{c}, q^{c}\right) . c$, for some prime $c$ dividing $d$. Moreover either $c=d=f+1=e$, or $c<d$ and $c$ divides $\operatorname{gcd}(d, e, f)$. In the former case, $G \leq \mathrm{GL}\left(1, q^{d}\right) \cdot d$ and $s=d=f+1$. As $s^{2}$ does not divide $|G|$, part (1) holds. In the latter case, since each of $r, s$ is at least $f+1>c$, it follows that $x^{c}, y^{c} \in \mathrm{GL}\left(\frac{d}{c}, q^{c}\right)$ and have orders divisible by $r, s$, respectively. Thus part (2) holds.

If $V$ is a $G$-module and $U$ is a $G$-invariant section of $V$ (that is to say, $U=V_{1} / V_{2}$ where $V_{2} \leq V_{1}$ are $G$-submodules of $V$ ), we denote by $G^{U}$ the group of automorphisms induced by $G$ on $U$. So $G^{U} \cong G / C_{G}(U)$, a factor group of $G$, and accordingly we denote by $x^{U}$ the image in $G^{U}$ of an element $x \in G$ under the relevant projection homomorphism.

For a prime $r$ and integer $n$, let $n_{r}$ denote the $r$-part of $n$, that is, the largest power of $r$ dividing $n$.

Remark 3.5. Let $G \leq \mathrm{GL}(d, q)$ and let $V=V(d, q)$ be the natural module for $G$. Suppose that $x$ is a $\operatorname{ppd}(q, e)$-element of $G$ of order divisible by a primitive prime divisor $r$ of $q^{e}-1$ and let $x_{0}$ be an element of order $r$ in $\langle x\rangle$. Then the following statements hold.
(1) Under the action of $\left\langle x_{0}\right\rangle,\left.V\right|_{\left\langle x_{0}\right\rangle}$ is a completely reducible $\mathbb{F}_{q}\left\langle x_{0}\right\rangle$ module, by Maschke's Theorem, and all the nontrivial irreducible submodules of $\left.V\right|_{\left\langle x_{0}\right\rangle}$ have dimension $e$ (see for instance [H, Theorem II.3.10]). In particular, it follows that there exists at least one $G$ composition factor $U$ of $V$ on which $x_{0}$, and hence also $x$, acts nontrivially. So $\left|x_{0}^{U}\right|=r$ and hence $x^{U}$ is a $\operatorname{ppd}(q, e)$-element of $G^{U} \leq$ $\operatorname{GL}\left(d_{0}, q\right)$ of order divisible by $\left|x_{0}^{U}\right|=r$ and satisfying $\left|x^{U}\right|_{r}=|x|_{r}$, where $d_{0}=\operatorname{dim}_{\mathbb{F}_{q}}(U) \geq e$.
(2) Suppose also that $y$ is a $\operatorname{ppd}(q, f)$-element of $G$ of order divisible by a primitive prime divisor $s$ of $q^{f}-1$, and that $d \geq e>f>d / 2$. Since both $e$ and $f$ are greater than $d / 2$, it follows by part (1) of this remark that there exists a unique $G$-composition factor $U$ of $V$, of dimension $d_{0}=\operatorname{dim}_{\mathbb{F}_{q}}(U) \geq e$, on which both $x$ and $y$ act nontrivially. Moreover $G^{U} \leq \operatorname{GL}\left(d_{0}, q\right)$, and $x^{U}$ is a $\operatorname{ppd}(q, e)$-element of $G^{U}$ of order satisfying $\left|x^{U}\right|_{r}=|x|_{r} \geq r$ and $y^{U}$ is a $\operatorname{ppd}(q, f)$-element of $G^{U}$ of order satisfying $\left|y^{U}\right|_{s}=|y|_{s} \geq s$.

We now apply Lemma 3.4 to linear groups of dimension large enough to allow the existence of primitive prime divisors for two distinct exponents, both greater than half the dimension of the linear group. It is convenient to deal separately with certain large exponent pairs.

Lemma 3.6. Let $j$ be a nonnegative integer, $\delta \in\{1,2\}$, and let $x, y \in$ $\mathrm{GL}(d, q)$ be such that $x$ is a $\operatorname{bppd}(q, e)$-element and $y$ is a $\operatorname{bppd}(q, f)$ element. Assume that e, $f, \delta$, and d satisfy the following conditions:

$$
e=d-j, f=d-j-\delta, d \geq 2 j+2 \delta+1
$$

If $\delta=1$, we assume in addition that $y$ is an $\operatorname{Lbpd}(q, f)$-element. Then $G=\langle x, y\rangle$ is nonsolvable.

Proof. Let $r \in \operatorname{bppd}(q, e)$ such that $r$ divides $|x|$, and let $s \in \operatorname{bppd}(q, f)$ such that $s$ divides $|y|$. As mentioned above, $r \equiv 1(\bmod e)$ and $s \equiv 1$ $(\bmod f)$. Moreover, $e>f=d-j-\delta \geq j+\delta+1 \geq 2$, so in particular both $r$ and $s$ are odd primes. As it is sufficient to prove that some subgroup of $G$ is nonsolvable, we may assume that $G=\langle x, y\rangle$, and $|x|,|y|$ are powers of $r$ and $s$, respectively. In particular, $|x|$ and $|y|$ are odd. If $\delta=1$, then in addition we assume that $y$ is an $\operatorname{Lbpd}(q, f)$ element.

By our assumptions, $j \leq \frac{d-1}{2}-\delta$, so $d \geq e>f=d-j-\delta \geq \frac{d+1}{2}>$ $\frac{d}{2}$. In the following, we denote by $V$ the natural module $V(d, q)$ for $\mathrm{GL}(d, q)$.

Case (1): $G$ is irreducible on $V$.
The conditions of Lemma 3.4 are satisfied. Applying that result we deduce that either $G$ is nonsolvable, or (1) $d=e=f+1$ is prime, $G$ is conjugate to a subgroup of $\mathrm{GL}\left(1, q^{d}\right) . d$, and $G$ contains no $\operatorname{Lbpd}(q, f)$ elements, or (2) there is a prime $c<d$ such that $c$ divides $\operatorname{gcd}(d, e, f)$, $G$ is conjugate to a subgroup of $\mathrm{GL}\left(d / c, q^{c}\right) \cdot c$, and $x^{c}$ and $y^{c}$ lie in $\operatorname{GL}\left(\frac{d}{c}, q^{c}\right)$ as a $\operatorname{bppd}\left(q^{c}, \frac{e}{c}\right)$-element and a $\operatorname{bppd}\left(q^{c}, \frac{f}{c}\right)$-element, respectively. If (1) holds, then $j=0, \delta=1$ and we have a contradiction, since $y$ is an $\operatorname{Lbpd}(q, f)$-element of $G$. Thus (2) holds. Since $c$ is a prime dividing $\operatorname{gcd}(d, e, f)$, and since $\operatorname{gcd}(e, f)=\operatorname{gcd}(e, e-f) \leq e-f=$ $\delta \leq 2$, it follows that $c=\delta=2$ and all of $d, e, f, j$ are even. Hence $G \leq \operatorname{GL}\left(d / 2, q^{2}\right) \cdot 2$, and since the orders of $x, y$ are odd, we conclude that $G \leq \mathrm{GL}\left(d / 2, q^{2}\right)$. Since $d$ and $j$ are even, our assumption that $d \geq 2 j+2 \delta+1=2 j+5$ implies $\frac{d}{2} \geq 2 \frac{j}{2}+3$. Thus replacing $(d, q, e, f, j, \delta)$ by $\left(\frac{d}{2}, q^{2}, \frac{e}{2}, \frac{f}{2}, \frac{j}{2}, \frac{\delta}{2}\right)$, all the conditions of the lemma hold, and $G$ is irreducible on $V\left(\frac{d}{2}, q^{2}\right)$. By the arguments above and since $\frac{\delta}{2}=1$, we conclude that $G$ must be nonsolvable.

Case (2): $G$ is reducible on $V$.

If $e=d$, then $|x|$ would be a multiple of a primitive prime divisor of $q^{d}-1$ and so $\langle x\rangle$ would act irreducibly on the natural module $V$. However $G$ is reducible, so we must have $e<d$ and $j \geq 1$. By Remark 3.5(2), there exists a unique $G$-composition factor $U$ of $V$ of dimension $d_{0}=\operatorname{dim}_{\mathbb{F}_{q}}(U)$, say, where $d>d_{0} \geq e=d-j$, such that $x^{U}$ and $y^{U}$ are $\operatorname{bppd}(q, e)$ - and $\operatorname{bppd}(q, f)$-elements of $G^{U}$, respectively, with $\left|x^{U}\right|_{r}=|x|_{r} \geq r,\left|y^{U}\right|_{s}=|y|_{s} \geq s$. In particular, if $y$ is an $\operatorname{Lbpd}(q, f)-$ element of $G$, then $y^{U}$ is an $\operatorname{Lbpd}(q, f)$-element of $G^{U}$.

We claim that the irreducible group $G^{U}=\left\langle x^{U}, y^{U}\right\rangle \leq \mathrm{GL}\left(d_{0}, q\right)$ induced by $G$ on $U$ satisfies the conditions of Lemma 3.6 with parameters $d_{0}, e, f, \delta$, relative to the integer $j_{0}:=j-d+d_{0}$. Note that $j>j_{0}=d_{0}-e \geq 0$ and $d_{0}=j_{0}+d-j \geq j_{0}+j+2 \delta+1>2 j_{0}+2 \delta+1 \geq 3$, and the conditions $e=d_{0}-j_{0}, f=d_{0}-j_{0}-\delta$ hold, by the definition of $j_{0}$. This proves the claim. Since $G^{U}$ is irreducible, it follows from Case (1) of this proof that $G^{U}$ is nonsolvable. Consequently, also $G$ is nonsolvable.

We can now prove Theorem B for classical simple groups of large dimension.

Proposition 3.7. Assume that $S$ is one of the following simple groups:
(1): $\operatorname{PSL}(d, q)$, with $d \geq 4$ and $(d, q) \neq(6,2)$;
(2): $\operatorname{PSp}(d, q)$ ( $d$ even) or $\operatorname{PSU}(d, q)$ ( $d$ odd), with $d \geq 5$ and $(d, q) \notin\{(5,2),(6,2),(8,2)\}$;
(3): $\mathrm{P} \Omega^{\circ}(d, q)$ (dq odd) or $\operatorname{PSU}(d, q)$ (d even), with $d \geq 7$;
(4): $\mathrm{P} \Omega^{ \pm}(d, q)$ ( $d$ even), with $d \geq 10$ and $(d, q) \neq(10,2)$.

Then there exist $a, b \in$ oe $(S)$ such that for every choice of $x, y \in S$ with $|x|=a$ and $|y|=b$, the group $\langle x, y\rangle$ is nonsolvable.

Proof. We work with the group $\hat{S}$ defined as follows:

- $\hat{S}=\mathrm{SL}(d, q) \leq \mathrm{GL}(d, q)$, when $S=\operatorname{PSL}(d, q)$;
- $\hat{S}=\operatorname{Sp}(d, q) \leq \operatorname{GL}(d, q)$, when $S=\operatorname{PSp}(d, q)$;
- $\hat{S}=\mathrm{SU}(d, q) \leq \operatorname{GL}\left(d, q^{2}\right)$, when $S=\operatorname{PSU}(d, q)$;
- $\hat{S}=\Omega^{\varepsilon}(d, q) \leq \operatorname{GL}(d, q)$, when $S=\operatorname{P} \Omega^{\varepsilon}(d, q)$, for $\varepsilon= \pm$ or $\circ$.

Let $\mathbb{Z}=\mathbb{Z}(\hat{S})$. We prove that there exist $a, b \in o e(\hat{S})$ with $\operatorname{gcd}(a b,|\mathbb{Z}|)$ $=1$ such that, for every $\hat{x} \in \hat{S}$ of order $a$ and every $\hat{y} \in \hat{S}$ of order $b$, the group $\langle\hat{x}, \hat{y}\rangle$ is nonsolvable. Since $a$ and $b$ are coprime to $|\mathbb{Z}|$, if $|\hat{x}|=a$ then also the order of $\hat{x} \mathbb{Z}$ in $S$ is equal to $a$, and similarly, if $|\hat{y}|=b$ then $|\hat{y} \mathbb{Z}|=b$. In addition, if $\langle\hat{x}, \hat{y}\rangle$ is nonsolvable then also $\langle x, y\rangle$ is nonsolvable, because it is a central factor of the nonsolvable group $\langle\hat{x}, \hat{y}\rangle$. Thus it is sufficient to work as described with the group
$\hat{S}$. In fact, each of $a$ and $b$ will be either a prime or the square of a prime, and it will be easy to check that $\operatorname{gcd}(a b,|\mathbb{Z}|)=1$.
(1): Let $\hat{S}=\operatorname{SL}(d, q)$, with $d \geq 4$. Suppose first that

$$
\begin{equation*}
(d, q) \notin\{(4,4),(5,2),(5,3),(6,2),(7,2),(7,3),(7,5), \tag{1.1}
\end{equation*}
$$

$$
(11,2),(13,2),(19,2)\}
$$

Then by Theorem 3.1 there exists $a \in \operatorname{bppd}(q, d)$, and by Proposition 3.2 there exists $b \in \operatorname{Lbpd}(q, d-1)$. Now $q^{d}-1$ and $q^{d-1}-1$ divide $\hat{S}$ and hence $a, b \in o e(\hat{S})$ by Remark 3.3, Also $\operatorname{gcd}(a b,|\mathbb{Z}|)=1$. Let now $\hat{x}, \hat{y} \in \hat{S}$ be such that $|\hat{x}|=a$ and $|\hat{y}|=b$. Then $\hat{x}$ is a $\operatorname{bppd}(q, d)-$ element of $\hat{S}$ and $\hat{y}$ is an $\operatorname{Lbpd}(q, d-1)$-element of $\hat{S}$. Since $d \geq 4$, the conditions of Lemma 3.6 are satisfied with $\delta=1, j=0$, and we conclude that $\langle\hat{x}, \hat{y}\rangle$ is nonsolvable.

We now consider the "isolated" cases left out of this proof and listed in (1.1). We deal with all these cases, except the excluded pair $(d, q)=$ $(6,2)$, by taking elements $\hat{x}, \hat{y} \in \hat{S}$ with $|\hat{x}|=a$ and $|\hat{y}|=b$, where $a, b$ are as in the following table. (Here $q_{i}$ denotes a prime in $\operatorname{bppd}(q, i)$, and, by Theorem 3.1, for all entries in the table such a prime $q_{i}$ exists.)

| $(d, q)$ | $(4,4)$ | $(5,2)$ | $(5,3)$ | $(7,2)$ | $(7,3)$ | $(7,5)$ | $(11,2)$ | $(13,2)$ | $(19,2)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $a$ | $q_{4}$ | $q_{5}$ | $q_{5}$ | $q_{7}$ | $q_{7}$ | $q_{7}$ | $q_{11}$ | $q_{13}$ | $q_{19}$ |
| $b$ | $q_{2}$ | $q_{3}$ | $q_{3}$ | $q_{5}$ | $q_{5}$ | $q_{5}$ | $q_{9}$ | $q_{11}$ | $q_{17}$ |

Thus $\hat{x}, \hat{y}$ satisfy all the conditions of Lemma 3.6 with $\delta=2, j=0$, and hence $\langle\hat{x}, \hat{y}\rangle$ is nonsolvable.
(2): If $\hat{S}=\operatorname{Sp}(d, q)$ with $d \geq 5$ and $d$ even, then the order of $\hat{S}$ is divisible by both $q^{d}-1$ and $q^{d-2}-1$. By Theorem 3.1, $\operatorname{bppd}(q, d) \neq \emptyset$ and $\operatorname{bppd}(q, d-2) \neq \emptyset$, since by our assumptions $d>d-2 \geq 4$ and $(q, d),(q, d-2) \neq(2,6)$. We take $a \in \operatorname{bppd}(q, d)$ and $b \in \operatorname{bppd}(q, d-2)$ and note that $a, b \in o e(\hat{S})$.

If $\hat{S}=\mathrm{SU}(d, q) \leq \mathrm{GL}\left(d, q^{2}\right)$ with $d \geq 5$ and $d$ odd, then the order of $\hat{S}$ is divisible by both $q^{d}+1$ and $q^{d-2}+1$. Here we take $a \in \operatorname{bppd}(q, 2 d)=$ $\operatorname{bppd}\left(q^{2}, d\right)$ and $b \in \operatorname{bppd}(q, 2 d-4)=\operatorname{bppd}\left(q^{2}, d-2\right)$. Note that both $\operatorname{bppd}(q, 2 d)$ and $\operatorname{bppd}(q, 2 d-4)$ are nonempty, since $(d, q) \neq(5,2)$. Also $a, b \in o e(\hat{S})$.

In both cases we apply Lemma 3.6 with $\delta=2, j=0$. It follows that in each of these two cases, $\langle\hat{x}, \hat{y}\rangle$ is nonsolvable for every choice of $\hat{x}, \hat{y} \in \hat{S}$ with $|\hat{x}|=a$ and $|\hat{y}|=b$.
(3): If $\hat{S}=\Omega^{\circ}(d, q)$, with $d \geq 7$ and $d q$ odd, then both $q^{d-1}-1$ and $q^{d-3}-1$ divide the order of $\hat{S}$ and each has a basic primitive prime divisor by Theorem 3.1. We take $a \in \operatorname{bppd}(q, d-1)$ and $b \in$ $\operatorname{bppd}(q, d-3)$ and note that $a, b \in o e(\hat{S})$.

If $\hat{S}=\mathrm{SU}(d, q) \leq \mathrm{GL}\left(d, q^{2}\right)$ with $d \geq 7$ and $d$ even, then the order of $\hat{S}$ is divisible by both $q^{d-1}+1$ and $q^{d-3}+1$. We take $a \in \operatorname{bppd}(q, 2 d-$ $2)=\operatorname{bppd}\left(q^{2}, d-1\right)$ and $b \in \operatorname{bppd}(q, 2 d-6)=\operatorname{bppd}\left(q^{2}, d-3\right)$, noting that both $\operatorname{bppd}\left(q^{2}, d-1\right)$ and $\operatorname{bppd}\left(q^{2}, d-3\right)$ are nonempty, by Theorem 3.1, since $d-1>d-3 \geq 4$ and $q^{2}>2$. Thus $a, b \in o e(\hat{S})$.

In both cases we apply Lemma 3.6 with $\delta=2, j=1$. It follows that in each of these two cases, $\langle\hat{x}, \hat{y}\rangle$ is nonsolvable for every choice of $\hat{x}, \hat{y} \in \hat{S}$ with $|\hat{x}|=a$ and $|\hat{y}|=b$.
(4): Assume, finally, that $\hat{S}=\Omega^{ \pm}(d, q)$ with $d \geq 10, d$ even, and $(d, q) \neq(10,2)$. In this case both $q^{d-2}-1$ and $q^{d-4}-1$ divide the order of $\hat{S}$, and each has a basic primitive prime divisor by Theorem 3.1, since $d-2>d-4 \geq 6$ and $(d, q) \neq(10,2)$. We take $a \in \operatorname{bppd}(q, d-2)$ and $b \in \operatorname{bppd}(q, d-4)$, and note that $a, b \in o e(\hat{S})$. By Lemma3.6 with $\delta=2, j=2$, we conclude that $\langle\hat{x}, \hat{y}\rangle$ is nonsolvable for every choice of $\hat{x}, \hat{y} \in \hat{S}$ with $|\hat{x}|=a$ and $|\hat{y}|=b$.

## 4. Proof of Theorem B

Proposition 3.7 does not cover the classical groups in small dimensions, when the conditions of Lemma 3.6 do not hold. For most of these cases we use the following result, which deals with the case where there is at least one $\operatorname{Lbpd}(q, e)$-element in $G \leq \operatorname{GL}(d, q)$, with $e>d / 2$.
Lemma 4.1. Let $G$ be an irreducible subgroup of $\mathrm{GL}(d, q)$, with $d \geq 3$. Assume that $G$ contains an $\operatorname{Lbpd}(q, e)$-element for some integer $e$ with $d \geq e>d / 2$. Assume also that

$$
(d, q) \notin\{(3,2),(3,3),(3,4),(4,2),(4,3)\}
$$

Then either $G$ is conjugate to a subgroup of $\mathrm{GL}\left(d / c, q^{c}\right) . c$ for some prime divisor $c$ of $\operatorname{gcd}(d, e)$, or $G$ is nonsolvable.
Proof. This lemma follows from [NP2, Theorem 3.1]. Note that the cases (b), (d) and (c)(i) of that theorem do not occur, because of the assumption that the $\operatorname{bppd}(q, e)$-element is large. So either $G$ is of classical type (part (a) of [NP2, Theorem 3.1]) and hence is nonsolvable because of the assumptions on $(d, q)$ (see the details in the proof of Lemma 3.4, case (a)), or $G$ is of nearly simple type (part (e) of NP2, Theorem 3.1]) and hence is nonsolvable, or $G$ is of extension field type
(part (c)(ii) of [NP2, Theorem 3.1]), that is to say, $G$ is conjugate to a subgroup of $\operatorname{GL}\left(d / c, q^{c}\right) . c$ for some prime divisor $c$ of $\operatorname{gcd}(d, e)$.

We now complete the proof of Theorem B for classical groups of Lie type.

Proposition 4.2. Theorem B holds for all classical finite simple groups of Lie type.

Proof. (1): Assume first that $S=\operatorname{PSL}(d, q)$ with $d \geq 2$.
Suppose first that $d=2$ and $q \leq 7$. Theorem B holds for the groups $\operatorname{PSL}(2,4) \cong \operatorname{PSL}(2,5) \cong A_{5}$ by Proposition 2.1. Also, for $S=$ PSL $(2,7)$, the maximal subgroups of order divisible by 7 have order 21 , so $\langle x, y\rangle=S$ whenever $x, y \in S$ with $|x|=2$ and $|y|=7$.

Next we consider $S=\operatorname{PSL}(2, q)$ with $q \geq 8$. Take $a=(q+1) / k$ and $b=(q-1) / k$, where $k=\operatorname{gcd}(q-1,2)$. Then $a \geq 5, b \geq 4$ and $a, b \in o e(S)$ (see Theorems II.8.3 and II.8.4 in $[\mathrm{H}]$ ). The classification of the subgroups of $\operatorname{PSL}(2, q)$ (see Theorem II.8.28 in $[\mathrm{H}]$ ) implies that $\langle x, y\rangle=S$ whenever $x, y \in S$ with $|x|=a$ and $|y|=b$.

Suppose next that $S=\operatorname{PSL}(3, q)$. The group $\operatorname{PSL}(3,2) \cong \operatorname{PSL}(2,7)$ has been dealt with already. If $S=\operatorname{PSL}(3, q)$ with $q=3$ or 4 , then taking $x, y \in S$ with $(|x|,|y|)=(13,2)$ or $(7,5)$, respectively, we find that $S=\langle x, y\rangle$ (see ATLAS, pp. 13,23]). Hence we may assume that $q \geq 5$. By Proposition 3.2, there exists $a \in \operatorname{Lbpd}(q, 3)$. If $q=3^{k}$, then $k>1$ and we take $b \in \operatorname{ppd}(q, 2)$ which, by Theorem 3.1, is nonempty. On the other hand, if $q$ is not a power of 3 , then we choose $b=p$ (recall that $q$ is a power of $p$ ). As in the proof of Proposition 3.7, we work with $\hat{S}=\mathrm{SL}(3, q) \leq \mathrm{GL}(3, q)$ in order to apply Lemma 4.1. (We shall often do this throughout the proof without further reference.) Let $\hat{x}, \hat{y} \in \hat{S}$, with $|\hat{x}|$ a multiple of $a$ and $|\hat{y}|$ a multiple of $b$, and let $X=\langle\hat{x}, \hat{y}\rangle$. Since $\hat{x} \in X, X$ is an irreducible subgroup of $\mathrm{GL}(3, q)$, and since $\hat{y} \in X,|X|$ does not divide |GL $\left(1, q^{3}\right) \cdot 3 \mid$. Hence, by Lemma 4.1, $X$ is nonsolvable. It follows that $\langle x, y\rangle$ is nonsolvable for every $x, y \in S$ with $|x|=a$ and $|y|=b$.

Thus we may assume that $d \geq 4$. For these groups $S$, Theorem B follows from Proposition 3.7(1), unless $(d, q)=(6,2)$. For $S=\operatorname{PSL}(6,2) \cong$ $\mathrm{GL}(6,2)$, consider $a=31 \in \operatorname{Lbpd}(2,5)$ and $b=7 \in \operatorname{bppd}(2,3)$, and note that $a, b \in o e(S)$. Let $x, y \in \operatorname{GL}(6,2)$ with $|x|=31$ and $|y|=7$, and set $X=\langle x, y\rangle$. If $X$ is reducible on $V=V(6,2)$, then, since $x \in X$, $X$ acts irreducibly on some $X$-composition factor $U$ of $V$ of dimension 5 and $X^{U}=\left\langle x^{U}, y^{U}\right\rangle \leq \operatorname{GL}(5,2)$. By Remark 3.5(2), $\left|x^{U}\right|=|x|=31$ and $\left|y^{U}\right|=|y|=7$, and hence $X^{U}$ is nonsolvable by Lemma 3.6 applied with $\delta=2, j=0$. Consequently, also $X$ is nonsolvable. If $X$
is irreducible, then by Lemma 4.1 (applied with $d=6, e=5$ ), $X$ is nonsolvable, since $\operatorname{gcd}(6,5)=1$.
(2): Next let $S=\operatorname{PSp}(d, q)^{\prime}$ with $d \geq 4$ and $d$ even (noting that $\operatorname{PSp}(2, q) \cong \operatorname{PSL}(2, q)$ has been dealt with above).

First consider $d=4$. The result for $\operatorname{PSp}(4,2)^{\prime} \cong A_{6}$ follows from Proposition 2.1. If $S=\operatorname{PSp}(4,3)$, we have $5,9 \in o e(S)$ and no maximal subgroup of $S$ contains elements of both orders 5 and 9 (see ATLAS, p. 26]). Hence $S=\langle x, y\rangle$ for all $x, y \in S$ with $|x|=5$ and $|y|=9$. If $S=\operatorname{PSp}(4,4)$, then $5,17 \in o e(S)$ and (see ATLAS, p. 44]) the only maximal subgroups of $S$ containing an element of order 17 are of the form $\operatorname{PSL}(2,16): 2$, and every subgroup of such a group of order divisible by both 17 and 5 contains $\operatorname{PSL}(2,16)$. Hence $\langle x, y\rangle$ is nonsolvable whenever $x, y \in S$ with $|x|=17$ and $|y|=5$.

So suppose that $q \geq 5$ and take $a \in \operatorname{Lbpd}(q, 4)$, which is nonempty by Proposition 3.2, and $b=\left(q^{2}-1\right) / \operatorname{gcd}(2, q-1)$. Note that $a, b \in o e(S)$, since $\operatorname{PSp}\left(2, q^{2}\right) \cong \operatorname{PSL}\left(2, q^{2}\right)$ is isomorphic to a subgroup of $S$. We consider $\hat{S}=\mathrm{Sp}(4, q) \leq \operatorname{GL}(4, q)$. Let $\hat{x}, \hat{y} \in \hat{S}$, with $|\hat{x}|$ a multiple of $a$ and $|\hat{y}|$ a multiple of $b$, and let $X=\langle\hat{x}, \hat{y}\rangle$. Then $\hat{x}$ is an $\operatorname{Lbpd}(q, 4)-$ element of $X$, and in particular $\hat{x}$ acts irreducibly on $V(4, q)$. Hence $X$ is an irreducible subgroup of $\operatorname{GL}(4, q)$. By Lemma 4.1, either $X$ is nonsolvable or $X$ is (conjugate to) a subgroup of $\mathrm{GL}\left(2, q^{2}\right) \cdot 2$. Assume the latter. Then $\left\langle\hat{x}^{2}, \hat{y}^{2}\right\rangle \leq \operatorname{GL}\left(2, q^{2}\right),\left|\hat{x}^{2}\right|$ is a multiple of the large primitive divisor $a$ of $q^{4}-1$ and $\left|\hat{y}^{2}\right|$ is a multiple of $b / \operatorname{gcd}(b, 2) \geq 6$. Hence, by the classification of the subgroups of $\operatorname{PSL}\left(2, q^{2}\right)$ (see [H, Theorem II.8.27]), we conclude that $\left\langle\hat{x}^{2}, \hat{y}^{2}\right\rangle \geq \mathrm{SL}\left(2, q^{2}\right)$ and hence, in particular, $X=\langle\hat{x}, \hat{y}\rangle$ is nonsolvable.

By part (2) of Proposition 3.7, we are left with the following cases: $\operatorname{PSp}(6,2)$ and $\operatorname{PSp}(8,2)$. If $S=\operatorname{PSp}(6,2)$, then $15,7 \in o e(S)$, and for all $x, y \in S$ with $|x|=15,|y|=7$, the group $X=\langle x, y\rangle$ is nonsolvable. This is seen as follows: one checks from [ATLAS, p. 46] that each maximal subgroup of $S$ of order divisible by 35 is isomorphic to $S_{8}$. Moreover, a subgroup of $S_{8}$ generated by two elements, of orders 15 and 7 , contains $A_{8}$, and in fact is equal to $A_{8}$. Thus, $X=S$ or $X \cong A_{8}$.

Finally, let $S=\operatorname{PSp}(8,2) \cong \operatorname{Sp}(8,2)<\mathrm{GL}(8,2)$. Then $17,7 \in o e(S)$. By [ATLAS, p.123], each maximal subgroup of $S$ of order divisible by $17 \cdot 7$ is isomorphic to $\mathrm{P} \Omega^{-}(8,2): 2$. By [ATLAS, p. 89], each maximal subgroup of $\mathrm{P} \Omega^{-}(8,2)$ of order divisible by 17 is isomorphic to $\operatorname{PSL}(2,16): 2$, and so contains no elements of order 7. Thus $\langle x, y\rangle$ is nonsolvable whenever $x, y \in S$ with $|x|=17$ and $|y|=7$.
(3): Assume now that $S=\operatorname{PSU}(d, q)$, with $d \geq 3$ and $d$ odd.

By Proposition 3.7, we need only consider the cases $S=\operatorname{PSU}(3, q)$ with $q \geq 3$ (since $\operatorname{PSU}(3,2) \cong 3^{2}: Q_{8}$ is solvable), and $S=\operatorname{PSU}(5,2)$. By ATLAS, pp. 72-73], if $S=\operatorname{PSU}(5,2)$, then $11,15 \in o e(S)$ and each maximal subgroup of $S$ of order divisible by 11 is isomorphic to $\operatorname{PSL}(2,11)$ and contains no elements of order 15 . Thus if $x, y \in S$ with $|x|=11$ and $|y|=15$, then $\langle x, y\rangle=S$.

Therefore we may assume that $S=\operatorname{PSU}(3, q)$ with $q \geq 3$. By ATLAS, pp. 14, 34], if $S=\operatorname{PSU}(3, q)$ with $q \in\{3,5\}$, then $7,8 \in o e(S)$ and each maximal subgroup of $S$ of order divisible by 7 is isomorphic to $\operatorname{PSL}(2,7)$ if $q=3$, and to $A_{7}$ if $q=5$, and neither of these groups contains an element of order 8. Thus if $x, y \in S$ with $|x|=7$ and $|y|=8$, then $\langle x, y\rangle=S$. So we may assume that $q \neq 2,3,5$. Then by Proposition [3.1, $\operatorname{Lbpd}(q, 6) \neq \emptyset$. Let $a \in \operatorname{Lbpd}(q, 6)$, and note that $a \in o e(S)$ by Remark 3.3. Also let $b=p$ if $p \neq 3$, and $b=$ $(q-1) / 2$ if $p=3$ (recall that $q$ is a power of $p$ ), and note that $b \in$ $o e(S)$ since $\operatorname{PSU}(2, q) \cong \operatorname{PSL}(2, q)$ is isomorphic to a subgroup of $S$. Now let $\hat{S}=\mathrm{SU}(3, q) \leq \mathrm{GL}\left(3, q^{2}\right)$, and note that $a, b \in o e(\hat{S})$ and that $\operatorname{gcd}(a b,|\mathbb{Z}(\hat{S})|)=1$. Consider $\hat{x}, \hat{y} \in \hat{S}$ with $|\hat{x}|$ a multiple of $a$ and $|\hat{y}|$ a multiple of $b$, and let $X=\langle\hat{x}, \hat{y}\rangle$. Since $a$ is a primitive prime divisor of $\left(q^{2}\right)^{3}-1, X$ is an irreducible subgroup of GL $\left(3, q^{2}\right)$. Thus by Lemma 4.1, either $X$ is nonsolvable or $X$ is a subgroup of $X_{0}=\operatorname{GL}\left(1, q^{3}\right) .3$. Suppose that $X \leq X_{0}$. If $p \neq 3$ then $b=p$ does not divide $\left|X_{0}\right|$. Hence $p=3$ and $b=(q-1) / 2$. However $\hat{x} \in X_{0}$ and $|\hat{x}|$ is a multiple of $a \in \operatorname{Lbpd}(q, 6)$, so $|\hat{x}|$ does not divide $\left|X_{0}\right|$, a contradiction. Hence $X$ is nonsolvable and, consequently, $\langle x, y\rangle$ is nonsolvable whenever $x, y \in S$ with $|x|=a$ and $|y|=b$.
(4): Assume now that $S=\mathrm{P} \Omega^{\circ}(d, q)$, with $d$ odd and $d \geq 3$.

Since $\operatorname{P} \Omega^{\circ}\left(2 m+1,2^{k}\right) \cong \operatorname{PSp}\left(2 m, 2^{k}\right)$, for all $m$ and $k$, we may assume that $q$ is odd. Also since $\operatorname{P} \Omega^{\circ}(3, q) \cong \operatorname{PSL}(2, q)$ and $\operatorname{P} \Omega^{\circ}(5, q) \cong$ $\operatorname{PSp}(4, q)$, we may assume that $d \geq 7$. In this case Theorem B follows from Proposition 3.7.
(5): Assume now that $S=\operatorname{PSU}(d, q)$, with $d$ even.

Since $\operatorname{PSU}(2, q) \cong \operatorname{PSL}(2, q)$, we may assume that $d \geq 4$, and it follows by Proposition 3.7 that we only have to check dimensions $d=4$ and $d=6$.

Let $S=\operatorname{PSU}(4, q)$. Since $\operatorname{PSU}(4,2) \cong \operatorname{PSp}(4,3)$, we may assume that $q \geq 3$. For $S=\operatorname{PSU}(4,3)$, it follows from ATLAS, p. 52-53] that $7,9 \in o e(S)$ and each maximal subgroup of $S$ of order divisible by 7 is isomorphic to $\operatorname{PSL}(3,4), \operatorname{PSU}(3,3)$ or $A_{7}$, and hence contains no elements of order 9 . Thus $\langle x, y\rangle=S$ whenever $x, y \in S$ with $|x|=7$ and $|y|=9$.

Thus we may assume that $S=\operatorname{PSU}(4, q)$ with $q \geq 4$. Then, by Theorem 3.1 and Proposition 3.2, both $\operatorname{Lbpd}\left(q^{2}, 3\right)$ and $\operatorname{bppd}(q, 4)$ are nonempty. Let $a \in \operatorname{Lbpd}\left(q^{2}, 3\right)$ and $b \in \operatorname{bppd}(q, 4)$, and note that, by Remark 3.3, $a, b \in o e(S)$ (since $\left(q^{4}-1\right)\left(q^{3}+1\right)$ divides $\left.|S|\right)$. Consider $\hat{S}=\mathrm{SU}(4, q) \leq \mathrm{GL}\left(4, q^{2}\right)$ acting on the natural module $V=V\left(4, q^{2}\right)$. Observe that $\operatorname{gcd}(a b,|\mathbb{Z}(\hat{S})|)=1$ and hence $a, b \in o e(\hat{S})$. Let $\hat{x}, \hat{y} \in \hat{S}$ with $|\hat{x}|$ a multiple of $a$ and $|\hat{y}|$ a multiple of $b$, and let $X=\langle\hat{x}, \hat{y}\rangle$. By Remark 3.5, there exists an $X$-composition factor $U$ of $V$ such that both $\hat{x}$ and $\hat{y}$ act nontrivially on $U$, with $\operatorname{dim}_{\mathbb{F}_{q^{2}}}(U) \geq 3$. Moreover $\hat{x}^{U}$ is an $\operatorname{Lbpd}\left(q^{2}, 3\right)$-element and $\hat{y}^{U}$ is an $\operatorname{Lbpd}\left(q^{2}, 2\right)$-element of $X^{U}$ (since $b \geq 5)$. Assume first that $X$ acts reducibly on $V$. Then $\operatorname{dim}_{\mathbb{F}_{q^{2}}}(U)=$ 3 , and hence, by Lemma [3.6 with $\delta=1, j=0$, the group $X^{U}$ is nonsolvable. Thus also $X$ is nonsolvable. Therefore we may assume that $X$ is an irreducible subgroup of GL $\left(4, q^{2}\right)$. Then, by Lemma 4.1, $X$ is nonsolvable, as $a \in \operatorname{Lbpd}\left(q^{2}, 3\right)$ and the extension field case cannot occur because $\operatorname{gcd}(4,3)=1$.

Finally let $S=\operatorname{PSU}(6, q)$. If $S=\operatorname{PSU}(6,2)$, then by ATLAS, pp. 39,115], $7,11 \in o e(S)$, each maximal subgroup of $S$ of order divisible by $7 \cdot 11$ is isomorphic to $M_{22}$, and $M_{22}$ has no maximal subgroup of order divisible by $7 \cdot 11$. Hence $\langle x, y\rangle=S$ or $\langle x, y\rangle \cong M_{22}$ whenever $x, y \in S$ with $|x|=7$ and $|y|=11$. Thus we may assume that $q>2$. Then, by Theorem 3.1 and Proposition [3.2, both $\operatorname{Lbpd}\left(q^{2}, 5\right)$ and $\operatorname{bppd}(q, 6)$ are nonempty. Let $a \in \operatorname{Lbpd}\left(q^{2}, 5\right)$ and $b \in \operatorname{bppd}(q, 6)$. Since $\left(q^{5}+1\right)\left(q^{3}+1\right)$ divides $|S|$, it follows by Remark 3.3 that $a, b \in o e(S)$. Consider now $\hat{S}=\mathrm{SU}(6, q) \leq \mathrm{GL}\left(6, q^{2}\right)$ acting on the natural module $V=V\left(6, q^{2}\right)$. Since $\operatorname{gcd}(a b,|Z(\hat{S})|)=1$, it follows that $a, b \in o e(\hat{S})$. Let $\hat{x}, \hat{y} \in \hat{S}$ with $|\hat{x}|$ a multiple of $a$ and $|\hat{y}|$ a multiple of $b$, and let $X=\langle\hat{x}, \hat{y}\rangle$. If $X$ is an irreducible subgroup of $\operatorname{GL}\left(6, q^{2}\right)$, then we conclude by Lemma 4.1 that $X$ is nonsolvable, since $a \in \operatorname{Lbpd}\left(q^{2}, 5\right)$ and the extension field case cannot occur because $\operatorname{gcd}(6,5)=1$. So we may assume that $X$ acts reducibly on $V$. By Remark 3.5(2), there exists an $X$-composition factor $U$ of $V$ such that both $\hat{x}$ and $\hat{y}$ act nontrivially on $U$, with $\operatorname{dim}_{\mathbb{F}_{q^{2}}}(U) \geq 5$. It follows that $\operatorname{dim}_{\mathbb{F}_{q^{2}}}(U)=5$. Moreover, $\hat{x}^{U}$ is an $\operatorname{Lbpd}\left(q^{2}, 5\right)$-element and $\hat{y}^{U}$ is $a \operatorname{bppd}\left(q^{2}, 3\right)$-element of $X^{U}$. Hence by Lemma 3.6 with $\delta=2, j=0$, the group $X^{U}$ is nonsolvable and so is $X$.
(6): Let $S=\mathrm{P} \Omega^{-}(d, q)$, with $d$ even and $d \geq 4$.

Since $\operatorname{P} \Omega^{-}(4, q) \cong \operatorname{PSL}\left(2, q^{2}\right)$ and $\mathrm{P} \Omega^{-}(6, q) \cong \operatorname{PSU}(4, q)$, we may assume that $d \geq 8$. Hence we are left, by Proposition 3.7, only with the cases $\mathrm{P} \Omega^{-}(8, q)$, for $q \geq 2$, and $\mathrm{P} \Omega^{-}(10,2)$.

Let $S=\mathrm{P} \Omega^{-}(8, q)$. For $S=\mathrm{P} \Omega^{-}(8,2)$, it follows from ATLAS, p. 89] that $7,17 \in o e(S)$ and each maximal subgroup of $S$ of order divisible by 17 is isomorphic to $\operatorname{PSL}(2,16): 2$, and hence contains no elements of order 7 . Thus $\langle x, y\rangle=S$ whenever $x, y \in S$ with $|x|=7$ and $|y|=17$. So we may assume that $q>2$. Then, by Theorem 3.1, both $\operatorname{bppd}(q, 8)$ and $\operatorname{bppd}(q, 6)$ are nonempty. Let $a \in \operatorname{bppd}(q, 8)$ and $b \in$ $\operatorname{bppd}(q, 6)$. Since $\left(q^{4}+1\right)\left(q^{6}-1\right)$ divides $|S|$, it follows by Remark 3.3 that $a, b \in o e(S)$. Consider now $\hat{S}=\Omega^{-}(8, q) \leq G L(8, q)$. Since $\operatorname{gcd}(a b,|\mathbb{Z}(\hat{S})|)=1$, it follows that $a, b \in o e(\hat{S})$. Let $\hat{x}, \hat{y} \in \hat{S}$ with $|\hat{x}|$ a multiple of $a$ and $|\hat{y}|$ a multiple of $b$, and let $X=\langle\hat{x}, \hat{y}\rangle$. Then $X$ is nonsolvable by Lemma 3.6 with $\delta=2, j=0$. Hence $\langle x, y\rangle$ is nonsolvable whenever $x, y \in S$ with $|x|=a$ and $|y|=b$.

Finally, let $S=\mathrm{P} \Omega^{-}(10,2) \leq \mathrm{GL}(10,2)$. Consider $11 \in \operatorname{bppd}(2,10)$ and $17 \in \operatorname{bppd}(2,8)$, and note that $11,17 \in o e(S)$. Then by Lemma 3.6 with $\delta=2, j=0$, the subgroup $\langle x, y\rangle$ is nonsolvable whenever $x, y \in S$ with $|x|=11$ and $|y|=17$.
(7): Let $S=\mathrm{P} \Omega^{+}(d, q)$, with $d$ even.

Since $S$ is not nonabelian simple if $d=2$ and $d=4$, and $\mathrm{P}^{+}(6, q) \cong$ $\operatorname{PSL}(4, q)$, we may assume that $d \geq 8$. Then by Proposition 3.7, we only need to consider the cases $\mathrm{P} \Omega^{+}(8, q)$, for $q \geq 2$, and $\mathrm{P} \Omega^{+}(10,2)$.

Let $S=\mathrm{P} \Omega^{+}(8, q)$. For $S=\mathrm{P} \Omega^{+}(8,2)$, it follows from ATLAS, p. 85] that $7,15 \in o e(S)$ and each maximal subgroup of $S$ of order divisible by 35 is isomorphic to $\operatorname{PSp}(6,2), 2^{6}: A_{8}$ or $A_{9}$. We have shown above that a subgroup of $\operatorname{PSp}(6,2)$ which contains elements of orders 15 and 7 is nonsolvable. Also, in the last paragraph of the proof of Proposition 2.1, we saw that a subgroup of $A_{9}$ containing elements of orders 7 and 5 is $A_{d}$ for some $d \geq 7$. Thus also a subgroup of $2^{6}: A_{8}$ containing such elements has a composition factor $A_{7}$ or $A_{8}$. Hence $\langle x, y\rangle$ is nonsolvable whenever $x, y \in S$ with $|x|=7$ and $|y|=15$. Thus we may assume that $q>2$.

For $S=\mathrm{P} \Omega^{+}(8,3)$, it follows from ATLAS, pp. 140-141 and 54-55] that $7,15 \in$ oe $(S)$, each maximal subgroup of $S$ of order divisible by 7 is isomorphic to $\mathrm{P} \Omega^{\circ}(7,3), \mathrm{P} \Omega^{+}(8,2)$ or $2 \cdot \operatorname{PSU}(4,3) \cdot 2^{2}$, and the last of these groups contains no elements of order 15. Let $x, y \in S$ with $|x|=7$ and $|y|=15$ and let $X=\langle x, y\rangle$. Assume that $X<S$ and let $M$ be a maximal subgroup of $S$ containing $X$. Then $M$ is $\mathrm{P} \Omega^{\circ}(7,3)$ or $\mathrm{P} \Omega^{+}(8,2)$. Suppose first that $M=\mathrm{P} \Omega^{\circ}(7,3)$. Observe that $5,7 \in o e(M), 7 \in \operatorname{bppd}(3,6)$ and $5 \in \operatorname{bppd}(3,4)$. By Lemma 3.6 with $\delta=2, j=1$, every subgroup of $\hat{M}=\Omega^{\circ}(7,3) \leq \mathrm{GL}(7,3)$ containing elements of orders 7 and 5 is nonsolvable. Hence in this case $X$ is nonsolvable. On the other hand, if $M=\mathrm{P} \Omega^{+}(8,2)$, then by the
previous paragraph any subgroup of $M$ containing elements of orders 7 and 15 is nonsolvable. Thus also in this case $X$ is nonsolvable. So we may assume that $q \geq 4$.

Next we deal with $S=\mathrm{P} \Omega^{+}(8,5)$. Notice that both $7 \in \operatorname{bppd}(5,6)$ and $13 \in \operatorname{bppd}(5,4)$ belong to $o e(S)$. Let $X=\langle\hat{x}, \hat{y}\rangle \leq \Omega^{+}(8,5) \leq$ GL $(8,5)$, where $|\hat{x}|$ is divisible by 7 and $|\hat{y}|$ is divisible by 13 . We shall prove that $X$ is nonsolvable. Now $\hat{x}$ is a $\operatorname{bppd}(5,6)$-element of $X$ (even if not a large one), and $\hat{y}$ is an $\operatorname{Lbpd}(5,4)$-element of $X$. Assume first that $X$ acts reducibly on $V=V(8,5)$. By Remark 3.5(2) there exists an $X$-composition factor $U$ of $V$ of dimension $d_{0}=\operatorname{dim}_{\mathbb{F}_{5}}(U) \geq 6$ such that $\left|\hat{x}^{U}\right|_{7} \geq 7$ and $\left|\hat{y}^{U}\right|_{13} \geq 13$. Thus $d_{0} \in\{6,7\}$, and by Lemma 3.6 the group $X^{U}$ is nonsolvable in both cases $d_{0}=6($ taking $\delta=2, j=0)$ and $d_{0}=7($ taking $\delta=2, j=1)$.

Thus we may assume that $X$ is an irreducible subgroup of GL $(8,5)$. Observe that here we cannot use Lemma 3.6, or even Lemma 4.1, because $\operatorname{Lbpd}(5,6)=\emptyset$. So we have to apply [NP2, Theorem 3.1] directly (with $d=8, e=6$ ), checking each of the cases (a)-(e) of that theorem. If $X$ is in case (a) or (e), then $X$ is nonsolvable. Since $7 \cdot 13$ divides $|X|$, neither of the cases (b) nor (d) holds for $X$. So we may assume that case (c) holds for $X$, which implies that $X$ is (isomorphic to) a subgroup of GL $\left(4,5^{2}\right) \cdot 2$. Hence $X_{0}:=\left\langle\hat{x}^{2}, \hat{y}^{2}\right\rangle \leq \mathrm{GL}\left(4,5^{2}\right)$. Observe that $X_{0}$ acts irreducibly on $V\left(4,5^{2}\right)$, as it acts irreducibly on $V$. Now, $7 \in \operatorname{Lbpd}\left(5^{2}, 3\right)$, and hence we can apply Lemma 4.1. Note that $X_{0}$ is not a subgroup of $\mathrm{GL}\left(2,5^{4}\right) \cdot 2$, because $\left|\mathrm{GL}\left(2,5^{4}\right) \cdot 2\right|$ is not divisible by 7 . Therefore we conclude that $X_{0}$ is nonsolvable, and hence also $X$ is nonsolvable.

Now consider $S=\mathrm{P} \Omega^{+}(8, q)$ with $q \neq 2,3,5$. By Theorem 3.1 and Proposition [3.2, both $\operatorname{Lbpd}(q, 6)$ and $\operatorname{bppd}(q, 4)$ are nonempty. Let $a \in \operatorname{Lbpd}(q, 6)$ and $b \in \operatorname{bppd}(q, 4)$. Since $\left(q^{6}-1\right)\left(q^{4}-1\right)$ divides $|S|$, it follows by Remark 3.3 that $a, b \in o e(S)$. Consider now $\hat{S}=\Omega^{+}(8, q) \leq$ $\mathrm{GL}(8, q)$ acting on on $V=V(8, q)$. Since $\operatorname{gcd}(a b,|\mathbb{Z}(\hat{S})|)=1$, it follows that $a, b \in o e(\hat{S})$. Let $\hat{x}, \hat{y} \in \hat{S}$ with $|\hat{x}|$ a multiple of $a$ and $|\hat{y}|$ a multiple of $b$, and let $X=\langle\hat{x}, \hat{y}\rangle$. Now $\hat{x}$ is an $\operatorname{Lbpd}(q, 6)$-element and $\hat{y}$ is a $\operatorname{bppd}(q, 4)$-element of $X$. By Remark 3.5(2), there is an $X$-composition factor $U$ of $V$ with $d_{0}=\operatorname{dim}_{\mathbb{F}_{q}}(U) \in\{6,7,8\}$ such that $\left|\hat{x}^{U}\right|_{a}=|\hat{x}|_{a} \geq a$ and $\left|\hat{y}^{U}\right|_{b}=|\hat{y}|_{b} \geq b$. It follows by Lemma 3.6 that $X^{U}$, and hence also $X$, is nonsolvable if $d_{0}=6$ (taking $\delta=2, j=$ 0 ), and also if $d_{0}=7$ (taking $\delta=2, j=1$ ). Thus we may assume that $d_{0}=8$, that is, $X$ is irreducible on $V$. Then by Lemma 4.1, either $X$ is nonsolvable, as required, or $X$ is (isomorphic to) a subgroup of GL $\left(4, q^{2}\right) \cdot 2$. Suppose that $X \leq \operatorname{GL}\left(4, q^{2}\right) \cdot 2$ and consider $X_{0}=$
$\left\langle\hat{x}^{2}, \hat{y}^{2}\right\rangle \leq \mathrm{GL}\left(4, q^{2}\right)$. By the above argument applied to $X_{0}$, we may assume that $X_{0}$ acts irreducibly on $V=V\left(4, q^{2}\right)$. It follows then, by Lemma 4.1, that either $X_{0}$ is nonsolvable or $X_{0}$ is (isomorphic to) a subgroup of $\operatorname{GL}\left(2, q^{4}\right) .2$. However, since $a \in \operatorname{Lbpd}(q, 6), a$ is coprime to $\left|\mathrm{GL}\left(2, q^{4}\right) \cdot 2\right|=2 q^{4}\left(q^{4}-1\right)\left(q^{8}-1\right)$, and we conclude that $X_{0}$, and hence also $X$, is nonsolvable.

Finally let $S=\mathrm{P} \Omega^{+}(10,2)$. By [ATLAS, p. 147], $17,31 \in o e(S)$ and no maximal subgroup of $S$ has order divisible by both 17 and 31. Hence $\langle x, y\rangle=S$ whenever $x, y \in S$ with $|x|=17$ and $|y|=31$.

We now prove Theorem B for the exceptional finite simple groups of Lie type. We make use of five papers. The first is the paper [FS] of Feit and Seitz. They prove, in [FS, Theorem 3.1], the existence of certain self-centralizing cyclic maximal tori in simple groups of Lie type. The second is the paper (W) of Wiegel. He gives, in W, Table 1], a list of cyclic maximal tori in exceptional groups of Lie type, with some small cases excluded, and in [W] Section 4] he determines the maximal subgroups containing these tori, for each such group. We were kindly informed by Frank Lübeck, in a letter, that Weigel's list of cyclic maximal tori is correct without the extra conditions on $q$ or $k$, with only one exception: namely the group $G_{2}(2)$, which has elements of order $q^{2}-q+1=3$ in several classes of maximal tori. We shall refer to [L] concerning this important information. The third is the paper [GM] of Guralnick and Malle, which is still in preparation. The authors kindly informed us that their paper contains important information about maximal subgroups of $E_{7}(2)_{s c}$ and $E_{7}(3)_{s c}$. The forth is the paper [MT] of Moretò and Tiep. We use [MT, Lemma 2.3], in a slightly 'extended' form, which was kindly approved by Pham Tiep. They prove, in Lemma 2.3, that each exceptional simple group of Lie type contains elements $s_{1}$ and $s_{2}$ of prime orders $p_{1}$ and $p_{2}$, respectively, such that their centralizers have suitable orders. The 'extended' version of this lemma states not only that such elements exist, but also that the centralizers of every element of order $p_{1}$ or $p_{2}$ are of the same suitable orders. Finally, the fifth paper is the paper [GK] by Guralnick and Kantor, which provides in [GK, Proposition 6.2] information concerning elements of the groups excluded in [ W$]$ and of the sporadic subgroups, contained in a unique or small number of maximal subgroups.

We recall that every finite simple group of Lie type occurs as a composition factor of the group of fixed points $G_{F}$, under a Frobenius map $F: G \rightarrow G$ of a connected reductive algebraic group $G$ over the algebraic closure $\overline{\mathbb{F}_{q}}$ of a field $\mathbb{F}_{q}$ of order $q$.

If we choose $G$ to be simply connected, then every finite simple exceptional group of Lie type is a quotient $G_{F} / \mathbb{Z}\left(G_{F}\right)$. Moreover, $\mathbb{Z}\left(G_{F}\right)=1$ unless $G$ is of type $E_{6},{ }^{2} E_{6}$ or $E_{7}$. The following facts are used repeatedly.

Lemma 4.3. Let $S=G_{F} / \mathbb{Z}\left(G_{F}\right), q$ be as above, and suppose that $S$ has a cyclic maximal torus $T$ of order divisible by a prime $p$, such that $|S: T|$ is coprime to $p$, and $C_{S}(y)=T$ for $y \in T$ of order $|T|_{p}$. Then for each $x \in S$ with $|x|=|T|$, the subgroup $\langle x\rangle$ is conjugate to $T$ in $S$, and in particular it is a maximal torus of order $|T|$.

Proof. By assumption $T$ has a unique Sylow $p$-subgroup, say $P=\langle y\rangle$, and $P$ is a Sylow $p$-subgroup of $S$. Let $x \in S$ with $|x|=|T|$. Then $\langle x\rangle$ contains a subgroup $P_{0}$ of order $|P|$, so by Sylow's Theorem $P_{0}^{g}=P$ for some $g \in S$. Then $\langle x\rangle^{g} \leq C_{S}(y)$ which by assumption is equal to $T$. It follows that $\langle x\rangle^{g}=T$.

The following is an immediate corollary of Lemma 4.3.
Corollary 4.4. Let $S, T$ be as in Lemma 4.3, let $a=|T| \in o e(S)$, and suppose that $b \in o e(S)$ is such that each maximal subgroup of $S$ containing $T$ has order coprime to $b$. Then for each $x, y \in S$ with $|x|=a$ and $|y|=b$, the group $\langle x, y\rangle=S$ and, in particular, is nonsolvable.

We now prove
Proposition 4.5. Theorem $B$ holds for all exceptional finite simple groups of Lie type.

Proof. In the following, we denote by $\pi(n)$ the set of prime divisors of the positive integer $n$ and by $\Phi_{k}(x)$ the $k$-th cyclotomic polynomial. We consider the exceptional groups, beginning with those of smallest Lie rank. Our basic proof strategy is to choose $a, b \in o e(S)$, where possible, so that the hypotheses of Corollary 4.4 hold. Then we have immediately that Theorem B holds for $a, b$. We call this 'the standard argument'.
(1): Let $S={ }^{2} B_{2}(q)$, with $q=2^{2 n+1}$ and $n \geq 1$.

Then $|S|=q^{2}(q-1)\left(q^{2}+1\right)$. Write $r=2^{n+1}$, so $q^{2}+1=(q+r+$ 1) $(q-r+1)$. Since ${ }^{2} B_{2}(2)$ is a Frobenius group of order 20 , the field order $q$ is at least 8 .

Let $p \in \operatorname{ppd}(q, 4)$, which is nonempty. Since $q^{2}+1=(q+r+1)(q-r+$ 1), the prime $p$ divides $a:=q+\varepsilon r+1$ where $\varepsilon= \pm 1$. By [FS, Theorem 3.1], $S$ has a cyclic maximal torus $T$ of order $a$ and we note that $|S: T|$ is coprime to $p$. By comparing orders, we deduce from the 'extended'
[MT, Lemma 2.3] that $C_{S}(y)=T$ for $y \in T$ of order $|T|_{p}$. By [S], the only maximal subgroup of $S$ containing $T$ is its normaliser, of order $4 a$. Then the standard argument applies for $a$ and any $b \in \pi(q-\varepsilon r+1)$, since $b \neq 2$ and $\operatorname{gcd}(q+r+1, q-r+1)=1$, so $b$ does not divide $4 a$.
(2): Let $S={ }^{2} G_{2}(q)^{\prime}$, with $q=3^{2 n+1}$ and $n \geq 1$.

Then $|S|=q^{3}(q-1)\left(q^{3}+1\right)$. Write $r=3^{n+1}$, so $q^{3}+1=(q+1)(q+$ $r+1)(q-r+1)$. Since ${ }^{2} G_{2}(3)^{\prime} \cong \operatorname{PSL}(2,8)$ has been already treated in Proposition 4.2, we may assume that $q \geq 27$.

Let $p \in \operatorname{ppd}(q, 6)$, which is nonempty. Then $p$ divides $a=q+\varepsilon r+1$ where $\varepsilon= \pm 1$. By [FS, Theorem 3.1], $S$ has a cyclic maximal torus $T$ of order $a$ and we note that $|S: T|$ is coprime to $p$. As in (1), $C_{S}(y)=T$ for $y \in T$ of order $|T|_{p}$, and by [K] and [LN], the only maximal subgroup of $S$ containing $T$ is its normaliser, of order $6 a$. Let $b \in \pi(q-\varepsilon r+1)$. Then $b \in o e(S)$, but $b$ does not divide $6 a$, because $b \neq 2,3$ and $\operatorname{gcd}(q+r+1, q-r+1)=1$. Thus the standard argument applies.
(3a): Let $S={ }^{2} F_{4}(2)^{\prime}$, the Tits group.
Then $|S|=2^{11} \cdot 3^{3} \cdot 5^{2} \cdot 13$. By ATLAS3], $13,10 \in o e(S)$ and each maximal subgroup of $S$ of order divisible by 130 is isomorphic to $\operatorname{PSL}(2,25)$, which contains no elements of order 10. Hence $\langle x, y\rangle=S$ whenever $x, y \in S$ with $|x|=13$ and $|y|=10$.
(3b): Let $S={ }^{2} F_{4}(q)$, with $q=2^{2 n+1}$ and $n \geq 1$.
Then $|S|=q^{12}\left(q^{6}+1\right)\left(q^{4}-1\right)\left(q^{3}+1\right)(q-1)$. Write $r=2^{n+1}$, so

$$
\frac{q^{6}+1}{q^{2}+1}=q^{4}-q^{2}+1=\left(q^{2}+r q+q+r+1\right)\left(q^{2}-r q+q-r+1\right)
$$

and $\operatorname{gcd}\left(q^{2}+r q+q+r+1, q^{2}-r q+q-r+1\right)$ divides $\left(q^{2}+q+1\right)(q-1)=$ $q^{3}-1$.

Let $p \in \operatorname{ppd}(q, 12)$, which is nonempty. Then $p$ divides $a:=q^{2}+\varepsilon r q+$ $q+\varepsilon r+1$, where $\varepsilon= \pm 1$, and $|S| / a$ is coprime to $p$. By [FS, Theorem 3.1], $S$ has a cyclic maximal torus $T$ of order $a$, and arguing as in (1), the hypotheses of Lemma 4.3 hold for $T$. By [M], the only maximal subgroup of $S$ containing $T$ is $N_{S}(T)$ of order $12 a$. Let $b \in \operatorname{ppd}(q, 6)$, which is nonempty. Then $b$ divides $q^{3}+1$ and hence $b \in o e(S)$ and $b$ does not divide $a$. Also $b \geq 7$ and so $b$ does not divide 12. It follows that $b$ does not divide $12 a$, and hence, the standard argument applies.
(4): Let $S=G_{2}(q)$, with $q>2$.

Then $|S|=q^{6}\left(q^{6}-1\right)\left(q^{2}-1\right)$. Since $G_{2}(2)^{\prime} \cong \operatorname{PSU}(3,3)$ has been already treated, we may assume that $q \geq 3$. First we deal with $G_{2}(3)$ and $G_{2}(4)$, which were excluded in W.

Let $S=G_{2}(q)$ with $q=3$ or 4 . Then by [ATLAS, pp. 60-61,97], $a, 13 \in o e(S)$, where $a=7$ if $q=3$ and $a=5$ if $q=4$, and each
maximal subgroup $M$ of $S$ of order divisible by $13 a$ is isomorphic to $\operatorname{PSL}(2,13)$ if $q=3$, and to $\operatorname{PSU}(3,4): 2$ if $q=4$. In either case, the derived group $M^{\prime}$ is generated by any pair of its elements with one of order $a$ and the other of order 13. Hence, if $x, y \in S$ with $|x|=a$ and $|y|=13$, then $\langle x, y\rangle$ is nonsolvable.

Let, now, $S=G_{2}(q)$, with $q \geq 5$ and let $p \in \operatorname{ppd}(q, 6)$, which is nonempty. Since $q^{3}+1=\left(q^{2}-q+1\right)(q+1), p$ divides $a:=q^{2}-q+1=$ $\Phi_{6}(q)$ and $|S| / a$ is coprime to $p$. By [W, Table I], $S$ has a cyclic maximal torus $T$ of order $a$, and arguing as in (1), the hypotheses of Lemma 4.3 hold for $T$. By [W, Section 4], each maximal subgroup $M$ of $S$ containing $T$ is isomorphic to $\mathrm{SU}(3, q) .2$ and hence $|M|=2 q^{3}\left(q^{3}+\right.$ 1) $\left(q^{2}-1\right)$. Let $b \in \operatorname{ppd}(q, 3)$, which is nonempty. Then $b \in o e(S)$ and $\operatorname{gcd}(b, 2 q)=1$. Since $b$ divides $q^{3}-1$, also $\operatorname{gcd}\left(b, q^{3}+1\right)=1$, and it follows that $b$ does not divide $|M|$. Hence, the standard argument applies.
(5): Let $S={ }^{3} D_{4}(q)$, with $q \geq 2$.

Then $|S|=q^{12}\left(q^{8}+q^{4}+1\right)\left(q^{6}-1\right)\left(q^{2}-1\right)$, where $q^{8}+q^{4}+1=$ $\left(q^{4}-q^{2}+1\right)\left(q^{4}+q^{2}+1\right)$.

Let $p \in \operatorname{ppd}(q, 12)$, which is nonempty. Since $q^{6}+1=\left(q^{4}-q^{2}+\right.$ 1) $\left(q^{2}+1\right)$, $p$ divides $a=q^{4}-q^{2}+1=\Phi_{12}(q)$ and $|S| / a$ is coprime to $p$. By [W, Table I], there exist a cyclic maximal torus $T$ of $S$ of order $a$, and arguing as in (1), the hypotheses of Lemma 4.3 hold for $T$. By [W, Section 4], the only maximal subgroup of $S$ containing $T$ is $N_{S}(T)$ of order $4 a$. Let $b \in \operatorname{ppd}(q, 6)$ if $q \neq 2$ and let $b=7$ if $q=2$. Then $b \neq 2, b \in o e(S)$ and $\operatorname{gcd}\left(b, q^{4}-q^{2}+1\right)=1$, since $b$ divides $q^{6}-1$ and $q^{4}-q^{2}+1$ divides $q^{6}+1$. Thus $b$ does not divide $4 a$, and hence, the standard argument applies.
(6): Let $S=F_{4}(q)$, with $q \geq 2$.

Then $|S|=q^{24}\left(q^{12}-1\right)\left(q^{8}-1\right)\left(q^{6}-1\right)\left(q^{2}-1\right)$. First we deal with $F_{4}(2)$ and $F_{4}(3)$, which were excluded in [W].

Let first $S=F_{4}(2)$. Notice that $13,17 \in o e(S)$ and by GK, Proposition 6.2], the maximal subgroups of $S$ of order divisible by 17 are isomorphic to $\operatorname{PSp}(8,2)$, which is of order not divisible by 13. Hence, the standard argument applies.

Let now $S=F_{4}(3)$. Notice that both $73 \in \operatorname{ppd}(3,12)$ and $41 \in$ $\operatorname{ppd}(3,8)$ belong to $o e(S)$. By [GK, Proposition 6.2], the maximal subgroups of $S$ of order divisible by 73 are isomorphic to ${ }^{3} D_{4}(3) .3$, which is of order not divisible by 41 (see [ATLAS, p. 241]). Hence, the standard argument applies.

Let, finally, $S=F_{4}(q)$, with $q \geq 4$ and let $p \in \operatorname{ppd}(q, 12)$. It follows, as in (5), that $p$ divides $a=q^{4}-q^{2}+1=\Phi_{12}(q)$ and $|S| / a$ is coprime to $p$. By [W], Table I], $S$ has a cyclic maximal torus $T$ of
order $a$ and arguing as in (1), the hypotheses of Lemma 4.3 hold for $T$. By [W, Section 4], every maximal subgroup $M$ of $S$ containing $T$ is isomorphic to ${ }^{3} D_{4}(q) .3$ and hence $|M|=3 q^{12}\left(q^{8}+q^{4}+1\right)\left(q^{6}-\right.$ 1) $\left(q^{2}-1\right)$. Let $b \in \operatorname{ppd}(q, 8)$, which is nonempty. Then $b \in o e(S)$, but $b$ does not divide $|M|$, since $\operatorname{gcd}(b, 3 q)=1, \operatorname{gcd}\left(b,\left(q^{6}-1\right)\left(q^{2}-1\right)\right)=1$ and $\operatorname{gcd}\left(q^{8}+q^{4}+1, b\right)$ divides $\operatorname{gcd}\left(q^{12}-1, q^{8}-1\right)=q^{4}-1$, so also $\operatorname{gcd}\left(q^{8}+q^{4}+1, b\right)=1$. Hence, the standard argument applies.
(7): Let $S={ }^{2} E_{6}(q)$, with $q \geq 2$.

Then $|S|=\frac{1}{d} q^{36}\left(q^{12}-1\right)\left(q^{9}+1\right)\left(q^{8}-1\right)\left(q^{6}-1\right)\left(q^{5}+1\right)\left(q^{2}-1\right)$, and $d=\operatorname{gcd}(3, q+1)$. Moreover, $S=\hat{S} / \mathbb{Z}(\hat{S})$, where $\hat{S}=G_{F}$, with $G$ a simply connected algebraic group of exceptional type ${ }^{2} E_{6}$ and $|\mathbb{Z}(\hat{S})|=d$.

Let $p \in \operatorname{ppd}(q, 18)$, which is nonempty. Since $q^{9}+1=\left(q^{3}+1\right)\left(q^{6}-\right.$ $q^{3}+1$ ), $p$ divides $a=q^{6}-q^{3}+1$. By [W, Table 1] and [L], $\hat{S}$ has a cyclic maximal torus $T$ of order $a$ and arguing as in (1), the hypotheses of Lemma 4.3 hold for $T$. Note that $\mathbb{Z}(\hat{S}) \leq T$.

It follows by [W, Section 4] for $q \geq 4$ and by [GK, Theorem 6.2] for $q=2,3$, that every maximal subgroup $M$ of $\hat{S}$ containing $T$ is isomorphic to $\operatorname{PSU}\left(3, q^{3}\right) .3$ and hence $|M|=\frac{3}{d} q^{9}\left(q^{9}+1\right)\left(q^{6}-1\right)$.

Let $b \in \operatorname{ppd}(q, 12)$, which is nonempty, and note that $b \geq 12+1=13$. Then $b$ divides $q^{6}+1$ and $b \neq 3$. Hence $b \in o e(\hat{S})$ and $b$ does not divide $|M|$. Thus, by the standard argument, if $x \in \hat{S}$ is of order $a$ and $y \in \hat{S}$ is of order $b$, then $\langle x, y\rangle=\hat{S}$.

If $d=1$, then Theorem B holds for $S=\hat{S}$. So suppose that $d=3$. Then $q \equiv-1(\bmod 3), a=q^{6}-q^{3}+1 \equiv 3(\bmod 9), \operatorname{gcd}(3, b)=1$ and since $\mathbb{Z}(\hat{S}) \leq\langle x\rangle$ for each $x \in \hat{S}$ of order $a$, it follows that 3 divides $a$. Thus $a / 3, b \in o e(S)$. Let $z=\hat{z} \mathbb{Z}(\hat{S})$ be an arbitrary element of $S$ of order $a / 3$ and let $w=\hat{w} \mathbb{Z}(\hat{S})$ be an arbitrary element of $S$ of order $b$, where $\hat{z}, \hat{w}$ are elements of $\hat{S}$. Since $\operatorname{gcd}(a / 3,3)=\operatorname{gcd}(b, 3)=1$ and $|\mathbb{Z}(\hat{S})|=3$, we may always choose $\hat{z}$ of order $a$ and $\hat{w}$ of order $b$. Since, as shown above, $\langle\hat{z}, \hat{w}\rangle=\hat{S}$, it follows that $\langle z, w\rangle=S$.
(8): Let $S=E_{6}(q)$, with $q \geq 2$.

Then $|S|=\frac{1}{d} q^{36}\left(q^{12}-1\right)\left(q^{9}-1\right)\left(q^{8}-1\right)\left(q^{6}-1\right)\left(q^{5}-1\right)\left(q^{2}-1\right)$, where $d=\operatorname{gcd}(3, q-1)$. Moreover, $S=\hat{S} / \mathbb{Z}(\hat{S})$, where $\hat{S}=G_{F}$, with $G$ a simply connected algebraic group of exceptional type $E_{6}$ and $|\mathbb{Z}(\hat{S})|=d$.

Let $p \in \operatorname{ppd}(q, 9)$, which is nonempty. Since $q^{9}-1=\left(q^{3}-1\right)\left(q^{6}+q^{3}+\right.$ 1), $p$ divides $a=q^{6}+q^{3}+1$. By [W, Table 1 ], $\hat{S}$ has a cyclic maximal torus $T$ of order $a$, and arguing as in (1), the hypotheses of Lemma 4.3 hold for $T$. In particular, $\mathbb{Z}(\hat{S}) \leq T$. By [W, Section 4], every maximal
subgroup $M$ of $\hat{S}$ containing $T$ is isomorphic to $\operatorname{SL}\left(3, q^{3}\right) .3$ and hence $|M|=3 q^{9}\left(q^{9}-1\right)\left(q^{6}-1\right)$.

Let $b \in \operatorname{ppd}(q, 12)$, which is nonempty, and note that $b \geq 12+1=13$. Then $b \in o e(\hat{S}), b$ divides $q^{6}+1$ and $b \neq 3$. It follows that $b$ does not divide $|M|$. Hence, by the standard argument, if $x \in \hat{S}$ is of order $a$ and $y \in \hat{S}$ is of order $b$, then $\langle\hat{x}, \hat{y}\rangle=\hat{S}$.

If $d=1$, then Theorem B holds for $S=\hat{S}$. So suppose that $d=3$. Then it follows using the same proof as for ${ }^{2} E_{6}(q)$ that if $z=\hat{z} \mathbb{Z}(\hat{S})$ has order $a / 3$ and $w=\hat{w} \mathbb{Z}(\hat{S})$ has order $b$, where $\hat{z}, \hat{w}$ are elements of $\hat{S}$, then $\langle z, w\rangle=S$.
(9): Let $S=E_{7}(q)$, with $q \geq 2$. Then

$$
|S|=\frac{1}{d} q^{63} \prod_{i \in I}\left(q^{i}-1\right), \quad \text { with } \quad I=\{2,6,8,10,12,14,18\}
$$

where $d=\operatorname{gcd}(2, q-1)$. Moreover, $S=\hat{S} / \mathbb{Z}(\hat{S})$, where $\hat{S}=G_{F}$, with $G$ a simply connected algebraic group of exceptional type $E_{7}$ and $|\mathbb{Z}(\hat{S})|=d$.

Let $p \in \operatorname{ppd}(q, 18)$, which is nonempty. Since $q^{9}+1=\left(q^{3}+1\right)\left(q^{6}-\right.$ $\left.q^{3}+1\right), p$ divides $a=(q+1)\left(q^{6}-q^{3}+1\right)$. By [W, Table 1] and [L], $S$ has a cyclic maximal torus $T$ of order $a$, and arguing as in (1), the hypotheses of Lemma 4.3 hold for $T$. In particular, $\mathbb{Z}(\hat{S}) \leq T$.

By [W, Section 4] for $q \geq 4$ and by [GM, Proposition 2.11] for $q=2,3$, every maximal subgroup $M$ of $\hat{S}$ containing $T$ is isomorphic to $\left(Z_{q+1} .^{2} E_{6}(q)\right) .2$ and hence $|M|=\frac{2}{d_{1}}(q+1) q^{36}\left(q^{12}-1\right)\left(q^{9}+1\right)\left(q^{8}-\right.$ 1) $\left(q^{6}-1\right)\left(q^{5}+1\right)\left(q^{2}-1\right)$, where $d_{1}=\operatorname{gcd}(3, q-1)$.

Let $b \in \operatorname{ppd}(3,14)$, which is nonempty. Then $b \in o e(\hat{S}), b$ divides $q^{7}+1$ and $b \neq 2$. Since $\operatorname{gcd}\left(q^{9}+1, q^{7}+1\right)$ divides $q^{2}-1$, it follows that $b$ does not divide $|M|$. Hence, by the standard argument, if $x \in \hat{S}$ is of order $a$ and $y \in \hat{S}$ is of order $b$, then $\langle\hat{x}, \hat{y}\rangle=\hat{S}$.

If $d=1$, then Theorem B holds for $S=\hat{S}$. So suppose that $d=2$. Then $q$ is odd and since $\mathbb{Z}(\hat{S}) \leq\langle x\rangle$ for each $x \in \hat{S}$ of order $a$, it follows that $a$ is even, and $a / 2, b \in o e(S)$. Let $z=\hat{z} \mathbb{Z}(\hat{S})$ be an arbitrary element of $S$ of order $a / 2$ and let $w=\hat{w} \mathbb{Z}(\hat{S})$ be an arbitrary element of $S$ of order $b$, where $\hat{z}, \hat{w}$ are elements of $\hat{S}$. Since $\operatorname{gcd}(b, 2)=1$ and $|\mathbb{Z}(\hat{S})|=2$, it follows that we may choose $\hat{w}$ of order $b$. Now consider $\hat{z}$. Let $L=\langle\hat{z}, \mathbb{Z}(\hat{S})\rangle$. Then $L$ is abelian and it contains an element $\hat{u}$ of order $p$. By [MT, Lemma 2.3], $\left|\mathbb{C}_{\hat{S}}(\hat{u})\right|=a$ and as shown in (1), this centralizer is a cyclic maximal torus of $\hat{S}$. Consequently, as $L \leq \mathbb{C}_{\hat{S}}(\hat{u})$,
we may also choose $\hat{z}$ of order $a$. Since, as shown above, $\langle\hat{z}, \hat{w}\rangle=\hat{S}$, it follows that $\langle z, w\rangle=S$.
(10): Let $S=E_{8}(q)$, with $q \geq 2$. Then

$$
|S|=q^{120} \prod_{i \in I}\left(q^{i}-1\right), \quad \text { with } \quad I=\{2,8,12,14,18,20,24,30\}
$$

Let $p \in \operatorname{ppd}(q, 30)$, which is nonempty. Since

$$
\begin{align*}
q^{15}+1=\left(q^{2}-q+1\right)\left(q^{5}+1\right)\left(q^{8}+\right. & \left.q^{7}-q^{5}-q^{4}-q^{3}+q+1\right)  \tag{1}\\
& =\left(q^{2}-q+1\right)\left(q^{5}+1\right) \Phi_{30}(q)
\end{align*}
$$

$p$ divides $a=\Phi_{30}(q)$ and $|S| / a$ is coprime to $p$. By [W, Table I], $S$ has a cyclic maximal torus $T$ of order $a$, and arguing as in (1), the hypotheses of Lemma 4.3 hold for $T$. By [W, Section 4], the only maximal subgroup of $S$ containing $T$ is $N_{S}(T)$ of order $30 a$.

Let $b \in \operatorname{ppd}(q, 24)$, which is nonempty, and note that $b \equiv 1(\bmod 24)$, and hence that $b \geq 25$ and $b$ does not divide 30 . Now $b$ divides $q^{12}+1$ and $a$ divides $q^{15}+1$, and since $\operatorname{gcd}\left(q^{12}+1, q^{15}+1\right)$ divides $q^{3}-1$, it follows that $b$ does not divide $a$. Therefore, $b$ does not divide $\left|N_{S}(T)\right|=30 a$, so the standard argument applies.

We are now ready to complete the proof of Theorem B, which we state again:

Theorem B. Let $S$ be a nonabelian finite simple group. Then there exist $a, b \in o e(S)$, such that every pair of elements of $S$ of order $a$ and $b$, respectively, generates a nonsolvable subgroup of $S$.

Proof. This follows from Propositions [2.1, 2.2, 4.2, 4.5 and from the classification of the finite simple groups.

## 5. Proof of Theorem A

We finally show that Theorem A follows from Theorem B. First, we restate Theorem A.

Theorem A. Let $G$ be a finite group. Assume that for every $x, y \in G$ there exists an element $g \in G$ such that $\left\langle x, y^{g}\right\rangle$ is solvable. Then $G$ is solvable.

Proof. Suppose that the hypothesis holds for a group $G$. This is clearly equivalent to assuming that, for all pairs $C, D$ of conjugacy classes of a group $G$ there exist elements $x \in C$ and $y \in D$ such that $\langle x, y\rangle$ is solvable.

We claim that this property is inherited by factor groups: let $N$ be a normal subgroup of $G$ and write $\bar{G}=G / N$. Since "overbar" is a
homomorphism, it sends conjugacy classes of $G$ onto conjugacy classes of $\bar{G}$. Hence, given two conjugacy classes $\bar{C}$ and $\bar{D}$ of $\bar{G}$, we may assume that $C$ and $D$ are conjugacy classes of $G$. So, by our assumption, there exist $x \in C$ and $y \in D$ such that $\langle x, y\rangle$ is solvable. Hence, $\bar{x} \in \bar{C}$, $\bar{y} \in \bar{D}$ and $\langle\bar{x}, \bar{y}\rangle \leq \overline{\langle x, y\rangle}$ is solvable. Thus the claim is proved.

Theorem A holds trivially if $|G|=1$. Suppose inductively that $|G|>1$ and that Theorem A holds for groups of orders less than $|G|$. Let $M$ be a minimal normal subgroup of $G$. Then by induction, $G / M$ is solvable. If $G$ has distinct minimal normal subgroups $M_{1}, M_{2}$, then $G$ is isomorphic to a subgroup of the solvable group $G / M_{1} \times G / M_{2}$. Thus we may assume that $G$ has a unique minimal normal subgroup $M$. If $M$ is solvable, then $G$ is solvable as well. We show that this must be the case: suppose to the contrary that $M$ is nonsolvable. The characteristically simple group $M$ is a direct product of isomorphic simple groups. We hence identify $M$ with the direct power $S^{k}$ of a nonabelian simple group $S$. By Theorem B, there exist $a, b \in o e(S)$ such that for every choice of elements $x, y \in S$ with $|x|=a$ and $|y|=b$, the group $\langle x, y\rangle$ is nonsolvable. In particular, $\left\langle x^{\alpha}, y^{\beta}\right\rangle$ is nonsolvable for all $\alpha, \beta \in$ Aut $(S)$. Consider now the diagonal elements $u=(x, x, \ldots, x)$ and $w=(y, y, \ldots, y)$ of $M$. Recalling that $G$ can be embedded in the wreath product of $\operatorname{Aut}(S)$ by a (solvable) subgroup of the symmetric group $S_{k}$, we see that for every $g, h \in G$ we have $u^{g}=\left(x^{\alpha_{1}}, x^{\alpha_{2}}, \ldots, x^{\alpha_{k}}\right)$ and $w^{h}=\left(y^{\beta_{1}}, y^{\beta_{2}}, \ldots, y^{\beta_{k}}\right)$, where $\alpha_{i}, \beta_{i} \in \operatorname{Aut}(S)$ for $i=1,2, \ldots, k$. Observe also that $\left\langle u^{g}, w^{h}\right\rangle$ is a subdirect subgroup of the direct product

$$
\prod_{i=1}^{k}\left\langle x^{\alpha_{i}}, y^{\beta_{i}}\right\rangle
$$

However $\left\langle x^{\alpha_{i}}, y^{\beta_{i}}\right\rangle$ is a nonsolvable subgroup of $S$, for every $i=1, \ldots, k$. It follows that $\left\langle u^{g}, w^{h}\right\rangle$ is nonsolvable for every choice of $g$ and $h$ in $G$, which is the required contradiction.

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