A NEW SOLVABILITY CRITERION FOR FINITE GROUPS

SILVIO DOLFI, MARCEL HERZOG, AND CHERYL E. PRAEGER

ABSTRACT. In 1968, John Thompson proved that a finite group G is solvable if and only if every 2-generator subgroup of G is solvable. In this paper, we prove that solvability of a finite group G is guaranteed by a seemingly weaker condition: G is solvable if for all conjugacy classes C and D of G, there exist $x \in C$ and $y \in D$ for which $\langle x, y \rangle$ is solvable. We also prove the following property of finite nonabelian simple groups, which is the key tool for our proof of the solvability criterion: if G is a finite nonabelian simple group, then there exist two integers a and b which represent orders of elements in G and for all elements $x, y \in G$ with |x| = a and |y| = b, the subgroup $\langle x, y \rangle$ is nonsolvable.

1. INTRODUCTION

John G. Thompson's famous 'N-group paper' [T] of 1968 included the following important solvability criterion for finite groups:

A finite group is solvable if and only if every pair of its elements generates a solvable group.

P. Flavell [F] gave a relatively simple proof of Thompson's result in 1995. We prove that solvability of finite groups is guaranteed by a seemingly weaker condition than the solvability of all its 2-generator subgroups.

Theorem A. Let G be a finite group such that, for all $x, y \in G$, there exists an element $g \in G$ for which $\langle x, y^g \rangle$ is solvable. Then G is solvable.

Theorem A can be rephrased as the following equivalent result.

²⁰⁰⁰ Mathematics Subject Classification. 20D10, 20F16.

Key words and phrases. Solvable groups, finite simple groups.

The first author is grateful to the School of Mathematics and Statistics of the University of Western Australia for its hospitality and support, while the investigation was carried out. He was partially supported by the MIUR project "Teoria dei gruppi e applicazioni". The third author was supported by Federation Fellowship FF0776186 of the Australian Research Council.

Theorem A'. Let G be a finite group such that, for all conjugacy classes C and D of G (possibly C = D), there exist $x \in C$ and $y \in D$ for which $\langle x, y \rangle$ is solvable. Then G is solvable.

Our second main result, which is the key tool for proving Theorem A, deals with the nonsolvability of certain 2-generator subgroups of finite nonabelian simple groups. For a finite group G let

$$oe(G) = \{m \mid \exists g \in G \text{ with } |g| = m\}$$

denote the set of element orders of G. Using the classification of finite simple groups, we prove the following theorem.

Theorem B. Let G be a finite nonabelian simple group. Then there exist $a, b \in oe(G)$, such that, for all $x, y \in G$ with |x| = a, |y| = b, the subgroup $\langle x, y \rangle$ is nonsolvable.

Theorem B was proved separately for alternating groups, sporadic groups, classical groups of Lie type and exceptional groups of Lie type in Propositions 2.1, 2.2, 4.2, 4.5, respectively. In view of Propositions 2.1, 2.2, we state the following conjecture.

Conjecture. If G is a finite nonabelian simple group, then there exist two distinct primes $p, q \in oe(G)$, such that, for all $x, y \in G$ with |x| = p, |y| = q, the subgroup $\langle x, y \rangle$ is nonsolvable (or, maybe, even nonabelian simple).

The authors are grateful to Frank Lübeck, Pham Tiep and Thomas Wiegel for supplying us with very important information concerning simple groups of Lie type. We are also grateful to Bob Guralnick and Gunter Malle for conveying to us results from their paper [GM] prior to its publication.

1.1. Other generalisations of Thompson's theorem. Several other 'Thompson-like' results have appeared in the literature recently. We mention here four such theorems. In the first three results, solvability of all 2-generator subgroups is replaced by a weaker condition restricting the required set of solvable 2-generator subgroups, in different ways from our generalisation.

In 2000, Guralnick and Wilson [GW] obtained a solvability criterion by restricting the proportion of 2-generator subgroups required to be solvable.

Theorem 1.1. A finite group is solvable if and only if more than $\frac{11}{30}$ of the pairs of elements of G generate a solvable subgroup.

In addition they proved similar results showing that the properties of nilpotency and having odd order are also guaranteed if a sufficient proportion of element pairs generate subgroups with these properties, namely more than $\frac{1}{2}$ for nilpotency, and more than $\frac{11}{30}$ for having odd order.

In contrast to this, in a paper published in 2009, Gordeev, Grunewald, Kunyavskii and Plotkin [GGKP] proved a solvability criterion which involved 2-generation within each conjugacy class. This result was also proved independently by Guest in [G, Corollary 1].

Theorem 1.2. A finite group G is solvable if and only if, for each conjugacy class C of G, each pair of elements of C generates a solvable subgroup.

A stronger result of this type was obtained recently by Kaplan and Levy in [KL, Theorem 4]. Their criterion involves only a limited 2generation within the conjugacy classes of elements of odd prime-power order.

Theorem 1.3. A finite group G is solvable if and only if for every $x, y \in G$, with x a p-element for some odd prime p and y a 2-element, the group $\langle x, x^y \rangle$ is solvable.

Our requirement, while ranging over all conjugacy classes, requires only *existence* of a solvable 2-generator subgroup with one generator from each of two (possibly equal) classes. We know of no similar criteria in this respect.

The forth result we draw attention to is in a 2006 paper of Guralnick, Kunyavski, Plotkin and Shalev [GKPS]. They proved that membership of the solvable radical of a finite group is characterised by solvability of certain 2-generator subgroups. (The solvable radical R(G) of a finite group G is the largest solvable normal subgroup of G.)

Theorem 1.4. For a finite group G, the solvable radical R(G) coincides with the set of all elements $x \in G$ with the property: "for any $y \in G$, the subgroup $\langle x, y \rangle$ is solvable".

2. Alternating and sporadic simple groups

Theorem B for the alternating groups follows from the following proposition.

Proposition 2.1. If $m \ge 5$, then there exist distinct primes p and q satisfying $m/2 such that, for all <math>x, y \in A_m$ with |x| = p and |y| = b, the subgroup $\langle x, y \rangle \cong A_d$ for some $d \ge 5$. In particular, $\langle x, y \rangle$ is nonsolvable.

Proof. First we remark that if n is a positive integer and $\pi(n)$ denotes the number of primes at most n, then the following is known (see, for example [R, p.188]):

$$\pi(2n) - \pi(n) > n/(3\ln n)$$
 for $n \ge 5$.

In particular, if $n \ge 17$, then

$$\pi(2n) - \pi(n) > 17/(3\ln 17) > 2,$$

which implies that $\pi(2n) - \pi(n) \geq 3$. Thus, if $n \geq 17$, then there are at least 3 distinct primes p, r and q satisfying n . $In particular, <math>n . Hence, if <math>m \geq 34$, then there exist primes p and q such that m/2 , and $elements <math>x, y \in A_m$ of order p and q, respectively. Let x, y be any such elements and let $H = \langle x, y \rangle$. Let Δ be the support of H, that is, the subset of $\{1, 2, \ldots, m\}$ consisting of all the points moved by H, and let $d := |\Delta|$. Since q > p > m/2, it follows that H is transitive on Δ , and that $q \leq d \leq m < 2p$. By [Wi, Theorem 8.4], H is primitive on Δ . Moreover, since $d \geq q > p + 3 > 5$ and H contains only even permutations, it follows from a theorem of C. Jordan dating from 1873, see [Wi, Theorem 13.9], that $H \cong A_d$ for some $d \geq 5$, as claimed.

It remains to deal with A_m , for $5 \le m \le 33$. In each case we will choose primes p and q such that $m/2 \le p < q \le m$, and consider the subgroup $H = \langle x, y \rangle$ generated by elements x and y of A_m of order pand q, respectively. Denote by Δ the support of H, and let $d = |\Delta|$. In all cases $d \ge 5$ since $d \ge q > p \ge 3$, and hence A_d is nonsolvable.

For $17 \leq m \leq 33$, and for $11 \leq m \leq 13$, let q be the largest prime such that $q \leq m$, and let p be the smallest prime such that p > m/2. Then $p \leq q - 3$, and the argument above shows that $H \cong A_d$.

If $14 \leq m \leq 16$, let q = 13, p = 11. If d = 13, then $H \leq A_{13}$ and since by the [ATLAS, p. 104] no maximal subgroup of A_{13} has order divisible by $11 \cdot 13$, it follows that $H = A_{13}$. If d > 13, then as before, H is a primitive group on Δ , and since d = p + k with $k \geq 3$, it follows by [Wi, Theorem 13.9] that $H \cong A_d$.

If $7 \le m \le 10$, let q = 7, p = 5. It follows from the lists of maximal subgroups of A_d in [ATLAS, pp. 10, 22, 37, 48] that $H \cong A_d$. If m = 5, 6, let q = 5, p = 3. It follows from the lists of maximal subgroups of these groups in [ATLAS, pp. 2, 4] that $H = A_5$ if m = 5, and that $H \cong A_5$ or A_6 if m = 6. The proof of Proposition 2.1 is complete.

Theorem B for the sporadic simple groups follows from the following proposition. For compactness of notation, we write $L_2(q)$ instead of PSL(2,q) in Table 1.

S	p	q	$\langle x, y \rangle$	S	p	q	$\langle x,y angle$
M_{11}	2	11	$M_{11}, L_2(11)$	M_{12}	2	11	$M_{12}, M_{11}, L_2(11)$
M_{22}	2	11	$M_{22}, L_2(11)$	M_{23}	2	23	M_{23}
M_{24}	2	23	$M_{24}, L_2(23)$	J_1	5	19	J_1
J_2	2	7	$J_2, L_2(7)$	J_3	2	19	$J_3, L_2(19)$
J_4	3	43	J_4	HS	2	11	$HS, M_{11}, L_2(11), M_{22}$
He	7	17	He	McL	2	11	$McL, M_{11}, M_{12}, L_2(11)$
Suz	11	13	Suz	Ly	3	67	Ly
Ru	3	29	Ru	O'N	2	31	O'N
Co_1	13	23	Co_1	Co_2	2	23	Co_2, M_{23}
Co_3	2	23	Co_3, M_{23}	Fi_{22}	11	13	Fi_{22}
$F_{i_{23}}$	2	23	$Fi_{23}, M_{23}, L_2(23)$	Fi'_{24}	3	29	Fi'_{24}
HN	5	19	HN	Th	19	31	Th
B	2	47	В	M	2	59	$M, L_2(59)$

TABLE 1. Results table for Proposition 2.2

Proposition 2.2. Let S be a sporadic simple group as in one of the rows of Table 1. Then for the primes p, q in the corresponding row of Table 1, $p, q \in oe(S)$ and, for all $x, y \in S$ with |x| = p and |y| = q, the subgroup $\langle x, y \rangle$ is one of the nonabelian simple groups in the row for S and column labeled $\langle x, y \rangle$ of Table 1.

Proof. The proof uses heavily the lists of maximal subgroups of the sporadic simple groups and of other simple groups, which appear in [ATLAS] and in [ATLAS3]. For all sporadic simple groups our results follow from a close examination of these lists. In particular the groups listed in the column labeled $\langle x, y \rangle$ are the only subgroups which could possibly be generated by two elements, the first of order p and the second of order q.

As an example, we describe in detail our treatment of the Higman-Sims sporadic group HS of order $2^{9} \cdot 3^{2} \cdot 5^{3} \cdot 7 \cdot 11$. Having checked the maximal subgroups of HS, we choose the primes p = 2 and q = 11. If the subgroup $X = \langle x, y \rangle$ is not equal to HS, then X is contained in a maximal subgroup of HS of order divisible by 11. We see from [ATLAS, p. 80] that the only such maximal subgroups of HS are the simple groups M_{22} and M_{11} . Suppose first that X is a subgroup of M_{11} . If $X \neq M_{11}$, then it is contained in a maximal subgroup of M_{11} of order divisible by 11. By [ATLAS, p. 18], each such maximal subgroup of M_{11} is isomorphic to the simple group $L_2(11)$. If $X \neq L_2(11)$, then X is contained in a maximal subgroup of $L_2(11)$ of order divisible by 11. However, such maximal subgroups have order $5 \cdot 11$, which is not divisible by 2. Thus we are left with the possibilities: X = HS, $X = M_{11}$, $X = L_2(11)$, or X is a subgroup of M_{22} . If X is a proper subgroup of M_{22} , then X is contained in a maximal subgroup of M_{22} of order divisible by 11. By [ATLAS, p. 30], the only such maximal subgroup of M_{22} is the simple group $L_2(11)$, which we have already examined. Thus, finally, X is one of the simple groups HS, M_{11} , $L_2(11)$ or M_{22} , as in Table 1.

Thus Proposition 2.2 is proved.

3. PRIMITIVE PRIME DIVISORS

In the following, $q = p^k$ is a power of a prime p. For any positive integer e, we say that a prime r is a *primitive prime divisor* of $q^e - 1$ if r divides $q^e - 1$ and r does not divide $q^i - 1$ for any positive integer i < e. Observe that then e is the order of q modulo the prime r; so e divides r - 1 and, in particular, $r \ge e + 1$. The set of primitive prime divisors of $q^e - 1$ will be denoted by ppd(q, e).

We say that a prime r is a *basic* primitive prime divisor of $q^e - 1$, if r is a primitive prime divisor of $p^{ke} - 1$, that is to say, if $r \in ppd(p, ek)$. We denote by bppd(q, e) the set of basic primitive prime divisors of $q^e - 1$. Note that $bppd(q, e) \subseteq ppd(q, e)$, and that the inclusion can be strict; for example, $ppd(2^2, 3) = \{7\}$, but $bppd(2^2, 3) = ppd(2, 6) = \emptyset$. The following result of Zsigmondy [Z] will be used frequently.

Theorem 3.1. Let $q \ge 2$ and $e \ge 2$. Then $bppd(q, e) = \emptyset$ if and only if one of the following holds.

(i): q is a Mersenne prime, e = 2, and here ppd(q, 2) = Ø;
(ii): (q, e) = (2, 6), and here ppd(2, 6) = Ø;
(iii): (q, e) ∈ {(4, 3), (8, 2)}, and here ppd(4, 3) = {7}, and ppd(8, 2) = {3}.

Next, we define the set Lpd(q, e) of large primitive divisors of $q^e - 1$ to be the set consisting of primes $r \in ppd(q, e)$ such that r > e + 1 together with the integer $(e+1)^2$ if $e+1 \in ppd(q, e)$ and $(e+1)^2$ divides $q^e - 1$.

Finally, we define the set Lbpd(q, e) of large basic primitive divisors of $q^e - 1$ to be the set consisting of primes $r \in bppd(q, e)$ such that r > e + 1, together with the integer $(e + 1)^2$ if $e + 1 \in bppd(q, e)$ and $(e + 1)^2$ divides $q^e - 1$.

The following observation will be very useful in the sequel.

Proposition 3.2. Let $q \ge 2$ and $e \ge 3$. Then $Lbpd(q, e) = \emptyset$ if and only if (q, e) is one of the following:

(2, 4), (2, 6), (2, 10), (2, 12), (2, 18), (3, 4), (3, 6), (4, 3), (5, 6).

Proof. Let Lpd, Lbpd, bppd, ppd denote the sets Lpd(q, e), Lbpd(q, e), bppd(q, e), ppd(q, e), respectively. We prove first that $\text{Lbpd} \neq \emptyset$ if and only if both bppd $\neq \emptyset$ and $\text{Lpd} \neq \emptyset$. If $\text{Lbpd} \neq \emptyset$, then clearly bppd $\neq \emptyset$ and $\text{Lpd} \neq \emptyset$. Conversely, suppose that bppd $\neq \emptyset$ and $\text{Lpd} \neq \emptyset$. Let r be the largest element of bppd. Then $r \in \text{ppd}(p, ke)$, so $r \ge ke+1 \ge e+1$. If r > e+1, then $r \in \text{Lbpd}$, so we may assume that r = e+1. Then k = 1 and bppd $= \text{ppd} = \{e+1\}$. Let $s \in \text{Lpd}$. Since $\text{ppd} = \{e+1\}$, it follows that $s = (e+1)^2$ and s divides $q^e - 1$, whence $s \in \text{Lbpd}$. Thus in both cases Lbpd is nonempty, as required.

Hence Lbpd is empty if and only if either Lpd is empty or bppd is empty. Now assume that $q \ge 2$ and $e \ge 3$. By [NP1, Theorem 2.2], the set Lpd is empty only for q = 2 and $e \in \{4, 6, 10, 12, 18\}$, for q = 3and $e \in \{4, 6\}$, and for (q, e) = (5, 6). Moreover, by Theorem 3.1, the set bppd is empty only if $(q, e) = \{(2, 6), (4, 3)\}$. So Lbpd is empty precisely for the values of (q, e) listed in the proposition.

Remark 3.3. Let G be a subgroup of GL(d, q) and let $m \in Lpd(q, e)$ (or $m \in Lbpd(q, e)$) with $d \ge e > d/2$. If m divides |G|, then $m \in oe(G)$. In fact, either m = r or $m = r^2 = (e + 1)^2$, where r is a (basic) primitive prime divisor of $q^e - 1$. Since e > d/2, a Sylow r-subgroup of GL(d, q) is cyclic and so G has elements of order m.

We say that an element $g \in G$ is a bppd(q, e)-element if the order of g is divisible by some element of bppd(q, e). Similarly, we say that $g \in G$ is an Lbpd(q, e)-element if the order of g is divisible by some element of Lbpd(q, e).

We use results from [NP1] and [NP2] to deal with subgroups of linear groups containing one or two "big" ppd-elements. We observe here that *basic* primitive prime divisors will be relevant in order to exclude examples of "subfield type" (see case (d) in the proof of Lemma 3.4), while *large* primitive divisors will be relevant in the proof of Lemma 4.1. Our first lemma deals with irreducible subgroups of GL(d, q) for $d \geq 3$.

Lemma 3.4. Let G be an irreducible subgroup of GL(d,q), with $d \ge 3$. Assume that $x, y \in G$ are such that x is a bppd(q,e)-element and y is a bppd(q, f)-element, with $d \ge e > f > d/2$. Then either G is nonsolvable or one of the following holds.

- (1): d = e = f + 1 is prime, G is conjugate to a subgroup of $GL(1, q^d).d$, and G contains no Lbpd(q, f)-elements;
- (2): There exists a prime c dividing gcd(d, e, f) such that G is conjugate to a subgroup of $GL(\frac{d}{c}, q^c)$.c and the elements x^c and y^c lie in $GL(\frac{d}{c}, q^c)$ as a $bppd(q^c, \frac{e}{c})$ -element and a $bppd(q^c, \frac{f}{c})$ -element, respectively.

Proof. By assumption |x| is a multiple of an element $r \in \text{bppd}(q, e)$ and |y| is a multiple of an element $s \in \text{bppd}(q, f)$. Suppose first that d = 3 and $q \leq 4$. Then $3 \geq e > f > 3/2$ and so e = 3, f = 2 and $\text{bppd}(q, 3) \neq \emptyset$, $\text{bppd}(q, 2) \neq \emptyset$. By Theorem 3.1, $q \neq 3, 4$. Hence, q = 2and so $r = 7, s = 3, G \leq \text{GL}(3, 2)$ and $\text{Lbpd}(q, f) = \emptyset$. By [ATLAS, p.3], either G = GL(3, 2) is nonsolvable or $G = \langle x, y \rangle \cong \text{GL}(1, 8).3$ and part (1) holds. Thus if d = 3 we may assume that $q \geq 5$.

Next we claim that, if (d, q) = (4, 2) or (4, 3), then G is nonsolvable. In these cases $d = 4 \ge e > f > 2$ and so e = 4, f = 3 and bppd $(q, 4) \ne \emptyset$, bppd $(q, 3) \ne \emptyset$. If (d, q) = (4, 3), then $G \le \text{GL}(4, 3) \cong 2.\text{PSL}(4, 3).2$ (see [ATLAS, pp. 68 and 69]) and r = 5, s = 13. Since no proper subgroup of PSL(4, 3) is divisible by $5 \cdot 13$, PSL(4, 3) is a section of G and hence G is nonsolvable, as claimed. If (d, q) = (4, 2), then $G \le \text{GL}(4, 2) \cong \text{PSL}(4, 2)$ and r = 5, s = 7. It follows from [ATLAS, pp. 10 and 22] that $G \cong A_7$ or A_8 and in particular G is nonsolvable. Thus we may assume further that $(d, q) \ne (4, 2)$ or (4, 3).

Note that $q = p^k$ for some prime p and $k \ge 1$. Since $r \in ppd(p, ek)$, $s \in ppd(p, fk)$ and $e > f \ge 2$, neither r nor s divides q - 1. Moreover, if $i \le d$ and $h \le \frac{k}{2}$, then neither r nor s divides $p^{ih} - 1$ (since $ih \le \frac{dk}{2} < fk < ek$).

We apply [NP1, Theorem 4.7]. First we show that the cases (a), (c), (d) and (e) of [NP1, Theorem 4.7], if they occur, imply that G is nonsolvable.

(a): (Classical type) G has a normal subgroup Ω , where Ω is one of the following groups: SL(d,q), Sp(d,q) (d even), $SU(d,q_0)$ ($q = q_0^2$), $\Omega^{\pm}(d,q)$ (d even), and $\Omega^{\circ}(d,q)$ (d odd). Because of our assumption that $(d,q) \neq (4,2), (4,3)$ or (3,q) with $q \leq 4$, each of these classical groups is nonsolvable and so G is nonsolvable in case (a).

(c): (Nearly Simple Groups) Here G is nearly simple, and hence nonsolvable.

(d): (Subfield type) Denote by Z the subgroup of scalar matrices of $\operatorname{GL}(d,q)$ and let $Z \circ \operatorname{GL}(d,q_0)$ denote a group which can be defined over a subfield modulo scalars. In case (d), G is conjugate to a subgroup of $Z \circ \operatorname{GL}(d,q_0)$ for some *proper* subfield \mathbb{F}_{q_0} of \mathbb{F}_q , say $q = q_0^a$ and $q_0 = p^h$, with k = ah and $a \ge 2$. Since $x \in G$ and r does not divide q - 1 (as noted above), it follows that r divides $|\operatorname{GL}(d,q_0)|$. Hence r divides $p^{ht} - 1$ for some $t \le d$. Since $h \le \frac{k}{2}$, this contradicts our observation above, so case (d) does not arise.

(e): (Imprimitive type) Here G preserves a direct-sum decomposition $V = U_1 \oplus \cdots \oplus U_d$ with $\dim(U_i) = 1$ for $i = 1, \ldots, d$ and G induces A_d or S_d on the set $\{U_1, \ldots, U_d\}$. Since A_d is nonsolvable for $d \ge 5$,

we may assume that $d \leq 4$. In particular, |G| divides $(q-1)^d d!$ and since the primes r, s do not divide q-1, each of them divides d!. In particular, $r, s \leq d \leq 4$. However, $e > f \geq 2$, so $r \geq e+1 \geq 4$, which implies that r = 4, a contradiction.

This leaves case (b) of [NP1, Theorem 4.7], and in that case G is conjugate to a subgroup of $\operatorname{GL}(\frac{d}{c}, q^c).c$, for some prime c dividing d. Moreover either c = d = f + 1 = e, or c < d and c divides $\operatorname{gcd}(d, e, f)$. In the former case, $G \leq \operatorname{GL}(1, q^d).d$ and s = d = f + 1. As s^2 does not divide |G|, part (1) holds. In the latter case, since each of r, s is at least f + 1 > c, it follows that $x^c, y^c \in \operatorname{GL}(\frac{d}{c}, q^c)$ and have orders divisible by r, s, respectively. Thus part (2) holds. \Box

If V is a G-module and U is a G-invariant section of V (that is to say, $U = V_1/V_2$ where $V_2 \leq V_1$ are G-submodules of V), we denote by G^U the group of automorphisms induced by G on U. So $G^U \cong G/C_G(U)$, a factor group of G, and accordingly we denote by x^U the image in G^U of an element $x \in G$ under the relevant projection homomorphism.

For a prime r and integer n, let n_r denote the r-part of n, that is, the largest power of r dividing n.

Remark 3.5. Let $G \leq \operatorname{GL}(d,q)$ and let V = V(d,q) be the natural module for G. Suppose that x is a $\operatorname{ppd}(q,e)$ -element of G of order divisible by a primitive prime divisor r of $q^e - 1$ and let x_0 be an element of order r in $\langle x \rangle$. Then the following statements hold.

(1) Under the action of $\langle x_0 \rangle$, $V|_{\langle x_0 \rangle}$ is a completely reducible $\mathbb{F}_q \langle x_0 \rangle$ module, by Maschke's Theorem, and all the nontrivial irreducible submodules of $V|_{\langle x_0 \rangle}$ have dimension e (see for instance [H, Theorem II.3.10]). In particular, it follows that there exists at least one Gcomposition factor U of V on which x_0 , and hence also x, acts nontrivially. So $|x_0^U| = r$ and hence x^U is a ppd(q, e)-element of $G^U \leq$ $\operatorname{GL}(d_0, q)$ of order divisible by $|x_0^U| = r$ and satisfying $|x^U|_r = |x|_r$, where $d_0 = \dim_{\mathbb{F}_q}(U) \geq e$.

(2) Suppose also that y is a ppd(q, f)-element of G of order divisible by a primitive prime divisor s of $q^f - 1$, and that $d \ge e > f > d/2$. Since both e and f are greater than d/2, it follows by part (1) of this remark that there exists a unique G-composition factor U of V, of dimension $d_0 = \dim_{\mathbb{F}_q}(U) \ge e$, on which both x and y act nontrivially. Moreover $G^U \le \operatorname{GL}(d_0, q)$, and x^U is a $\operatorname{ppd}(q, e)$ -element of G^U of order satisfying $|x^U|_r = |x|_r \ge r$ and y^U is a $\operatorname{ppd}(q, f)$ -element of G^U of order satisfying $|y^U|_s = |y|_s \ge s$. We now apply Lemma 3.4 to linear groups of dimension large enough to allow the existence of primitive prime divisors for two distinct exponents, both greater than half the dimension of the linear group. It is convenient to deal separately with certain large exponent pairs.

Lemma 3.6. Let j be a nonnegative integer, $\delta \in \{1, 2\}$, and let $x, y \in \text{GL}(d,q)$ be such that x is a bppd(q,e)-element and y is a bppd(q,f)-element. Assume that e, f, δ , and d satisfy the following conditions:

$$e = d - j, \ f = d - j - \delta, \ d \ge 2j + 2\delta + 1.$$

If $\delta = 1$, we assume in addition that y is an Lbpd(q, f)-element. Then $G = \langle x, y \rangle$ is nonsolvable.

Proof. Let $r \in \text{bppd}(q, e)$ such that r divides |x|, and let $s \in \text{bppd}(q, f)$ such that s divides |y|. As mentioned above, $r \equiv 1 \pmod{e}$ and $s \equiv 1 \pmod{f}$. Moreover, $e > f = d - j - \delta \ge j + \delta + 1 \ge 2$, so in particular both r and s are odd primes. As it is sufficient to prove that some subgroup of G is nonsolvable, we may assume that $G = \langle x, y \rangle$, and |x|, |y| are powers of r and s, respectively. In particular, |x| and |y| are odd. If $\delta = 1$, then in addition we assume that y is an Lbpd(q, f)-element.

By our assumptions, $j \leq \frac{d-1}{2} - \delta$, so $d \geq e > f = d - j - \delta \geq \frac{d+1}{2} > \frac{d}{2}$. In the following, we denote by V the natural module V(d,q) for $\operatorname{GL}(d,q)$.

Case (1): G is irreducible on V.

The conditions of Lemma 3.4 are satisfied. Applying that result we deduce that either G is nonsolvable, or (1) d = e = f + 1 is prime, G is conjugate to a subgroup of $GL(1, q^d).d$, and G contains no Lbpd(q, f)elements, or (2) there is a prime c < d such that c divides gcd(d, e, f), G is conjugate to a subgroup of $GL(d/c, q^c).c$, and x^c and y^c lie in $\operatorname{GL}(\frac{d}{c}, q^c)$ as a bppd $(q^c, \frac{e}{c})$ -element and a bppd $(q^c, \frac{f}{c})$ -element, respectively. If (1) holds, then $j = 0, \delta = 1$ and we have a contradiction, since y is an Lbpd(q, f)-element of G. Thus (2) holds. Since c is a prime dividing gcd(d, e, f), and since $gcd(e, f) = gcd(e, e - f) \leq e - f =$ $\delta \leq 2$, it follows that $c = \delta = 2$ and all of d, e, f, j are even. Hence $G \leq \operatorname{GL}(d/2, q^2).2$, and since the orders of x, y are odd, we conclude that $G \leq \operatorname{GL}(d/2, q^2)$. Since d and j are even, our assumption that $d \ge 2j+2\delta+1 = 2j+5$ implies $\frac{d}{2} \ge 2\frac{j}{2}+3$. Thus replacing (d, q, e, f, j, δ) by $(\frac{d}{2}, q^2, \frac{e}{2}, \frac{f}{2}, \frac{j}{2}, \frac{\delta}{2})$, all the conditions of the lemma hold, and G is irreducible on $V(\frac{d}{2}, q^2)$. By the arguments above and since $\frac{\delta}{2} = 1$, we conclude that G must be nonsolvable.

Case (2): G is reducible on V.

If e = d, then |x| would be a multiple of a primitive prime divisor of $q^d - 1$ and so $\langle x \rangle$ would act irreducibly on the natural module V. However G is reducible, so we must have e < d and $j \ge 1$. By Remark 3.5(2), there exists a unique G-composition factor U of V of dimension $d_0 = \dim_{\mathbb{F}_q}(U)$, say, where $d > d_0 \ge e = d - j$, such that x^U and y^U are bppd(q, e)- and bppd(q, f)-elements of G^U , respectively, with $|x^U|_r = |x|_r \ge r, |y^U|_s = |y|_s \ge s$. In particular, if y is an $\mathrm{Lbpd}(q, f)$ element of G, then y^U is an $\mathrm{Lbpd}(q, f)$ -element of G^U .

We claim that the irreducible group $G^U = \langle x^U, y^U \rangle \leq \operatorname{GL}(d_0, q)$ induced by G on U satisfies the conditions of Lemma 3.6 with parameters d_0, e, f, δ , relative to the integer $j_0 := j - d + d_0$. Note that $j > j_0 = d_0 - e \geq 0$ and $d_0 = j_0 + d - j \geq j_0 + j + 2\delta + 1 > 2j_0 + 2\delta + 1 \geq 3$, and the conditions $e = d_0 - j_0, f = d_0 - j_0 - \delta$ hold, by the definition of j_0 . This proves the claim. Since G^U is irreducible, it follows from Case (1) of this proof that G^U is nonsolvable. Consequently, also G is nonsolvable.

We can now prove Theorem B for classical simple groups of large dimension.

Proposition 3.7. Assume that S is one of the following simple groups:

(1): PSL(d,q), with d ≥ 4 and (d,q) ≠ (6,2);
(2): PSp(d,q) (d even) or PSU(d,q) (d odd), with d ≥ 5 and (d,q) ∉ {(5,2), (6,2), (8,2)};
(3): PΩ°(d,q) (dq odd) or PSU(d,q) (d even), with d ≥ 7;
(4): PΩ[±](d,q) (d even), with d > 10 and (d,q) ≠ (10,2).

Then there exist $a, b \in oe(S)$ such that for every choice of $x, y \in S$ with |x| = a and |y| = b, the group $\langle x, y \rangle$ is nonsolvable.

Proof. We work with the group \hat{S} defined as follows:

- $\hat{S} = \operatorname{SL}(d, q) \leq \operatorname{GL}(d, q)$, when $S = \operatorname{PSL}(d, q)$;
- $\hat{S} = \operatorname{Sp}(d, q) \leq \operatorname{GL}(d, q)$, when $S = \operatorname{PSp}(d, q)$;
- $\hat{S} = SU(d,q) \le GL(d,q^2)$, when S = PSU(d,q);
- $\hat{S} = \Omega^{\varepsilon}(d,q) \leq \operatorname{GL}(d,q)$, when $S = P\Omega^{\varepsilon}(d,q)$, for $\varepsilon = \pm$ or \circ .

Let $\mathbb{Z} = \mathbb{Z}(\hat{S})$. We prove that there exist $a, b \in oe(\hat{S})$ with $gcd(ab, |\mathbb{Z}|) = 1$ such that, for every $\hat{x} \in \hat{S}$ of order a and every $\hat{y} \in \hat{S}$ of order b, the group $\langle \hat{x}, \hat{y} \rangle$ is nonsolvable. Since a and b are coprime to $|\mathbb{Z}|$, if $|\hat{x}| = a$ then also the order of $\hat{x}\mathbb{Z}$ in S is equal to a, and similarly, if $|\hat{y}| = b$ then $|\hat{y}\mathbb{Z}| = b$. In addition, if $\langle \hat{x}, \hat{y} \rangle$ is nonsolvable then also $\langle x, y \rangle$ is nonsolvable, because it is a central factor of the nonsolvable group $\langle \hat{x}, \hat{y} \rangle$. Thus it is sufficient to work as described with the group

 \hat{S} . In fact, each of a and b will be either a prime or the square of a prime, and it will be easy to check that $gcd(ab, |\mathbb{Z}|) = 1$.

(1): Let $\hat{S} = SL(d, q)$, with $d \ge 4$. Suppose first that

$$(1.1) \quad (d,q) \notin \{(4,4), (5,2), (5,3), (6,2), (7,2), (7,3), (7,5), \\(11,2), (13,2), (19,2)\}.$$

Then by Theorem 3.1 there exists $a \in \text{bppd}(q, d)$, and by Proposition 3.2 there exists $b \in \text{Lbpd}(q, d-1)$. Now $q^d - 1$ and $q^{d-1} - 1$ divide \hat{S} and hence $a, b \in oe(\hat{S})$ by Remark 3.3. Also $\text{gcd}(ab, |\mathbb{Z}|) = 1$. Let now $\hat{x}, \hat{y} \in \hat{S}$ be such that $|\hat{x}| = a$ and $|\hat{y}| = b$. Then \hat{x} is a bppd(q, d)-element of \hat{S} and \hat{y} is an Lbpd(q, d-1)-element of \hat{S} . Since $d \geq 4$, the conditions of Lemma 3.6 are satisfied with $\delta = 1, j = 0$, and we conclude that $\langle \hat{x}, \hat{y} \rangle$ is nonsolvable.

We now consider the "isolated" cases left out of this proof and listed in (1.1). We deal with all these cases, except the excluded pair (d, q) =(6, 2), by taking elements $\hat{x}, \hat{y} \in \hat{S}$ with $|\hat{x}| = a$ and $|\hat{y}| = b$, where a, bare as in the following table. (Here q_i denotes a prime in bppd(q, i), and, by Theorem 3.1, for all entries in the table such a prime q_i exists.)

(d,q)	(4, 4)	(5, 2)	(5, 3)	(7, 2)	(7,3)	(7, 5)	(11, 2)	(13, 2)	(19, 2)
a	q_4	q_5	q_5	q_7	q_7	q_7	q_{11}	q_{13}	q_{19}
b	q_2	q_3	q_3	q_5	q_5	q_5	q_9	q_{11}	q_{17}

Thus \hat{x}, \hat{y} satisfy all the conditions of Lemma 3.6 with $\delta = 2, j = 0$, and hence $\langle \hat{x}, \hat{y} \rangle$ is nonsolvable.

(2): If $\hat{S} = \text{Sp}(d,q)$ with $d \ge 5$ and d even, then the order of \hat{S} is divisible by both $q^d - 1$ and $q^{d-2} - 1$. By Theorem 3.1, $\text{bppd}(q,d) \ne \emptyset$ and $\text{bppd}(q,d-2) \ne \emptyset$, since by our assumptions $d > d-2 \ge 4$ and $(q,d), (q,d-2) \ne (2,6)$. We take $a \in \text{bppd}(q,d)$ and $b \in \text{bppd}(q,d-2)$ and note that $a, b \in oe(\hat{S})$.

If $\hat{S} = \mathrm{SU}(d,q) \leq \mathrm{GL}(d,q^2)$ with $d \geq 5$ and d odd, then the order of \hat{S} is divisible by both $q^d + 1$ and $q^{d-2} + 1$. Here we take $a \in \mathrm{bppd}(q, 2d) = \mathrm{bppd}(q^2, d)$ and $b \in \mathrm{bppd}(q, 2d - 4) = \mathrm{bppd}(q^2, d - 2)$. Note that both $\mathrm{bppd}(q, 2d)$ and $\mathrm{bppd}(q, 2d - 4)$ are nonempty, since $(d,q) \neq (5,2)$. Also $a, b \in \mathrm{oe}(\hat{S})$.

In both cases we apply Lemma 3.6 with $\delta = 2, j = 0$. It follows that in each of these two cases, $\langle \hat{x}, \hat{y} \rangle$ is nonsolvable for every choice of $\hat{x}, \hat{y} \in \hat{S}$ with $|\hat{x}| = a$ and $|\hat{y}| = b$.

(3): If $\hat{S} = \Omega^{\circ}(d,q)$, with $d \geq 7$ and dq odd, then both $q^{d-1} - 1$ and $q^{d-3} - 1$ divide the order of \hat{S} and each has a basic primitive prime divisor by Theorem 3.1. We take $a \in \text{bppd}(q, d-1)$ and $b \in \text{bppd}(q, d-3)$ and note that $a, b \in oe(\hat{S})$.

If $\hat{S} = \mathrm{SU}(d,q) \leq \mathrm{GL}(d,q^2)$ with $d \geq 7$ and d even, then the order of \hat{S} is divisible by both $q^{d-1} + 1$ and $q^{d-3} + 1$. We take $a \in \mathrm{bppd}(q, 2d - 2) = \mathrm{bppd}(q^2, d-1)$ and $b \in \mathrm{bppd}(q, 2d-6) = \mathrm{bppd}(q^2, d-3)$, noting that both $\mathrm{bppd}(q^2, d-1)$ and $\mathrm{bppd}(q^2, d-3)$ are nonempty, by Theorem 3.1, since $d-1 > d-3 \geq 4$ and $q^2 > 2$. Thus $a, b \in oe(\hat{S})$.

In both cases we apply Lemma 3.6 with $\delta = 2, j = 1$. It follows that in each of these two cases, $\langle \hat{x}, \hat{y} \rangle$ is nonsolvable for every choice of $\hat{x}, \hat{y} \in \hat{S}$ with $|\hat{x}| = a$ and $|\hat{y}| = b$.

(4): Assume, finally, that $\hat{S} = \Omega^{\pm}(d,q)$ with $d \ge 10$, d even, and $(d,q) \ne (10,2)$. In this case both $q^{d-2}-1$ and $q^{d-4}-1$ divide the order of \hat{S} , and each has a basic primitive prime divisor by Theorem 3.1, since $d-2 > d-4 \ge 6$ and $(d,q) \ne (10,2)$. We take $a \in \text{bppd}(q,d-2)$ and $b \in \text{bppd}(q,d-4)$, and note that $a, b \in oe(\hat{S})$. By Lemma 3.6 with $\delta = 2, j = 2$, we conclude that $\langle \hat{x}, \hat{y} \rangle$ is nonsolvable for every choice of $\hat{x}, \hat{y} \in \hat{S}$ with $|\hat{x}| = a$ and $|\hat{y}| = b$.

4. Proof of Theorem B

Proposition 3.7 does not cover the classical groups in small dimensions, when the conditions of Lemma 3.6 do not hold. For most of these cases we use the following result, which deals with the case where there is at least one Lbpd(q, e)-element in $G \leq \text{GL}(d, q)$, with e > d/2.

Lemma 4.1. Let G be an irreducible subgroup of GL(d,q), with $d \ge 3$. Assume that G contains an Lbpd(q,e)-element for some integer e with $d \ge e > d/2$. Assume also that

$$(d,q) \notin \{(3,2), (3,3), (3,4), (4,2), (4,3)\}.$$

Then either G is conjugate to a subgroup of $\operatorname{GL}(d/c, q^c).c$ for some prime divisor c of $\operatorname{gcd}(d, e)$, or G is nonsolvable.

Proof. This lemma follows from [NP2, Theorem 3.1]. Note that the cases (b), (d) and (c)(i) of that theorem do not occur, because of the assumption that the bppd(q, e)-element is large. So either G is of classical type (part (a) of [NP2, Theorem 3.1]) and hence is nonsolvable because of the assumptions on (d, q) (see the details in the proof of Lemma 3.4, case (a)), or G is of nearly simple type (part (e) of [NP2, Theorem 3.1]) and hence is nonsolvable, or G is of extension field type

(part (c)(ii) of [NP2, Theorem 3.1]), that is to say, G is conjugate to a subgroup of $\operatorname{GL}(d/c, q^c).c$ for some prime divisor c of $\operatorname{gcd}(d, e)$.

We now complete the proof of Theorem B for classical groups of Lie type.

Proposition 4.2. Theorem B holds for all classical finite simple groups of Lie type.

Proof. (1): Assume first that S = PSL(d, q) with $d \ge 2$.

Suppose first that d = 2 and $q \leq 7$. Theorem B holds for the groups $PSL(2,4) \cong PSL(2,5) \cong A_5$ by Proposition 2.1. Also, for S = PSL(2,7), the maximal subgroups of order divisible by 7 have order 21, so $\langle x, y \rangle = S$ whenever $x, y \in S$ with |x| = 2 and |y| = 7.

Next we consider S = PSL(2, q) with $q \ge 8$. Take a = (q + 1)/kand b = (q - 1)/k, where k = gcd(q - 1, 2). Then $a \ge 5$, $b \ge 4$ and $a, b \in oe(S)$ (see Theorems II.8.3 and II.8.4 in [H]). The classification of the subgroups of PSL(2, q) (see Theorem II.8.28 in [H]) implies that $\langle x, y \rangle = S$ whenever $x, y \in S$ with |x| = a and |y| = b.

Suppose next that S = PSL(3, q). The group $PSL(3, 2) \cong PSL(2, 7)$ has been dealt with already. If S = PSL(3, q) with q = 3 or 4, then taking $x, y \in S$ with (|x|, |y|) = (13, 2) or (7, 5), respectively, we find that $S = \langle x, y \rangle$ (see [ATLAS, pp. 13,23]). Hence we may assume that $q \ge 5$. By Proposition 3.2, there exists $a \in Lbpd(q, 3)$. If $q = 3^k$, then k > 1 and we take $b \in ppd(q, 2)$ which, by Theorem 3.1, is nonempty. On the other hand, if q is not a power of 3, then we choose b = p (recall that q is a power of p). As in the proof of Proposition 3.7, we work with $\hat{S} = SL(3,q) \le GL(3,q)$ in order to apply Lemma 4.1. (We shall often do this throughout the proof without further reference.) Let $\hat{x}, \hat{y} \in \hat{S}$, with $|\hat{x}|$ a multiple of a and $|\hat{y}|$ a multiple of b, and let $X = \langle \hat{x}, \hat{y} \rangle$. Since $\hat{x} \in X, X$ is an irreducible subgroup of GL(3,q), and since $\hat{y} \in X, |X|$ does not divide $|GL(1,q^3).3|$. Hence, by Lemma 4.1, X is nonsolvable. It follows that $\langle x, y \rangle$ is nonsolvable for every $x, y \in S$ with |x| = a and |y| = b.

Thus we may assume that $d \ge 4$. For these groups S, Theorem B follows from Proposition 3.7(1), unless (d,q) = (6,2). For $S = PSL(6,2) \cong$ GL(6,2), consider $a = 31 \in Lbpd(2,5)$ and $b = 7 \in bppd(2,3)$, and note that $a, b \in oe(S)$. Let $x, y \in GL(6,2)$ with |x| = 31 and |y| = 7, and set $X = \langle x, y \rangle$. If X is reducible on V = V(6,2), then, since $x \in X$, X acts irreducibly on some X-composition factor U of V of dimension 5 and $X^U = \langle x^U, y^U \rangle \leq GL(5,2)$. By Remark 3.5(2), $|x^U| = |x| = 31$ and $|y^U| = |y| = 7$, and hence X^U is nonsolvable by Lemma 3.6 applied with $\delta = 2, j = 0$. Consequently, also X is nonsolvable. If X is irreducible, then by Lemma 4.1 (applied with d = 6, e = 5), X is nonsolvable, since gcd(6, 5) = 1.

(2): Next let S = PSp(d,q)' with $d \ge 4$ and d even (noting that $PSp(2,q) \cong PSL(2,q)$ has been dealt with above).

First consider d = 4. The result for $PSp(4, 2)' \cong A_6$ follows from Proposition 2.1. If S = PSp(4, 3), we have $5, 9 \in oe(S)$ and no maximal subgroup of S contains elements of both orders 5 and 9 (see [ATLAS, p. 26]). Hence $S = \langle x, y \rangle$ for all $x, y \in S$ with |x| = 5 and |y| = 9. If S = PSp(4, 4), then $5, 17 \in oe(S)$ and (see [ATLAS, p. 44]) the only maximal subgroups of S containing an element of order 17 are of the form PSL(2, 16) : 2, and every subgroup of such a group of order divisible by both 17 and 5 contains PSL(2, 16). Hence $\langle x, y \rangle$ is nonsolvable whenever $x, y \in S$ with |x| = 17 and |y| = 5.

So suppose that $q \geq 5$ and take $a \in \text{Lbpd}(q, 4)$, which is nonempty by Proposition 3.2, and $b = (q^2 - 1)/\gcd(2, q - 1)$. Note that $a, b \in oe(S)$, since $\text{PSp}(2, q^2) \cong \text{PSL}(2, q^2)$ is isomorphic to a subgroup of S. We consider $\hat{S} = \text{Sp}(4, q) \leq \text{GL}(4, q)$. Let $\hat{x}, \hat{y} \in \hat{S}$, with $|\hat{x}|$ a multiple of a and $|\hat{y}|$ a multiple of b, and let $X = \langle \hat{x}, \hat{y} \rangle$. Then \hat{x} is an Lbpd(q, 4)element of X, and in particular \hat{x} acts irreducibly on V(4, q). Hence X is an irreducible subgroup of GL(4, q). By Lemma 4.1, either X is nonsolvable or X is (conjugate to) a subgroup of $\text{GL}(2, q^2).2$. Assume the latter. Then $\langle \hat{x}^2, \hat{y}^2 \rangle \leq \text{GL}(2, q^2), |\hat{x}^2|$ is a multiple of the large primitive divisor a of $q^4 - 1$ and $|\hat{y}^2|$ is a multiple of $b/\gcd(b, 2) \geq 6$. Hence, by the classification of the subgroups of $\text{PSL}(2, q^2)$ (see [H, Theorem II.8.27]), we conclude that $\langle \hat{x}^2, \hat{y}^2 \rangle \geq \text{SL}(2, q^2)$ and hence, in particular, $X = \langle \hat{x}, \hat{y} \rangle$ is nonsolvable.

By part (2) of Proposition 3.7, we are left with the following cases: PSp(6,2) and PSp(8,2). If S = PSp(6,2), then $15, 7 \in oe(S)$, and for all $x, y \in S$ with |x| = 15, |y| = 7, the group $X = \langle x, y \rangle$ is nonsolvable. This is seen as follows: one checks from [ATLAS, p. 46] that each maximal subgroup of S of order divisible by 35 is isomorphic to S_8 . Moreover, a subgroup of S_8 generated by two elements, of orders 15 and 7, contains A_8 , and in fact is equal to A_8 . Thus, X = S or $X \cong A_8$.

Finally, let $S = PSp(8, 2) \cong Sp(8, 2) < GL(8, 2)$. Then $17, 7 \in oe(S)$. By [ATLAS, p. 123], each maximal subgroup of S of order divisible by $17 \cdot 7$ is isomorphic to $P\Omega^{-}(8, 2) : 2$. By [ATLAS, p. 89], each maximal subgroup of $P\Omega^{-}(8, 2)$ of order divisible by 17 is isomorphic to PSL(2, 16) : 2, and so contains no elements of order 7. Thus $\langle x, y \rangle$ is nonsolvable whenever $x, y \in S$ with |x| = 17 and |y| = 7.

(3): Assume now that S = PSU(d, q), with $d \ge 3$ and d odd.

By Proposition 3.7, we need only consider the cases S = PSU(3, q)with $q \ge 3$ (since $PSU(3, 2) \cong 3^2 : Q_8$ is solvable), and S = PSU(5, 2). By [ATLAS, pp. 72-73], if S = PSU(5, 2), then 11, 15 $\in oe(S)$ and each maximal subgroup of S of order divisible by 11 is isomorphic to PSL(2, 11) and contains no elements of order 15. Thus if $x, y \in S$ with |x| = 11 and |y| = 15, then $\langle x, y \rangle = S$.

Therefore we may assume that S = PSU(3,q) with $q \ge 3$. By [ATLAS, pp. 14, 34], if S = PSU(3, q) with $q \in \{3, 5\}$, then $7, 8 \in oe(S)$ and each maximal subgroup of S of order divisible by 7 is isomorphic to PSL(2,7) if q = 3, and to A_7 if q = 5, and neither of these groups contains an element of order 8. Thus if $x, y \in S$ with |x| = 7 and |y| = 8, then $\langle x, y \rangle = S$. So we may assume that $q \neq 2, 3, 5$. Then by Proposition 3.1, $Lbpd(q, 6) \neq \emptyset$. Let $a \in Lbpd(q, 6)$, and note that $a \in oe(S)$ by Remark 3.3. Also let b = p if $p \neq 3$, and b =(q-1)/2 if p=3 (recall that q is a power of p), and note that $b \in$ oe(S) since $PSU(2,q) \cong PSL(2,q)$ is isomorphic to a subgroup of S. Now let $\hat{S} = SU(3,q) \leq GL(3,q^2)$, and note that $a, b \in oe(\hat{S})$ and that $gcd(ab, |\mathbb{Z}(\hat{S})|) = 1$. Consider $\hat{x}, \hat{y} \in \hat{S}$ with $|\hat{x}|$ a multiple of a and $|\hat{y}|$ a multiple of b, and let $X = \langle \hat{x}, \hat{y} \rangle$. Since a is a primitive prime divisor of $(q^2)^3 - 1$, X is an irreducible subgroup of $GL(3, q^2)$. Thus by Lemma 4.1, either X is nonsolvable or X is a subgroup of $X_0 = \operatorname{GL}(1,q^3).3$. Suppose that $X \leq X_0$. If $p \neq 3$ then b = p does not divide $|X_0|$. Hence p = 3 and b = (q - 1)/2. However $\hat{x} \in X_0$ and $|\hat{x}|$ is a multiple of $a \in \text{Lbpd}(q, 6)$, so $|\hat{x}|$ does not divide $|X_0|$, a contradiction. Hence X is nonsolvable and, consequently, $\langle x, y \rangle$ is nonsolvable whenever $x, y \in S$ with |x| = a and |y| = b.

(4): Assume now that $S = P\Omega^{\circ}(d, q)$, with d odd and $d \ge 3$.

Since $P\Omega^{\circ}(2m + 1, 2^k) \cong PSp(2m, 2^k)$, for all m and k, we may assume that q is odd. Also since $P\Omega^{\circ}(3, q) \cong PSL(2, q)$ and $P\Omega^{\circ}(5, q) \cong PSp(4, q)$, we may assume that $d \ge 7$. In this case Theorem B follows from Proposition 3.7.

(5): Assume now that S = PSU(d, q), with d even.

Since $PSU(2,q) \cong PSL(2,q)$, we may assume that $d \ge 4$, and it follows by Proposition 3.7 that we only have to check dimensions d = 4 and d = 6.

Let S = PSU(4, q). Since $\text{PSU}(4, 2) \cong \text{PSp}(4, 3)$, we may assume that $q \ge 3$. For S = PSU(4, 3), it follows from [ATLAS, p. 52-53] that $7, 9 \in oe(S)$ and each maximal subgroup of S of order divisible by 7 is isomorphic to PSL(3, 4), PSU(3, 3) or A_7 , and hence contains no elements of order 9. Thus $\langle x, y \rangle = S$ whenever $x, y \in S$ with |x| = 7 and |y| = 9.

Thus we may assume that S = PSU(4, q) with $q \ge 4$. Then, by Theorem 3.1 and Proposition 3.2, both $Lbpd(q^2, 3)$ and bppd(q, 4) are nonempty. Let $a \in \text{Lbpd}(q^2, 3)$ and $b \in \text{bppd}(q, 4)$, and note that, by Remark 3.3, $a, b \in oe(S)$ (since $(q^4 - 1)(q^3 + 1)$ divides |S|). Consider $\hat{S} = SU(4,q) \leq GL(4,q^2)$ acting on the natural module $V = V(4,q^2)$. Observe that $gcd(ab, |\mathbb{Z}(\hat{S})|) = 1$ and hence $a, b \in oe(\hat{S})$. Let $\hat{x}, \hat{y} \in \hat{S}$ with $|\hat{x}|$ a multiple of a and $|\hat{y}|$ a multiple of b, and let $X = \langle \hat{x}, \hat{y} \rangle$. By Remark 3.5, there exists an X-composition factor U of V such that both \hat{x} and \hat{y} act nontrivially on U, with $\dim_{\mathbb{F}_{q^2}}(U) \geq 3$. Moreover \hat{x}^U is an Lbpd $(q^2, 3)$ -element and \hat{y}^U is an Lbpd $(q^2, 2)$ -element of X^U (since $b \geq 5$). Assume first that X acts reducibly on V. Then $\dim_{\mathbb{F}_{q^2}}(U) =$ 3, and hence, by Lemma 3.6 with $\delta = 1, j = 0$, the group X^U is nonsolvable. Thus also X is nonsolvable. Therefore we may assume that X is an irreducible subgroup of $GL(4, q^2)$. Then, by Lemma 4.1, X is nonsolvable, as $a \in Lbpd(q^2, 3)$ and the extension field case cannot occur because gcd(4,3) = 1.

Finally let S = PSU(6,q). If S = PSU(6,2), then by [ATLAS, pp. 39, 115], 7, 11 $\in oe(S)$, each maximal subgroup of S of order divisible by $7 \cdot 11$ is isomorphic to M_{22} , and M_{22} has no maximal subgroup of order divisible by 7 · 11. Hence $\langle x, y \rangle = S$ or $\langle x, y \rangle \cong M_{22}$ whenever $x, y \in S$ with |x| = 7 and |y| = 11. Thus we may assume that q > 2. Then, by Theorem 3.1 and Proposition 3.2, both $Lbpd(q^2, 5)$ and bppd(q, 6) are nonempty. Let $a \in Lbpd(q^2, 5)$ and $b \in \text{bppd}(q, 6)$. Since $(q^5 + 1)(q^3 + 1)$ divides |S|, it follows by Remark 3.3 that $a, b \in oe(S)$. Consider now $\hat{S} = SU(6, q) \leq GL(6, q^2)$ acting on the natural module $V = V(6, q^2)$. Since $gcd(ab, |Z(\hat{S})|) = 1$, it follows that $a, b \in oe(\hat{S})$. Let $\hat{x}, \hat{y} \in \hat{S}$ with $|\hat{x}|$ a multiple of a and $|\hat{y}|$ a multiple of b, and let $X = \langle \hat{x}, \hat{y} \rangle$. If X is an irreducible subgroup of $GL(6, q^2)$, then we conclude by Lemma 4.1 that X is nonsolvable, since $a \in \text{Lbpd}(q^2, 5)$ and the extension field case cannot occur because gcd(6,5) = 1. So we may assume that X acts reducibly on V. By Remark 3.5(2), there exists an X-composition factor U of V such that both \hat{x} and \hat{y} act nontrivially on U, with $\dim_{\mathbb{F}_{q^2}}(U) \geq 5$. It follows that $\dim_{\mathbb{F}_{q^2}}(U) = 5$. Moreover, \hat{x}^U is an $\mathrm{Lbpd}(q^2, 5)$ -element and \hat{y}^U is a bppd $(q^2, 3)$ -element of X^U . Hence by Lemma 3.6 with $\delta = 2, j = 0$, the group X^U is nonsolvable and so is X.

(6): Let $S = P\Omega^{-}(d, q)$, with d even and $d \ge 4$.

Since $P\Omega^{-}(4,q) \cong PSL(2,q^2)$ and $P\Omega^{-}(6,q) \cong PSU(4,q)$, we may assume that $d \geq 8$. Hence we are left, by Proposition 3.7, only with the cases $P\Omega^{-}(8,q)$, for $q \geq 2$, and $P\Omega^{-}(10,2)$.

Let $S = P\Omega^{-}(8, q)$. For $S = P\Omega^{-}(8, 2)$, it follows from [ATLAS, p. 89] that 7, 17 $\in oe(S)$ and each maximal subgroup of S of order divisible by 17 is isomorphic to PSL(2, 16) : 2, and hence contains no elements of order 7. Thus $\langle x, y \rangle = S$ whenever $x, y \in S$ with |x| = 7 and |y| = 17. So we may assume that q > 2. Then, by Theorem 3.1, both bppd(q, 8) and bppd(q, 6) are nonempty. Let $a \in bppd(q, 8)$ and $b \in$ bppd(q, 6). Since $(q^4 + 1)(q^6 - 1)$ divides |S|, it follows by Remark 3.3 that $a, b \in oe(S)$. Consider now $\hat{S} = \Omega^{-}(8, q) \leq GL(8, q)$. Since $gcd(ab, |\mathbb{Z}(\hat{S})|) = 1$, it follows that $a, b \in oe(\hat{S})$. Let $\hat{x}, \hat{y} \in \hat{S}$ with $|\hat{x}|$ a multiple of a and $|\hat{y}|$ a multiple of b, and let $X = \langle \hat{x}, \hat{y} \rangle$. Then X is nonsolvable by Lemma 3.6 with $\delta = 2, j = 0$. Hence $\langle x, y \rangle$ is nonsolvable whenever $x, y \in S$ with |x| = a and |y| = b.

Finally, let $S = P\Omega^{-}(10, 2) \leq GL(10, 2)$. Consider $11 \in bppd(2, 10)$ and $17 \in bppd(2, 8)$, and note that $11, 17 \in oe(S)$. Then by Lemma 3.6 with $\delta = 2, j = 0$, the subgroup $\langle x, y \rangle$ is nonsolvable whenever $x, y \in S$ with |x| = 11 and |y| = 17.

(7): Let $S = P\Omega^+(d,q)$, with d even.

Since S is not nonabelian simple if d = 2 and d = 4, and $P\Omega^+(6, q) \cong$ PSL(4, q), we may assume that $d \ge 8$. Then by Proposition 3.7, we only need to consider the cases $P\Omega^+(8, q)$, for $q \ge 2$, and $P\Omega^+(10, 2)$.

Let $S = P\Omega^+(8, q)$. For $S = P\Omega^+(8, 2)$, it follows from [ATLAS, p. 85] that 7, 15 $\in oe(S)$ and each maximal subgroup of S of order divisible by 35 is isomorphic to PSp(6, 2), $2^6 : A_8$ or A_9 . We have shown above that a subgroup of PSp(6, 2) which contains elements of orders 15 and 7 is nonsolvable. Also, in the last paragraph of the proof of Proposition 2.1, we saw that a subgroup of A_9 containing elements of orders 7 and 5 is A_d for some $d \ge 7$. Thus also a subgroup of $2^6 : A_8$ containing such elements has a composition factor A_7 or A_8 . Hence $\langle x, y \rangle$ is nonsolvable whenever $x, y \in S$ with |x| = 7 and |y| = 15. Thus we may assume that q > 2.

For $S = P\Omega^+(8,3)$, it follows from [ATLAS, pp. 140–141 and 54–55] that 7, 15 $\in oe(S)$, each maximal subgroup of S of order divisible by 7 is isomorphic to $P\Omega^{\circ}(7,3)$, $P\Omega^+(8,2)$ or 2 · PSU(4,3) · 2², and the last of these groups contains no elements of order 15. Let $x, y \in S$ with |x| = 7 and |y| = 15 and let $X = \langle x, y \rangle$. Assume that X < Sand let M be a maximal subgroup of S containing X. Then M is $P\Omega^{\circ}(7,3)$ or $P\Omega^+(8,2)$. Suppose first that $M = P\Omega^{\circ}(7,3)$. Observe that 5, $7 \in oe(M)$, $7 \in bppd(3,6)$ and $5 \in bppd(3,4)$. By Lemma 3.6 with $\delta = 2, j = 1$, every subgroup of $\hat{M} = \Omega^{\circ}(7,3) \leq GL(7,3)$ containing elements of orders 7 and 5 is nonsolvable. Hence in this case X is nonsolvable. On the other hand, if $M = P\Omega^+(8,2)$, then by the previous paragraph any subgroup of M containing elements of orders 7 and 15 is nonsolvable. Thus also in this case X is nonsolvable. So we may assume that $q \ge 4$.

Next we deal with $S = P\Omega^+(8,5)$. Notice that both $7 \in \text{bppd}(5,6)$ and $13 \in \text{bppd}(5,4)$ belong to oe(S). Let $X = \langle \hat{x}, \hat{y} \rangle \leq \Omega^+(8,5) \leq$ GL(8,5), where $|\hat{x}|$ is divisible by 7 and $|\hat{y}|$ is divisible by 13. We shall prove that X is nonsolvable. Now \hat{x} is a bppd(5,6)-element of X (even if not a large one), and \hat{y} is an Lbpd(5,4)-element of X. Assume first that X acts reducibly on V = V(8,5). By Remark 3.5(2) there exists an X-composition factor U of V of dimension $d_0 = \dim_{\mathbb{F}_5}(U) \geq 6$ such that $|\hat{x}^U|_7 \geq 7$ and $|\hat{y}^U|_{13} \geq 13$. Thus $d_0 \in \{6,7\}$, and by Lemma 3.6 the group X^U is nonsolvable in both cases $d_0 = 6$ (taking $\delta = 2, j = 0$) and $d_0 = 7$ (taking $\delta = 2, j = 1$).

Thus we may assume that X is an irreducible subgroup of GL(8, 5). Observe that here we cannot use Lemma 3.6, or even Lemma 4.1, because Lbpd(5,6) = \emptyset . So we have to apply [NP2, Theorem 3.1] directly (with d = 8, e = 6), checking each of the cases (a)–(e) of that theorem. If X is in case (a) or (e), then X is nonsolvable. Since $7 \cdot 13$ divides |X|, neither of the cases (b) nor (d) holds for X. So we may assume that case (c) holds for X, which implies that X is (isomorphic to) a subgroup of GL(4, 5²) \cdot 2. Hence $X_0 := \langle \hat{x}^2, \hat{y}^2 \rangle \leq \text{GL}(4, 5^2)$. Observe that X_0 acts irreducibly on $V(4, 5^2)$, as it acts irreducibly on V. Now, $7 \in \text{Lbpd}(5^2, 3)$, and hence we can apply Lemma 4.1. Note that X_0 is not a subgroup of GL(2, 5⁴) \cdot 2, because $|\text{GL}(2, 5^4) \cdot 2|$ is not divisible by 7. Therefore we conclude that X_0 is nonsolvable, and hence also X is nonsolvable.

Now consider $S = P\Omega^+(8, q)$ with $q \neq 2, 3, 5$. By Theorem 3.1 and Proposition 3.2, both Lbpd(q, 6) and bppd(q, 4) are nonempty. Let $a \in Lbpd(q, 6)$ and $b \in bppd(q, 4)$. Since $(q^6 - 1)(q^4 - 1)$ divides |S|, it follows by Remark 3.3 that $a, b \in oe(S)$. Consider now $\hat{S} = \Omega^+(8, q) \leq$ GL(8, q) acting on on V = V(8, q). Since $gcd(ab, |\mathbb{Z}(\hat{S})|) = 1$, it follows that $a, b \in oe(\hat{S})$. Let $\hat{x}, \hat{y} \in \hat{S}$ with $|\hat{x}|$ a multiple of a and $|\hat{y}|$ a multiple of b, and let $X = \langle \hat{x}, \hat{y} \rangle$. Now \hat{x} is an Lbpd(q, 6)-element and \hat{y} is a bppd(q, 4)-element of X. By Remark 3.5(2), there is an X-composition factor U of V with $d_0 = \dim_{\mathbb{F}_q}(U) \in \{6, 7, 8\}$ such that $|\hat{x}^U|_a = |\hat{x}|_a \geq a$ and $|\hat{y}^U|_b = |\hat{y}|_b \geq b$. It follows by Lemma 3.6 that X^U , and hence also X, is nonsolvable if $d_0 = 6$ (taking $\delta = 2, j =$ 0), and also if $d_0 = 7$ (taking $\delta = 2, j = 1$). Thus we may assume that $d_0 = 8$, that is, X is irreducible on V. Then by Lemma 4.1, either X is nonsolvable, as required, or X is (isomorphic to) a subgroup of $GL(4, q^2).2$. Suppose that $X \leq GL(4, q^2).2$ and consider $X_0 =$ $\langle \hat{x}^2, \hat{y}^2 \rangle \leq \operatorname{GL}(4, q^2)$. By the above argument applied to X_0 , we may assume that X_0 acts irreducibly on $V = V(4, q^2)$. It follows then, by Lemma 4.1, that either X_0 is nonsolvable or X_0 is (isomorphic to) a subgroup of $\operatorname{GL}(2, q^4).2$. However, since $a \in \operatorname{Lbpd}(q, 6)$, a is coprime to $|\operatorname{GL}(2, q^4).2| = 2q^4(q^4 - 1)(q^8 - 1)$, and we conclude that X_0 , and hence also X, is nonsolvable.

Finally let $S = P\Omega^+(10, 2)$. By [ATLAS, p. 147], 17, $31 \in oe(S)$ and no maximal subgroup of S has order divisible by both 17 and 31. Hence $\langle x, y \rangle = S$ whenever $x, y \in S$ with |x| = 17 and |y| = 31.

We now prove Theorem B for the exceptional finite simple groups of Lie type. We make use of five papers. The first is the paper [FS] of Feit and Seitz. They prove, in [FS, Theorem 3.1], the existence of certain self-centralizing cyclic maximal tori in simple groups of Lie type. The second is the paper [W] of Wiegel. He gives, in [W, Table 1], a list of cyclic maximal tori in exceptional groups of Lie type, with some small cases excluded, and in [W, Section 4] he determines the maximal subgroups containing these tori, for each such group. We were kindly informed by Frank Lübeck, in a letter, that Weigel's list of cyclic maximal tori is correct without the extra conditions on qor k, with only one exception: namely the group $G_2(2)$, which has elements of order $q^2 - q + 1 = 3$ in several classes of maximal tori. We shall refer to [L] concerning this important information. The third is the paper [GM] of Guralnick and Malle, which is still in preparation. The authors kindly informed us that their paper contains important information about maximal subgroups of $E_7(2)_{sc}$ and $E_7(3)_{sc}$. The forth is the paper |MT| of Moretò and Tiep. We use |MT, Lemma 2.3|, in a slightly 'extended' form, which was kindly approved by Pham Tiep. They prove, in Lemma 2.3, that each exceptional simple group of Lie type contains elements s_1 and s_2 of prime orders p_1 and p_2 , respectively, such that their centralizers have suitable orders. The 'extended' version of this lemma states not only that such elements exist, but also that the centralizers of every element of order p_1 or p_2 are of the same suitable orders. Finally, the fifth paper is the paper [GK] by Guralnick and Kantor, which provides in [GK, Proposition 6.2] information concerning elements of the groups excluded in [W] and of the sporadic subgroups, contained in a unique or small number of maximal subgroups.

We recall that every finite simple group of Lie type occurs as a composition factor of the group of fixed points G_F , under a Frobenius map $F: G \to G$ of a connected reductive algebraic group G over the algebraic closure $\overline{\mathbb{F}_q}$ of a field \mathbb{F}_q of order q. If we choose G to be simply connected, then every finite simple exceptional group of Lie type is a quotient $G_F/\mathbb{Z}(G_F)$. Moreover, $\mathbb{Z}(G_F) = 1$ unless G is of type E_6 , 2E_6 or E_7 . The following facts are used repeatedly.

Lemma 4.3. Let $S = G_F/\mathbb{Z}(G_F)$, q be as above, and suppose that S has a cyclic maximal torus T of order divisible by a prime p, such that |S:T| is coprime to p, and $C_S(y) = T$ for $y \in T$ of order $|T|_p$. Then for each $x \in S$ with |x| = |T|, the subgroup $\langle x \rangle$ is conjugate to T in S, and in particular it is a maximal torus of order |T|.

Proof. By assumption T has a unique Sylow p-subgroup, say $P = \langle y \rangle$, and P is a Sylow p-subgroup of S. Let $x \in S$ with |x| = |T|. Then $\langle x \rangle$ contains a subgroup P_0 of order |P|, so by Sylow's Theorem $P_0^g = P$ for some $g \in S$. Then $\langle x \rangle^g \leq C_S(y)$ which by assumption is equal to T. It follows that $\langle x \rangle^g = T$.

The following is an immediate corollary of Lemma 4.3.

Corollary 4.4. Let S, T be as in Lemma 4.3, let $a = |T| \in oe(S)$, and suppose that $b \in oe(S)$ is such that each maximal subgroup of S containing T has order coprime to b. Then for each $x, y \in S$ with |x| = a and |y| = b, the group $\langle x, y \rangle = S$ and, in particular, is non-solvable.

We now prove

Proposition 4.5. Theorem B holds for all exceptional finite simple groups of Lie type.

Proof. In the following, we denote by $\pi(n)$ the set of prime divisors of the positive integer n and by $\Phi_k(x)$ the k-th cyclotomic polynomial. We consider the exceptional groups, beginning with those of smallest Lie rank. Our basic proof strategy is to choose $a, b \in oe(S)$, where possible, so that the hypotheses of Corollary 4.4 hold. Then we have immediately that Theorem B holds for a, b. We call this 'the standard argument'.

(1): Let $S = {}^{2}B_{2}(q)$, with $q = 2^{2n+1}$ and $n \ge 1$.

Then $|S| = q^2(q-1)(q^2+1)$. Write $r = 2^{n+1}$, so $q^2 + 1 = (q+r+1)(q-r+1)$. Since ${}^2B_2(2)$ is a Frobenius group of order 20, the field order q is at least 8.

Let $p \in \text{ppd}(q, 4)$, which is nonempty. Since $q^2+1 = (q+r+1)(q-r+1)$, the prime p divides $a := q + \varepsilon r + 1$ where $\varepsilon = \pm 1$. By [FS, Theorem 3.1], S has a cyclic maximal torus T of order a and we note that |S:T| is coprime to p. By comparing orders, we deduce from the 'extended'

[MT, Lemma 2.3] that $C_S(y) = T$ for $y \in T$ of order $|T|_p$. By [S], the only maximal subgroup of S containing T is its normaliser, of order 4a. Then the standard argument applies for a and any $b \in \pi(q - \varepsilon r + 1)$, since $b \neq 2$ and $\gcd(q + r + 1, q - r + 1) = 1$, so b does not divide 4a.

(2): Let $S = {}^{2}G_{2}(q)'$, with $q = 3^{2n+1}$ and $n \ge 1$.

Then $|S| = q^3(q-1)(q^3+1)$. Write $r = 3^{n+1}$, so $q^3 + 1 = (q+1)(q+r+1)(q-r+1)$. Since ${}^2G_2(3)' \cong \mathrm{PSL}(2,8)$ has been already treated in Proposition 4.2, we may assume that $q \ge 27$.

Let $p \in \text{ppd}(q, 6)$, which is nonempty. Then p divides $a = q + \varepsilon r + 1$ where $\varepsilon = \pm 1$. By [FS, Theorem 3.1], S has a cyclic maximal torus T of order a and we note that |S : T| is coprime to p. As in (1), $C_S(y) = T$ for $y \in T$ of order $|T|_p$, and by [K] and [LN], the only maximal subgroup of S containing T is its normaliser, of order 6a. Let $b \in \pi(q - \varepsilon r + 1)$. Then $b \in oe(S)$, but b does not divide 6a, because $b \neq 2,3$ and gcd(q + r + 1, q - r + 1) = 1. Thus the standard argument applies.

(3a): Let $S = {}^{2}F_{4}(2)'$, the Tits group.

Then $|S| = 2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$. By [ATLAS3], $13, 10 \in oe(S)$ and each maximal subgroup of S of order divisible by 130 is isomorphic to PSL(2, 25), which contains no elements of order 10. Hence $\langle x, y \rangle = S$ whenever $x, y \in S$ with |x| = 13 and |y| = 10.

(3b): Let $S = {}^{2}F_{4}(q)$, with $q = 2^{2n+1}$ and $n \ge 1$.

Then $|S| = q^{12}(q^6 + 1)(q^4 - 1)(q^3 + 1)(q - 1)$. Write $r = 2^{n+1}$, so

$$\frac{q^{6}+1}{q^{2}+1} = q^{4}-q^{2}+1 = (q^{2}+rq+q+r+1)(q^{2}-rq+q-r+1)$$

and $gcd(q^2+rq+q+r+1, q^2-rq+q-r+1)$ divides $(q^2+q+1)(q-1) = q^3 - 1$.

Let $p \in ppd(q, 12)$, which is nonempty. Then p divides $a := q^2 + \varepsilon rq + q + \varepsilon r + 1$, where $\varepsilon = \pm 1$, and |S|/a is coprime to p. By [FS, Theorem 3.1], S has a cyclic maximal torus T of order a, and arguing as in (1), the hypotheses of Lemma 4.3 hold for T. By [M], the only maximal subgroup of S containing T is $N_S(T)$ of order 12a. Let $b \in ppd(q, 6)$, which is nonempty. Then b divides $q^3 + 1$ and hence $b \in oe(S)$ and b does not divide a. Also $b \geq 7$ and so b does not divide 12a. It follows that b does not divide 12a, and hence, the standard argument applies.

(4): Let $S = G_2(q)$, with q > 2.

Then $|S| = q^6(q^6 - 1)(q^2 - 1)$. Since $G_2(2)' \cong \text{PSU}(3,3)$ has been already treated, we may assume that $q \geq 3$. First we deal with $G_2(3)$ and $G_2(4)$, which were excluded in [W].

Let $S = G_2(q)$ with q = 3 or 4. Then by [ATLAS, pp. 60–61,97], $a, 13 \in oe(S)$, where a = 7 if q = 3 and a = 5 if q = 4, and each

maximal subgroup M of S of order divisible by 13a is isomorphic to PSL(2, 13) if q = 3, and to PSU(3, 4) : 2 if q = 4. In either case, the derived group M' is generated by any pair of its elements with one of order a and the other of order 13. Hence, if $x, y \in S$ with |x| = a and |y| = 13, then $\langle x, y \rangle$ is nonsolvable.

Let, now, $S = G_2(q)$, with $q \ge 5$ and let $p \in \text{ppd}(q, 6)$, which is nonempty. Since $q^3 + 1 = (q^2 - q + 1)(q + 1)$, p divides $a := q^2 - q + 1 = \Phi_6(q)$ and |S|/a is coprime to p. By [W, Table I], S has a cyclic maximal torus T of order a, and arguing as in (1), the hypotheses of Lemma 4.3 hold for T. By [W, Section 4], each maximal subgroup Mof S containing T is isomorphic to SU(3, q).2 and hence $|M| = 2q^3(q^3 + 1)(q^2 - 1)$. Let $b \in \text{ppd}(q, 3)$, which is nonempty. Then $b \in oe(S)$ and gcd(b, 2q) = 1. Since b divides $q^3 - 1$, also $gcd(b, q^3 + 1) = 1$, and it follows that b does not divide |M|. Hence, the standard argument applies.

(5): Let $S = {}^{3}D_{4}(q)$, with $q \ge 2$.

Then $|S| = q^{12}(q^8 + q^4 + 1)(q^6 - 1)(q^2 - 1)$, where $q^8 + q^4 + 1 = (q^4 - q^2 + 1)(q^4 + q^2 + 1)$.

Let $p \in ppd(q, 12)$, which is nonempty. Since $q^6 + 1 = (q^4 - q^2 + 1)(q^2 + 1)$, p divides $a = q^4 - q^2 + 1 = \Phi_{12}(q)$ and |S|/a is coprime to p. By [W, Table I], there exist a cyclic maximal torus T of S of order a, and arguing as in (1), the hypotheses of Lemma 4.3 hold for T. By [W, Section 4], the only maximal subgroup of S containing T is $N_S(T)$ of order 4a. Let $b \in ppd(q, 6)$ if $q \neq 2$ and let b = 7 if q = 2. Then $b \neq 2, b \in oe(S)$ and $gcd(b, q^4 - q^2 + 1) = 1$, since b divides $q^6 - 1$ and $q^4 - q^2 + 1$ divides $q^6 + 1$. Thus b does not divide 4a, and hence, the standard argument applies.

(6): Let $S = F_4(q)$, with $q \ge 2$.

Then $|S| = q^{24}(q^{12} - 1)(q^8 - 1)(q^6 - 1)(q^2 - 1)$. First we deal with $F_4(2)$ and $F_4(3)$, which were excluded in [W].

Let first $S = F_4(2)$. Notice that $13, 17 \in oe(S)$ and by [GK, Proposition 6.2], the maximal subgroups of S of order divisible by 17 are isomorphic to PSp(8, 2), which is of order not divisible by 13. Hence, the standard argument applies.

Let now $S = F_4(3)$. Notice that both $73 \in \text{ppd}(3, 12)$ and $41 \in \text{ppd}(3, 8)$ belong to oe(S). By [GK, Proposition 6.2], the maximal subgroups of S of order divisible by 73 are isomorphic to ${}^{3}D_4(3).3$, which is of order not divisible by 41 (see [ATLAS, p. 241]). Hence, the standard argument applies.

Let, finally, $S = F_4(q)$, with $q \ge 4$ and let $p \in \text{ppd}(q, 12)$. It follows, as in (5), that p divides $a = q^4 - q^2 + 1 = \Phi_{12}(q)$ and |S|/a is coprime to p. By [W, Table I], S has a cyclic maximal torus T of

order a and arguing as in (1), the hypotheses of Lemma 4.3 hold for T. By [W, Section 4], every maximal subgroup M of S containing T is isomorphic to ${}^{3}D_{4}(q).3$ and hence $|M| = 3q^{12}(q^{8} + q^{4} + 1)(q^{6} - q^{6})$ 1) $(q^2 - 1)$. Let $b \in ppd(q, 8)$, which is nonempty. Then $b \in oe(S)$, but b does not divide |M|, since gcd(b, 3q) = 1, $gcd(b, (q^6 - 1)(q^2 - 1)) = 1$ and $gcd(q^8 + q^4 + 1, b)$ divides $gcd(q^{12} - 1, q^8 - 1) = q^4 - 1$, so also $gcd(q^8 + q^4 + 1, b) = 1$. Hence, the standard argument applies.

(7): Let $S = {}^{2}E_{6}(q)$, with $q \ge 2$. Then $|S| = \frac{1}{d}q^{36}(q^{12} - 1)(q^{9} + 1)(q^{8} - 1)(q^{6} - 1)(q^{5} + 1)(q^{2} - 1)$, and $d = \gcd(3, q+1)$. Moreover, $S = \hat{S}/\mathbb{Z}(\hat{S})$, where $\hat{S} = G_F$, with G a simply connected algebraic group of exceptional type ${}^{2}E_{6}$ and $|\mathbb{Z}(S)| = d.$

Let $p \in ppd(q, 18)$, which is nonempty. Since $q^9 + 1 = (q^3 + 1)(q^6 - q^3)$ $q^{3} + 1$, p divides $a = q^{6} - q^{3} + 1$. By [W, Table 1] and [L], \hat{S} has a cyclic maximal torus T of order a and arguing as in (1), the hypotheses of Lemma 4.3 hold for T. Note that $\mathbb{Z}(\hat{S}) \leq T$.

It follows by [W, Section 4] for $q \ge 4$ and by [GK, Theorem 6.2] for q = 2, 3, that every maximal subgroup M of \hat{S} containing T is isomorphic to $PSU(3, q^3)$.3 and hence $|M| = \frac{3}{d}q^9(q^9+1)(q^6-1)$.

Let $b \in ppd(q, 12)$, which is nonempty, and note that $b \ge 12+1 = 13$. Then b divides $q^6 + 1$ and $b \neq 3$. Hence $b \in oe(\hat{S})$ and b does not divide |M|. Thus, by the standard argument, if $x \in \hat{S}$ is of order a and $y \in \hat{S}$ is of order b, then $\langle x, y \rangle = \hat{S}$.

If d = 1, then Theorem B holds for $S = \hat{S}$. So suppose that d = 3. Then $q \equiv -1 \pmod{3}$, $a = q^6 - q^3 + 1 \equiv 3 \pmod{9}$, gcd(3, b) = 1 and since $\mathbb{Z}(\hat{S}) \leq \langle x \rangle$ for each $x \in \hat{S}$ of order a, it follows that 3 divides a. Thus $a/3, b \in oe(S)$. Let $z = \hat{z}\mathbb{Z}(\hat{S})$ be an arbitrary element of S of order a/3 and let $w = \hat{w}\mathbb{Z}(\hat{S})$ be an arbitrary element of S of order b, where \hat{z}, \hat{w} are elements of \hat{S} . Since gcd(a/3,3) = gcd(b,3) = 1 and $|\mathbb{Z}(\hat{S})| = 3$, we may always choose \hat{z} of order a and \hat{w} of order b. Since, as shown above, $\langle \hat{z}, \hat{w} \rangle = \hat{S}$, it follows that $\langle z, w \rangle = S$.

(8): Let $S = E_6(q)$, with $q \ge 2$. Then $|S| = \frac{1}{d}q^{36}(q^{12} - 1)(q^9 - 1)(q^8 - 1)(q^6 - 1)(q^5 - 1)(q^2 - 1)$, where $d = \gcd(3, q - 1)$. Moreover, $S = \hat{S}/\mathbb{Z}(\hat{S})$, where $\hat{S} = G_F$, with G a simply connected algebraic group of exceptional type E_6 and $|\mathbb{Z}(S)| = d.$

Let $p \in ppd(q, 9)$, which is nonempty. Since $q^9 - 1 = (q^3 - 1)(q^6 + q^3 + q^3)$ 1), p divides $a = q^6 + q^3 + 1$. By [W, Table 1], \hat{S} has a cyclic maximal torus T of order a, and arguing as in (1), the hypotheses of Lemma 4.3 hold for T. In particular, $\mathbb{Z}(\hat{S}) < T$. By [W, Section 4], every maximal

subgroup M of \hat{S} containing T is isomorphic to $SL(3, q^3).3$ and hence $|M| = 3q^9(q^9 - 1)(q^6 - 1).$

Let $b \in \operatorname{ppd}(q, 12)$, which is nonempty, and note that $b \ge 12+1 = 13$. Then $b \in oe(\hat{S})$, b divides $q^6 + 1$ and $b \ne 3$. It follows that b does not divide |M|. Hence, by the standard argument, if $x \in \hat{S}$ is of order a and $y \in \hat{S}$ is of order b, then $\langle \hat{x}, \hat{y} \rangle = \hat{S}$.

If d = 1, then Theorem B holds for $S = \hat{S}$. So suppose that d = 3. Then it follows using the same proof as for ${}^{2}E_{6}(q)$ that if $z = \hat{z}\mathbb{Z}(\hat{S})$ has order a/3 and $w = \hat{w}\mathbb{Z}(\hat{S})$ has order b, where \hat{z}, \hat{w} are elements of \hat{S} , then $\langle z, w \rangle = S$.

(9): Let $S = E_7(q)$, with $q \ge 2$. Then

$$|S| = \frac{1}{d}q^{63} \prod_{i \in I} (q^i - 1), \quad \text{with} \quad I = \{2, 6, 8, 10, 12, 14, 18\}$$

where $d = \gcd(2, q - 1)$. Moreover, $S = \hat{S}/\mathbb{Z}(\hat{S})$, where $\hat{S} = G_F$, with G a simply connected algebraic group of exceptional type E_7 and $|\mathbb{Z}(\hat{S})| = d$.

Let $p \in \text{ppd}(q, 18)$, which is nonempty. Since $q^9 + 1 = (q^3 + 1)(q^6 - q^3 + 1)$, p divides $a = (q + 1)(q^6 - q^3 + 1)$. By [W, Table 1] and [L], S has a cyclic maximal torus T of order a, and arguing as in (1), the hypotheses of Lemma 4.3 hold for T. In particular, $\mathbb{Z}(\hat{S}) \leq T$.

By [W, Section 4] for $q \ge 4$ and by [GM, Proposition 2.11] for q = 2, 3, every maximal subgroup M of \hat{S} containing T is isomorphic to $(Z_{q+1} \cdot {}^2E_6(q)).2$ and hence $|M| = \frac{2}{d_1}(q+1)q^{36}(q^{12}-1)(q^9+1)(q^8-1)(q^6-1)(q^5+1)(q^2-1)$, where $d_1 = \gcd(3, q-1)$.

Let $b \in \text{ppd}(3, 14)$, which is nonempty. Then $b \in oe(\hat{S})$, b divides $q^7 + 1$ and $b \neq 2$. Since $\text{gcd}(q^9 + 1, q^7 + 1)$ divides $q^2 - 1$, it follows that b does not divide |M|. Hence, by the standard argument, if $x \in \hat{S}$ is of order a and $y \in \hat{S}$ is of order b, then $\langle \hat{x}, \hat{y} \rangle = \hat{S}$.

If d = 1, then Theorem B holds for $S = \hat{S}$. So suppose that d = 2. Then q is odd and since $\mathbb{Z}(\hat{S}) \leq \langle x \rangle$ for each $x \in \hat{S}$ of order a, it follows that a is even, and $a/2, b \in oe(S)$. Let $z = \hat{z}\mathbb{Z}(\hat{S})$ be an arbitrary element of S of order a/2 and let $w = \hat{w}\mathbb{Z}(\hat{S})$ be an arbitrary element of S of order b, where \hat{z}, \hat{w} are elements of \hat{S} . Since gcd(b, 2) = 1 and $|\mathbb{Z}(\hat{S})| = 2$, it follows that we may choose \hat{w} of order b. Now consider \hat{z} . Let $L = \langle \hat{z}, \mathbb{Z}(\hat{S}) \rangle$. Then L is abelian and it contains an element \hat{u} of order p. By [MT, Lemma 2.3], $|\mathbb{C}_{\hat{S}}(\hat{u})| = a$ and as shown in (1), this centralizer is a cyclic maximal torus of \hat{S} . Consequently, as $L \leq \mathbb{C}_{\hat{S}}(\hat{u})$, we may also choose \hat{z} of order *a*. Since, as shown above, $\langle \hat{z}, \hat{w} \rangle = \hat{S}$, it follows that $\langle z, w \rangle = S$.

(10): Let
$$S = E_8(q)$$
, with $q \ge 2$. Then

$$|S| = q^{120} \prod_{i \in I} (q^i - 1), \quad \text{with} \quad I = \{2, 8, 12, 14, 18, 20, 24, 30\} .$$

Let $p \in ppd(q, 30)$, which is nonempty. Since

(1)
$$q^{15} + 1 = (q^2 - q + 1)(q^5 + 1)(q^8 + q^7 - q^5 - q^4 - q^3 + q + 1)$$

= $(q^2 - q + 1)(q^5 + 1)\Phi_{30}(q),$

p divides $a = \Phi_{30}(q)$ and |S|/a is coprime to p. By [W, Table I], S has a cyclic maximal torus T of order a, and arguing as in (1), the hypotheses of Lemma 4.3 hold for T. By [W, Section 4], the only maximal subgroup of S containing T is $N_S(T)$ of order 30a.

Let $b \in \text{ppd}(q, 24)$, which is nonempty, and note that $b \equiv 1 \pmod{24}$, and hence that $b \geq 25$ and b does not divide 30. Now b divides $q^{12} + 1$ and a divides $q^{15} + 1$, and since $\text{gcd}(q^{12} + 1, q^{15} + 1)$ divides $q^3 - 1$, it follows that b does not divide a. Therefore, b does not divide $|N_S(T)| = 30a$, so the standard argument applies. \Box

We are now ready to complete the proof of Theorem B, which we state again:

Theorem B. Let S be a nonabelian finite simple group. Then there exist $a, b \in oe(S)$, such that every pair of elements of S of order a and b, respectively, generates a nonsolvable subgroup of S.

Proof. This follows from Propositions 2.1, 2.2, 4.2, 4.5 and from the classification of the finite simple groups. \Box

5. Proof of Theorem A

We finally show that Theorem A follows from Theorem B. First, we restate Theorem A.

Theorem A. Let G be a finite group. Assume that for every $x, y \in G$ there exists an element $g \in G$ such that $\langle x, y^g \rangle$ is solvable. Then G is solvable.

Proof. Suppose that the hypothesis holds for a group G. This is clearly equivalent to assuming that, for all pairs C, D of conjugacy classes of a group G there exist elements $x \in C$ and $y \in D$ such that $\langle x, y \rangle$ is solvable.

We claim that this property is inherited by factor groups: let N be a normal subgroup of G and write $\overline{G} = G/N$. Since "overbar" is a homomorphism, it sends conjugacy classes of \overline{G} onto conjugacy classes of \overline{G} . Hence, given two conjugacy classes \overline{C} and \overline{D} of \overline{G} , we may assume that C and D are conjugacy classes of G. So, by our assumption, there exist $x \in C$ and $y \in D$ such that $\langle x, y \rangle$ is solvable. Hence, $\overline{x} \in \overline{C}$, $\overline{y} \in \overline{D}$ and $\langle \overline{x}, \overline{y} \rangle \leq \overline{\langle x, y \rangle}$ is solvable. Thus the claim is proved.

Theorem A holds trivially if |G| = 1. Suppose inductively that |G| > 1 and that Theorem A holds for groups of orders less than |G|. Let M be a minimal normal subgroup of G. Then by induction, G/M is solvable. If G has distinct minimal normal subgroups M_1, M_2 , then G is isomorphic to a subgroup of the solvable group $G/M_1 \times G/M_2$. Thus we may assume that G has a unique minimal normal subgroup M. If M is solvable, then G is solvable as well. We show that this must be the case: suppose to the contrary that M is nonsolvable. The characteristically simple group M is a direct product of isomorphic simple groups. We hence identify M with the direct power S^k of a nonabelian simple group S. By Theorem B, there exist $a, b \in oe(S)$ such that for every choice of elements $x, y \in S$ with |x| = a and |y| = b, the group $\langle x, y \rangle$ is nonsolvable. In particular, $\langle x^{\alpha}, y^{\beta} \rangle$ is nonsolvable for all $\alpha, \beta \in$ Aut(S). Consider now the diagonal elements u = (x, x, ..., x) and $w = (y, y, \dots, y)$ of M. Recalling that G can be embedded in the wreath product of Aut(S) by a (solvable) subgroup of the symmetric group S_k , we see that for every $g, h \in G$ we have $u^g = (x^{\alpha_1}, x^{\alpha_2}, \dots, x^{\alpha_k})$ and $w^h = (y^{\beta_1}, y^{\beta_2}, \dots, y^{\beta_k})$, where $\alpha_i, \beta_i \in \operatorname{Aut}(S)$ for $i = 1, 2, \dots, k$. Observe also that $\langle u^g, w^h \rangle$ is a subdirect subgroup of the direct product

$$\prod_{i=1}^k \langle x^{\alpha_i}, y^{\beta_i} \rangle.$$

However $\langle x^{\alpha_i}, y^{\beta_i} \rangle$ is a nonsolvable subgroup of S, for every $i = 1, \ldots, k$. It follows that $\langle u^g, w^h \rangle$ is nonsolvable for every choice of g and h in G, which is the required contradiction.

References

- [ATLAS] J.H. Conway, R.S. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson, Atlas of Finite Groups, Clarendon Press, Oxford, 1985.
- [ATLAS3] R.A. Wilson et al., Atlas of Group Representations, http://brauer.maths.qmul.ac.uk/Atlas/v3/
- [BGK] T. Breuer, R. Guralnick and W. Kantor, Probabilistic generation of finite simple groups, II, J. Algebra 320 (2008), 443-494.
- [F] P. Flavell, Finite groups in which every two elements generate a soluble group Invent. Math. 121 (1995), 279-285.
- [FS] W. Feit and G. M. Seitz, On finite rational groups and related topics Illinois J. Math 33 (1988), 103-131.

- [GGKP] N. Gordeev, F. Grunewald, B. Kunyavskii and E. Plotkin, Baer-Suzuki theorem for solvable radical of a finite group, *Comptes Rendus Acad. Sci. Paris*, *Ser I.* 347 (2009), 217-222.
- [G] Simon Guest, A solvable version of the Baer–Suzuki Theorem, submitted. arXiv:0902.1738v1 [math.GR], 10 February 2009.
- [GK] R. Guralnick and W. Kantor, Probabilistic generation of finite simple groups, J. Algebra 234 (2000), 743-792.
- [GKPS] R. Guralnick, B. Kunyavskii, E. Plotkin and A. Shalev, Thompson-like characterizations of the solvable radical, J. Algebra 300 (2006), 363-375.
- [GM] R. Guralnick and G. Malle, Uniform triples and fixed point spaces, in preparation.
- [GPPS] R. Guralnick, T. Pentilla, C. Praeger and J. Saxl, Linear groups with order having certain large prime divisors, *Proc. London Math. Soc.* 78 (1999), 167-214.
- [GW] R. Guralnick and J. Wilson, The probability of generating a finite soluble group, Proc. London Math. Soc. (3) 81 (2000), 405-427.
- [H] B. Huppert, Endliche Gruppen I, Springer, Berlin, 1967.
- [K] P. Kleidman, The maximal subgroups of the finite Steinerg triality groups ${}^{3}D_{4}(q)$ and their automorphism groups, J. Algebra **117** (1988), 30-71.
- [KL] G. Kaplan and D. Levy, Solvability of finite groups via conditions on products of 2-elements and odd *p*-elements, *Bull. Austral. Math. Soc.*, to appear.
- [L] F. Lübeck, Private communication.
- [LM] F. Lübeck and G. Malle, (2,3)-generation of exceptional groups, J. London Math.Soc. 59 (1999), 109-122.
- [LN] V. M. Levchuck and Y. N. Nuzhin, Structure of Ree groups, Algebra Logika 24 (1985), 26-41.
- [M] G. Malle, The maximal subgroups of ${}^{2}F_{4}(q^{2})$, J. Algebra **139** (1991), 52-69.
- [MT] A. Moretó and P. H. Tiep, Prime divisors of character degrees, J. Group Theory 11 (2008), 341-356.
- [NP1] A. Niemeyer and C. Praeger, A recognition algorithm for classical groups over finite fields, Proc. London Math. Soc. 77 (1998), 117-169.
- [NP2] A. Niemeyer and C. Praeger, A recognition algorithm for nongeneric classical groups over finite fields, J. Austral. Math. Soc. 67 (1999), 223-253.
- [R] P. Ribenboim, The book of prime number records. Second edition, Springer-Verlag, New York, 1989.
- [S] M. Suzuki, On a class of doubly transitive groups, Ann. Math 75 (1962), 105-145.
- [T] J. Thompson, Nonsolvable finite groups all of whose local subgroups are solvable, Bull. Amer. Math. Soc. 74 (1968), 383-437.
- [W] T. Weigel, Generation of exceptional groups of Lie-type, Geometriae Dedicata 41 (1992), 63-87.
- [Wi] H. Wielandt, Permutation Groups, Academic Press, New York-London, 1964.
- [Z] K.Zsigmondy, Zur Theorie der Potenzreste, Monatsh. Math. Phys. 3 (1892), 265-284.

Silvio Dolfi, Dipartimento di Matematica U. Dini, Università degli Studi di Firenze, viale Morgagni 67/a, 50134 Firenze, Italy.

E-mail address: dolfi@math.unifi.it

MARCEL HERZOG, DEPARTMENT OF MATHEMATICS, RAYMOND AND BEVERLY SACKLER FACULTY OF EXACT SCIENCES, TEL AVIV UNIVERSITY, TEL AVIV, ISRAEL.

E-mail address: herzogm@post.tau.ac.il

CHERYL E. PRAEGER, SCHOOL OF MATHEMATICS AND STATISTICS, THE UNIVERSITY OF WESTERN AUSTRALIA, 35 STIRLING HIGHWAY, CRAWLEY, WA 6009, AUSTRALIA

E-mail address: praeger@maths.uwa.edu.au