# Glauberman correspondents and extensions of nilpotent block algebras 

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#### Abstract

The main purpose of this paper is to prove that the extensions of a nilpotent block algebra and its Glauberman correspondent block algebra are Morita equivalent under an additional group-theoretic condition (see Theorem 1.6); in particular, Harris and Linckelman's theorem and Koshitani and Michler's theorem are covered (see Theorems 7.5 and 7.6). The ingredient to carry out our purpose is the two main results in Külshammer and Puig's work Extensions of nilpotent blocks; we actually revisited them, giving completely new proofs of both and slightly improving the second one (see Theorems 3.5 and 3.14).


## 1. Introduction

1.1. Let $\mathcal{O}$ be a complete discrete valuation ring with an algebraically closed residue field $k$ of characteristic $p$ and a quotient field $\mathcal{K}$ of characteristic 0 . In addition, $\mathcal{K}$ is also assumed to be big enough for all finite groups that we consider below. Let $H$ be a finite group. We denote by $\operatorname{Irr}_{\mathcal{K}}(H)$ the set of all irreducible characters of $H$ over $\mathcal{K}$. Let $A$ be another finite group and assume that there is a group homomorphism $A \rightarrow \operatorname{Aut}(H)$. Such a group $H$ with an $A$-action is called an $A$-group. We denote by $H^{A}$ the subgroup of all $A$-fixed elements in $H$. Clearly $A$ acts on $\operatorname{Irr}_{\mathcal{K}}(H)$. We denote by $\operatorname{Irr}_{\mathcal{K}}(H)^{A}$ the set of all $A$-fixed elements in $\operatorname{Irr}_{\mathcal{K}}(H)$. Assume that $A$ is solvable and the order of $A$ is coprime to the order of $H$. By [11, Theorem 13.1], there is a bijection

$$
\pi(H, A): \operatorname{Irr}_{\mathcal{K}}(H)^{A} \rightarrow \operatorname{Irr}_{\mathcal{K}}\left(H^{A}\right)
$$

such that
1.1.1. For any normal subgroup $B$ of $A$, the bijection $\pi(H, B)$ maps $\operatorname{Irr}_{\mathcal{K}}(H)^{A}$ to $\operatorname{Irr}_{\mathcal{K}}\left(H^{B}\right)^{A}$, and in $\operatorname{Irr}_{\mathcal{K}}(H)^{A}$ we have

$$
\pi(H, A)=\pi\left(H^{B}, A / B\right) \circ \pi(H, B)
$$

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1.1.2. If $A$ is a $q$-group for some prime $q$, then for any $\chi \in \operatorname{Irr}_{\mathcal{K}}(H)^{A}$, the corresponding irreducible character $\pi(H, A)(\chi)$ of $G^{A}$ is the unique irreducible constituent of $\operatorname{Res}_{H^{A}}^{H}(\chi)$ occurring with a multiplicity coprime to $q$.
The character $\pi(H, A)(\chi)$ of $H^{A}$ is called the Glauberman correspondent of the character $\chi$ of $H$.
1.2. For any central idempotent $c$ of $\mathcal{O} H$, we denote by $\operatorname{Irr}_{\mathcal{K}}(H, c)$ the set of all irreducible characters of $H$ provided by some $\mathcal{K} H c$-module. Let $b$ be a block of $H$ - namely $b$ is a primitive central idempotent of $\mathcal{O H}$; then $\mathcal{O H b}$ is called the block algebra corresponding to $b$. Assume that $A$ stabilizes the block $b$ and centralizes a defect group of $b$. Then, by [26, Proposition 1 and Theorem 1], $A$ stabilizes all characters of $\operatorname{Irr}_{\mathcal{K}}(H, b)$ and there is a unique block $w(b)$ of $\mathcal{O}\left(H^{A}\right)$ such that

$$
\operatorname{Irr}_{\mathcal{K}}\left(H^{A}, w(b)\right)=\pi(H, A)\left(\operatorname{Irr}_{\mathcal{K}}(H, b)\right) ;
$$

moreover ,there is a perfect isometry (see [6])

$$
R_{H}^{b}: \mathcal{R}_{\mathcal{K}}(H, b) \rightarrow \mathcal{R}_{\mathcal{K}}\left(H^{A}, w(b)\right)
$$

such that $R_{H}^{b}(\chi)= \pm \pi(H, A)(\chi)$ for any $\chi \in \operatorname{Irr}_{\mathcal{K}}(H, b)$, where we denote by $\mathcal{R}_{\mathcal{K}}(H, b)$ and $\mathcal{R}_{\mathcal{K}}\left(H^{A}, w(b)\right)$ the additive groups generated by $\operatorname{Irr}_{\mathcal{K}}(H, b)$ and $\operatorname{Irr}_{\mathcal{K}}\left(H^{A}, w(b)\right)$. Such a block $w(b)$ is called the Glauberman correspondent of $b$ (see [26]). Since a perfect isometry between blocks is often nothing but the character-theoretic 'shadow' of a derived equivalence, it seems reasonable to ask whether there is a derived equivalence between a block and its Glauberman correspondent. In the last few years, some Morita equivalences between $b$ and $w(b)$ were found in the cases where $H$ is $p$-solvable or the defect groups of $b$ are normal in $H$, which supply Glauberman correspondences from $\operatorname{Irr}_{\mathcal{K}}(H, b)$ to $\operatorname{Irr}_{\mathcal{K}}\left(H^{A}, w(b)\right)$ (see [9], [12] and [10]); moreover, all these Morita equivalences between $b$ and $w(b)$ are basic in the sense of [20].
1.3. By induction, the groups $H$ and $H^{A}$ and the blocks $b$ and $w(b)$ in the main results of [9], [12] and [10] can be reduced to the situation where, for some $A$-stable normal subgroup $K$ of $H$, we have $H=H^{A} \cdot K$, the block $b$ is an $H$-stable block of $K$ with trivial or central defect group, and the block $w(b)$ is an $H^{A}$-stable block of $K^{A}$ with trivial or central defect group. Recall that the block $b$ of $H$ is called nilpotent (see [18]) if the quotient group $N_{H}\left(R_{\varepsilon}\right) / C_{H}(R)$ is a $p$-group for any local pointed group $R_{\varepsilon}$ on $\mathcal{O} H b$. Blocks with trivial or central defect group are nilpotent and therefore in
these situations $\mathcal{O} H b$ and $\mathcal{O}\left(H^{A}\right) w(b)$ are extensions of the nilpotent block algebras $\mathcal{O} K b$ and $\mathcal{O} K^{A} w(b)$ respectively. Külshammer and Puig already precisely described the algebraic structure of extensions of nilpotent block algebras (see [14] or Section 3 below) and these results can be applied to blocks of $p$-solvable groups (see [19]) and to blocks with normal defect groups (see $[24,13]$ ). Thus, it is reasonable to seek a common generalization of the main results of $[9,12,10]$ in the setting of extensions of nilpotent block algebras.
1.4. Let $G$ be another finite $A$-group having $H$ as an $A$-stable normal subgroup and consider the $A$-action on $H$ induced by the $A$-group $G$. We assume that $A$ stabilizes $b$ and denote by $N$ the stabilizer of $b$ in $G$. Clearly $N$ is $A$-stable. Set

$$
c=\operatorname{Tr}_{N}^{G}(b) \text { and } \alpha=\{c\}
$$

then the idempotent $c$ is $A$-stable and $\alpha$ is an $A$-stable point of $G$ on the group algebra $\mathcal{O} H$ (the action of $G$ on $\mathcal{O H}$ is induced by conjugation). In particular, $G_{\alpha}$ is a pointed group on $\mathcal{O} H$. Let $P$ be a defect group of $G_{\alpha}$; then, by [14, Proposition 5.3], $Q=P \cap H$ is a defect group of the block $b$ of $H$.

Theorem 1.5. Assume that $A$ centralizes $P$, that $A$ is solvable and that the orders of $G$ and $A$ are coprime. Set $w(c)=\operatorname{Tr}_{N^{A}}^{G^{A}}(w(b))$ and $w(\alpha)=\{w(c)\}$. Then, $w(\alpha)$ is a point of $G^{A}$ on the group algebra $\mathcal{O}\left(H^{A}\right)$ and $P$ is a defect group of the pointed group $\left(G^{A}\right)_{w(\alpha)}$ on $\mathcal{O}\left(H^{A}\right)$. Moreover, if $G=H \cdot G^{A}$ and the block $b$ of $H$ is nilpotent, we have

$$
\operatorname{Irr}_{\mathcal{K}}(G, c)=\operatorname{Irr}_{\mathcal{K}}(G, c)^{A} \text { and } \pi(G, A)\left(\operatorname{Irr}_{\mathcal{K}}(G, c)\right)=\operatorname{Irr}_{\mathcal{K}}\left(G^{A}, w(c)\right)
$$

The following theorem shows that there is a "basic" Morita equivalence between $\mathcal{O} G c$ and $\mathcal{O} G^{A} w(c)$; that is to say, this Morita equivalence induces basic Morita equivalences [20] between corresponding block algebras.

Theorem 1.6. Assume that $A$ centralizes $P$, that $A$ is solvable and that the orders of $G$ and $A$ are coprime. Set $w(c)=\operatorname{Tr}_{N^{A}}^{G^{A}}(w(b))$. Assume that $G=$ $G^{A} \cdot H$ and that the block $b$ is nilpotent. Then, there is an $\mathcal{O}\left(H \times H^{A}\right)$-module $M$ inducing a basic Morita equivalence between $\mathcal{O} H b$ and $\mathcal{O}\left(H^{A}\right) w(b)$, which can be extended to the inverse image $K$ in $N \times N^{A}$ of the "diagonal" subgroup of $N / H \times N^{A} / H^{A}$ in such a way that $\operatorname{Ind}_{K}^{G \times G^{A}}(M)$ induces a "basic" Morita equivalence between $\mathcal{O} G c$ and $\mathcal{O}\left(G^{A}\right) w(c)$.

Remark 1.7. Since $G=H \cdot G^{A}$, we have $N=H \cdot N^{A}$ and then the inclusion $N^{A} \subset N$ induces a group isomorphism $N / H \cong N^{A} / H^{A}$.

We use pointed groups introduced by Lluis Puig. For more details on pointed groups, readers can see [15] or Paragraph 2.5 below. In Section 2, we introduce some notation and terminology. Section 3 revisits Külshammer and Puig's main results on extensions of nilpotent blocks; the proof of the existence and uniqueness of the finite group $L$ (see [14, Theorem 1.8] and Theorem 3.5 below) is dramatically simplified; actually, Corollary 3.14 below slightly improves [14, Theorem 1.12]; explicitly, $S_{\gamma}$ in Corollary 3.14 is unique up to determinant one. With the Glauberman correspondents of blocks due to Watanabe, in Section 4 we define Glauberman correspondents of extensions of blocks and compare the local structures of extensions of blocks and their Glauberman correspondents.

By Puig's structure theorem of nilpotent blocks, there is a bijection between the sets of irreducible characters of the nilpotent block $b$ of $H$ and of its defect $Q$; in Section 5 , for a suitable local point $\delta$ of $Q$, we prove that this bijection preserves $N_{G}\left(Q_{\delta}\right)$-actions on these sets. As a consequence, we obtain an $N_{G}\left(Q_{\delta}\right)$-stable irreducible character $\chi$ of $H$ such that $\chi$ lifts the unique irreducible Brauer character of the nilpotent block $b$ of $H$ and that the Glauberman correspondent character $\pi(H, A)(\chi)$ lifts the unique irreducible Brauer character of the Glauberman correspondent block $w(b)$ of $H^{A}$ (see Lemma 5.6).

Obviously, $N$ stabilizes the unique simple module in the nilpotent block $b$ of $H$; with this $N$-stable $\mathcal{O} H b$-simple module, we construct an $A$-stable $k^{*}$-group $\hat{\bar{N}}^{k}$ (see 2.3 and 3.13 below); since $N^{A}$ stabilizes the unique simple module of the nilpotent block $w(b)$ of $H^{A}$, a $k^{*}$-group ${\widehat{N^{A}}}^{k}$ is similarly constructed. In Section 6, we prove that ${\widehat{N^{A}}}^{k}$ and $\left(\hat{\bar{N}}^{k}\right) A$ are isomorphic as $k^{*}$-groups (see Theorem 6.4). In Section 7, we use the improved version of Külshammer and Puig's main result to prove our main theorem 1.6.

## 2. Notation and terminology

2.1. Throughout this paper, all $\mathcal{O}$-modules are $\mathcal{O}$-free finitely generated - except in 2.4 below; all $\mathcal{O}$-algebras have identity elements, but their subalgebras need not have the same identity element. Let $\mathcal{A}$ be an $\mathcal{O}$ algebra; we denote by $\mathcal{A}^{\circ}, \mathcal{A}^{*}, Z(\mathcal{A}), J(\mathcal{A})$ and $1_{\mathcal{A}}$ the opposite $\mathcal{O}$-algebra of $\mathcal{A}$, the multiplicative group of all invertible elements of $\mathcal{A}$, the center of $\mathcal{A}$, the radical of $\mathcal{A}$ and the identity element of $\mathcal{A}$ respectively. Sometimes we write 1 instead of $1_{\mathcal{A}}$. For any abelian group $V, \mathrm{id}_{V}$ denotes the identity automorphism on $V$. Let $\mathcal{B}$ be an $\mathcal{O}$-algebra; a homomorphism $\mathcal{F}: \mathcal{A} \rightarrow \mathcal{B}$ of $\mathcal{O}$-algebras is said to be an embedding if $\mathcal{F}$ is injective and we have

$$
\mathcal{F}(\mathcal{A})=\mathcal{F}\left(1_{\mathcal{A}}\right) \mathcal{B} \mathcal{F}\left(1_{\mathcal{A}}\right)
$$

Let $S$ be a set and $G$ be a group acting on $S$. For any $g \in G$ and $s \in S$, we write the action of $g$ on $s$ as $s \cdot g$.
2.2. Let $X$ be a finite group. An $X$-interior $\mathcal{O}$-algebra $\mathcal{A}$ is an $\mathcal{O}$-algebra $\mathcal{A}$ endowed with a group homomorphism $\rho: X \rightarrow \mathcal{A}^{*}$; for any $x, y \in X$ and $a \in \mathcal{A}$, we write $\rho(x) a \rho(y)$ as $x \cdot a \cdot y$ or xay if there is no confusion. Let $\varrho: Y \rightarrow X$ be a group homomorphism; the $\mathcal{O}$-algebra $\mathcal{A}$ with the group homomorphism $\rho \circ \varrho: Y \rightarrow \mathcal{A}^{*}$ is an $Y$-interior $\mathcal{O}$-algebra and we denote it by $\operatorname{Res}_{\varrho}(\mathcal{A})$. Let $\mathcal{A}^{\prime}$ be another $X$-interior $\mathcal{O}$-algebra; an $\mathcal{O}$-algebra homomorphism $\mathcal{F}: \mathcal{A} \rightarrow \mathcal{A}^{\prime}$ is said to be a homomorphism of $X$-interior $\mathcal{O}$-algebras if for any $x, y \in X$ and any $a \in \mathcal{A}$, we have $\mathcal{F}(x a y)=x \mathcal{F}(a) y$. The tensor product $\mathcal{A} \otimes_{\mathcal{O}} \mathcal{A}^{\prime}$ of $\mathcal{A}$ and $\mathcal{A}^{\prime}$ is an $X$-interior $\mathcal{O}$-algebra with the group homomorphism

$$
X \rightarrow\left(\mathcal{A} \otimes_{\mathcal{O}} \mathcal{A}^{\prime}\right)^{*}, \quad x \mapsto x 1_{\mathcal{A}} \otimes x 1_{\mathcal{A}^{\prime}}
$$

Let $Z$ be a subgroup of $X$ and let $\mathcal{B}$ be an $\mathcal{O} Z$-interior algebra. Obviously, the left and right multiplications by $\mathcal{O Z}$ on $\mathcal{B}$ define an $(\mathcal{O Z}, \mathcal{O} Z)$-bimodule structure on $\mathcal{B}$. Set

$$
\operatorname{Ind}_{Z}^{X}(\mathcal{B})=\mathcal{O} X \otimes_{\mathcal{O Z}} \mathcal{B} \otimes_{\mathcal{O Z}} \mathcal{O} X
$$

and then this the $(\mathcal{O} X, \mathcal{O} X)$-bimodule $\operatorname{Ind}_{Z}^{X}(\mathcal{B})$ becomes an $X$-interior $\mathcal{O}$ algebra with the product

$$
(x \otimes b \otimes y)\left(x^{\prime} \otimes b^{\prime} \otimes y^{\prime}\right)= \begin{cases}x \otimes b \cdot y x^{\prime} \cdot b^{\prime} \otimes y^{\prime} & \text { if } y x^{\prime} \in Z \\ 0 & \text { otherwise }\end{cases}
$$

for any $x, y, x^{\prime}, y^{\prime} \in X$ and any $b, b^{\prime} \in \mathcal{B}$, and with the homomorphism $\mathcal{O} X \rightarrow \operatorname{Ind}_{Z}^{X}(\mathcal{B})$ mapping $x \in X$ onto $\sum_{y} x y \otimes 1 \otimes y^{-1}$, where $y$ runs over a set of representatives for left cosets of $Z$ in $X$.
2.3. A $k^{*}$-group with $k^{*}$-quotient $X$ is a group $\hat{X}$ endowed with an injective group homomorphism $\theta: k^{*} \rightarrow Z(\hat{X})$ together with an isomorphism $\hat{X} / \theta\left(k^{*}\right) \cong X$; usually we omit to mention $\theta$ and the quotient $X=\hat{X} / \theta\left(k^{*}\right)$ is called the $k^{*}$-quotient of $\hat{X}$, writing $\lambda \cdot \hat{x}$ instead of $\theta(\lambda) \hat{x}$ for any $\lambda \in k^{*}$ and any $\hat{x} \in \hat{X}$. We denote by $\hat{Y}$ the inverse image of $Y$ in $\hat{X}$ for any subset $Y$ of $X$ and, if no precision is needed, we often denote by $\hat{x}$ some lifting of an element $x \in X$. We denote by $\hat{X}^{\circ}$ the $k^{*}$-group with the same underlying group $\hat{X}$ endowed with the group homomorphism $\theta^{-1}: k^{*} \rightarrow Z(\hat{X}), \lambda \mapsto \theta(\lambda)^{-1}$. Let $\vartheta: Z \rightarrow X$ be a group homomorphism; we denote by $\operatorname{Res}_{\vartheta}(\hat{X})$ the $k^{*}$-group formed by the group of pairs $(\hat{x}, y) \in \hat{X} \times Z$ such that $\vartheta(y)$ is the image of $\hat{x}$ in $X$, endowed with the group homomorphism mapping $\lambda \in k^{*}$ on $(\theta(\lambda), 1)$; up to suitable identifications, $Z$ is the $k^{*}$-quotient of $\operatorname{Res}_{\vartheta}(\hat{X})$. Let $\hat{U}$ be another $k^{*}$-group with $k^{*}$-quotient $U$. A group homomorphism $\phi: \hat{X} \rightarrow \hat{U}$ is a homomorphism of $k^{*}$-groups if $\phi(\lambda \cdot \hat{x})=\lambda \cdot \phi(\hat{x})$ for any $\lambda \in k^{*}$ and $\hat{x} \in \hat{X}$. For more details on $k^{*}$-groups, please see [17, §5].
2.4. Let $\hat{X}$ be a $k^{*}$-group with $k^{*}$-quotient $X$. By [25, Charpter II, Proposition 8], there exists a canonical decomposition $\mathcal{O}^{*} \cong(1+J(\mathcal{O})) \times k^{*}$, thus $k^{*}$ can be canonically regarded as a subgroup of $\mathcal{O}^{*}$. Set

$$
\mathcal{O}_{*} \hat{X}=\mathcal{O} \otimes_{\mathcal{O} k^{*}} \mathcal{O} \hat{X}
$$

where the left $\mathcal{O} k^{*}$-module $\mathcal{O} \hat{X}$ and the right $\mathcal{O} k^{*}$-module $\mathcal{O}$ are defined by the left and right multiplication by $k^{*}$ on $\hat{X}$ and $\mathcal{O}^{*}$ respectively. It is straightforward to verify that the $\mathcal{O}$-module $\mathcal{O}_{*} \hat{X}$ is an $\mathcal{O}$-algebra with the distributive multiplication

$$
\left(a_{1} \otimes \hat{x}_{1}\right)\left(a_{2} \otimes \hat{x}_{2}\right)=a_{1} a_{2} \otimes \hat{x}_{1} \hat{x}_{2}
$$

for any $a_{1}, a_{2} \in \mathcal{O}$ and any $\hat{x}_{1}, \hat{x}_{2} \in \hat{X}$.
2.5. Let $\mathcal{A}$ be an $X$-algebra over $\mathcal{O}$; that is to say, $\mathcal{A}$ is endowed with a group homomorphism $\psi: X \rightarrow \operatorname{Aut}(\mathcal{A})$, where $\operatorname{Aut}(\mathcal{A})$ is the group of all $\mathcal{O}$-automorphisms of $A$; usually, we omit to mention $\psi$. For any subgroup $Y$ of $X$, we denote by $\mathcal{A}^{Y}$ the $\mathcal{O}$-subalgebra of all $Y$-fixed elements in $\mathcal{A}$.

A pointed group $Y_{\beta}$ on $\mathcal{A}$ consists of a subgroup $Y$ of $X$ and of an $\left(\mathcal{A}^{Y}\right)^{*}$ conjugate class $\beta$ of primitive idempotents of $\mathcal{A}^{Y}$. We often say that $\beta$ is a point of $Y$ on $\mathcal{A}$. Obviously, $X$ acts on the set of all pointed groups on $\mathcal{A}$ by the equality $\left(Y_{\beta}\right)^{x}=Y_{\psi\left(x^{-1}\right)(\beta)}^{x}$ and we denote by $N_{X}\left(Y_{\beta}\right)$ the stabilizer of $Y_{\beta}$ in $X$ for any pointed group $Y_{\beta}$ on $\mathcal{A}$. Another pointed group $Z_{\gamma}$ is said contained in $Y_{\beta}$ if $Z \leq Y$ and there exist some $i \in \beta$ and $j \in \gamma$ such that $i j=j i=j$. For a subgroup $U$ of $G$, set

$$
\mathcal{A}(U)=k \otimes_{\mathcal{O}}\left(\mathcal{A}^{U} / \sum_{V} \mathcal{A}_{V}^{U}\right)
$$

where $V$ runs over the set of proper subgroups of $U$ and $\mathcal{A}_{V}^{U}$ is the image of the relative trace map $\operatorname{Tr}_{V}^{U}: \mathcal{A}^{V} \rightarrow \mathcal{A}^{U}$; the canonical surjective homomorphism $\operatorname{Br}_{U}^{\mathcal{A}}: \mathcal{A}^{U} \rightarrow \mathcal{A}(U)$ is called the Brauer homomorphism of the $X$-algebra $\mathcal{A}$ at $U$. When $\mathcal{A}$ is equal to the group algebra $\mathcal{O} X$, the homomorphism $k C_{X}(U) \rightarrow \mathcal{A}(U)$ sending $x \in C_{X}(U)$ onto the image of $x$ in $\mathcal{A}(U)$ is an isomorphism, through which we identify $\mathcal{A}(U)$ with $k C_{X}(U)$. A pointed group $U_{\gamma}$ on $\mathcal{A}$ is said local if the image of $\gamma$ in $\mathcal{A}(U)$ is not equal to $\{0\}$, which forces $U$ to be a $p$-group; then, a local pointed group $U_{\gamma}$ is said a defect pointed group of a pointed group $Y_{\beta}$ on $\mathcal{A}$ if $U_{\gamma} \leq Y_{\beta}$ and we have $\beta \subset \operatorname{Tr}_{U}^{Z}\left(\mathcal{A}^{U} \cdot \gamma \cdot A^{U}\right)$, where $\mathcal{A}^{U} \cdot \gamma \cdot A^{U}$ is the ideal of $\mathcal{A}^{U}$ generated by $\gamma$. Let $c$ be a block of $X$; then $\{c\}$ is a point of $X$ on $\mathcal{O} X$ and if $P_{\gamma}$ is a defect pointed group of $X_{\{c\}}$ then $P$ is a defect group of $c$.

## 3. Extensions of nilpotent blocks revisited

In this section, we assume that $\mathcal{O}$ is a complete discrete valuation ring with an algebraically closed residue field of characteristic $p$.
3.1. Let $G$ be a finite group, $H$ be a normal subgroup of $G$ and $b$ be a block of $H$ over $\mathcal{O}$. Denote by $N$ the stabilizer of $b$ in $G$ and set $\bar{N}=N / H$. Obviously, $\beta=\{b\}$ is a point of $H$ and $N$ on $\mathcal{O} H$ and there is a unique pointed group $G_{\alpha}$ on $\mathcal{O H}$ such that

$$
H_{\beta} \leq N_{\beta} \leq G_{\alpha}
$$

Let $Q_{\delta}$ be a defect pointed group of $H_{\beta}$ and $P_{\gamma}$ be a defect pointed group of $N_{\beta}$ such that $Q_{\delta} \leq P_{\gamma}$; by [14, Proposition 5.3], we have $Q=P \cap H$ and, since we have $[14,1.7]$

$$
\mathcal{O} G \operatorname{Tr}_{N}^{G}(b) \cong \operatorname{Ind}_{N}^{G}(\mathcal{O} N b)
$$

it is easily checked that $P_{\gamma}$ is also a defect pointed group of $G_{\alpha}[16,1.12]$. Assume that the block $b$ is nilpotent; it follows from [14, Proposition 6.5] that $b$ remains a nilpotent block of $H \cdot R$ for any subgroup $R$ of $P$, and from [14, Theorem 6.6] that there is a unique local point $\varepsilon$ of $R$ on $\mathcal{O H}$ such that $R_{\varepsilon} \leq P_{\gamma}$.
3.2. Set $\mathcal{A}=\mathcal{O} N b$ and $\mathcal{B}=\mathcal{O} H b$. Choosing $j \in \delta$ and $i \in \gamma$ such that $i j=j i=j$, we set

$$
\mathcal{A}_{\gamma}=(\mathcal{O} G)_{\gamma}=i \mathcal{A} i, \mathcal{B}_{\gamma}=(\mathcal{O} H)_{\gamma}=i \mathcal{B} i \text { and } \mathcal{B}_{\delta}=(\mathcal{O} H)_{\delta}=j \mathcal{B} j .
$$

Then $\mathcal{A}_{\gamma}$ is a $P$-interior algebra with the group homomorphism $P \rightarrow \mathcal{A}_{\gamma}^{*}$ mapping $u$ onto $u i$ for any $u \in P, \mathcal{B}_{\gamma}$ is a $P$-stable subalgebra of $\mathcal{A}_{\gamma}$ and $\mathcal{B}_{\delta}$ is a $Q$-interior algebra with the group homomorphism $Q \rightarrow \mathcal{B}_{\delta}^{*}$ mapping $v \in Q$ onto $v j$ for any $v \in Q$. Clearly $\mathcal{A}$ is an $N / H$-graded algebra with the $\bar{x}$-component $\mathcal{O} H x b$, where $\bar{x} \in N / H$ and $x$ is a representative of $\bar{x}$ in $N$. Since $i$ belongs to the 1 -component $\mathcal{B}, \mathcal{A}_{\gamma}$ is an $N / H$-graded algebra with the $\bar{x}$-component $i(\mathcal{O} H x) i$.
3.3. In [14] Külshammer and Puig describe the structure of any block of $G$ lying over $b$ in terms of a new finite group $L$ which need not be involved in $G$ [14, Theorem 1.8]. More explicitly, $L$ is a group extension of $\bar{N}$ by $Q$ holding strong uniqueness properties. In order to prove these properties, in [14] the group $L$ is exhibited inside a suitable $\mathcal{O}$-algebra [14, Theorem 8.13], demanding a huge effort. But, as a matter of fact, these properties can be obtained directly from the so-called local structure of $G$ over $\mathcal{O H b}$, a fact that we only have understood recently. Then, with these uniqueness properties in hand, the second main result [14, Theorem 1.12] follows quite easily. With the notation and framework of [14], we completely develop both new proofs.
3.4. Denote by $\mathcal{E}_{(b, H, G)}$ the category - called the extension category associated to $G, H$ and $b$ - where the objects are all the subgroups of $P$ and, for any pair of subgroups $R$ and $T$ of $P$, the morphisms from $T$ to $R$ are the pairs $\left(\psi_{x}, \bar{x}\right)$ formed by an injective group homomorphism $\psi_{x}: T \rightarrow R$ and an element $\bar{x}$ of $\bar{N}$ both determined by an element $x \in N$ fulfilling $T_{\nu} \leq\left(R_{\varepsilon}\right)^{x}$ where $\varepsilon$ and $\nu$ are the respective local points of $R$ and $T$ on $\mathcal{O} H$ determined by $P_{\gamma}$ - in general, we should consider the $(b, N)$-Brauer pairs over the $p$-permutation $N$-algebra $\mathcal{O H b}$ [5, Definition 1.6 and Theorem 1.8] but, in our situation, they coincide with the local pointed groups over this
$N$-algebra. The composition in $\mathcal{E}_{(b, H, G)}$ is determined by the composition of group homomorphisms and by the product in $\bar{N}$.

Theorem 3.5. There is a triple formed by a finite group $L$ and by two group homomorphisms

$$
\tau: P \longrightarrow L \quad \text { and } \quad \bar{\pi}: L \longrightarrow \bar{N}
$$

such that $\tau$ is injective, that $\bar{\pi}$ is surjective, that we have $\operatorname{Ker}(\bar{\pi})=\tau(Q)$ and $\bar{\pi}(\tau(u))=\bar{u}$ for any $u \in P$, and that these homomorphisms induce an equivalence of categories

$$
\mathcal{E}_{(b, H, G)} \cong \mathcal{E}_{(1, \tau(Q), L)} .
$$

Moreover, for another such a triple $L^{\prime}, \tau^{\prime}$ and $\bar{\pi}^{\prime}$, there is a group isomorphism $\lambda: L \cong L^{\prime}$, unique up to conjugation, fulfilling

$$
\lambda \circ \tau=\tau^{\prime} \quad \text { and } \quad \bar{\pi}^{\prime} \circ \lambda=\bar{\pi} .
$$

Proof: Set $Z=Z(Q), M=N_{G}\left(Q_{\delta}\right)$ and $\mathcal{E}=\mathcal{E}_{(b, H, G)}$, denote by $\mathcal{E}(R, T)$ the set of $\mathcal{E}$-morphisms from $T$ to $R$, and write $\mathcal{E}(R)$ instead of $\mathcal{E}(R, R)$; by the very definition of the category $\mathcal{E}$, we have the exact sequence

$$
1 \longrightarrow C_{H}(Q) \longrightarrow M \longrightarrow \mathcal{E}(Q) \longrightarrow 1 ;
$$

it is clear that $M$ contains $P$ and that we have $C_{H}(Q) \cap P=Z$; moreover, denoting by $\mathcal{E}_{P}(Q)$ the image of $P$ in $\mathcal{E}(Q)$, it is easily checked from [14, Proposition 5.3] that $\mathcal{E}_{P}(Q)$ is a Sylow $p$-subgroup of $\mathcal{E}(Q)$.

We claim that the element $\bar{h}$ induced by $P$ in the second cohomology group $\mathbb{H}^{2}\left(\mathcal{E}_{P}(Q), Z\right)$ belongs to the image of $\mathbb{H}^{2}(\mathcal{E}(Q), Z)$. Indeed, according to in [7, Ch. XII, Theorem 10.1], it suffices to prove that, for any subgroup $R$ of $P$ containing $Z$ and any $\left(\varphi_{x}, \bar{x}\right) \in \mathcal{E}(Q)$ such that

$$
\left(\varphi_{x}, \bar{x}\right) \circ \mathcal{E}_{R}(Q) \circ\left(\varphi_{x}, \bar{x}\right)^{-1} \leq \mathcal{E}_{P}(Q)
$$

the restriction $\operatorname{res}_{\left(\varphi_{x}, \bar{x}\right)}(\bar{h})$ of $\bar{h}$ via the conjugation by $\left(\varphi_{x}, \bar{x}\right)$ and the element of $\mathbb{H}^{2}\left(\mathcal{E}_{R}(Q), Z\right)$ determined by $R$ coincide; actually, we may assume that $R$ contains $Q$. Thus, $x$ normalizes $Q_{\delta}$ and inclusion 3.5.3 forces

$$
C_{H}(Q) \cdot R \leq\left(C_{H}(Q) \cdot P\right)^{x} ;
$$

in particular, respectively denoting by $\lambda$ and $\mu$ the points of $C_{H}(Q) \cdot P$ and $C_{H}(Q) \cdot R$ on $\mathcal{O} H$ such that $\left(C_{H}(Q) \cdot P\right)_{\lambda}$ and $\left(C_{H}(Q) \cdot R\right)_{\mu}$ contain $Q_{\delta}[18$, Lemma 3.9], by uniqueness we have

$$
\left(C_{H}(Q) \cdot R\right)_{\mu} \leq\left(C_{H}(Q) \cdot P\right)_{\lambda}
$$

and, with the notation above, it follows from [14, Proposition 3.5] that $P_{\gamma}$ and $R_{\varepsilon}$ are defect pointed groups of the respective pointed groups $\left(C_{H}(Q) \cdot P\right)_{\lambda}$ and $\left(C_{H}(Q) \cdot R\right)_{\mu}$; consequently, there is $z \in C_{H}(Q)$ fulfilling $R_{\varepsilon} \leq\left(P_{\gamma}\right)^{z x}$ [15, Theorem 1.2]. That is to say, the conjugation by $z x$ induces a group homomorphism $R \rightarrow P$ mapping $Z$ onto $Z$ and inducing the element $\left(\psi_{z x}, \overline{z x}\right)$ of $\mathcal{E}(P, R)$ which extends $\left(\varphi_{x}, \bar{x}\right)$, so that the map

$$
\operatorname{res}_{\left(\varphi_{x}, \bar{x}\right)}: \mathbb{H}^{2}\left(\mathcal{E}_{P}(Q), Z\right) \longrightarrow \mathbb{H}^{2}\left(\mathcal{E}_{R}(Q), Z\right)
$$

sends $\bar{h}$ to the element of $\mathbb{H}^{2}\left(\mathcal{E}_{R}(Q), Z\right)$ determined by $R$ [7, Chap. XIV, Theorem 4.2].

In particular, the corresponding element of $\mathbb{H}^{2}(\mathcal{E}(Q), Z)$ determines a group extension

$$
1 \longrightarrow Z \xrightarrow{\tau} L \xrightarrow{\pi} \mathcal{E}(Q) \longrightarrow 1
$$

and, since $\bar{h} \in \mathbb{H}^{2}\left(\mathcal{E}_{P}(Q), Z\right)$ is the image of this element, there is a group extension homomorphism $\tau: P \rightarrow L$ [7, Chap. XIV, Theorem 4.2]; it is clear that $\tau$ is injective and, since $\mathcal{E}_{P}(Q)$ is a Sylow $p$-subgroup of $\mathcal{E}(Q), \operatorname{Im}(\tau)$ is a Sylow $p$-subgroup of $L$; moreover, since $N=H \cdot M$ [15, Theorem 1.2], we have

$$
\bar{N} \cong M / C_{H}(Q) \cdot Q \cong \mathcal{E}(Q) / \mathcal{E}_{Q}(Q) ;
$$

in particular, $\pi$ determines a group homomorphism $\bar{\pi}: L \rightarrow \bar{N}$ and, since $\tau$ is a group extension homomorphism, we get $\bar{\pi}(\tau(u))=\bar{u}$ for any $u \in P$ and may choose $\pi$ in such a way that we have

$$
y \tau(v) y^{-1}=\tau\left(\varphi_{x}(v)\right)
$$

for any $y \in L$ and any $v \in Q$ where $\pi(y)=\left(\varphi_{x}, \bar{x}\right)$ for some $x \in N$. Then, we claim that, up to a suitable modification of our choice of $\tau$, the group $L$ endowed with $\tau$ and $\bar{\pi}$ fulfills the conditions above; set $\hat{\mathcal{E}}=\mathcal{E}_{(1, \tau(Q), L)}$ for short.

For any pair of subgroups $R$ and $T$ of $P$ containing $Q$, since we have $H \cap R=Q=H \cap T$, denoting by $\varepsilon$ and $\nu$ the respective local points of $R$ and
$T$ such that $P_{\gamma}$ contains $R_{\varepsilon}$ and $T_{\nu}$, these local pointed groups contain $Q_{\delta}$ and, in particular, any $\mathcal{E}$-morphism

$$
\left(\psi_{x}, \bar{x}\right): T \longrightarrow R
$$

determines an element $\left(\varphi_{x}, \bar{x}\right)$ of $\mathcal{E}(Q)$ fulfilling

$$
\left(\varphi_{x}, \bar{x}\right) \circ \mathcal{E}_{T}(Q) \circ\left(\varphi_{x}, \bar{x}\right)^{-1} \subset \mathcal{E}_{R}(Q)
$$

Thus, for any $y \in L$ such that $\pi(y)=\left(\varphi_{x}, \bar{x}\right)$, we have

$$
y \tau(T) y^{-1} \leq \tau(R)
$$

more precisely, for any $w \in T$ and any $v \in Q$, from equality 3.5.4 we get

$$
y \tau\left(v^{w}\right) y^{-1}=\tau\left(\varphi_{x}\left(v^{w}\right)\right)=\tau\left(\varphi_{x}(v)\right)^{\tau\left(\psi_{x}(w)\right)}
$$

moreover, since $x T x^{-1} \leq R$, we have

$$
\bar{\pi}\left(y \tau(w) y^{-1}\right)=\bar{x} \bar{w} \bar{x}^{-1}=\bar{\pi}\left(\tau\left(\psi_{x}(w)\right)\right)
$$

Hence, for any $w \in T$ and a suitable $\theta_{x}(w) \in Z$, we get

$$
y \tau\left(w \theta_{x}(w)\right) y^{-1}=\tau\left(\psi_{x}(w)\right) .
$$

Conversely, since $R$ and $T$ have a unique (local) point on $\mathcal{O} Q$, any $\hat{\mathcal{E}}$-morphism from $T$ to $R$ induced by an element $y$ of $L$ determines an element $\pi(y)=\left(\varphi_{x}, \bar{x}\right)$ of $\mathcal{E}(Q)$, for some $x \in N$, which still fulfills

$$
\left(\varphi_{x}, \bar{x}\right) \circ \mathcal{E}_{T}(Q) \circ\left(\varphi_{x}, \bar{x}\right)^{-1} \subset \mathcal{E}_{R}(Q)
$$

thus, as above, $x$ normalizes $Q_{\delta}$ and this inclusion forces

$$
C_{H}(Q) \cdot T \leq\left(C_{H}(Q) \cdot R\right)^{x} .
$$

Once again, respectively denoting by $\lambda$ and $\mu$ the points of $C_{H}(Q) \cdot R$ and $C_{H}(Q) \cdot T$ on $\mathcal{O H}$ such that $\left(C_{H}(Q) \cdot R\right)_{\lambda}$ and $\left(C_{H}(Q) \cdot T\right)_{\mu}$ contain $Q_{\delta}[18$, Lemma 3.9], and by $\varepsilon$ and $\nu$ the local points of $R$ and $T$ on $\mathcal{O} H$ such that $P_{\gamma}$ contains $R_{\varepsilon}$ and $T_{\nu}$, it follows from [14, Proposition 3.5] that $R_{\varepsilon}$ and $T_{\nu}$ are defect pointed groups of the respective pointed groups $\left(C_{H}(Q) \cdot R\right)_{\lambda}$ and $\left(C_{H}(Q) \cdot T\right)_{\mu}$; since by uniqueness we have

$$
\left(C_{H}(Q) \cdot T\right)_{\mu} \leq\left(C_{H}(Q) \cdot R\right)_{\lambda},
$$

there is $z \in C_{H}(Q)$ fulfilling $T_{\nu} \leq\left(R_{\varepsilon}\right)^{z x}$ [15, Theorem 1.2]. That is to say, the conjugation by $z x$ induces a group homomorphism $\psi_{z x}: T \rightarrow R$ mapping $Z$ onto $Z$ and inducing the element $\left(\psi_{z x}, \overline{z x}\right)$ of $\mathcal{E}(R, T)$ which extends $\left(\varphi_{x}, \bar{x}\right)$; hence, as above, for any $w \in T$ and a suitable $\theta_{y}(w) \in Z$ we get

$$
y \tau\left(w \theta_{y}(w)\right) y^{-1}=\tau\left(\psi_{z x}(w)\right) .
$$

We claim that, for a suitable choice of $\tau$, the elements $\theta_{x}(w)$ and $\theta_{y}(w)$ are always trivial; then, the equivalence of categories 3.5 .2 will be an easy consequence of the above correspondences. Above, for any $y \in L$ such that $\tau(T) \subset \tau(R)^{y}$ we have found an element $\left(\psi_{y}, \bar{\pi}(y)\right) \in \mathcal{E}(R, T)$ lifting $\pi(y)$ in such a way that, for any $w \in T$, we have

$$
\tau\left(w \theta_{y}(w)\right)=\tau\left(\psi_{y}(w)\right)^{y}
$$

for a suitable $\theta_{y}(w) \in Z$; note that, according to equality 3.5.4, for any $v \in Q$ we have $\theta_{y}(v)=1$, and whenever $y$ belongs to $\tau(R)$ we may choose $\psi_{y}$ in such a way that $\theta_{y}(w)=1$.

In this situation, for any $w, w^{\prime} \in T$, we get

$$
\begin{aligned}
\tau\left(w w^{\prime} \theta_{y}\left(w w^{\prime}\right)\right) & =\tau\left(\psi_{y}\left(w w^{\prime}\right)\right)^{y} \\
& =\tau\left(\psi_{y}(w)\right)^{y} \tau\left(\psi_{y}\left(w^{\prime}\right)\right)^{y} \\
& =\tau\left(w \theta_{y}(w)\right) \tau\left(w^{\prime} \theta_{y}\left(w^{\prime}\right)\right) \\
& =\tau\left(w \theta_{y}(w) w^{\prime} \theta_{y}\left(w^{\prime}\right)\right) \\
& =\tau\left(w w^{\prime} \theta_{y}(w)^{w^{\prime}} \theta_{y}\left(w^{\prime}\right)\right)
\end{aligned}
$$

and therefore, since $\tau$ is injective, we still get

$$
\theta_{y}\left(w w^{\prime}\right)=\theta_{y}(w)^{w^{\prime}} \theta_{y}\left(w^{\prime}\right)
$$

in particular, for any $z \in Z$ we have

$$
\theta_{y}(w z)=\theta_{y}(w)^{z} \theta_{y}(z)=\theta_{y}(w) .
$$

In other words, the map $\theta_{y}$ determines a $Z$-valued 1-cocycle from the image $\tilde{T}$ of $T$ in $\widetilde{\operatorname{Aut}}(Q)=\operatorname{Out}(Q)$.

Actually, the cohomology class $\bar{\theta}_{y}$ of this 1-cocycle does not depend on the choice of $\psi_{y}$; indeed, if another choice $\psi_{y}^{\prime}$ determines $\theta_{y}^{\prime}: T \rightarrow Z$ then we
clearly have $\psi_{y}^{\prime}(T)=\psi_{y}(T)$ and, according to our argument above, there is $z \in C_{H}(Q)$ such that

$$
\left(T_{\nu}\right)^{z}=T_{\nu} \quad \text { and } \quad \psi_{y}^{\prime}=\psi_{y} \circ \kappa_{z}
$$

where $\kappa_{z}: T \rightarrow T$ denotes the conjugation by $z$; actually, we still have

$$
[z, T] \leq H \cap T=Q
$$

But, since $T_{\nu}$ is a defect pointed group of $\left(C_{H}(Q) \cdot T\right)_{\mu}$ and, according to [4, Theorem 1.2] and [14, Proposition 6.5], $\mu$ determines a nilpotent block of the group $C_{H}(Q) \cdot T$, we have $N_{C_{H}(Q) \cdot T}\left(T_{\nu}\right)=C_{H}(T) \cdot T$. Thus, $z$ belongs to $Z \cdot C_{H}(T)$ and we actually may assume that $z$ belongs to $Z$.

In this case, it follows from equality 3.5.6 applied twice that

$$
\begin{aligned}
\tau\left(w \theta_{y}^{\prime}(w)\right) & =\tau\left(\psi_{y}^{\prime}(w)\right)^{y} \\
& =\tau\left(\psi_{y}\left(z w z^{-1}\right)\right)^{y} \\
& =\tau\left(\left(z w z^{-1}\right) \theta_{y}\left(z w z^{-1}\right)\right)
\end{aligned}
$$

for any $w \in T$ and, since $\theta_{y}\left(z w z^{-1}\right)=\theta_{y}(w)$ and $\tau$ is injective, we get

$$
\theta_{y}^{\prime}(w) \theta_{y}(w)^{-1}=w^{-1} z w z^{-1}=\left(z^{-1}\right)^{w} z
$$

Consequently, denoting by $\mathcal{T}_{L}$ the category where the objects are the subgroups of $\tau(P)$ and the set of morphisms $\mathcal{T}_{L}(\tau(R), \tau(T))$ from $\tau(T)$ to $\tau(R)$ is just the corresponding transporter in $L$, the correspondence sending an element $y \in \mathcal{T}_{L}(\tau(R), \tau(T))$ to the cohomology class $\bar{\theta}_{y}$ of $\theta_{y}$ determines a map

$$
\bar{\theta}_{R, T}: \mathcal{T}_{L}(\tau(R), \tau(T)) \longrightarrow \mathbb{H}^{1}(\tilde{T}, Z)
$$

Moreover, if $U$ is a subgroup of $P$ containing $Q$ and $t$ an element of $L$ fulfilling $\tau(U) \subset \tau(T)^{t}$, as above we can choose $\left(\psi_{t}, \bar{\pi}(t)\right) \in \mathcal{E}(T, U)$ lifting $\pi(t)$ in such a way that, for any $u \in U$, we have

$$
\tau\left(u \theta_{t}(u)\right)=\tau\left(\psi_{t}(u)\right)^{t}
$$

for a suitable $\theta_{t}(u) \in Z$; then, the composition $\left(\psi_{y}, \bar{\pi}(y)\right) \circ\left(\psi_{t}, \bar{\pi}(t)\right)$ lifts $\pi(y t)$ and, for any $u \in U$, we may assume that (cf. 3.5.4)

$$
\begin{aligned}
\tau\left(u \theta_{y t}(u)\right) & =\tau\left(\left(\psi_{y} \circ \psi_{t}\right)(u)\right)^{y t} \\
& =\tau\left(\psi_{t}(u) \theta_{y}\left(\psi_{t}(u)\right)\right)^{t} \\
& =\tau\left(u \theta_{t}(u)\right) \tau\left(\theta_{y}\left(\psi_{t}(u)\right)\right)^{t} \\
& =\tau\left(u \theta_{t}(u) \pi(t)^{-1}\left(\theta_{y}\left(\psi_{t}(u)\right)\right)\right)
\end{aligned}
$$

finally, since $\tau$ is injective, using additive notation in $Z$ we get

$$
\theta_{y t}(u)=\theta_{t}(u)+\pi(t)^{-1}\left(\theta_{y}\left(\psi_{t}(u)\right)\right)
$$

Hence, denoting by $\tilde{t}$ the image of $t$ in $\widetilde{\operatorname{Aut}}(Q)$ and by $\psi_{\tilde{t}}: \tilde{U} \rightarrow \tilde{T}$ and $\mathcal{Z}(\tilde{t}): Z \cong Z$ the corresponding group homomorphisms, we get the 1-cocycle condition

$$
\bar{\theta}_{y t}=\bar{\theta}_{t}+\mathbb{H}^{1}\left(\psi_{\tilde{t}}, \mathcal{Z}(\tilde{t})\right)\left(\bar{\theta}_{y}\right) ;
$$

in particular, since $\theta_{y}(w)=0$ whenever $y \in \tau(R)$, it is easily checked from this condition that $\bar{\theta}_{y}$ only depends on the class of $y$ in the exterior quotient

$$
\tilde{\mathcal{T}}_{L}(\tau(R), \tau(T))=\tau(R) \backslash \mathcal{T}_{L}(\tau(R), \tau(T))
$$

Thus, respectively denoting by $\tilde{L}, \tilde{R}, \tilde{T}$ and $\tilde{P}$ the images of $L, \tau(R), \tau(T)$ and $\tau(P)$ in $\widetilde{\operatorname{Aut}}(Q)$, the map $\bar{\theta}_{R, T}$ above admits a factorization

$$
\tilde{\bar{\theta}}_{\tilde{R}, \tilde{T}}: \tilde{\mathcal{T}}_{\tilde{L}}(\tilde{R}, \tilde{T}) \longrightarrow \mathbb{H}^{1}(\tilde{T}, Z) .
$$

That is to say, let us consider the exterior quotient $\tilde{\mathcal{T}}_{\tilde{L}}$ of the category $\mathcal{T}_{\tilde{L}}$ and the contravariant functor

$$
\mathfrak{h}_{Z}^{1}: \tilde{\mathcal{T}}_{\tilde{L}} \longrightarrow \mathfrak{A} \mathfrak{b}
$$

to the category of Abelian groups $\mathfrak{A} \mathfrak{b}$ mapping $\tilde{T}$ on $\mathbb{H}^{1}(\tilde{T}, Z)$; then, identifying the $\tilde{\mathcal{T}}_{\tilde{L}}$-morphism $\tilde{y} \in \tilde{\mathcal{T}}_{\tilde{L}}(\tilde{R}, \tilde{T})$ with the obvious $\tilde{\mathcal{T}}_{\tilde{L}}$-chain $\Delta_{1} \longrightarrow \tilde{\mathcal{T}}_{\tilde{L}}-$ the functor from the category $\Delta_{1}$, formed by the objects 0 and 1 and a nonidentity morphism $0 \bullet 1$ from 0 to 1 , mapping 0 on $T, 1$ on $R$ and $0 \bullet 1$ on $\tilde{y}$ — the family $\bar{\theta}=\left\{\bar{\theta}_{\tilde{y}}\right\}_{\tilde{y}}$, where $\tilde{y}$ runs over the set of all the $\tilde{\mathcal{T}}_{\tilde{L}}$-morphisms,
defines a 1-cocycle from $\tilde{\mathcal{T}}_{\tilde{L}}$ to $\mathfrak{h}_{Z}^{1}$ since equalities 3.5 .7 guarantee that the differential map sends $\bar{\theta}$ to zero.

We claim that this 1 -cocycle is a 1-coboundary; indeed, for any subgroup $\tilde{R}$ of $\tilde{P}$, choose a set of representatives $\tilde{X}_{\tilde{R}} \subset \tilde{L}$ for the set of double classes $\tilde{P} \backslash \tilde{L} / \tilde{R}$ and, for any $\tilde{n} \in \tilde{X}_{\tilde{R}}$, set $\tilde{R}_{\tilde{n}}=\tilde{R} \cap P^{\tilde{n}}$, consider the $\tilde{\mathcal{T}}_{\tilde{L}}$-morphisms $\tilde{n}: \tilde{R}_{\tilde{n}} \rightarrow \tilde{P}$ and $\tilde{\imath}_{\tilde{R}_{\tilde{n}}}^{\tilde{R}}: \tilde{R}_{\tilde{n}} \rightarrow \tilde{R}$ respectively determined by $\tilde{n}$ and by the trivial element of $\tilde{L}$, and denote by

$$
\left(\mathfrak{h}_{Z}^{1}\right)^{\circ}\left(\tilde{\imath}_{\tilde{R}_{\tilde{n}}^{\tilde{N}}}\right): \mathbb{H}^{1}\left(\tilde{R}_{\tilde{n}}, Z\right) \longrightarrow \mathbb{H}^{1}(\tilde{R}, Z)
$$

the corresponding transfer homomorphism [7, Ch. XII, §8]; then, setting

$$
\bar{\sigma}_{\tilde{R}}=\frac{|P|}{|L|} \cdot \sum_{\tilde{n} \in \tilde{X}_{\tilde{R}}}\left(\left(\mathfrak{h}_{Z}^{1}\right)^{\circ}\left(\tilde{\tau}_{\tilde{R}_{\tilde{n}}}^{\tilde{R}}\right)\right)\left(\bar{\theta}_{\tilde{n}}\right),
$$

we claim that, for any $\tilde{y} \in \overline{\mathcal{T}}_{\tilde{L}}(\tilde{R}, \tilde{T})$, we have

$$
\bar{\theta}_{\tilde{y}}=\bar{\sigma}_{\tilde{T}}-\left(\mathfrak{h}_{Z}^{1}(\tilde{y})\right)\left(\bar{\sigma}_{\tilde{R}}\right) .
$$

Indeed, note that $\mathfrak{h}_{Z}^{1}(\tilde{y})$ is the composition of the restriction via the $\tilde{\mathcal{T}}_{\tilde{L}}$-morphism

$$
\tilde{\imath}_{\tilde{y} \tilde{y} \tilde{y}^{-1}}: \tilde{y} \tilde{T} \tilde{y}^{-1} \longrightarrow \tilde{R}
$$

determined by the trivial element of $L$, with the conjugation determined by $\tilde{y}$, which we denote by $\mathfrak{h}_{Z}^{1}\left(\tilde{y}_{*}\right)$; thus, by the corresponding Mackey equalities [7, Ch. XII, Proposition 9.1], we get

$$
\begin{gathered}
\left.\mathfrak{h}_{Z}^{1}(\tilde{y})\left(\sum_{\tilde{n} \in \tilde{X}_{R}}\left(\mathfrak{h}_{Z}^{1}\right)^{\circ}\left(\tilde{\imath}_{\tilde{R}_{\tilde{n}}}^{\tilde{R}}\right)\right)\left(\bar{\theta}_{\tilde{n}}\right)\right) \\
\left.=\mathfrak{h}_{Z}^{1}\left(\tilde{y}_{*}\right)\left(\sum_{\tilde{n} \in \tilde{X}_{\tilde{R}}} \sum_{\tilde{r} \in \tilde{Y}_{\tilde{n}}}\left(\mathfrak{h}_{Z}^{1}\right)^{\circ}\left(\begin{array}{l}
\tilde{y_{\tilde{P}}^{\tilde{T}} \tilde{y} \tilde{r}} \cap \tilde{y} \tilde{T} \tilde{y}^{-1}
\end{array}\right) \circ \mathfrak{h}_{Z}^{1}(\tilde{r})\right)\left(\bar{\theta}_{\tilde{n}}\right)\right) \\
\left.=\quad \sum_{\tilde{n} \in \tilde{X}_{\tilde{R}}} \sum_{\tilde{r} \in \tilde{Y}_{\tilde{n}}}\left(\mathfrak{h}_{Z}^{1}\right)^{\circ}\left(\tilde{\imath}_{\tilde{P} \tilde{\tilde{n}} \tilde{r} \tilde{y} \cap \tilde{T}}\right) \circ \mathfrak{h}_{Z}^{1}(\tilde{r} \tilde{y})\right)\left(\bar{\theta}_{\tilde{n}}\right),
\end{gathered}
$$

where, for any $\tilde{n} \in X_{\tilde{R}}$, the subset $\tilde{Y}_{\tilde{n}} \subset \tilde{R}$ is a set of representatives for the set of double classes $\tilde{R}_{\tilde{n}} \backslash \tilde{R} / \tilde{y} \tilde{T} \tilde{y}^{-1}$ and, for any $\tilde{r} \in \tilde{Y}_{\tilde{n}}$, we consider the $\tilde{\mathcal{T}}_{\tilde{L}}$-morphisms

$$
\tilde{r}: \tilde{P} \tilde{n} \tilde{r} \cap \tilde{y} \tilde{T} \tilde{y}^{-1} \longrightarrow \tilde{R}_{\tilde{n}} \quad \text { and } \quad \tilde{r} \tilde{y}: \tilde{P}^{\tilde{n} \tilde{r} \tilde{y}} \cap \tilde{T} \longrightarrow \tilde{R}_{\tilde{n}} .
$$

Moreover, setting $\tilde{m}=\tilde{n} \tilde{r} \tilde{y}$ for $\tilde{n} \in \tilde{X}_{\tilde{R}}$ and $\tilde{r} \in \tilde{Y}_{\tilde{n}}$, since we assume that $\theta_{\tilde{r}}=0$, it follows from equality 3.5.7 that

$$
\left(\mathfrak{h}_{Z}^{1}(\tilde{r} \tilde{y})\right)\left(\bar{\theta}_{\tilde{n}}\right)=\bar{\theta}_{\tilde{m}}-\bar{\theta}_{\tilde{r} \tilde{y}}=\bar{\theta}_{\tilde{m}}-\left(\mathfrak{h}_{Z}^{1}\left(\tilde{i}_{\tilde{T}_{\tilde{m}}}^{\tilde{T}}\right)\right)\left(\bar{\theta}_{\tilde{y}}\right) ;
$$

thus, choosing $\tilde{X}_{\tilde{T}}=\bigsqcup_{\tilde{n} \in \tilde{X}_{\tilde{R}}} \tilde{n} \tilde{Y}_{\tilde{n}} \tilde{y}$, we get [7, Ch. XII, $\left.\S 8 .(6)\right]$

$$
\begin{aligned}
\bar{\sigma}_{\tilde{T}}-\left(\mathfrak{h}_{Z}^{1}(\tilde{y})\right)\left(\bar{\sigma}_{\tilde{R}}\right) & =\frac{|P|}{|L|} \cdot \sum_{\tilde{m} \in \tilde{X}_{\tilde{T}}}\left(\left(\mathfrak{h}_{Z}^{1}\right)^{\circ}\left(\tilde{\imath}_{\tilde{T}_{\tilde{m}}}^{\tilde{T}}\right)\right)\left(\bar{\theta}_{\tilde{m}}-\left(\mathfrak{h}_{Z}^{1}(\tilde{r} \tilde{y})\right)\left(\bar{\theta}_{\tilde{n}}\right)\right) \\
& =\frac{|P|}{|L|} \cdot \sum_{\tilde{m} \in \tilde{X}_{\tilde{T}}}\left(\left(\mathfrak{h}_{Z}^{1}\right)^{0}\left(\tilde{\imath}_{\tilde{T}_{\tilde{m}}}^{\tilde{T}}\right)\right)\left(\left(\mathfrak{h}_{Z}^{1}\left(\tilde{i}_{\tilde{T}_{\tilde{m}}}^{\tilde{T}}\right)\right)\left(\bar{\theta}_{\tilde{y}}\right)\right) \\
& =\sum_{\tilde{m} \in \tilde{X}_{\tilde{T}}} \frac{\left|\tilde{T} / \tilde{T}_{\tilde{m}}\right|}{|\tilde{L} / \tilde{P}|} \cdot \bar{\theta}_{\tilde{y}}=\bar{\theta}_{\tilde{y}} .
\end{aligned}
$$

In particular, for any subgroup $\tilde{R}$ of $\tilde{P}$, we get

$$
\bar{\sigma}_{\tilde{R}}=\left(\mathfrak{h}_{Z}^{1}\left(\tilde{\imath}_{\tilde{R}}^{\tilde{R}}\right)\right)\left(\bar{\sigma}_{\tilde{P}}\right)
$$

and the element $\bar{\sigma}_{\tilde{P}} \in \mathbb{H}^{1}(\tilde{P}, Z)$ can be lifted to a 1-cocycle $\sigma_{\tilde{P}}: \tilde{P} \rightarrow Z$ which determines a group automorphism $\sigma: P \cong P$ mapping $u \in P$ on $u \sigma_{\tilde{P}}(\tilde{u})$ where $\tilde{u}$ denotes the image of $u$ in $\tilde{P}$; moreover, according to equality 3.5.8, in 3.5.5 we may choose

$$
\theta_{y}(w)=\sigma_{\tilde{P}}(\tilde{w})(\pi(y))^{-1}\left(\sigma_{\tilde{P}}\left(\widetilde{\psi_{y}(w)}\right)\right)^{-1}
$$

Hence, replacing $\tau$ by $\hat{\tau}=\tau \circ \sigma$, the maps $\pi$ and $\hat{\tau}$ still fulfill the conditons above and, for any $w \in T$, in equality 3.5.6 we get

$$
\begin{aligned}
\tau\left(\psi_{y}(w)\right)^{y} & =\tau\left(w \theta_{y}(w)\right) \\
& =\tau\left(w\left(w^{-1} \sigma(w)\right)(\pi(y))^{-1}\left(\psi_{y}(w)^{-1} \sigma\left(\psi_{y}(w)\right)\right)^{-1}\right) \\
& =\tau\left(\sigma(w)(\pi(y))^{-1}\left(\sigma\left(\psi_{y}(w)\right)^{-1} \psi_{y}(w)\right)\right) \\
& =\hat{\tau}(w) \tau\left(\sigma\left(\psi_{y}(w)\right)^{-1} \psi_{y}(w)\right)^{y} \\
& =\hat{\tau}(w) \hat{\tau}\left(\psi_{y}(w)^{-1}\right)^{y} \tau\left(\psi_{y}(w)\right)^{y}
\end{aligned}
$$

so that, as announced, we obtain

$$
\hat{\tau}\left(\psi_{y}(w)\right)^{y}=\hat{\tau}(w)
$$

In conclusion, we get a functor from $\hat{\mathcal{E}}$ to $\mathcal{E}$ mapping any $\hat{\mathcal{E}}$-morphism

$$
\left(\kappa_{y}, \bar{y}\right): \hat{\tau}(T) \longrightarrow \hat{\tau}(R)
$$

induced by an element $y$ of $L$, where $\kappa_{y}$ denotes the corresponding conjugation by $y$ which actually fulfills $\hat{\tau}(Q \cdot T) \leq(\hat{\tau}(Q \cdot R))^{y}$, on the $\mathcal{E}$-morphism

$$
\left(\psi_{y}, \bar{\pi}(y)\right): T \longrightarrow R
$$

where $\psi_{y}: T \rightarrow R$ is the group homomorphism determined by the equality

$$
\hat{\tau}_{R} \circ \psi_{y}=\kappa_{y} \circ \hat{\tau}_{T}
$$

$\hat{\tau}_{R}$ and $\hat{\tau}_{T}$ denoting the respective restrictions of $\hat{\tau}$ to $R$ and $T$; indeed, it is clear that this correspondence maps the composition of $\hat{\mathcal{E}}$-morphisms on the corresponding composition of $\mathcal{E}$-morphisms. Moreover, it is clear that this functor is faithful, and it follows from our argument above that any $\mathcal{E}$-morphism

$$
\left(\psi_{x}, \bar{x}\right): T \longrightarrow R
$$

comes from an $\hat{\mathcal{E}}$-morphism from $\hat{\tau}(T)$ to $\hat{\tau}(R)$.
Moreover, for another triple $L^{\prime}, \tau^{\prime}$ and $\bar{\pi}^{\prime}$ fulfilling the above conditions, the corresponding equivalences of categories 3.5.2 induce an equivalence of categories

$$
\hat{\mathcal{E}} \cong \mathcal{E}_{\left(1, \tau^{\prime}(Q), L^{\prime}\right)}=\mathcal{E}^{\prime}
$$

in particular, we have a group homomorphism

$$
\bar{\sigma}: L \longrightarrow \hat{\mathcal{E}}(\hat{\tau}(Q)) \cong \mathcal{E}^{\prime}\left(\tau^{\prime}(Q)\right) \cong L^{\prime} / \tau^{\prime}(Z)
$$

and we claim that Lemma 3.6 below applies to the finite groups $L$ and $L^{\prime}$, with the Sylow $p$-subgroup $\hat{\tau}(P)$ of $L$, the Abelian normal $p$-group $\tau^{\prime}(Z)$ of $L^{\prime}$ and the group homomorphism $\bar{\sigma}: L \rightarrow L^{\prime} / \tau^{\prime}(Z)$ above; indeed, the group homomorphism $\hat{\tau}(P) \rightarrow L^{\prime}$ mapping $\hat{\tau}(u)$ on $\tau^{\prime}(u)$, for any $u \in P$, clearly lifts the restriction of $\bar{\sigma}$ and it is easily checked from the equivalence 3.5 .9 that it fulfills condition 3.6 .1 below. Consequently, the last statement immediately follows from this lemma. We are done.

Lemma 3.6. Let $L$ be a finite group, $M$ a group, $Z$ a normal Abelian $p^{\prime}$-divisible subgroup of $M$ and $\bar{\sigma}: L \rightarrow \bar{M}=M / Z$ a group homomorphism. Assume that, for a Sylow p-subgroup $P$ of $L$, there exists a group homomorphism $\tau: P \rightarrow M$ lifting the restriction of $\bar{\sigma}$ to $P$ and fulfilling the following condition
3.6.1 For any subgroup $R$ of $P$ and any $x \in L$ such that $R^{x} \subset P$, there is $y \in M$ such that $\bar{\sigma}(x)=\bar{y}$ and $\tau\left(u^{x}\right)=\tau(u)^{y}$ for any $u \in R$.
Then, there is a group homomorphism $\sigma: L \rightarrow M$ lifting $\bar{\sigma}$ and extending $\tau$. Moreover, if $\sigma^{\prime}: L \rightarrow M$ is a group homomorphism which lifts $\bar{\sigma}$ and extends $\tau$, then there is $z \in Z$ such that $\sigma^{\prime}(x)=\sigma(x)^{z}$ for any $x \in L$.

Proof: It is clear that $\bar{\sigma}$ determines an action of $L$ on $Z$ and it makes sense to consider the cohomology groups $\mathbb{H}^{n}(L, Z)$ and $\mathbb{H}^{n}(P, Z)$ for any $n$ in $\mathbb{N}$. But, $M$ determines an element $\bar{\mu}$ of $\mathbb{H}^{2}(\bar{M}, Z)$ [7, Chap. XIV, Theorem 4.2] and if there is a group homomorphism $\tau: P \rightarrow M$ lifting the restriction of $\bar{\sigma}$ then the corresponding image of $\bar{\mu}$ in $\mathbb{H}^{2}(P, Z)$ has to be zero [7, Chap. XIV, Theorem 4.2]; thus, since the restriction map

$$
\mathbb{H}^{2}(L, Z) \longrightarrow \mathbb{H}^{2}(P, Z)
$$

is injective [7, Ch. XII, Theorem 10.1], we also get

$$
\left(\mathbb{H}^{2}\left(\bar{\sigma}, \mathrm{id}_{Z}\right)\right)(\bar{\mu})=0
$$

and therefore there is a group homomorphism $\sigma: L \rightarrow M$ lifting $\bar{\sigma}$.
At this point, the difference between $\tau$ and the restriction of $\sigma$ to $P$ defines a 1-cocycle $\theta: P \rightarrow Z$ and, for any subgroup $R$ of $P$ and any $x \in L$ such that $R^{x} \subset P$, it follows from condition 3.6.1 that, for a suitable $y \in M$ fulfilling $\bar{y}=\bar{\sigma}(x)$, for any $u \in R$ we have

$$
\begin{aligned}
\theta\left(u^{x}\right) & =\tau\left(u^{x}\right)^{-1} \sigma\left(u^{x}\right) \\
& =\tau\left(u^{-1}\right)^{y} \sigma(u)^{\sigma(x)} \\
& =\tau\left(u^{-1}\right)^{y} \tau(u)^{\sigma(x)} \theta(u)^{\sigma(x)} \\
& =\left(\left(y \sigma(x)^{-1}\right)^{-1}\left(y \sigma(x)^{-1}\right)^{\tau(u)} \theta(u)\right)^{\sigma(x)}
\end{aligned}
$$

consequently, since the map sending $u \in R$ to

$$
\left(y \sigma(x)^{-1}\right)^{-1}\left(y \sigma(x)^{-1}\right)^{\tau(u)} \in Z
$$

is a 1 -coboundary, the cohomology class $\bar{\theta}$ of $\theta$ is $L$-stable, and it follows again from [7, Ch. XII, Theorem 10.1] that it is the restriction of a suitable element $\bar{\eta} \in \mathbb{H}^{1}(L, Z)$; then, it suffices to modify $\sigma$ by a representative of $\bar{\eta}$ to get a new group homomorphism $\sigma^{\prime}: L \rightarrow M$ lifting $\bar{\sigma}$ and extending $\tau$.

Now, if $\sigma^{\prime}: L \rightarrow M$ is a group homomorphism which lifts $\bar{\sigma}$ and extends $\tau$, the element $\sigma^{\prime}(x) \sigma(x)^{-1}$ belongs to $Z$ for any $x \in L$ and thus, we get a 1-cocycle $\lambda: L \rightarrow Z$ mapping $x \in L$ on $\sigma^{\prime}(x) \sigma(x)^{-1}$, which vanish over $P$; hence, it is a 1 -coboundary [7, Ch. XII, Theorem 10.1] and therefore there exists $z \in Z$ such that

$$
\lambda(x)=z^{-1} \sigma(x) z \sigma(x)^{-1}
$$

so that we have $\sigma^{\prime}(x)=\sigma(x)^{z}$ for any $x \in L$. We are done.
3.7. Since $Q$ normalizes a unitary full matrix $\mathcal{O}$-subalgebra $T$ of $\mathcal{B}_{\delta}$ such that [18, Theorem 1.6]

$$
\mathcal{B}_{\delta} \cong T Q \quad \text { and } \quad \operatorname{rank}_{\mathcal{O}}(T) \equiv 1 \bmod p
$$

the action of $Q$ on $T$ admits a unique lifting to a group homomorphism [18, 1.8]

$$
Q \longrightarrow \operatorname{Ker}\left(\operatorname{det}_{T}\right)
$$

hence, we have

$$
\mathcal{B}_{\delta} \cong T \otimes_{\mathcal{O}} \mathcal{O} Q
$$

and therefore $\mathcal{B}_{\delta}$ admits a unique two-sided ideal $\mathfrak{n}_{\delta}$ such that, considering $\mathcal{B}_{\delta} / \mathfrak{n}_{\delta}$ as a $Q$-interior $\mathcal{O}$-algebra, there is an isomorphism

$$
\mathcal{B}_{\delta} / \mathfrak{n}_{\delta} \cong T
$$

of $Q$-interior $\mathcal{O}$-algebras. Then, a canonical embedding $f_{\delta}: \mathcal{B}_{\delta} \rightarrow \operatorname{Res}_{Q}^{H}(\mathcal{B})$ $\left[\begin{array}{ll}18, & 2.8\end{array}\right]$ and the ideal $\mathfrak{n}_{\delta}$ determine a two-sided ideal $\mathfrak{n}$ of $\mathcal{B}$ such that $S=\mathcal{B} / \mathfrak{n}$ is also a full matrix $\mathcal{O}$-algebra.

Proposition 3.8. With the notation above, the action of $N$ on $\mathcal{B}$ stabilizes $\mathfrak{n}$.

Proof: Since we have $N=H \cdot N_{G}\left(Q_{\delta}\right)$, for the first statement we may consider $x \in N_{G}\left(Q_{\delta}\right)$; then, denoting by $\sigma_{x}$ the automorphism of $Q$ induced by the conjugation by $x$, it is clear that the isomorphism

$$
f_{x}: \operatorname{Res}_{\sigma_{x}}\left(\operatorname{Res}_{Q}^{H}(\mathcal{B})\right) \cong \operatorname{Res}_{Q}^{H}(\mathcal{B})
$$

of $Q$-interior algebras mapping $a \in \mathcal{B}$ on $a^{x}$ induces a commutative diagram of exterior homomorphisms of $Q$-interior algebras [18, 2.8]

moreover, the uniqueness of $\mathfrak{n}_{\delta}$ clearly implies that this ideal is stabilized by $\left(\tilde{f}_{x}\right)_{\delta}$; consequently, $\mathfrak{n}$ is still stabilized by $\tilde{f}_{x}$.
3.9. In particular, $N$ acts on the full matrix $\mathcal{O}$-algebra $S$ and therefore the action on $S$ of any element $x \in N$ can be lifted to a suitable $s_{x} \in S^{*}$; thus, setting $\mathrm{r}=\operatorname{rank}_{\mathcal{O}}(S)$, denoting by $\bar{H}$ the image of $H$ in $S^{*}$ and considering a finite extension $\mathcal{O}^{\prime}$ of $\mathcal{O}$ containing the group $U$ of $|H|$-th roots of unity and the r-th roots of $\operatorname{det}_{S}\left(s_{x}\right)$ for any $x \in N$, since r divides $|H|$, the pull-back

$$
\begin{array}{ccc}
N & \longrightarrow & \operatorname{Aut}\left(\mathcal{O}^{\prime} \otimes \mathcal{O} S\right) \\
\uparrow & & \uparrow \\
\hat{N} & \longrightarrow & (U \otimes \bar{H}) \cdot \operatorname{Ker}\left(\operatorname{det}_{\mathcal{O}^{\prime} \otimes \mathcal{O} S}\right)
\end{array}
$$

determines a central extension $\hat{N}$ of $N$ by $U$, which clearly does not depend on the choice of $\mathcal{O}^{\prime}$; moreover, the inclusion $H \leq N$ and the structural group homomorphism $H \rightarrow\left(\mathcal{O}^{\prime} \otimes_{\mathcal{O}} S\right)^{*}$ induces an injective group homomorphism $H \rightarrow \hat{N}$ with an image which is a normal subgroup of $\check{N}$ and has a trivial intersection with the image of $U$ - we identify this image with $H$ and set

$$
\hat{N}=\hat{N} / H .
$$

We will consider the $H$-interior $N$-algebras (see [21, 2.1])

$$
\hat{\mathcal{A}}=S^{\circ} \otimes_{\mathcal{O}} \mathcal{A} \quad \text { and } \quad \hat{\mathcal{B}}=S^{\circ} \otimes_{\mathcal{O}} \mathcal{B}
$$

and note that $\mathcal{O}^{\prime} \otimes \hat{\mathcal{A}}$ actually has an $\hat{N}$-interior algebra structure.
3.10. On the other hand, since $b$ is also a nilpotent block of the group $H \cdot P$, it is easily checked that [18, 1.9]

$$
\mathcal{O}(H \cdot P) b / J(\mathcal{O}(H \cdot P) b) \cong k \otimes_{\mathcal{O}} S ;
$$

moreover, since the inclusion map $\mathcal{O} H \rightarrow \mathcal{O}(H \cdot P)$ is a semicovering of $P$-algebras [14, Example 3.9, 3.10 and Theorem 3.16], we can identify $\gamma$
with a local point of $P$ on $\mathcal{O}(H \cdot P) b$. Set $\mathcal{O}(H \cdot P)_{\gamma}=i(\mathcal{O}(H \cdot P)) i$ and $S_{\gamma}=\bar{\imath} S \bar{\imath}$, where $\bar{\imath}$ is the image of $i$ in $S$; then, as in 3.7 above, we have an isomorphism of $P$-interior algebras [18, Theorem 1.6]

$$
\mathcal{O}(H \cdot P)_{\gamma} \cong S_{\gamma} P
$$

$S_{\gamma}$ is actually a Dade $P$-algebra - namely, a full matrix $P$-algebra over $\mathcal{O}$ where $P$ stabilizes an $\mathcal{O}$-basis containing the unity element - such that $\operatorname{rank}_{\mathcal{O}}\left(S_{\gamma}\right) \equiv 1 \bmod p$, and the action of $P$ on $S_{\gamma}$ can be uniquely lifted to a group homomorphism $P \rightarrow \operatorname{Ker}\left(\operatorname{det}_{S_{\gamma}}\right)$ [18, 1.8], so that isomorphism 3.10.1 becomes

$$
\mathcal{O}(H \cdot P)_{\gamma} \cong S_{\gamma} \otimes_{\mathcal{O}} \mathcal{O} P
$$

Proposition 3.11. With the notation above, the structural homomorphism $\mathcal{B}_{\gamma} \rightarrow S_{\gamma}$ of $P$-algebras is a strict semicovering.

Proof: It follows from isomorphism 3.10.2 that the canonical homomorphism of $P$-algebras
3.11.1

$$
\mathcal{O}(H \cdot P)_{\gamma} \longrightarrow S_{\gamma}
$$

admits a $P$-algebra section mapping $s \in S_{\gamma}$ on the image of $s \otimes 1$ by the inverse of that isomorphism, which proves that the $P$-interior algebra homomorphism 3.11.1 is a covering [18, 4.14 and Example 4.25]; thus, since the inclusion map $\mathcal{O} H \rightarrow \mathcal{O}(H \cdot P)$ is a semicovering of $P$-algebras, the canonical homomorphism of $P$-algebras

$$
\mathcal{B}_{\gamma}=(\mathcal{O H})_{\gamma} \longrightarrow S_{\gamma}
$$

remains a semicovering [14, Proposition 3.13]; moreover, since $\mathfrak{n} \leq J(\mathcal{B})$, it is a strict semicovering [14, 3.10].
3.12. Consequently, it easily follows from [14, Theorem 3.16] and [18, Proposition 5.6] that we still have a strict semicovering homomorphism of $P$-algebras

$$
\left(S_{\gamma}\right)^{\circ} \otimes_{\mathcal{O}} \mathcal{B}_{\gamma} \longrightarrow\left(S_{\gamma}\right)^{\circ} \otimes_{\mathcal{O}} S_{\gamma} \cong \operatorname{End}_{\mathcal{O}}\left(S_{\gamma}\right) ;
$$

hence, denoting by $\hat{\gamma}$ the local point of $P$ over $\left(S_{\gamma}\right)^{\circ} \otimes_{\mathcal{O}} \mathcal{B}_{\gamma}$ determined by $\gamma$, the image of $\hat{\gamma}$ in $\left(S_{\gamma}\right)^{\circ} \otimes_{\mathcal{O}} S_{\gamma}$ is contained in the corresponding local point of $P$ and therefore we get a strict semicovering homomorphism [18, 5.7]

$$
\hat{\mathcal{B}}_{\hat{\gamma}} \longrightarrow \mathcal{O} \cong\left(\left(S_{\gamma}\right)^{\circ} \otimes_{\mathcal{O}} S_{\gamma}\right)_{\hat{\gamma}}
$$

of $P$-algebras; that is to say, any $\hat{\imath} \in \hat{\gamma}$ is actually a primitive idempotent in $\hat{\mathcal{B}}$ and therefore, for any local pointed group $R_{\hat{\varepsilon}}$ over $\hat{\mathcal{B}}$ contained in $P_{\hat{\gamma}}$, it also belongs to $\hat{\varepsilon}$; in particular, denoting by $\hat{\delta}$ the local point of $Q$ over $\left(S_{\gamma}\right)^{\circ} \otimes_{\mathcal{O}} \mathcal{B}_{\gamma}$ determined by $\delta$, we clearly have $\hat{\mathcal{B}}_{\hat{\delta}}=\hat{\mathfrak{\mathcal { B }}} \hat{\imath} \cong \mathcal{O} Q$ (cf. 3.7.1).
3.13. As in $[14,2.11]$, we consider the $P$-interior algebra $\hat{\mathcal{A}}_{\hat{\gamma}}=\hat{\imath} \hat{\mathcal{A}} \hat{\imath}$; since $\mathcal{A}$ is an $N / H$-graded algebra, $\hat{\mathcal{A}}_{\hat{\gamma}}$ is also an $N / H$-graded algebra. On the other hand, since $\mathcal{O}^{\prime} / J\left(\mathcal{O}^{\prime}\right) \cong k$, we get a group homomorphism $\varpi: U \rightarrow k^{*}$ and, setting $\Delta_{\varpi}(U)=\left\{\left(\varpi(\xi), \xi^{-1}\right)\right\}_{\xi \in U}$, we obtain the obvious $k^{*}$-group

$$
\hat{\bar{N}}^{k}=\left(k^{*} \times \hat{\bar{N}}\right) / \Delta_{\varpi}(U) ;
$$

then, with the notation of Theorem 3.5, we set [17, 5.7]

$$
\hat{L}=\operatorname{Res}_{\bar{\pi}}\left(\hat{\bar{N}}^{k}\right) ;
$$

thus, $\mathcal{O}_{*} \hat{L}^{\circ}$ becomes a $P$-interior algebra via the lifting $\hat{\tau}: P \rightarrow \hat{L}^{\circ}$ of the group homomorphism $\tau: P \rightarrow L$, and it has an obvious $L / \tau(Q)$-graded algebra structure. The group homomorphism $\bar{\pi}$ induces a group isomorphism $L / \tau(Q) \cong N / H$, through which we identify $L / \tau(Q)$ and $N / H$, so that $\mathcal{O}_{*} \hat{L}^{\circ}$ becomes an $N / H$-graded algebra.

Theorem 3.14. With the notation above, we have a P-interior and $N / H$ graded algebra isomorphism $\hat{\mathcal{A}}_{\hat{\gamma}} \cong \mathcal{O}_{*} \hat{L}^{\circ}$.

Proof: Choosing $\hat{\imath} \in \hat{\gamma}$, we consider the groups

$$
M=N_{(\hat{\imath} \hat{\mathcal{A}} \hat{\imath})^{*}}(Q \cdot \hat{\imath}) / k^{*} \cdot \hat{\imath} \quad \text { and } \quad Z=\left((\hat{\imath} \hat{\mathcal{B}} \hat{\imath})^{Q}\right)^{*} / k^{*} \cdot \hat{\imath} \cong 1+J(Z(\mathcal{O} Q))
$$

it is clear that $Z$ is a normal Abelian $p^{\prime}$-divisible subgroup of $M$, and we set $\bar{M}=M / Z$. In order to apply Lemma 3.6, let $R$ be a subgroup of $P$ and $y$ an element of $L$ such that $\tau(R) \leq \tau(P)^{y}$; since $\tau(Q)$ is normal in $L$, we actually may assume that $R$ contains $Q$. According to the equivalence of categories 3.5.2, denoting by $\varepsilon$ the unique local point of $R$ on $\mathcal{B}$ fulfilling $R_{\varepsilon} \leq P_{\gamma}$ [14, Theorem 6.6], there is $x_{y} \in N$ such that
3.14.1

$$
\bar{x}_{y}=\bar{\pi}(y) \quad, \quad R_{\varepsilon} \leq\left(P_{\gamma}\right)^{x_{y}} \quad \text { and } \quad \tau\left({ }^{x_{y}} v\right)={ }^{y} \tau(v) \text { for any } v \in R ;
$$

in particular, $x_{y}$ normalizes $Q_{\delta}$.

By Proposition 3.11, a local pointed group $R_{\varepsilon}$ on $\mathcal{B}$ such that

$$
Q_{\delta} \leq R_{\varepsilon} \leq P_{\gamma}
$$

determines a local pointed group $R_{\tilde{\varepsilon}}$ on $S$ through the composition

$$
\mathcal{B}_{\gamma} \longrightarrow S_{\gamma} \hookrightarrow S
$$

(see [14, Proposition 3.15]). Since $S_{\gamma}$ has a $P$-stable $\mathcal{O}$-basis, $S_{\varepsilon}$ still has a $R$-stable $\mathcal{O}$-basis and, by [18, Theorem 5.3], there are unique local pointed groups $R_{\tilde{\varepsilon}}$ on $S_{\varepsilon}$ and $R_{\hat{\varepsilon}}$ on $\hat{\mathcal{B}}$ such that $\hat{l}(\tilde{l} \otimes l)=\hat{l}=(\tilde{l} \otimes l) \hat{l}$ for suitable $l \in \varepsilon, \tilde{l} \in \tilde{\varepsilon}$ and $\hat{l} \in \hat{\varepsilon} ;$. then, we claim that $R_{\hat{\varepsilon}} \leq\left(P_{\hat{\gamma}}\right)^{x_{y}}$ and that $x_{y}$ stabilizes $Q_{\hat{\delta}}$. Indeed, since $\left(R_{\varepsilon}\right)^{x_{y}^{-1}} \leq P_{\gamma}$, we have $\left(R_{\tilde{\varepsilon}}\right)^{x_{y}^{-1}} \leq P_{\tilde{\gamma}}$ and then it follows from [18, Proposition 5.6] that we have $\left(R_{\hat{\varepsilon}}\right)^{x_{y}^{-1}} \leq P_{\hat{\gamma}}$ or, equivalently, $R_{\hat{\varepsilon}} \leq\left(P_{\hat{\gamma}}\right)^{x_{y}}$; moreover, since $\delta$ is the unique local point of $Q$ such that $Q_{\delta}$ is contained in $P_{\gamma}$, again by [18, Proposition 5.6] we can easily conclude that $x_{y}$ stabilizes $Q_{\hat{\delta}}$.

In particular, since the image of $\hat{\imath}^{x_{y}}$ in $\hat{\mathcal{B}}\left(R_{\hat{\varepsilon}}\right)$ is not zero [14, 2.7] and since $\hat{\imath}$ is primitive in $\hat{\mathcal{B}}, \hat{\imath}^{x_{y}}$ belongs to $\hat{\varepsilon}$ and therefore, since $\hat{\imath}$ also belongs to $\hat{\varepsilon}$, there is $\hat{a}_{y} \in\left(\hat{\mathcal{B}}^{R}\right)^{*}$ such that $\hat{\imath}^{x_{y}}=\hat{\imath}^{\hat{a}_{y}}$; choose $s_{y} \in S^{*}$ lifting the action of $x_{y}$ on $S$ and set $\hat{x}_{y}=s_{y} \otimes x_{y}$, so that we have

$$
\hat{\imath}^{x_{y}}=\left(\hat{x}_{y}\right)^{-1} \hat{\imath} \hat{x}_{y}
$$

then, since $\hat{x}_{y}$ and $\hat{a}_{y}$ normalize $Q$, the element $\hat{x}_{y} \hat{a}_{y}^{-1}$ of $\hat{A}$ normalizes $Q \cdot \hat{\imath}$ and therefore $\hat{x}_{y} \hat{a}_{y}^{-1} \hat{\imath}$ determines an element $m_{y}$ of $M$. We claim that the image $\bar{m}_{y}$ of $m_{y}$ in $\bar{M}$ only depends on $y \in L$ and that, in the case where $R_{\varepsilon}=Q_{\delta}$, this correspondence determines a group homomorphism

$$
\bar{\sigma}: L \longrightarrow \bar{M}
$$

Indeed, if $x^{\prime} \in N$ still fulfills conditions 3.14.1 then we necessarily have $x^{\prime}=x_{y} z$ for some $z \in C_{H}(R)$ and therefore it suffices to choose the element $\hat{a}_{y} \cdot z$ of $\left(\hat{B}^{R}\right)^{*}$ in the definition above. On the other hand, if $\hat{a}^{\prime} \in\left(\hat{B}^{R}\right)^{*}$ still fulfills $\hat{\imath}^{\hat{x}_{y}}=\hat{\imath}^{\hat{a}^{\prime}}$ then we clearly have $\hat{a}^{\prime}=\hat{c} \hat{a}_{y}$ for some $\hat{c} \in\left(\hat{B}^{R}\right)^{*}$ centralizing $\hat{\imath}$, so that $\hat{c} \hat{\imath}$ belongs to $(\hat{\imath} \hat{B} \hat{\imath})^{Q}$; hence, the image of $\hat{x}_{y} \hat{a}_{y}^{-1} \hat{c}^{-1} \hat{\imath}$ in $\bar{M}$ coincides with $\bar{m}_{y}$. Moreover, in the case where $R_{\varepsilon}=Q_{\delta}$, for any element $y^{\prime}$ in $L$ we clearly can choose $\hat{x}_{y y^{\prime}}=\hat{x}_{y} \hat{x}_{y^{\prime}}$; then, we have

$$
\hat{\imath}^{\hat{x}_{y y^{\prime}}}=\left(\hat{\imath} \hat{a}_{y}\right)^{\hat{x}_{y^{\prime}}}=\hat{\imath}^{\hat{x}_{y^{\prime}} \cdot\left(\hat{a}_{y}\right)^{\hat{x}} y^{\prime}}=\hat{\imath}^{\hat{a}_{y^{\prime}}}\left(\hat{a}_{y}\right)^{\hat{x}_{y^{\prime}}}
$$

and therefore, since $\hat{a}_{y^{\prime}}\left(\hat{a}_{y}\right)^{\hat{x}_{y^{\prime}}}$ still belongs to $\left(\hat{B}^{Q}\right)^{*}$, we clearly can choose $\hat{a}_{y y^{\prime}}=\hat{a}_{y^{\prime}}\left(\hat{a}_{y}\right)^{\hat{x}_{y^{\prime}}}$, so that we get

$$
\hat{x}_{y y^{\prime}} \cdot \hat{a}_{y y^{\prime}}^{-1} \hat{\imath}=\hat{x}_{y} \hat{x}_{y^{\prime}} \cdot\left(\hat{a}_{y^{\prime}}\left(\hat{a}_{y}\right)^{\hat{x}_{y^{\prime}}}\right)^{-1} \hat{\imath}=\left(\hat{x}_{y} \cdot \hat{a}_{y}^{-1} \hat{\imath}\right)\left(\hat{x}_{y^{\prime}} \cdot \hat{a}_{y^{\prime}}^{-1} \hat{\imath}\right)
$$

which implies that $\bar{m}_{y y^{\prime}}=\bar{m}_{y} \bar{m}_{y^{\prime}}$. This proves our claim.
In particular, for any $u \in P$, we can choose $x_{\tau(u)}=u$ and $\hat{a}_{\tau(u)}=1$; moreover, since the action of $P$ on $S_{\gamma}$ can be lifted to a unique group homomorphism $\varrho: P \rightarrow \operatorname{Ker}\left(\operatorname{det}_{S_{\gamma}}\right)[18,1.8]$, we may choose $\hat{x}_{\tau(u)}=\varrho(u) \otimes u$; then, it is clear that the correspondence $\tau^{*}$ mapping $\tau(u)$ on the image of $(\varrho(u) \otimes u) \hat{\imath}$ in $M$ defines a group homomorphism from $\tau(P) \leq L$ to $M$ lifting the corresponding restriction of $\bar{\sigma}$.

Finally, we claim that $\tau^{*}$ fulfills condition 3.6.1; indeed, coming back to the general inclusion $\tau(R) \leq \tau(P)^{y}$ above, we clearly have $\bar{\sigma}(y)=\bar{m}_{y}$ and, according to the right-hand equalities in 3.14.1, for any $v \in R$ we get

$$
\tau^{*}\left(\tau(v)^{y}\right)=v^{x_{y}} \cdot \hat{\imath}=(v \cdot \hat{\imath})^{m_{y}}=\tau^{*}(\tau(v))^{m_{y}} .
$$

Consequently, it follows from Lemma 3.6 that $\bar{\sigma}$ can be lifted to a group homomorphism $\sigma: L \rightarrow M$ extending $\tau^{*}$; moreover, the inverse image of $\sigma(L)$ in $N_{(\hat{\mathcal{A}} \hat{\hat{\imath}})}(Q \cdot \hat{\imath})$ is a $k^{*}$-group which is clearly contained in

$$
\hat{N} \cdot\left(\mathcal{O}^{\prime *} \otimes 1\right) \subset \mathcal{O}^{\prime} \otimes_{\mathcal{O}} \hat{\mathcal{A}}
$$

hence, according to definition 3.13.1, $\sigma$ still can be lifted to a $k^{*}$-group homomorphism

$$
\hat{\sigma}: \hat{L}^{\circ} \longrightarrow N_{(\hat{\imath} \hat{\mathcal{A}} \hat{\imath})^{*}}(Q \cdot \hat{\imath})
$$

mapping $\tau(u)$ on $u \cdot \hat{\imath}$ for any $u \in P$; hence, we get a $P$-interior and $N / H$ graded algebra homomorphism

$$
\mathcal{O}_{*} \hat{L}^{\circ} \longrightarrow \hat{\mathcal{A}}_{\hat{\gamma}}
$$

We claim that homomorphism 3.14.2 is an isomorphism.
Indeed, denoting by $X \leq N_{G}\left(Q_{\delta}\right)$ a set of representatives for $\bar{N}=N / H$, it is clear that we have

$$
\mathcal{A}=\bigoplus_{x \in X} x \cdot \mathcal{B}
$$

and therefore we still have

$$
\hat{\mathcal{A}}=S \otimes_{\mathcal{O}} \mathcal{A}=\bigoplus_{x \in X}\left(s_{x} \otimes x\right)\left(S \otimes_{\mathcal{O}} \mathcal{B}\right)=\bigoplus_{x \in X}\left(s_{x} \otimes x\right) \hat{\mathcal{B}} ;
$$

moreover, choosing as above an element $\hat{a}_{x} \in\left(\hat{\mathcal{B}}^{Q}\right)^{*}$ such that $\hat{\imath}^{x}=\hat{\imath}^{\hat{a}_{x}}$, it is clear that $\left(s_{x} \otimes x\right) \hat{a}_{x}^{-1} \hat{\mathcal{B}}=\left(s_{x} \otimes x\right) \hat{\mathcal{B}}$ for any $x \in X$ and therefore we get

$$
\hat{\mathcal{A}}_{\hat{\gamma}}=\hat{\imath} \hat{\mathcal{A}} \hat{\imath}=\bigoplus_{x \in X}\left(\left(s_{x} \otimes x\right) \hat{a}_{x}^{-1} \hat{\imath}\right)(\hat{\imath} \hat{\mathcal{B}} \hat{\imath}) ;
$$

thus, since we know that $\hat{\imath} \hat{\mathcal{B}} \hat{\imath} \cong \mathcal{O} Q$ and that $L / \tau(Q) \cong \bar{N}$, denoting by $Y \leq L$ a set of representatives for $L / \tau(Q)$ and by $\hat{y}$ a lifting of $y \in Y$ to $\hat{L}$, we still get

$$
\hat{\mathcal{A}}_{\hat{\gamma}} \cong \bigoplus_{y \in Y} \hat{\sigma}(\hat{y}) \mathcal{O} Q
$$

which proves that homomorphism 3.14.2 is an isomorphism.
Corollary 3.15. With the notation above, we have a $P$-interior and $N / H$ graded algebra isomorphism $\mathcal{A}_{\gamma} \cong S_{\gamma} \otimes_{\mathcal{O}} \mathcal{O}_{*} \hat{L}^{\circ}$.

Proof: Since $\hat{\mathcal{A}}=S^{\circ} \otimes_{\mathcal{O}} \mathcal{A}$ and we have a $P$-interior algebra embedding $\mathcal{O} \rightarrow S_{\gamma} \otimes_{\mathcal{O}} S_{\gamma}^{\circ}[18,5.7]$, we still have the following commutative diagram of exterior $P$-interior algebra embeddings and homomorphisms [14, 2.10]

moreover, since the unity element is primitive in $\left(S_{\gamma}\right)^{P}$ and the kernel of the canonical homomorphism

$$
\left(S_{\gamma} \otimes_{\mathcal{O}} \mathcal{O} Q\right)^{P} \longrightarrow\left(S_{\gamma}\right)^{P}
$$

is contained in the radical, the unity element is primitive in $\left(S_{\gamma} \otimes_{\mathcal{O}} \mathcal{O} Q\right)^{P}$ too; since $P$ has a unique local point over $S_{\gamma} \otimes_{\mathcal{O}} S_{\gamma}^{\circ} \otimes_{\mathcal{O}} \mathcal{A}_{\gamma}$ [18, Proposition 5.6], from diagram 3.15.1 we get the announced isomorphism.
3.16. Let us take advantage of this revision to correct the erroneous proof of $[14,1.15 .1]$. Indeed, as proved in Proposition 3.11 above, we have a strict covering of $Q$-interior $k$-algebras

$$
k \otimes_{\mathcal{O}} \mathcal{B}_{\delta} \longrightarrow k \otimes_{\mathcal{O}} S_{\delta}
$$

but not a strict covering $k \otimes_{\mathcal{O}} \mathcal{B} \longrightarrow k \otimes_{\mathcal{O}} S$ of $H$-interior $k$-algebras as stated in [14, 1.15]; however, it follows from [17, 2.14 .4 and Lemma 9.12] that the isomorphism $\mathcal{B}_{\delta}\left(Q_{\delta}\right) \cong S_{\delta}(Q)$ induced by homomorphism 3.16.1 [18, 4.14] forces the embedding $\mathcal{B}\left(Q_{\delta}\right) \rightarrow S\left(Q_{\bar{\delta}}\right)$ where $\bar{\delta}$ denotes the local point of $Q$ over $S$ determined by $\delta$; hence, we still have the isomorphism [14, 1.15.5] which allows us to complete the argument.

## 4. Extensions of Glauberman correspondents of blocks

In this section, we continue to use the notation in Paragraph 3.1, namely $\mathcal{O}$ is a complete discrete valuation ring with an algebraically closed residue field $k$ of characteristic $p$; moreover we assume that its quotient field $\mathcal{K}$ has characteristic 0 and is big enough for all finite groups that we will consider; this assumption is kept throughout the rest of this paper.
4.1. Let $A$ be a cyclic group of order $q$, where $q$ is a power of a prime. Assume that $G$ is an $A$-group, that $H$ is an $A$-stable normal subgroup of $G$ and that $b$ is $A$-stable. Note that, in this section, $b$ is not necessarily nilpotent. Assume that $A$ and $G$ have coprime orders; by [15, Theorem 1.2], $G$ acts transitively on the set of all defect groups of $G_{\alpha}$ and, obviously, $A$ also acts on this set; hence, since $A$ and $G$ have coprime orders, by [11, Lemma 13.8 and Corollary 13.9] $A$ stabilizes some defect group of $G_{\alpha}$ and $G^{A}$ acts transitively on the set of them. Similarly, $A$ stabilizes some defect group of $N_{\beta}$ and $N^{A}$ acts transitively on the set of them. Thus, we may assume that $A$ stabilizes $P \leq N$ and actually we ssume that $A$ centralizes $P$; recall that $Q=P \cap H$.
4.2. Clearly $H^{A}$ is normal in $G^{A}$. We claim that $N^{A}$ is the stabilizer of $w(b)$ in $G^{A}$. Indeed, for any $x \in G^{A}, b^{x}$ is a block of $H$ and $Q^{x}$ is a defect group of $b^{x}$; since $A$ stabilizes $b^{x}$ and centralizes $Q^{x}, w\left(b^{x}\right)$ makes sense. Note that $G$ acts on $\operatorname{Irr}_{\mathcal{K}}(H)$, that $G^{A}$ acts on $\operatorname{Irr}_{\mathcal{K}}\left(H^{A}\right)$ and that the Glauberman correspondence $\pi(G, A)$ is compatible with the obvious actions of $G^{A}$ on $\operatorname{Irr}_{\mathcal{K}}(H)$ and $\operatorname{Irr}_{\mathcal{K}}\left(H^{A}\right)$. So we have

$$
\begin{aligned}
\operatorname{Irr}_{\mathcal{K}}\left(H^{A}, w\left(b^{x}\right)\right) & =\pi(H, A)\left(\operatorname{Irr}_{\mathcal{K}}\left(H, b^{x}\right)\right) \\
& =\pi(H, A)\left(\operatorname{Irr}_{\mathcal{K}}(H, b)^{x}\right) \\
& =\left(\pi(H, A)\left(\operatorname{Irr}_{\mathcal{K}}(H, b)\right)\right)^{x} \\
& =\operatorname{Irr}_{\mathcal{K}}\left(H^{A}, w(b)\right)^{x} ;
\end{aligned}
$$

in particular, we get $w\left(b^{x}\right)=w(b)^{x}$ and therefore we have $w(b)^{x}=w(b)$ if and only if $x$ belongs to $N^{A}$. We set

$$
w(c)=\operatorname{Tr}_{N^{A}}^{G^{A}}(w(b)), w(\beta)=\{w(b)\} \text { and } w(\alpha)=\{w(c)\}
$$

Then $w(\beta)$ is a point of $N^{A}$ on $\mathcal{O}\left(H^{A}\right), w(\alpha)$ is a point of $G^{A}$ on $\mathcal{O}\left(H^{A}\right)$, we have $\left(N^{A}\right)_{w(\beta)} \leq\left(G^{A}\right)_{w(\alpha)}$ and any defect group of $\left(N^{A}\right)_{w(\beta)}$ is a defect group of $\left(G^{A}\right)_{w(\alpha)}$.
4.3. Let $\mathfrak{B}$ and $w(\mathfrak{B})$ be the respective sets of $A$-stable blocks of $G$ covering $b$ and of $G^{A}$ covering $w(b)$. Take $e \in \mathfrak{B}$; since $P$ is a defect group of $G_{\alpha}$ and $c$ fulfills $e c=e, e$ has a defect group contained in $P$ and therefore, since $A$ centralizes $P$, $e$ has a defect group centralized by $A$; hence, by [26, Proposition 1 and Theorem 1], $w(e)$ makes sense and $A$ stabilizes all the characters in $\operatorname{Irr}_{\mathcal{K}}(G, e)$; that is to say, $A$ stabilizes all the characters of blocks in $\mathfrak{B}$. Moreover, by [11, Theorem 13.29], $w(e)$ belongs to $w(\mathfrak{B})$.

Proposition 4.4. The map $w: \mathfrak{B} \rightarrow w(\mathfrak{B}), e \mapsto w(e)$ is bijective and we have

$$
\operatorname{Irr}_{\mathcal{K}}\left(G^{A}, w(c)\right)=\pi(G, A)\left(\operatorname{Irr}_{\mathcal{K}}(G, c)^{A}\right) .
$$

Proof. Assume that $g \in \mathfrak{B}$ and $w(e)=w(g)$; then there exist $\chi \in \operatorname{Irr}_{\mathcal{K}}(G, e)$ and $\phi \in \operatorname{Irr}_{\mathcal{K}}(G, g)$ such that $\pi(G, A)(\chi)=\pi(G, A)(\phi)$; but this contradicts the bijectivity of the Glauberman correspondence. Therefore the map $w$ is injective.

Take $h \in w(\mathfrak{B})$; then $h$ covers $w(b)$ and so there exist $\zeta \in \operatorname{Irr}_{\mathcal{K}}\left(G^{A}, h\right)$ and $\eta \in \operatorname{Irr}_{\mathcal{K}}\left(H^{A}, w(b)\right)$ such that $\eta$ is a constituent of $\operatorname{Res}_{H^{A}}^{G^{A}}(\zeta)$. Set

$$
\theta=(\pi(G, A))^{-1}(\zeta) \quad \text { and } \quad \vartheta=(\pi(H, A))^{-1}(\eta) \quad ;
$$

by [11, Theorem 13.29], $\vartheta$ is a constituent of $\operatorname{Res}_{H}^{G}(\theta)$; let $l$ be the block of $G$ acting as the identity map on a $\mathcal{K} G$-module affording $\theta$; then $l$ covers $b$ and we have $w(l)=h$. Finally, we have

$$
\begin{aligned}
\pi(G, A)\left(\operatorname{Irr}_{\mathcal{K}}(G, c)^{A}\right) & =\pi(G, A)\left(\cup_{e \in \mathfrak{B}} \operatorname{Irr}_{\mathcal{K}}(G, e)\right) \\
& =\cup_{w(e) \in w(\mathfrak{B})} \operatorname{Irr}_{\mathcal{K}}\left(G^{A}, w(e)\right) \\
& =\operatorname{Irr}_{\mathcal{K}}\left(G^{A}, w(c)\right)
\end{aligned}
$$

Proposition 4.5. $P$ is a defect group of the pointed group $\left(G^{A}\right)_{w(\alpha)}$.

Proof. It suffices to show that $P$ is a defect group of $\left(N^{A}\right)_{w(\beta)}$ (cf. 3.1); thus, without loss of generality, we can assume that $G=N$. Obviously, $A$ stabilizes $P \cdot H$ and $b$ is the unique block of $P \cdot H$ covering the block $b$ of $H$; since $P$ is a defect group of $G_{\alpha}$ and $N_{\beta}, P$ is maximal in $N$ such that $\operatorname{Br}_{P}^{\mathcal{O} H}(b) \neq 0$; thus $P$ is maximal in $P \cdot H$ such that $\operatorname{Br}_{P}^{\mathcal{O}(P \cdot H)}(b) \neq 0$; therefore $P$ is a defect group of $b$ as a block of $P \cdot H$. Since $A$ centralizes $P$, the Glauberman correspondent $b^{\prime}$ of $b$ as a block of $P \cdot H$ makes sense; moreover by Proposition 4.4, $b^{\prime}$ covers $w(b)$. Since $w(b)$ is the unique block of $P \cdot H^{A}$ covering the block $w(b)$ of $H^{A}, b^{\prime}=w(b)$, and then, by [26, Theorem 1], $P$ is a defect group of $w(b)$ as a block of $P \cdot H^{A}$; in particular, $\mathrm{Br}_{P}^{\mathcal{O}\left(H^{A}\right)}(w(b)) \neq 0$.

Since $P$ is a defect group of $G_{\alpha}$, by $[14$, Theorem 5.3] the image of $P$ in the quotient group $N / H$ is a Sylow $p$-subgroup of $N / H$; but, the inclusion $\operatorname{map} N^{A} \hookrightarrow N$ induces a group isomorphism $N^{A} / H^{A} \cong(N / H)^{A}$; hence, the image of $P$ in $N^{A} / H^{A}$ is a Sylow $p$-subgroup of $N^{A} / H^{A}$; then, by [14, Theorem 5.3] again, $P$ is a defect group of $\left(N^{A}\right)_{w(\alpha)}$.
4.6. We may assume that $A$ stabilizes $P_{\gamma}$; then $A$ stabilizes $Q_{\delta}$ too (see [14, Proposition 5.5]). Let $R$ be a subgroup such that $Q \leq R \leq P$ and $R_{\varepsilon}$ a local pointed group on $\mathcal{O} H$ contained in $P_{\gamma}$. Since $A$ stabilizes $P_{\gamma}$ and centralizes $P, A$ centralizes $R$ and then, by [14, Proposition 5.5], it stabilizes $R_{\varepsilon}$. Since $\operatorname{Br}_{R}^{\mathcal{O H}}(\varepsilon)$ is a point of $k C_{H}(R)$, then there is a unique block $b_{\varepsilon}$ of $\mathcal{O} C_{H}(R)$ such that $\operatorname{Br}_{R}^{\mathcal{O} H}\left(b_{\varepsilon} \varepsilon\right)=\operatorname{Br}_{R}^{\mathcal{O H}}(\varepsilon)$ and, by [28, Lemma 2.3], $C_{Q}(R)$ is a defect group of $b_{\varepsilon}$; in particular, $b_{\varepsilon}$ is nilpotent. Obviously, $A$ centralizes $C_{Q}(R)$ and, since $A$ stabilizes $R_{\varepsilon}$ and thus it stabilizes $b_{\varepsilon}$, $w\left(b_{\varepsilon}\right)$ makes sense; moreover, $w\left(b_{\varepsilon}\right)$ is nilpotent and, since we have

$$
C_{H^{A}}(R)=C_{C_{H}(R)}(A),
$$

there is a unique local point $w(\varepsilon)$ of $R$ on $\mathcal{O}\left(H^{A}\right)$ such that

$$
\operatorname{Br}_{R}^{\mathcal{O}\left(H^{A}\right)}\left(w(\varepsilon) w\left(b_{\varepsilon}\right)\right)=\operatorname{Br}_{R}^{\mathcal{O}\left(H^{A}\right)}(w(\varepsilon))
$$

Proposition 4.7. $P_{w(\gamma)}$ is a defect pointed group of $\left(G^{A}\right)_{w(\alpha)}$ and $Q_{w(\delta)}$ is a defect pointed group of $\left(H^{A}\right)_{\{w(b)\}}$.
Proof. By [21, Proposition 2.8], the inclusion map $\mathcal{O} H \hookrightarrow \mathcal{O}(P \cdot H)$ is actually a strict semicovering $P \cdot H$-algebra homomorphism; hence, $\gamma$ determines a unique local point $\gamma^{\prime}$ of $P$ on $\mathcal{O}(P \cdot H)$ such that $\gamma \subset \gamma^{\prime}$. Obviously, $b$ is a block of $P \cdot H$. Since $\beta$ is also a point of $P \cdot H$ on $\mathcal{O} H$ and $P_{\gamma}$ is also a defect
pointed group of the pointed group $(P \cdot H)_{\beta}$ on $\mathcal{O} H$, by [14, Corollary 6.3] $P_{\gamma^{\prime}}$ is a defect pointed group of the pointed group $(P \cdot H)_{\beta}$ on $\mathcal{O}(P \cdot H)$.

Let $b_{\gamma^{\prime}}$ be the block of $C_{P . H}(P)$ such that

$$
\operatorname{Br}_{P}^{\mathcal{O}(P \cdot H)}\left(b_{\gamma^{\prime}} \gamma^{\prime}\right)=\operatorname{Br}_{P}^{\mathcal{O}(P \cdot H)}\left(\gamma^{\prime}\right) ;
$$

then $Z(P)$ is a defect group of $b_{\gamma^{\prime}}$ and therefore $w\left(b_{\gamma^{\prime}}\right)$ makes sense. Obviously, $b_{\gamma^{\prime}}$ covers $b_{\gamma}$ and thus $w\left(b_{\gamma^{\prime}}\right)$ covers $w\left(b_{\gamma}\right)$ (see Proposition 4.4); but, since $w(b)$ is also the Glauberman correspondent of the block $b$ of $P \cdot H$ (see the first paragraph of the proof of Proposition 4.5), by [26, Proposition 4] we have

$$
\operatorname{Br}_{P}^{\mathcal{O}\left(P \cdot H^{A}\right)}\left(w(b) w\left(b_{\gamma^{\prime}}\right)\right)=\operatorname{Br}_{P}^{\mathcal{O}\left(P \cdot H^{A}\right)}\left(w\left(b_{\gamma^{\prime}}\right)\right) ;
$$

this forces $\operatorname{Br}_{P}^{\mathcal{O}\left(H^{A}\right)}\left(w(b) w\left(b_{\gamma}\right)\right)=\operatorname{Br}_{P}^{\mathcal{O}\left(H^{A}\right)}\left(w\left(b_{\gamma}\right)\right)$, which implies that

$$
P_{w(\gamma)} \leq\left(P \cdot H^{A}\right)_{w(\beta)} \leq\left(G^{A}\right)_{w(\alpha)}
$$

hence, by Proposition 4.5, $P_{w(\gamma)}$ is a defect pointed group of $\left(G^{A}\right)_{w(\alpha)}$.
The statement that $Q_{w(\delta)}$ is a defect pointed group of $\left(H^{A}\right)_{\{w(b)\}}$ is clear.
Lemma 4.8. Let $R_{\varepsilon}$ and $T_{\eta}$ be local pointed groups on $\mathcal{B}$ such that $R$ is normal in $T$ and that we have $Q_{\delta} \leq R_{\varepsilon} \leq P_{\gamma}$ and $Q_{\delta} \leq T_{\eta} \leq P_{\gamma}$. Then, we have $R_{\varepsilon} \leq T_{\eta}$ if and only if we have

$$
\operatorname{Br}_{T}^{\mathcal{O} C_{H}(R)}\left(b_{\eta} b_{\varepsilon}\right)=\operatorname{Br}_{T}^{\mathcal{O} C_{H}(R)}\left(b_{\gamma}\right) .
$$

Proof. Obviously, $\mathcal{B}$ is a $p$-permutation $P \cdot H$-algebra (see [5, Def. 1.1]) by $P \cdot H$-conjugation and $\left(T, \operatorname{Br}_{T}^{\mathcal{B}}\left(b_{\eta}\right)\right)$ and $\left(R, \operatorname{Br}_{R}^{\mathcal{B}}\left(b_{\varepsilon}\right)\right)$ are $(b, P \cdot H)$-Brauer pairs (see [5, Def. 1.6]). Moreover $T$ stabilizes $b_{\varepsilon}$, and $\eta$ and $\varepsilon$ are the unique local points of $T$ and $R$ on $\mathcal{B}$ (see [14, Proposition 5.5]) such that

$$
\operatorname{Br}_{T}^{\mathcal{B}}(\eta) \operatorname{Br}_{T}^{\mathcal{B}}\left(b_{\eta}\right)=\operatorname{Br}_{T}^{\mathcal{B}}(\eta) \text { and } \operatorname{Br}_{R}^{\mathcal{B}}(\varepsilon) \operatorname{Br}_{R}^{\mathcal{B}}\left(b_{\varepsilon}\right)=\operatorname{Br}_{R}^{\mathcal{B}}(\varepsilon) .
$$

Assume that $R_{\varepsilon} \leq T_{\eta}$; then, there are $h \in \eta$ and $l \in \varepsilon$ such that $h l=l=l h$; thus, we have

$$
\operatorname{Br}_{R}^{\mathcal{B}}(h l)=\operatorname{Br}_{R}^{\mathcal{B}}(l) \quad \text { and } \quad \operatorname{Br}_{R}^{\mathcal{B}}(h) \operatorname{Br}_{R}^{\mathcal{B}}\left(b_{\varepsilon}\right) \neq 0 .
$$

Then, it follows from [5, Def. 1.7] that

$$
\left(R, \operatorname{Br}_{R}^{\mathcal{B}}\left(b_{\varepsilon}\right)\right) \subset\left(T, \operatorname{Br}_{T}^{\mathcal{B}}\left(b_{\eta}\right)\right)
$$

and from [5, Theorem 1.8] that we have $\operatorname{Br}_{T}^{\mathcal{O} C_{H}(R)}\left(b_{\eta} b_{\varepsilon}\right)=\operatorname{Br}_{T}^{\mathcal{O} C_{H}(R)}\left(b_{\eta}\right)$.
Conversely, if we have

$$
\operatorname{Br}_{T}^{\mathcal{O} C_{H}(R)}\left(b_{\eta} b_{\varepsilon}\right)=\operatorname{Br}_{T}^{\mathcal{O} C_{H}(R)}\left(b_{\eta}\right)
$$

then, by [5, Theorem 1.8] we still have $\operatorname{Br}_{R}^{\mathcal{B}}\left(e_{\varepsilon} h\right)=\operatorname{Br}_{R}^{\mathcal{B}}(h)$ for any $h \in \eta$; hence, by the lifting theorem for idempotents, we get $R_{\varepsilon} \leq T_{\eta}$.

Let $\mathcal{R}$ be a Dedekind domain of characteristic $0, \pi$ be a finite set of prime numbers such that $l \mathcal{R} \neq \mathcal{R}$ for all $l \in \pi$, and $X$ and $Y$ be finite groups with $X$ acting on $Y$. We consider the group algebra $\mathcal{R} Y$ and set

$$
Z_{\mathrm{id}}(\mathcal{R} Y)=\oplus \mathcal{R} c
$$

where $c$ runs over all central primitive idempotents of $\mathcal{R} Y$. Obviously, $X$ acts on $Z_{\text {id }}(\mathcal{R} Y)$ and, in the case that $X$ is a solvable $\pi$-group, Lluis Puig exhibes a $\mathcal{R}$-algebra homomorphism $\mathcal{G l} l_{X}^{Y}: Z_{\text {id }}(\mathcal{R} X) \rightarrow Z_{\text {id }}\left(\mathcal{R} Y^{X}\right)$ (see [21, Theorem 4.6]), which unifies the usual Brauer homomorphism and the Glauberman correspondence of characters - called the Brauer-Glauberman correspondence.

Proposition 4.9. Let $R_{\varepsilon}$ and $T_{\eta}$ be local pointed groups on $\mathcal{B}$ such that $Q_{\delta} \leq R_{\varepsilon} \leq P_{\gamma}$ and that $Q_{\delta} \leq T_{\eta} \leq P_{\gamma}$. Then $R_{\varepsilon} \leq T_{\eta}$ and $R_{w(\varepsilon)} \leq T_{w(\eta)}$ are equivalent to each other.

Proof. By induction we can assume that $R$ is normal and maximal in $T$; in particular, the quotient $T / R$ is cyclic. In this case, it follows from Lemma 4.8 that the inclusion $R_{w(\varepsilon)} \leq T_{w(\eta)}$ is equivalent to

$$
\operatorname{Br}_{T}^{\mathcal{O} C_{H^{A}}(R)}\left(w\left(b_{\varepsilon}\right) w\left(b_{\eta}\right)\right)=\operatorname{Br}_{T}^{\mathcal{O} C_{H^{A}}(R)}\left(w\left(b_{\eta}\right)\right) .
$$

Let $\mathbb{Z}$ be the ring of all rational integers and $S$ be the complement set of $p \mathbb{Z} \cup q \mathbb{Z}$ in $\mathbb{Z}$; then $S$ is a multiplicatively closed set in $\mathbb{Z}$. We take the localization $S^{-1} \mathbb{Z}$ of $\mathbb{Z}$ at $S$ and regard it as a subring of $\mathcal{K}$; since we assume that $\mathcal{K}$ is big enough for all finite groups we consider, we can assume that $\mathcal{K}$ contains an $|H|$-th primitive root $\omega$ of unity and we set

$$
\mathcal{R}=\left(S^{-1} \mathbb{Z}\right)[\omega]
$$

Then $\mathcal{R}$ is a Dedekind domain (see [2, Example 2 in Page 96 and Exercise 1 in Page 99]) and given a prime $l$, we have $l \mathcal{R} \neq \mathcal{R}$ if and only if $l=p$ or
$l=q$. We consider the group algebra $\mathcal{R} C_{H}(R)$ and the obvious action of $(T \times A) / R \cong(T / R) \times A$ on it.

Since $\mathcal{R}$ contains an $|H|$-th primitive unity root $\omega$, the blocks $b_{\varepsilon}$, $b_{\eta}$, $w\left(b_{\varepsilon}\right)$ and $w\left(b_{\eta}\right)$ respectively belong to

$$
Z_{\mathrm{id}}\left(\mathcal{R} C_{H}(R)\right), \quad Z_{\mathrm{id}}\left(\mathcal{R} C_{H}(T)\right), \quad Z_{\mathrm{id}}\left(\mathcal{R} C_{H^{A}}(R)\right) \quad \text { and } \quad Z_{\mathrm{id}}\left(\mathcal{R} C_{H^{A}}(T)\right)
$$

(see [8, Charpter IV, Lemma 7.2]); then, by [21, Corollary 5.9], we have

$$
\mathcal{G} l_{A}^{C_{H}(R)}\left(b_{\varepsilon}\right)=w\left(b_{\varepsilon}\right) \quad \text { and } \quad \mathcal{G} l_{A}^{C_{H}(T)}\left(b_{\eta}\right)=w\left(b_{\eta}\right) .
$$

If $R_{\varepsilon} \leq T_{\eta}$, by Lemma 4.8 we have the equality

$$
\operatorname{Br}_{T}^{\mathcal{O} C_{H}(R)}\left(b_{\varepsilon} b_{\eta}\right)=\operatorname{Br}_{T}^{\mathcal{O} C_{H}(R)}\left(b_{\eta}\right)
$$

which is equivalent to $\mathcal{G} l_{T / R}^{C_{H}(R)}\left(b_{\varepsilon}\right) b_{\eta}=b_{\eta}$ (see [21, 4.6.1 and the proof of Corollary 3.6]). Then by [21, 4.6.2], we have

$$
\begin{aligned}
w\left(b_{\eta}\right) & =\mathcal{G} l_{A}^{C_{H}(T)}\left(b_{\eta}\right)=\mathcal{G} l_{A}^{C_{H}(T)}\left(\mathcal{G l} l_{T / R}^{C_{H}(R)}\left(b_{\varepsilon}\right) b_{\eta}\right) \\
& =\mathcal{G} l_{(T / R) \times A}^{C_{H}(R)}\left(b_{\varepsilon}\right) \mathcal{G} l_{A}^{C_{H}(T)}\left(b_{\eta}\right) \\
& =\mathcal{G} l_{T / R}^{C_{H A}(R)}\left(\mathcal{G} l_{A}^{C_{H}(R)}\left(b_{\varepsilon}\right)\right) \mathcal{G} l_{A}^{C_{H}(T)}\left(b_{\eta}\right) \\
& =\mathcal{G} l_{T / R}^{C_{H A}(R)}\left(w\left(b_{\varepsilon}\right)\right) w\left(b_{\eta}\right) .
\end{aligned}
$$

which is equivalent again to equality 4.9 .1 above (see [21, 4.6.1 and the proof of Corollary 3.6] and therefore it implies $R_{w(\varepsilon)} \leq T_{w(\eta)}$. The prove that $R_{w(\varepsilon)} \leq T_{w(\eta)}$ implies $R_{\varepsilon} \leq T_{\eta}$ is similar.
4.10. The assumptions and consequences above are very scattered; we collect them in this paragraph, so that readers can easily find them and we can conveniently quote them later. Let $A$ be a cyclic group of order $q$, where $q$ is a prime number; we assume that $G$ is an $A$-group, that $H$ is an $A$-stable normal subgroup of $G$, that $b$ is $A$-stable, that $A$ centralizes $P$ and stabilizes $P_{\gamma}$, and that $A$ and $G$ have coprime orders. Without loss of generality, we may assume that $P \leq N$. Then, $A$ centralizes $Q$ and stabilizes $Q_{\delta}$, so that the Glauberman correspondent $w(b)$ of the block $b$ makes sense; moreover, the block $w(b)$ determines two pointed group $\left(N^{A}\right)_{w(\beta)}$ and $\left(G^{A}\right)_{w(\alpha)}$ such that $\left(N^{A}\right)_{w(\beta)} \leq\left(G^{A}\right)_{w(\alpha)}$ (see them in Paragraph 4.2)., and the local pointed groups $P_{\gamma}$ and $Q_{\delta}$ determine respective defect pointed groups $P_{w(\gamma)}$
and $Q_{w(\delta)}$ of $\left(G^{A}\right)_{w(\alpha)}$ and $\left(H^{A}\right)_{w(\beta)}$ (see Paragraph 4.6 and Proposition 4.7); actually, by Proposition 4.9, we have $Q_{w(\delta)} \leq P_{w(\gamma)}$. Take $w(i) \in w(\gamma)$ and $w(j) \in w(\delta)$, and set

$$
\begin{aligned}
\left(\mathcal{O} G^{A}\right)_{w(\gamma)} & =w(i)\left(\mathcal{O} G^{A}\right) w(i),\left(\mathcal{O} H^{A}\right)_{w(\gamma)}=w(i)\left(\mathcal{O} H^{A}\right) w(i) \\
& \text { and }\left(\mathcal{O} H^{A}\right)_{w(\delta)}=w(j)\left(\mathcal{O} H^{A}\right) w(j)
\end{aligned}
$$

then, $\left(\mathcal{O} G^{A}\right)_{w(\gamma)}$ is a $P$-interior and $\left(N^{A} / H^{A}\right)$-graded algebra; moreover, the $Q$-interior algebra $\left(\mathcal{O} H^{A}\right)_{w(\delta)}$ with the group homomorphism

$$
Q \longrightarrow\left(\mathcal{O} H^{A}\right)_{w(\delta)}^{*} \quad, \quad u \mapsto u w(j)
$$

is a source algebra of the block algebra $\mathcal{O} H^{A} w(b)$ (see [15]).

## 5. A Lemma

From now on, we use the notation and assumption in Paragraphs 3.1, 3.2 and 4.10; in particular, we assume that the block $b$ of $H$ is nilpotent. Obviously, $N_{G}\left(Q_{\delta}\right)$ acts on $\operatorname{Irr}_{\mathcal{K}}(H, b)$ and $\operatorname{Irr}_{\mathcal{K}}(Q)$ via the corresponding conjugation conjugation. Since $b$ is nilpotent, there is an explicit bijection between $\operatorname{Irr}_{\mathcal{K}}(H, b)$ and $\operatorname{Irr}_{\mathcal{K}}(Q)$ (see [27, Theorem 52.8]); in this section, we will show that this bijection is compatible with the $N_{G}\left(Q_{\delta}\right)$-actions; our main purpose is to obtain Lemma 5.6 below as a consequence of this compatibility.
5.1. For any $x \in N_{G}\left(Q_{\delta}\right), x j x^{-1}$ belongs to $\delta$ and thus there is some invertible element $a_{x} \in \mathcal{B}^{Q}$ such that $x j x^{-1}=a_{x} j a_{x}^{-1}$; let us denote by $X$ the set of all elements $\left(a_{x}^{-1} x\right) j$ such that $a_{x}$ is invertible in $\mathcal{B}^{Q}$ and we have $x j x^{-1}=a_{x} j a_{x}^{-1}$ when $x$ runs over $N_{G}\left(Q_{\delta}\right)$. Set

$$
E_{G}\left(Q_{\delta}\right)=N_{G}\left(Q_{\delta}\right) / Q C_{H}(Q)
$$

then, the following equality

$$
\left(\left(a_{x}^{-1} x\right) j\right) \cdot\left(\left(a_{y}^{-1} y\right) j\right)=\left(\left(a_{x}^{-1} x a_{y}^{-1} x^{-1}\right) x y\right) j
$$

shows that $X$ is a group with respect to the multiplication and it is easily checked that $Q \cdot\left(\mathcal{B}_{\delta}^{Q}\right)^{*}$ is normal in $X$ and that the map

$$
E_{G}\left(Q_{\delta}\right) \longrightarrow X / Q\left(\mathcal{B}_{\delta}^{Q}\right)^{*}
$$

sending the coset of $x \in N_{G}\left(Q_{\delta}\right)$ in $N_{G}\left(Q_{\delta}\right) / Q C_{H}(Q)$ to the coset of $\left(a_{x}^{-1} x\right) j$ in $X / Q\left(\mathcal{B}_{\delta}^{Q}\right)^{*}$ is a group isomorphism.
5.2. We denote by $Y$ the set of all such elements $a_{x}^{-1} x$ when $x$ runs over $N_{G}\left(Q_{\delta}\right)$ and $a_{x}$ over the invertible element of $\mathcal{B}^{Q}$ such that $a_{x}^{-1} x$ commutes with $j$. As in 5.1, it is easily checked that $Y$ is a group with respect to the multiplication

$$
\left(a_{x}^{-1} x\right) \cdot\left(a_{y}^{-1} y\right)=\left(a_{x}^{-1} x a_{y}^{-1} x^{-1}\right) x y,
$$

that $Y$ normalizes $Q \cdot\left((\mathcal{O H})^{Q}\right)^{*}$ and that the map

$$
E_{G}\left(Q_{\delta}\right) \longrightarrow\left(Y \cdot Q \cdot\left(\mathcal{B}^{Q}\right)^{*}\right) /\left(Q \cdot\left(\mathcal{B}^{Q}\right)^{*}\right)
$$

sending the coset of $x \in N_{G}\left(Q_{\delta}\right)$ to the coset of $a_{x}^{-1} x$ in the right-hand quotient is a group isomorphism.
5.3. Let $I$ and $J$ be the sets of isomorphism classes of all simple $\mathcal{K} \otimes_{\mathcal{O}} \mathcal{B}$ and $\mathcal{K} \otimes_{\mathcal{O}} \mathcal{B}_{\delta}$-modules respectively. Cleraly, $Y$ acts on $I$; but, since $Y \cap\left(\mathcal{B}^{Q}\right)^{*}$ acts trivially on $I$, the action of $Y$ on $I$ induces an action of $E_{G}\left(Q_{\delta}\right)$ on $I$ through isomorphism 5.2.1; actually, this action coincides with the action of $E_{G}\left(Q_{\delta}\right)$ on $\operatorname{Irr}_{\mathcal{K}}(H, b)$ induced by the $N_{G}\left(Q_{\delta}\right)$-conjugation. Similarly, $X$ acts on $J$ and this action of $X$ on $J$ induces an action of $E_{G}\left(Q_{\delta}\right)$ on $J$ through isomorphism 5.1.1. But, by [15, Corollary 3.5], the functor $M \mapsto j \cdot M$ is an equivalence between the categories of finitely generated $\mathcal{B}$ - and $\mathcal{B}_{\delta}$-modules, which induces a bijection between the sets $I$ and $J$. Then, since $Y$ commutes with $j$ and the map

$$
Y \longrightarrow X \quad, \quad y \mapsto y j
$$

is a group homomorphism, it is easily checked that this bijection is compatible with the actions of $E_{G}\left(Q_{\delta}\right)$ on $I$ and $J$.
5.4. Recall that (cf. 3.7)

$$
\mathcal{B}_{\delta} \cong T \otimes_{\mathcal{O}} \mathcal{O} Q
$$

where $T=\operatorname{End}_{\mathcal{O}}(W)$ for an endo-permutation $\mathcal{O} Q$-module $W$ such that the determinant of the image of any element of $Q$ in is one; in this case, the $\mathcal{O} Q$-module $W$ with these properties is unique up to isomorphism. Then, for any simple $\mathcal{K} \otimes_{\mathcal{O}} \mathcal{B}_{\delta}$-module $V$ there is a $\mathcal{K} Q$-module $V_{W}$, unique up to isomorphism, such that

$$
V \cong W \otimes_{\mathcal{O}} V_{W}
$$

as $\mathcal{K} \otimes_{\mathcal{O}} \mathcal{B}_{\delta}$-modules; moreover the correspondence

$$
V \mapsto V_{W}
$$

determines a bijection between $J$ and the set of isomorphism classes of all simple $\mathcal{K} Q$-modules. Now, the composition of this bijection with the bijection between isomorphism classes in 5.3 is a bijection from $I$ to the set of isomorphism classes of all simple $\mathcal{K} Q$-modules; translating this bijection to characters, we obtain a bijection

$$
\operatorname{Irr}_{\mathcal{K}}(H, b) \longrightarrow \operatorname{Irr}_{\mathcal{K}}(Q) \quad, \quad \chi_{\lambda} \mapsto \lambda ;
$$

let us denote by $\chi \in \operatorname{Irr}_{\mathcal{K}}(H, b)$ the image ofthe trivial character of $Q$.
5.5. Moreover, the $N_{G}\left(Q_{\delta}\right)$-conjugation induces an action of $E_{G}\left(Q_{\delta}\right)$ on the set of isomorphism classes of all simple $\mathcal{K} Q$-modules and we claim that, for any simple $\mathcal{K} \otimes_{\mathcal{O}} \mathcal{B}_{\delta}$-module $V$ and any $\bar{x} \in E_{G}\left(Q_{\delta}\right)$, we have a $\mathcal{K} Q$ module isomorphism

$$
\bar{x}\left(V_{W}\right) \cong\left({ }^{\bar{x}} V\right)_{W} \quad ;
$$

in particular, bijection 5.4 .2 is compatible with the actions of $E_{G}\left(Q_{\delta}\right)$ on $J$ and on the set of isomorphism classes of simple $\mathcal{K} Q$-modules. Indeed, let $x$ be a lifting of $\bar{x}$ in $N_{G}\left(Q_{\delta}\right)$ and denote by $\varphi_{x}$ the isomorphism

$$
Q \cong Q \quad, \quad u \mapsto x u x^{-1} ;
$$

take a lifting $y=a_{x}^{-1} x j$ of $\bar{x}$ in $X$ through isomorphism 5.1.1; since the conjugation by $y$ stabilizes $\mathcal{B}_{\delta}$, the map

$$
f_{y}: \mathcal{B}_{\delta} \cong \operatorname{Res}_{\varphi_{x}}\left(\mathcal{B}_{\delta}\right) \quad, \quad a \mapsto y a y^{-1}
$$

is a $Q$-interior algebra isomorphism; then, by [18, Corollary 6.9], we can modify $y$ with a suitable element of $\left(\mathcal{B}_{\delta}^{Q}\right)^{*}$ in such a way that $f_{y}$ stabilizes $T$; in this case, the restriction of $f_{y}$ to $T$ has to be inner and thus we have $W \cong \operatorname{Res}_{f_{y}}(W)$ as T-modules. Moreover, since the action of $Q$ on $T$ can be uniquely lifted to a $Q$-interior algebra structure such that the determinant of the image of any $u \in Q$ in $T$ is one, $f_{y}$ also stabilizes the image of $Q$ in $T$; more precisely, $f_{y}$ maps the image of $u \in Q$ onto the image of $\varphi_{x}(u)$. The claim follows.

Lemma 5.6. With the notation above,
5.6.1. The irreducible character $\chi$ is $N_{G}\left(Q_{\delta}\right)$-stable and its restriction to the set $H_{p^{\prime}}$ of all p-regular elements of $H$ is the unique irreducible Brauer character of $H$.
5.6.2. The Glauberman correspondent $\phi$ of $\chi$ is $N_{G^{A}}\left(Q_{w(\delta)}\right)$-stable and its restriction to the set $H_{p^{\prime}}^{A}$ of all p-regular elements of $H^{A}$ is the unique irreducible Brauer character of $H^{A}$.
Proof. It follows from 5.3 and 5.5 that the bijection 5.4.3 is compatible with the actions of $E_{G}\left(Q_{\delta}\right)$ in $\operatorname{Irr}_{\mathcal{K}}(H, b)$ and $\operatorname{Irr}_{\mathcal{K}}(Q)$; hence, $\chi$ is $E_{G}\left(Q_{\delta}\right)$-stable and thus $N_{G}\left(Q_{\delta}\right)$-stable. Since $\phi$ is the unique irreducible constituent of Res $H_{H^{A}}^{H}(\chi)$ occurring with a multiplicity coprime to $q$ and $N_{G^{A}}\left(Q_{w(\delta)}\right)$ is contained in $N_{G}\left(Q_{\delta}\right), \phi$ has to be $N_{G^{A}}\left(Q_{w(\delta)}\right)$-stable. By the very definition of the bijection 5.4.3, the restriction of $\chi$ to $H_{p^{\prime}}$ is the unique Brauer character of $H$. Since the perfect isometry $R_{H}^{b}$ between $\mathcal{R}_{\mathcal{K}}(H, b)$ and $\mathcal{R}_{\mathcal{K}}\left(H^{A}, w(b)\right)$ maps $\psi \in I$ onto $\pm \pi(H, A)(\psi)$ and the blocks $b$ and $w(b)$ are nilpotent, by [3, Theorem 4.11] the decomposition matrices of $b$ and $w(b)$ are the same if the characters indexing their columns correspond to each other by the Glauberman correspondence; hence, the restriction of $\phi$ to $H_{p^{\prime}}^{A}$ is the unique Brauer character of $H^{A}$.

## 6. A $k^{*}$-group isomorphism $\left(\hat{\bar{N}}^{k}\right)^{A} \cong{\widehat{N^{A}}}^{k}$

6.1. Let $x H$ be an $A$-stable coset in $\bar{N}$. We consider the action of $H \rtimes A$ on $x H$ defined by the obvious action of $A$ on $x H$ and the right multiplication of $H$ on $x H$; since $A$ and $G$ have coprime orders, it follows from [11, Lemma 13.8 and Corollary 13.9] that $x H \cap N^{A}$ is non-empty and that $H^{A}$ acts transitively on it; consequently, we have $\bar{N}^{A}=\left(H \cdot N^{A}\right) / H$ and the inclusion $N^{A} \subset N$ induces a group isomorphism

$$
\overline{N^{A}} \cong \bar{N}^{A}=\left(H \cdot N^{A}\right) / H
$$

Note that if $G=H \cdot G^{A}$ then we have $\bar{N}^{A}=\bar{N}$.
6.2. It follows from Lemma 5.6 that $N=H \cdot N_{G}\left(Q_{\delta}\right)$ stabilizes $\chi$ and actually the central extension $\hat{\bar{N}}$ of $\bar{N}$ by $U$ in 3.9 above is nothing but the so-called Clifford extension of $\bar{N}$ over $\chi$; moreover, since $A$ and $U$ also have coprime orders, we can prove as above that $\hat{\bar{N}}^{A}$ is a central extension of $\bar{N}^{A}$ by $U$, which is the Clifford extension of $\bar{N}^{A}$ over $\chi$. Since the Glauberman
correspondent $w(b)$ is nilpotent, we can repeat all the above constructions for $G^{A}, H^{A}, w(b)$ and $N^{A}$; then, denoting by $U_{A}$ the group of $\left|H^{A}\right|$-th roots of unity, we obtain a central extension $\widehat{\widehat{N^{A}}}$ of $\overline{N^{A}}=\bar{N}^{A}$ by $U_{A}$, which is the Clifford extension of $\bar{N}^{A}$ over $\phi$; moreover, note that $U_{A}$ is contained in $U$.
6.3. At this point, it follows from [23, Corollary 4.16] that there is an extension group isomorphism

$$
\hat{N}^{A} \cong\left(U \times \widehat{N^{A}}\right) / \Delta_{-1}\left(U_{A}\right)
$$

where we are setting $\Delta_{-1}\left(U_{A}\right)=\left\{\left(\xi^{-1}, \xi\right)\right\}_{\xi \in U_{A}}$; moreover, according to [21, Remark 4.17], this isomorphism is defined by a sequence of Brauer homomorphisms - in different characteristics - and, in particular, it is quite clear that it maps any $y \in H \leq \hat{N}^{A}$ in the classes of $(1, y)$ in the righthand member, so that isomorphism 6.3.1 induces a new extension group isomorphism

$$
\hat{\bar{N}}^{A} \cong\left(U \times \widehat{\widehat{N^{A}}}\right) / \Delta_{-1}\left(U_{A}\right)
$$

Consequently, denoting by $\varpi_{A}: U_{A} \rightarrow k^{*}$ the restriction of $\varpi$, we get a $k^{*}$-group isomorphism

$$
\begin{aligned}
\left(\hat{\bar{N}}^{k}\right)^{A} & =\left(\left(k^{*} \times \hat{\bar{N}}\right) / \Delta_{\varrho}(U)\right)^{A} \cong\left(k^{*} \times \hat{\bar{N}}^{A}\right) / \Delta_{\varrho}(U) \\
& \cong\left(k^{*} \times \widehat{N^{A}}\right) / \Delta_{\varrho_{A}}\left(U_{A}\right)={\widehat{N^{A}}}^{k}
\end{aligned}
$$

as announced.
Remark 6.4. Note that if $G=H \cdot G^{A}$ then we have $\hat{\bar{N}}^{A}=\hat{\bar{N}}$.

## 7. Proofs of Theorems 1.5 and 1.6

7.1. The first statement in Theorem 1.5 follows from Propositions 4.4 and 4.5. From now on, we assume that the block $b$ of $H$ is nilpotent; thus, the Glauberman correspondent $w(b)$ is also nilpotent and $\left(\mathcal{O} G^{A}\right) w(c)$ is an extension of the nilpotent block algebra $\left(\mathcal{O} H^{A}\right) w(b)$. This section will be devoted to comparing the extensions $\mathcal{O} G c$ and $\mathcal{O} G^{A} w(c)$ of the nilpotent block algebras $\mathcal{O} H b$ and $\mathcal{O} H^{A} w(b)$. Applying Theorem 3.5 to the finite
groups $G^{A}$ and $H^{A}$ and the nilpotent block $w(b)$ of $H^{A}$, we get a finite group $L^{A}$ and respective injective and surjective group homomorphisms

$$
\tau^{A}: P \longrightarrow L^{A} \quad \text { and } \quad \bar{\pi}^{A}: L^{A} \longrightarrow \overline{N^{A}}
$$

such that $\bar{\pi}^{A}\left(\tau^{A}(u)\right)=\bar{u}$ for any $u \in P$, that $\operatorname{Ker}\left(\bar{\pi}^{A}\right)=\tau^{A}(Q)$ and that they induce an equivalence of categories

$$
\mathcal{E}_{\left(w(b), H^{A}, G^{A}\right)} \cong \mathcal{E}_{\left(1, \tau^{A}(Q), L^{A}\right)} .
$$

Similarly, we et $\widehat{L^{A}}=\operatorname{res}_{\pi^{A}}\left({\widehat{N^{A}}}^{k}\right)$ and denote by $\widehat{\tau^{A}}: P \rightarrow \widehat{L^{A}}$ the lifting of $\tau^{A}$; then, by Corollary 3.15, there is a $P$-interior full matrix algebra $w\left(S_{\gamma}\right)$ such that we have an isomorphism

$$
\left(\mathcal{O}\left(G^{A}\right)\right)_{w(\gamma)} \cong w\left(S_{\gamma}\right) \otimes_{\mathcal{O}} \mathcal{O}_{*}{\widehat{L^{A}}}^{\circ}
$$

of both $P$-interior and $N^{A} / H^{A}$-graded algebras.
Lemma 7.2. Assume that $G=H \cdot G^{A}$. Then we have $N=H \cdot N^{A}$, the inclusion $N^{A} \subset N$ induces a group isomorphism $\overline{N^{A}} \cong \bar{N}$ and there is a group isomorphism

$$
\sigma: L^{A} \cong L
$$

such that $\sigma \circ \tau^{A}=\tau$ and $\bar{\pi} \circ \sigma=\bar{\pi}^{A}$.
Proof. For any subgroups $R$ and $T$ of $P$ containing $Q$, let us denote by

$$
\mathcal{E}_{(b, H, G)}(R, T) \quad \text { and } \quad \mathcal{E}_{\left(w(b), H^{A}, G^{A}\right)}(R, T)
$$

the respective sets of $\mathcal{E}_{(b, H, G)^{-}}$and $\mathcal{E}_{\left(w(b), H^{A}, G^{A}\right)^{-}}$-morphisms from $T$ to $R$; since $A$ acts trivially in $\mathcal{E}_{(b, H, G)}(R, T)$, by [11, Lemma 13.8 and Corollary 13.9] each morphism in $\mathcal{E}_{(b, H, G)}(R, T)$ is induced by some element in $N^{A}$; moreover, if $T_{\nu}$ and $R_{\varepsilon}$ are local pointed groups contained in $P_{\gamma}$, it follows from Proposition 4.9 that we have $T_{\nu} \leq\left(R_{\varepsilon}\right)^{x}$ for some $x \in N^{A}$ if and only if we have $T_{w(\nu)} \leq\left(R_{w(\varepsilon)}\right)^{x}$. Therefore, we get

$$
\mathcal{E}_{(b, H, G)}(T, R)=\mathcal{E}_{\left(w(b), H^{A}, G^{A}\right)}(T, R) .
$$

At this point, it is easy to check that $L, \tau$ and $\bar{\pi}$ fulfill the conditions in Theorem 3.5 with respect to $G^{A}, H^{A}$ and the nilpotent block $w(b)$. Then this lemma follows from the uniqueness part in Theorem 3.5.

Lemma 7.3. Assume that $G=H \cdot G^{A}$. Then there is a $k^{*}$-group isomorphism $\hat{\sigma}: \widehat{L^{A}} \cong \hat{L}$ lifting $\sigma$ and fulfilling $\hat{\sigma} \circ \widehat{\tau^{A}}=\hat{\tau}$. In particular, we have

$$
\operatorname{Irr}_{\mathcal{K}}(G, c)=\operatorname{Irr}_{\mathcal{K}}(G, c)^{A} .
$$

Proof. The first statement is an easy consequence of 6.3 and Lemma 7.2; then, the last equality follows from Corollary 3.15.
7.4. Proof of Theorem 1.6. Firstly we consider the case where the block $b$ of $H$ is not stabilized by $G$; then we have an isomorphism

$$
\operatorname{Ind}_{N}^{G}(\mathcal{O} N b) \cong \mathcal{O} G b
$$

of $\mathcal{O} G$-interior algebras mapping $1 \otimes a \otimes 1$ onto $a$ for any $a \in \mathcal{O} N b$ and an isomorphism

$$
\operatorname{Ind}_{N^{A}}^{G^{A}}\left(\mathcal{O}\left(N^{A}\right) w(b)\right) \cong \mathcal{O}\left(G^{A}\right) w(b)
$$

of $\mathcal{O}\left(G^{A}\right)$-interior algebras mapping $1 \otimes a \otimes 1$ onto $a$ for any $a \in \mathcal{O}\left(N^{A}\right) w(b)$. Suppose that an $\mathcal{O}\left(N^{A} \times N\right)$-module $M$ induces a Morita equivalence from $\mathcal{O}\left(N^{A}\right) w(b)$ to $\mathcal{O} N b$. Then it is easy to see that the $\mathcal{O}\left(G^{A} \times G\right)$-module $\operatorname{Ind}_{N^{A} \times N}^{G^{A} \times G}(M)$ induces a Morita equivalence from $\mathcal{O} G c$ to $\mathcal{O}\left(G^{A}\right) w(c)$. So, we can assume that $G=N$ and then we have $G^{A}=N^{A}$.

By Corollary 3.15, there exists an isomorphism of both $(N / H)$-graded and $P$-interior algebras

$$
(\mathcal{O} G)_{\gamma} \cong S_{\gamma} \otimes_{\mathcal{O}} \mathcal{O}_{*} \hat{L}^{\circ}
$$

denote by $V_{\gamma}$ an $\mathcal{O} P$-module such that $\operatorname{End}_{\mathcal{O}}\left(V_{\gamma}\right) \cong S_{\gamma}$; choosing $i \in \gamma$ and assuming that $(\mathcal{O} G)_{\gamma}=i(\mathcal{O} G) i$, we know that the $\mathcal{O} G b \otimes_{\mathcal{O}}(\mathcal{O} G)_{\gamma}^{\circ}$-module $(\mathcal{O G}) i$ determines a Morita equivalence from $\mathcal{O} G b$ to $(\mathcal{O} G)_{\gamma}$, whereas the $(\mathcal{O} G)_{\gamma} \otimes_{\mathcal{O}} \mathcal{O}_{*} \hat{L}$-module $V_{\gamma} \otimes_{\mathcal{O}} \mathcal{O}_{*} \hat{L}^{\circ}$ determines a Morita equivalence from $(\mathcal{O} G)_{\gamma}$ to $\mathcal{O}_{*} \hat{L}^{\circ}$, so that the $\mathcal{O} G b \otimes_{\mathcal{O}} \mathcal{O}_{*} \hat{L}$ - module

$$
(\mathcal{O} G) i \otimes_{(\mathcal{O} G)_{\gamma}}\left(V_{\gamma} \otimes_{\mathcal{O}} \mathcal{O}_{*} \hat{L}^{\circ}\right) \cong(\mathcal{O} G) i \otimes_{S_{\gamma}} V_{\gamma}
$$

determines a Morita equivalence from $\mathcal{O} G b$ to $\mathcal{O}_{*} \hat{L}^{\circ}$.
Similarly, choosing $j \in \delta$ such that $j i=j=i j$, assuming that $j(\mathcal{O H}) j=$ $(\mathcal{O} H)_{\delta}$ and setting $j \cdot V_{\gamma}=V_{\delta}$, so that $S_{\delta}=\operatorname{End}_{\mathcal{O}}\left(V_{\delta}\right)$, the $\mathcal{O} H b \otimes_{\mathcal{O}} \mathcal{O} Q$ module

$$
(\mathcal{O H}) j \otimes_{(\mathcal{O H})_{\delta}}\left(V_{\delta} \otimes_{\mathcal{O}} \mathcal{O} Q\right) \cong(\mathcal{O} H) j \otimes_{S_{\delta}} V_{\delta}
$$

determines a Morita equivalence from $\mathcal{O H b}$ to $\mathcal{O} Q$.
Analogously, with evident notation, the $\mathcal{O}\left(G^{A}\right) w(b) \otimes_{\mathcal{O}} \mathcal{O}_{*} \widehat{L^{A}}$ - module

$$
\mathcal{O}\left(G^{A}\right) w(i) \otimes_{w\left(S_{\gamma}\right)} w\left(V_{\gamma}\right)
$$

determines a Morita equivalence from $\mathcal{O}\left(G^{A}\right)(b)$ to $\mathcal{O}_{*}{\widehat{L^{A}}}$, whereas the $\mathcal{O}\left(H^{A}\right) w(b) \otimes_{\mathcal{O}} \mathcal{O} Q$-module

$$
\mathcal{O}\left(H^{A}\right) w(j) \otimes_{w\left(S_{\delta}\right)} w\left(V_{\delta}\right)
$$

determines a Morita equivalence from $\mathcal{O}\left(H^{A}\right) w(b)$ to $\mathcal{O} Q$.
Consequently, identifying $\widehat{L^{A}}$ with $\hat{L}$ through the isomorphism $\hat{\sigma}$ (cf. Lemma 7.3), the $\mathcal{O}\left(G \times G^{A}\right)$-module

$$
D=\left((\mathcal{O} G) i \otimes_{S_{\gamma}} V_{\gamma}\right) \otimes_{\mathcal{O}_{*} \hat{L}}\left(w\left(V_{\gamma}\right)^{\circ} \otimes_{w\left(S_{\gamma}\right)} w(i) \mathcal{O}\left(G^{A}\right)\right)
$$

determines a Morita equivalence from $\mathcal{O} G b$ to $\mathcal{O}\left(G^{A}\right) w(b)$, whereas the $\mathcal{O}\left(H \times H^{A}\right)$-module

$$
M=\left((\mathcal{O H}) j \otimes_{S_{\delta}} V_{\delta}\right) \otimes_{\mathcal{O} Q}\left(w\left(V_{\delta}\right)^{\circ} \otimes_{w\left(S_{\delta}\right)} w(j) \mathcal{O}\left(H^{A}\right)\right)
$$

determines a Morita equivalence from $\mathcal{O} H b$ to $\mathcal{O}\left(H^{A}\right) w(b)$.
Moreover, since we have the obvious inclusions

$$
(\mathcal{O H}) j \subset(\mathcal{O} G) i \quad, \quad S_{\delta} \subset S_{\gamma} \quad \text { and } \quad V_{\delta} \subset V_{\gamma}
$$

it is easily checked that we have

$$
(\mathcal{O H}) j \otimes_{S_{\delta}} V_{\delta} \cong(\mathcal{O H}) i \otimes_{S_{\gamma}} V_{\gamma} \subset(\mathcal{O} G) i \otimes_{S_{\gamma}} V_{\gamma}
$$

in particular, we have an evident section

$$
(\mathcal{O} G) i \otimes_{S_{\gamma}} V_{\gamma} \longrightarrow(\mathcal{O H}) j \otimes_{S_{\delta}} V_{\delta}
$$

which is actually an $\mathcal{O} H b \otimes_{\mathcal{O}} \mathcal{O} Q$-module homomorphism. Similarly, we have a split $\mathcal{O}\left(H^{A}\right) w(b) \otimes_{\mathcal{O}} \mathcal{O} Q$-module monomorphism

$$
\mathcal{O}\left(H^{A}\right) w(j) \otimes_{w\left(S_{\delta}\right)} w\left(V_{\delta}\right) \longrightarrow \mathcal{O}\left(G^{A}\right) w(i) \otimes_{w\left(S_{\gamma}\right)} w\left(V_{\gamma}\right)
$$

In conclusion, the $\mathcal{O} H b \otimes_{\mathcal{O}} \mathcal{O} Q$ - and $\mathcal{O}\left(H^{A}\right) w(b) \otimes_{\mathcal{O}} \mathcal{O} Q$-module homomorphisms 7.4.2 and 7.4.3, together with the inclusion $\mathcal{O} Q \subset \mathcal{O} \hat{L}$, determine an $\mathcal{O}\left(H \times H^{A}\right)$-module homomorphism

$$
M \longrightarrow \operatorname{Res}_{H \times H^{A}}^{G \times G^{A}}(D)
$$

which actually admits a section too. Now, denoting by $K$ the inverse image in $G \times G^{A}$ of the "diagonal" subgroup of $(G / H) \times\left(G^{A} / H^{A}\right)$, we claim that the product by $K$ stabilizes the image of $M$ in $D$, so that $M$ can be extended to an $\mathcal{O} K$-module.

Actually, we have

$$
K=\left(H \times H^{A}\right) \cdot \Delta\left(N_{G^{A}}\left(Q_{\delta}\right)\right),
$$

so that it suffices to prove that the image of $M$ is stable by multiplication by $\Delta\left(N_{G^{A}}\left(Q_{\delta}\right)\right)$. Given $x \in N_{G^{A}}\left(Q_{\delta}\right)$, there are some invertible elements $a_{x} \in(\mathcal{O H})^{Q}$ and $b_{x} \in\left(\mathcal{O}\left(H^{A}\right)\right)^{Q}$ such that

$$
x j x^{-1}=a_{x} j a_{x}^{-1} \quad \text { and } \quad x w(j) x^{-1}=b_{x} w(j) b_{x}^{-1}
$$

and therefore $a_{x}^{-1} x$ and $b_{x}^{-1} x$ respectively centralize $j$ and $w(j)$, so that $a_{x}^{-1} x j$ and $b_{x}^{-1} x w(j)$ respectively belong to $(\mathcal{O} G)_{\delta}$ and to $\left(\mathcal{O} G^{A}\right)_{w(\delta)}$; but, according to isomorphisms 7.4.1 and 7.1.1, we have $G / H$ - and $G^{A} / H^{A}$-graded isomorphisms

$$
(\mathcal{O} G)_{\delta} \cong S_{\delta} \otimes_{\mathcal{O}} \mathcal{O}_{*} \hat{L}^{\circ} \quad \text { and } \quad\left(\mathcal{O} G^{A}\right)_{w(\delta)} \cong w\left(S_{\delta}\right) \otimes_{\mathcal{O}} \mathcal{O}_{*} \hat{L}^{\circ}
$$

where we are setting $w\left(S_{\delta}\right)=w(j) w\left(S_{\gamma}\right) w(j)$.
Hence, identifying with each other both members of these isomorphisms and modifying if necessary our choice of $a_{x}$, for some $s_{x} \in S_{\delta}, t_{x} \in w\left(S_{\delta}\right)$ and $\hat{y}_{x} \in \hat{L}^{\circ}$, we get

$$
a_{x}^{-1} x j=s_{x} \otimes \hat{y}_{x} \quad \text { and } \quad b_{x}^{-1} x w(j)=t_{x} \otimes \hat{y}_{x} .
$$

Thus, setting $w\left(V_{\delta}\right)=w(j) w\left(V_{\gamma}\right)$, for any $a \in(\mathcal{O} H) j$, any $b \in\left(\mathcal{O} H^{A}\right) w(j)$, any $v \in V_{\delta}$ and any $w \in w\left(V_{\delta}\right)$, in $D$ we have

$$
\begin{aligned}
(x, x) & \cdot(a \otimes v) \otimes(w \otimes b)=(x a \otimes v) \otimes\left(w \otimes b x^{-1}\right) \\
= & \left(x a x^{-1} a_{x}\left(a_{x}^{-1} x j\right) \otimes v\right) \otimes\left(w \otimes\left(w(j) x^{-1} b_{x}\right) b_{x}^{-1} x b x^{-1}\right) \\
& =\left(x a x^{-1} a_{x} \otimes s_{x} \cdot v\right) \cdot \hat{y}_{x} \otimes \hat{y}_{x}^{-1} \cdot\left(w \cdot t_{x}^{-1} \otimes b_{x}^{-1} x b x^{-1}\right) \\
& =\left(x a x^{-1} a_{x} \otimes s_{x} \cdot v\right) \otimes\left(w \cdot t_{x}^{-1} \otimes b_{x}^{-1} x b x^{-1}\right) ;
\end{aligned}
$$

since $x a x^{-1} a_{x}$ and $b_{x}^{-1} x b x^{-1}$ respectively belong to $(\mathcal{O H}) j$ and $w(j)\left(\mathcal{O} H^{A}\right)$, this proves our claim.

Finally, since homomorphism 7.4 .4 actually becomes an $\mathcal{O} K$-module homomorphism, it induces an $\mathcal{O}\left(G \times G^{A}\right)$-module homomorphism

$$
\operatorname{Ind}_{K}^{G \times G^{A}}(M) \longrightarrow D
$$

which is actually an isomorphism as it is easily checked. We are done.
The following theorem is due to Harris and Linckelmann (see [9]).
Theorem 7.5. Let $G$ be an $A$-group and assume that $G$ is a finite $p$ solvable group and $A$ is a solvable group of order prime to $|G|$. Let $b$ be an $A$-stable block of $G$ over $\mathcal{O}$ with a defect group $P$ centralized by $A$ and denote by $w(b)$ the Glauberman correspondent of the block $b$. Then the block algebras $\mathcal{O} G b$ and $\mathcal{O}\left(G^{A}\right) w(b)$ are basically Morita equivalent.
Proof. By [9, Theorem 5.1], we can assume that $b$ is a $G \rtimes A$-stable block of $\mathrm{O}_{p^{\prime}}(G)$, where $\mathrm{O}_{p^{\prime}}(G)$ is the maximal normal $p^{\prime}$-subgroup of $G$. Clearly $b$ as a block of $\mathrm{O}_{p^{\prime}}(G)$ is nilpotent and thus $\mathcal{O} G b$ is an extension of the nilpotent block algebra $\mathcal{O O}_{p^{\prime}}(G) b$. By [9, Theorem 5.1] again, $w(b)$ is a $G^{A}$-stable block of $\mathrm{O}_{p^{\prime}}\left(G^{A}\right)$ and thus is nilpotent; thus $\mathcal{O}\left(G^{A}\right) w(b)$ is an extension of the nilpotent block algebra $\mathcal{O O}_{p^{\prime}}\left(G^{A}\right) w(b)$. By [9, Theorem 4.1], $w(b)$ is also the Glauberman correspondent of $b$ as a block of $\mathrm{O}_{p^{\prime}}(G)$. Then, by Theorem 1.6, the block algebras $\mathcal{O} G b$ and $\mathcal{O}\left(G^{A}\right) w(b)$ are basically Morita equivalent.

The following theorem is due to Koshitani and Michler (see [12]).
Theorem 7.6. Let $G$ be an $A$-group and assume that $A$ is a solvable group of order prime to $|G|$. Let $b$ be an $A$-stable block of $G$ over $\mathcal{O}$ with a defect group $P$ centralized by $A$ and denote by $w(b)$ the Glauberman correspondent of the block $b$. Assume that $P$ is normal in $G$. Then, the block algebras $\mathcal{O} G b$ and $\mathcal{O}\left(G^{A}\right) w(b)$ have isomorphic source algebras.
Proof. Since $P$ is normal in $G$, by $[1,2.9]$ there is a block $b_{P}$ of $C_{G}(P)$ such that $b=\operatorname{Tr}_{G_{b_{P}}}^{G}\left(b_{P}\right)$, where $G_{b_{P}}$ is the stabilizer of $b_{P}$ in $G$. Since $A$ and $G$ have coprime orders, by [11, Lemma 13.8 and Corollary 13.9], $b_{P}$ can be chosen such that $A$ stabilizes $b_{P}$. Since $P$ is the unique defect group of $b, P$ has to be contained in $G_{b_{P}}$; then by [14, Proposition 5.3], the intersection $Z(P)=P \cap C_{G}(P)$ is the defect group of $b_{P}$ and, in particular, $b_{P}$ is nilpotent. Thus the block $\mathcal{O} G b$ is an extension of the nilpotent block algebra $\mathcal{O}\left(P \cdot C_{G}(P)\right) b_{P}$ and, in particular, we have $\bar{N} \cong E_{G}\left(P_{\gamma}\right)$.

The Glauberman correspondent of $b_{P}$ makes sense and by [26, Proposition 4], we have

$$
w(b)=\operatorname{Tr}_{\left(G^{A}\right)_{w\left(b_{P}\right)}}^{G^{A}}\left(w\left(b_{P}\right)\right) .
$$

Since $w\left(b_{P}\right)$ has defect group $Z(P)$, it is also nilpotent and thus $\mathcal{O}\left(G^{A}\right) w(b)$ is an extension of the nilpotent block algebra $\mathcal{O}\left(P \cdot C_{G^{A}}(P)\right) w\left(b_{P}\right)$; once again, we have $\bar{N}^{A} \cong E_{G^{A}}\left(P_{w(\gamma)}\right)$.

On the other hand, since $P$ is normal in $G$, it follows from [17, Proposition 14.6] that

$$
(\mathcal{O} G)_{\gamma} \cong \mathcal{O}_{*}\left(P \rtimes \hat{E}_{G}\left(P_{\gamma}\right)\right) \quad \text { and } \quad\left(\mathcal{O} G^{A}\right)_{w(\gamma)} \cong \mathcal{O}_{*}\left(P \rtimes \hat{E}_{G^{A}}\left(P_{w(\gamma)}\right)\right)
$$

but, it follows from 6.3 that we have a $k^{*}$-group isomorphism

$$
\hat{E}_{G}\left(P_{\gamma}\right) \cong \hat{E}_{G^{A}}\left(P_{w(\gamma)}\right)
$$

We are done.

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