# SPECTRUM OF PLANE CURVES VIA KNOT THEORY 

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#### Abstract

In this paper we use topological methods to study various semicontinuity properties of spectra of singular points of plane algebraic curves and of polynomials in two variables at infinity. Using the Seifert form and the Tristram-Levine signature of links, we reprove (in a slightly weaker version) a result obtained by Steenbrink and Varchenko on semicontinuity of spectra of singular points under deformation and results of Némethi and Sabbah on semicontinuity of spectrum at infinity. We also relate the spectrum at infinity of a polynomial with spectra of singular points of a chosen fiber.


## 1. Introduction

The Hodge spectrum of a local hypersurface isolated singularity $f:\left(\mathbb{C}^{n+1}, 0\right) \rightarrow(\mathbb{C}, 0)$ is the output of the mixed Hodge structure of the vanishing cohomology of the singular germ A, St1, St2, Var, Var2]. Usually, it is not topological, it is one of the finest analytic invariants of the germ. Although, it does not characterize the singularity completely, it gives extremely strong information about it. As it was conjectured by Arnold [A, and proved by Varchenko [Var, Var2] and Steenbrink [St2], the spectrum behaves semicontinuously under deformations, which makes it, for example, a very strong tool in attempts to solve the adjacency problem (i.e., to determine, which singularities can specialize to a given one).

A more precise picture is the following: the algebraic monodromy acts on the vanishing cohomology, this cohomology supports a mixed Hodge structure, which is polarized by the intersection form, and the Seifert form (which can be identified with the variation map). The equivariant Hodge numbers were codified by Steenbrink in the spectral pairs; if one deletes the information about the weight filtration, one gets the spectral numbers $S p(f)$. They are (in some normalization) rational numbers in the interval ( $0, n+1$ ). In the presence of a deformation $f_{t}$, where $t$ is the deformation parameter $t \in(\mathbb{C}, 0)$, the semicontinuity guarantees that $\left|S p\left(f_{0}\right) \cap I\right| \geq\left|S p\left(f_{t \neq 0}\right) \cap I\right|$ for the semicontinuity domain $I$. Arnold made his conjecture for $I=(-\infty, x]$, Steenbrink and Varchenko proved the statement for $I=(x, x+1$ ], which implies Arnold's conjecture. Additionally, Varchenko in Var for some cases verified the strongest version, namely semicontinuity for $I=(x, x+1)$.

The semicontinuity property (with any domain) cannot be extended to the spectral pairs, hence in studies targeting these kind of applications one usually works with spectrum only. This is what we will do in the present article as well.

On the other hand, (one of) the strongest topological invariants of $f$ is its Seifert form, for terminology see e.g. AGV. The relation between the Hodge invariants and Seifert form was established by the second author in [Nem2], proving that the collection of mod 2 spectral pairs are equivalent with the real Seifert form. In this way, the real Seifert form is in strong relationship with the mod 2 spectrum, that is with the collection of numbers

[^0]$x \bmod 2$ in $(0,2]$, where $x$ run over $S p(f)$. Clearly, for plane curve singularities, i.e. when $n=1$, by taking mod 2 reduction we loose no information.

Our primary goal is to extend the above correspondence for an arbitrary link $\left(S_{R}^{3}, L\right)$, where $S_{R}^{3}$ is the boundary of some ball with radius $R$ in $\mathbb{C}^{2}$, and $L$ is the intersection of $S_{R}^{3}$ with some affine algebraic curve $C$ in $\mathbb{C}^{2}$. The primary interest is the link at infinity of such affine curve (hence $R \gg 0$ ), but we also wish to develop a method to study any general $\left(S_{R}^{3}, L\right)$, for which the available methods in the literature are rather sparse.

Let us consider a complex polynomial map $F: \mathbb{C}^{2} \rightarrow \mathbb{C}$. For its topology at infinity see Neumann's article [Neu3]. Our first main result recovers the spectrum at infinity associated with the limit mixed Hodge structure at infinity (supported by the cohomology of the generic fiber) from the real Seifert form of the regular link at infinity associated with F. In particular, we reobtain the spectrum at infinity topologically, in pure link-theoretical language.

The key bridge which connects the link-theoretical language and invariants with the Hodge theoretical spectrum is the Tristram-Levine signature Tri, Le]. For example, for the weighted homogeneous singularity given by $\left\{x^{p}-y^{q}=0\right\}$ with $p$ and $q$ relative prime integers, the spectrum is $S p_{p, q}=\left\{\frac{i}{p}+\frac{j}{q}, 1 \leq i \leq p-1,1 \leq j \leq q-1\right\}$, while the TristramLevine signature function of the $(p, q)$-torus knot, evaluated at $e^{2 \pi i x}$ with $x \in(0,1), p q x \notin$ $\mathbb{Z}$, is equal to $2\left|S p_{p, q} \cap(x, x+1)\right|-(p-1)(q-1)$, see e.g. [Li]. In [BN] we made this relation rigorous, showing a direct translation between spectra of singularities and Tristram-Levine signatures of their links.

In this correspondence, what is really surprising - and this is the seconds main message of the article - is the fact that the semicontinuity of the mod 2 spectrum is topological: it can be recovered independently of analytic (Hodge theoretical) tools, it follows from pure link theory. More precisely, we prove that length one 'intervals' intersected by the mod 2 spectrum, namely sets of type $S p \cap(x, x+1)$ and $(S p \cap(0, x)) \cup(S p \cap(x+1,2])$, for $x \in[0,1]$, satisfy semicontinuty properties, whenever this is question is well-posed.

In this article we exemplify this by three cases: we recover the semicontinuity (in the above form, with slight assumptions) for deformations of local plane curve singularities, corresponding to results of Varchenko and Steenbrink, and also we establish a semicontinuity of the spectrum at infinity associated with a family of polynomials in two variables, in the spirit of [NS]. The third case targets a new phenomenon: in the context of an affine curve $C \subset \mathbb{C}^{2}$ we show a semicontinuity connecting the local spectra of the singularities of $C$ with the spectrum at infinity of $C$. In all these cases, the key link-theoretical ingredient is a Murasugi type inequality, which controls the modification of the Tristram-Levine signature under those type of surgeries which appears when we pass from $C \cap S_{r}^{3}$ to $C \cap S_{R}^{3}$ via Morse theory $(r<R)$. This was studied by the first author in [B0].

The organization of the paper is the following. In section 2 we review the theory of hermitian variation structures from Nem2, their relation with the spectrum, how can one associate such a structure to a link $[\mathrm{BN}]$, and how it connects the spectrum with the Tristram-Levine signatures [BN. We also recall some of the main results of B 0 about surgery inequalities of links of type $S_{R}^{3} \cap C$. Section 3 contains the study of the spectrum at infinity of a polynomial map in terms of the Seifert form at infinity. In section 4 we prove semicontinuity results regarding the spectrum.

For a finite set $A$, we denote by $|A|$ the cardinality of $A$.

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## 2. Variations structures of links

We recall in 2.1] the definition of an abstract hermitian variation structure and its spectra, and in 2.2 the definition of the hermitian variation structure and spectra associated with links in a three-sphere. Subsection 2.3 reviews the definition of mixed Hodge structures and their Hodge spectra. Finally, in 2.4 we draw a relationship observed in BN between spectra and Tristram-Levine signatures of links. In 2.5 we recall some results from Bo which are crucial ingredients in the proof of the semicontinuity results of the last section.
2.1. Hermitian variation structures were introduced in (Nem2], they are generalization of $\varepsilon$-symmetric isometric structures. Here we review the minimal basics, for more details see Nem2, Nem3].

Recall that a structure $\left(U=\mathbb{C}^{n} ; b, h\right)$, where $b$ is an $\varepsilon$-symmetric hermitian form on $U$ preserved by the automorphism $h$ of $U$, is called an isometric structure (for $\varepsilon= \pm 1$ ). The classification of isometric structures when $b$ is non-degenerate was established by Milnor [Mil] (see also [Neu1, Neu2]). Any $\varepsilon$-hermitian variation structure (in short $\varepsilon$-HVS) can be regarded as the isometric structure together with an operator $V: U^{*} \rightarrow U$ such that

$$
\begin{equation*}
\overline{V^{*}}=-\varepsilon V \overline{h^{*}} \text { and } V \circ \tilde{b}=h-\mathrm{Id}, \tag{2.1.1}
\end{equation*}
$$

where $\tilde{b}$ is the form $b$ regarded as map from $U$ to $U^{*}$. We denote it as $\mathcal{V}=(U ; b, h, V)$. Here * denotes the duality, while - the complex conjugation.

Definition 2.1.2. We say that the isometric structure $(U ; b, h)$ can be completed to a hermitian variation structure, if there exists $V: U^{*} \rightarrow U$ such that (2.1.1) is satisfied.

If $b$ is non-degenerate, then the isometric structure can be uniquely completed to a HVS: $V=(h-\mathrm{Id}) \circ \tilde{b}^{-1}$. In general, not every isometric structures can be completed (see e.g. (3.2.8)(c) below). Moreover, if a completion exists, in general, it is not unique (even if we restrict ourselves to non-degenerate matrices $V$, see e.g. [Nem2, (2.7.7)]).

A HVS is called simple if $V$ is an isomorphism. The classification of simple HVS's is established in Nem2. Each simple variation structure is a direct sum of indecomposable simple variation structures. Indecomposable structures can be listed: for each positive integer $k$, and for each $\lambda \in \mathbb{C}$ such that $0<|\lambda| \leq 1$ we have

- for $|\lambda|<1$ a unique simple indecomposable variation structure $\mathcal{V}_{\lambda}^{2 k}$;
- for $|\lambda|=1$ two simple indecomposable structures, denoted by $\mathcal{W}_{\lambda}^{k}(+1)$ and $\mathcal{W}_{\lambda}^{k}(-1)$.

This classification is a refinement of the Jordan block decomposition of the matrix $h$ (or of Milnor's classification of non-degenerate isometric structures). More precisely, the matrix $h$ corresponding to $\mathcal{W}_{\lambda}^{k}( \pm 1)$ is a single Jordan block of size $k$ and eigenvalue $\lambda$, while the one corresponding to $\mathcal{V}_{\lambda}^{2 k}$ has two Jordan blocks of size $k$ : one with eigenvalue $\lambda$, the other with eigenvalue $1 / \lambda$. For their precise form see [Nem2].

Write a simple variation structure $\mathcal{V}$ as a (unique) sum of the indecomposable ones:

$$
\begin{equation*}
\mathcal{V}=\bigoplus_{\substack{0<|\lambda|<1 \\ k \geq 1}} q_{\lambda}^{k} \cdot \mathcal{V}_{\lambda}^{2 k} \oplus \bigoplus_{\substack{|\lambda|=1 \\ k \geq 1, u= \pm 1}} p_{\lambda}^{k}(u) \cdot \mathcal{W}_{\lambda}^{k}(u) \tag{2.1.3}
\end{equation*}
$$

for certain non-negative integers $q_{\lambda}^{k}$ and $p_{\lambda}^{k}(u)$. Here we write $m \cdot \mathcal{V}$ for $\mathcal{V} \oplus \cdots \oplus \mathcal{V}(m$-times $)$.
The numbers $\left\{q_{\lambda}^{k}\right\}_{|\lambda|<1}$ and $\left\{p_{\lambda}^{k}( \pm 1)\right\}_{\lambda \in S^{1}}$ are called the $H$-numbers of the HVS $\mathcal{V}$.
Using H -numbers we can define the spectrum of $\mathcal{V}$. Sometimes, in order to emphasize the source of the definition, we call it $H V S$-spectrum.

Definition 2.1.4. ([Nem1] or BN, (2.3.1)-(2.3.3)]) Consider the H-numbers $\left\{q_{\lambda}^{k}\right\}_{|\lambda|<1}$ and $\left\{p_{\lambda}^{k}( \pm 1)\right\}_{\lambda \in S^{1}}$ of $\mathcal{V}$. The extended spectrum $E S p$ is the union $E S p=S p \cup I S p$, where
(a) $S p$, the spectrum, is a finite set of real numbers from the interval $(0,2]$ such that any real number $\alpha$ occurs in $S p$ precisely $s(\alpha)$ times, where

$$
s(\alpha)=\sum_{n=1}^{\infty} \sum_{u= \pm 1}\left(\frac{2 n-1-u(-1)^{\lfloor\alpha\rfloor}}{2} p_{\lambda}^{2 n-1}(u)+n p_{\lambda}^{2 n}(u)\right), \quad\left(e^{2 \pi i \alpha}=\lambda\right) .
$$

(b) $I S p$ is the set of complex numbers from $(0,2] \times i \mathbb{R}, I S p \cap \mathbb{R}=\emptyset$, where $z=\alpha+i \beta$ occurs in $I S p$ presizely $s(z)$ times, where

$$
s(z)= \begin{cases}\sum k \cdot q_{\lambda}^{k} & \text { if } \alpha \leq 1, \beta>0 \text { and } e^{2 \pi i z}=\lambda \\ \sum k \cdot q_{\lambda}^{k} & \text { if } \alpha>1, \beta<0 \text { and } e^{2 \pi i z}=1 / \bar{\lambda} \\ 0 & \text { if } \alpha \leq 1 \text { and } \beta<0, \text { or } \alpha>1 \text { and } \beta>0\end{cases}
$$

Since the size of a matrices corresponding to $\mathcal{V}_{\lambda}^{2 k}$ is $2 k$ and to $\mathcal{W}_{\lambda}^{k}( \pm 1)$ is $k$, one gets

$$
\begin{equation*}
|E S p|=\operatorname{dim} U=\operatorname{deg} \operatorname{det}(h-t \mathrm{Id}) . \tag{2.1.5}
\end{equation*}
$$

2.2. The HVS and spectrum of a link. The variation structure and $H$-numbers of a link in $S^{3}$ were defined in $[\mathrm{BN}$. Let us review shortly how the construction is performed.

Let $S$ be a Seifert matrix of a link $L$. (For the convention of its definition see 3.2.) By Keef's result Keeff $S$ is S-equivalent either to an empty matrix, or to a matrix $S^{\prime}$, which can be decomposed into a direct sum

$$
\begin{equation*}
S^{\prime}=S_{0} \oplus S_{\mathrm{ndeg}} \tag{2.2.1}
\end{equation*}
$$

where $S_{0}$ is a zero matrix and $S_{\text {ndeg }}$ is non-degenerate, that is $\operatorname{det} S_{\text {ndeg }} \neq 0$. Moreover, any two such non-degenerate models $S_{\text {ndeg }}$ of the same link, are congruent over $\mathbb{Q}$. The size of $S_{0}$ is also determined by $L$ (it is equal to $\operatorname{dim}\left(\operatorname{ker} S \cap \operatorname{ker} S^{T}\right)$ ), we will call it the irregularity of $L$, and we will denote it by

$$
\begin{equation*}
\operatorname{Irr}=\operatorname{Irr}(L):=\operatorname{size}\left(S_{0}\right) \tag{2.2.2}
\end{equation*}
$$

Let $n$ be the size of $S_{\text {ndeg }}$. The quadruple $\mathcal{V}=(U, b, h, V)$, where $U=\mathbb{C}^{n}, V=\left(S_{\text {ndeg }}^{T}\right)^{-1}$, $h=V S, b=S-S^{T}$, constitutes a HVS with the sign choice $\varepsilon=-1$. (Here ${ }^{T}$ denote transposition.) As changing a Seifert matrix results in congruency of $S_{\text {ndeg }}$, which leads to an isomorphism of variations structures, the structure $\mathcal{V}$ does not depend on the choice of a Seifert matrix, so it is a well-defined link invariant, called $\mathcal{V}_{L}$. Additionally, $\mathcal{V}_{L}$ is simple. Note that $\mathcal{V}_{L}$ is defined over the rational numbers $\mathbb{Q}$. The characteristic polynomial $\Delta^{h}=\operatorname{det}(h-t \mathrm{Id})$ of $h$ will be called the characteristic polynomial of the link. Its connection with Alexander polynomial is as follows (see e.g. [BN, §4]):

Lemma 2.2.3. Let $\mathcal{V}_{L}$ be as above. If the Alexander polynomial $\Delta$ of $L$ is non-zero then $\Delta=\Delta^{h}$ up to multiplication by an invertible element of $\mathbb{Q}\left[t, t^{-1}\right]$. If the Alexander polynomial is zero, then $\Delta^{h}$ is proportional to the first higher Alexander polynomial $\Delta_{k}$, which is not identically zero: $\Delta_{k}=0$ for $0 \leq k<\operatorname{Irr}$ and $\Delta_{\text {Irr }}=\Delta^{h}$ (up to an invertible element).

Definition 2.2.4. Consider the integers $\left\{q_{\lambda}^{k}\right\}_{|\lambda|<1}$ and $\left\{p_{\lambda}^{k}( \pm 1)\right\}_{\lambda \in S^{1}}$ provided by the direct sum decomposition (2.1.3) of $\mathcal{V}_{L}$. They are called the $H$-numbers of the link $L$. The associated (extended) spectrum is called the (extended) spectrum of the link.

From (2.1.5) one has $|E S p|=\operatorname{deg} \Delta^{h}$. Moreover, $S p \backslash \mathbb{Z}$ is symmetric with respect to 1 .
2.3. Mixed Hodge structures and their spectra. The name and definition of spectrum in Definition 2.1.4 is motivated by the fact that if $L$ is an algebraic link, i.e. the link of (local) isolated plane curve singularity, then $I S p$ is empty and $S p$ is the classical spectrum associated with the mixed Hodge structure of the vanishing cohomology (for this see e.g. (St1, St2, Var).

More generally, let $f:\left(\mathbb{C}^{n+1}, 0\right) \rightarrow(\mathbb{C}, 0)$ be the germ of an analytic function with isolated singularity at 0 , and let $Y$ be the Milnor fiber and $U=\widetilde{H}_{n}(Y, \mathbb{R})$. (For details regarding the Milnor fibration, see e.g. [Mi2, AGV, Nem2].) One takes the monodromy operator $h: U \rightarrow U$, the intersection form $b: U \times U \rightarrow \mathbb{R}$ and the variation operator $V: U^{*} \rightarrow U$. One checks (see e.g. AGV] or [Nem2, §5]) that the complexification of $(U ; b, h, V)$ constitutes a $(-1)^{n}-$ HVS. If $S$ is the Seifert matrix of the Milnor fibration, then at the level of matrices $V=\left(S^{T}\right)^{-1}$. Since $S$ is unimodular, $V$ will be isomorphism too, hence the variation structure is simple. For plane curves $\varepsilon=-1$, hence $h=\left(S^{T}\right)^{-1} S$ and $b=S-S^{T}$. The structure $(U ; b, h, V) \otimes \mathbb{C}$ is called the 'homological HVS' of the germ.

There is dual a HVS, the 'cohomological HVS' associated with the germ, which sits on $H^{*}:=\widetilde{H}^{n}(Y, \mathbb{C})$. Additionally, $\widetilde{H}^{n}(Y, \mathbb{C})$ carries a limit mixed Hodge structure with Hodge filtration $F$ and weight filtration $W$ such that the semisimple part $h_{s s}^{*}$ of the cohomological monodromy operator acts on $\left(H^{*}, F, W\right)$. They define spectral pairs. In order to eliminate any confusion about the existing different normalizations, we provide some details.

One considers the generalized $\lambda$-eigenspaces $U_{\lambda}^{*}$ for all the eigenvalues $\lambda$ of the GaussManin monodromy operator $h_{G M}=\left(h_{s s}^{*}\right)^{-1}$ and the equivariant (Gauss-Manin) Hodge numbers $h_{\lambda}^{p, q}:=\operatorname{dim} G r_{F}^{p} G r_{p+q}^{W} U_{\lambda}^{*}$.

Then these numbers can be codified in a different way in the collection of Hodge spectral pairs of $\left(U^{*}, F, W ; h_{s s}^{*}\right)$. This is a collection of pairs $(\alpha, w)$ from $\mathbb{R} \times \mathbb{N}$ defined by

$$
\begin{equation*}
\operatorname{Spp}_{G M}(f)=\sum_{(\alpha, w)} h_{\exp (-2 \pi i \alpha)}^{n+[-\alpha], w+s-n-[-\alpha]}(\alpha, w) \in \mathbb{N}[\mathbb{R} \times \mathbb{N}] \tag{2.3.1}
\end{equation*}
$$

where $s=1$ if $\lambda=\exp (-2 \pi i \alpha)=1$ and $s=0$ otherwise.
This can be transformed in several ways. If by some geometric reason, one wishes to emphasize more the cohomological monodromy operator $h_{s s}^{*}\left(\right.$ instead of $\left.h_{G M}\right)$, one considers

$$
\begin{equation*}
S p p_{*}(f)=\sum_{(\alpha, w)} h_{\exp (2 \pi i \alpha)}^{n+[-\alpha], w+s-n-[-\alpha]}(\alpha, w) \in \mathbb{N}[\mathbb{R} \times \mathbb{N}] \tag{2.3.2}
\end{equation*}
$$

If we forget the weight filtration, then from the equivariant Hodge filtration one can read the Hodge spectrum, namely

$$
\begin{equation*}
\left.S p_{*}(f)=\sum \alpha \in \mathbb{N}[\mathbb{R}] \quad \text { (the sum over the spectral pairs }(\alpha, w) \text { of } S p p_{*}(f)\right) . \tag{2.3.3}
\end{equation*}
$$

Any spectral number $\alpha$ is in the interval $(-1, n)$. Another normalizations of the spectrum identifies it in the interval $(0, n+1): S p_{\text {MHS }}(f)$ is the collection of numbers $(\alpha+1)$ where $\alpha$ runs over the entries of $S p_{*}(f)$.

The identification of the Hodge invariants with the associated hermitian variation structure goes through the crucial polarization property of the mixed Hodge structure. In this way, the cohomological hermitian variation structure of $f$ can be obtained from $\left(U^{*}, F, W\right)$ by collapsing the Hodge filtration mod 2, having the collapsed spectral numbers in ( $-1,1$ ]. The corresponding H -numbers are, in fact, the equivariant primitive Hodge numbers of $\left(U^{*}, F, W\right)$ under this collapsing procedure. Usually, the homological and cohomological HVS's do not agree, in the case $\varepsilon=(-1)^{n}=-1$ they differ by a sign: $\mathcal{V}_{\text {coh }}=-\mathcal{V}_{\text {hom }}$. This explains the two slightly different definition of the spectral numbers (2.1.4) (a) and (2.3.2). Nevertheless, one has the following identification:

Proposition 2.3.4 ([Nem2, (6.5)]). The $H V S$-spectrum $S p_{\mathrm{HVS}}$ is a mod 2 reduction of the Hodge spectrum $S p_{\text {MHS }}$ considered in $(0,2]$. In other words

$$
S p_{\mathrm{HVS}}=\left\{x \bmod 2: x \in S p_{\mathrm{MHS}}\right\} .
$$

Therefore, for a gem of an isolated plane curve singularity one gets $S p_{\mathrm{HVS}}=S p_{\mathrm{MHS}}$. That means, that the Hodge spectrum can completely be described from the (real) Seifert form of the link. This is the model of our further investigation.
2.4. Spectrum of a link and the Tristram-Levine signature. The Tristram-Levine signature (defined first in [Tri, Le]) turn out to be a knot-theoretic counterpart of the spectrum of singular points. We recall how can they be explicitly expressed from the spectrum of the link.

Definition 2.4.1. Let $L$ be a link and $S$ its Seifert matrix. The Tristram-Levine signature function is the mapping from $S^{1} \backslash\{1\}=\{\zeta \in \mathbb{C}:|\zeta|=1, \zeta \neq 1\}$ to $\mathbb{Z}$ given by

$$
\sigma_{L}(\zeta)=\text { signature }\left[(1-\zeta) S+(1-\bar{\zeta}) S^{T}\right] .
$$

The nullity $n_{L}(\zeta)$ is the nullity of the same form $(1-\zeta) S+(1-\bar{\zeta}) S^{T}$, while the normalized nullity, $\tilde{n}_{L}(\zeta)$, is defined as $n_{L}(\zeta)$ - Irr. For completeness we extend the definitions for $\zeta=1$ too. First, we set $\sigma_{L}(1)=0$. Then notice that for any $\zeta \neq 1, \widetilde{n}_{L}(\zeta)$ equals to the multiplicity of the root of $\Delta^{h}$ at $\zeta$. We define $\widetilde{n}_{L}(1)$ by this characterization for $\zeta=1$.

We have the following relation between H -numbers, signatures and nullities of the link.
Proposition 2.4.2 ([BN, (4.4.6) and (4.4.9)]). Let $S p=S p_{\mathrm{HVS}}$ be the real part of the spectrum as defined in Definition 2.1.4. Let $x \in(0,1)$ and $\zeta=e^{2 \pi i x}$. Then

$$
\begin{aligned}
& \sigma(\zeta)=-|S p \cap(x, x+1)|+|S p \backslash[x, x+1]|+\sum_{n=1}^{\infty} \sum_{u= \pm 1} u p_{\zeta}^{2 n}(u) \\
& \widetilde{n}(\zeta)=\sum_{k, u} p_{\zeta}^{k}(u) .
\end{aligned}
$$

In particular,

$$
\begin{equation*}
-\sigma(\zeta)+\widetilde{n}(\zeta) \geq|S p \cap(x, x+1)|-|S p \backslash[x, x+1]| . \tag{2.4.3}
\end{equation*}
$$

Remark 2.4.4. In the cases $x \in\{0,1\}$, the inequality (2.4.3) still holds. Indeed, the left hand side is non-negative, while the right hand side is non-positive (since $S p \backslash \mathbb{Z}$ is symmetric). Moreover, if 1 is not a root of $\Delta^{h}$, then (2.4.3) is an equality for $x=1$.

Let us denote

$$
D=|S p \cap\{x, x+1\}| \geq 0 .
$$

Assume that $\Delta^{h}$, the characteristic polynomial of the link, has no roots outside the unit circle. Then $\operatorname{deg} \Delta^{h}=|S p|=|S p \cap(x, x+1)|+|S p \backslash[x, x+1]|+D$, hence one also has

$$
\begin{equation*}
\operatorname{deg} \Delta^{h}-\sigma(\zeta)+\widetilde{n}(\zeta)=2|S p \cap(x, x+1)|+\sum_{\substack{k \text { odd } \\ u= \pm 1}} p_{\zeta}^{k}(u)+\sum_{k \text { even }} 2 p_{\zeta}^{k}(-1)+D . \tag{2.4.5}
\end{equation*}
$$

For any $x \in[0,1]$, parallel to the set $S p \cap(x, x+1)$, we will also consider the set $S p \backslash[x, x+1]=S p \cap(0, x)+S p \cap(1+x, 2]$. These two types cover all the 'length one open intervals' of the mod 2 spectrum.

The following corollary will be used extensively in the sequel.

Corollary 2.4.6. Let $L$ be a link and $\Delta^{h}$ its characteristic polynomial. Assume that $\Delta^{h}$ has no roots outside the unit circle. If $\zeta=e^{2 \pi i x}$ is not a root of $\Delta^{h}$ then

$$
|S p \cap(x, x+1)|=\frac{1}{2}\left(\operatorname{deg} \Delta^{h}-\sigma(\zeta)\right) \quad \text { and }|S p \backslash[x, x+1]|=\frac{1}{2}\left(\operatorname{deg} \Delta^{h}+\sigma(\zeta)\right) .
$$

Moreover, for arbitrary $x \in[0,1]$ :

$$
\begin{align*}
& \frac{1}{2}\left(\operatorname{deg} \Delta^{h}-\sigma(\zeta)+\widetilde{n}(\zeta)\right) \geq|S p \cap(x, x+1)| \\
& \frac{1}{2}\left(\operatorname{deg} \Delta^{h}+\sigma(\zeta)+\widetilde{n}(\zeta)\right) \geq|S p \backslash[x, x+1]| \tag{2.4.7}
\end{align*}
$$

2.5. Morse theory of plane curves. For any $\xi \in \mathbb{C}^{2}$ and $r>0$ let $B(\xi, r)$ be the ball centered at $\xi$ and with radius $r$, also $S^{3}(\xi, r):=\partial B(\xi, r)$. For an algebraic curve $C$ sitting in $\mathbb{C}^{2}$, we write $(C \cap B(\xi, r))^{\wedge}$ for the normalization of $C \cap B(\xi, r)$, and the genus of $C \cap B(\xi, r)$ is the genus of its normalization.

For any link $L$, we denote by $c_{L}$ its number of components, and we set

$$
\begin{aligned}
w_{L}(\zeta) & :=-\sigma_{L}(\zeta)+1-c_{L}+n_{L}(\zeta) \\
-u_{L}(\zeta) & :=\sigma_{L}(\zeta)+1-c_{L}+n_{L}(\zeta) .
\end{aligned}
$$

Remark 2.5.1. The convention used in [B0] is that $n_{L}$ is the dimension of the kernel of $(1-\zeta) S+(1-\bar{\zeta}) S^{T}$ increased by 1 , this explains the formal differences compared with [Bo].

We will also fix $\zeta \in S^{1} \backslash\{1\}$. Let us begin by citing a result from [B0].
Proposition 2.5.2 ([Bo, Proposition 6.8]). Let $\xi$ be a generic point of $\mathbb{C}^{2}$ and $r_{0}<r_{1}$ two values such that the intersections $L_{i}:=C \cap S^{3}\left(\xi, r_{i}\right)$ are transverse ( $i=0,1$ ). With the notations $c_{i}=c_{L_{i}}, g_{i}=$ the genus of $C_{i}:=C \cap B\left(\xi, r_{i}\right)$ and $k_{i}=$ the number of connected components of $C_{i}{ }^{\wedge}$, one has

$$
\begin{align*}
w_{L_{1}}(\zeta)-\sum w_{L_{k}^{\text {sing }}}(\zeta)-w_{L_{0}}(\zeta) & \geq-2\left(g_{1}-g_{0}+c_{1}-c_{0}-k_{1}+k_{0}\right), \\
-\left(u_{L_{1}}(\zeta)-\sum u_{L_{k}^{\text {sing }}}(\zeta)-u_{L_{0}}(\zeta)\right) & \geq-2\left(g_{1}-g_{0}+c_{1}-c_{0}-k_{1}+k_{0}\right), \tag{2.5.3}
\end{align*}
$$

where $L_{k}^{\text {sing }}$ are the links of singularities of $C$, which lie in $B\left(\xi, r_{1}\right) \backslash B\left(\xi, r_{0}\right)$.
We use Proposition 2.5.2 in two special cases.
Corollary 2.5.4. Let $C_{0}$ and $C_{1}$ be as in 2.5.2. If $C_{01}=C_{1} \backslash C_{0}$ is smooth then

$$
\begin{aligned}
-\sigma_{L_{1}}(\zeta)+n_{L_{1}}(\zeta)-\left(-\sigma_{L_{0}}(\zeta)+n_{L_{0}}(\zeta)\right) & \geq \chi\left(C_{01}\right) . \\
\sigma_{L_{1}}(\zeta)+n_{L_{1}}(\zeta)-\left(\sigma_{L_{0}}(\zeta)+n_{L_{0}}(\zeta)\right) & \geq \chi\left(C_{01}\right) .
\end{aligned}
$$

Proof. Use the definition of $w$, (2.5.3) and $C_{01} \wedge=C_{01}$ for the first inequality. For the second one we use $-u$ instead of $w$.

The other important application is if $r_{0}$ is small, so that $L_{0}$ is an unknot.
Proposition 2.5.5. Fix $r$ such that the intersection $C \cap S(\xi, r)$ is transverse, and set $L:=C \cap S(\xi, r)$. Let $C_{\text {smooth }}$ be the smoothing of $C \cap B(\xi, r)$ (e.g. if $C$ is given by $F^{-1}(0)$ for some reduced polynomial, then $C_{\text {smooth }}$ can be taken as $F^{-1}(\varepsilon) \cap B(\xi, r)$ for $\varepsilon$ sufficiently small). Let $z_{1}, \ldots, z_{k}$ be the singular points of $C \cap B(\xi, r)$ with links $L_{1}^{\text {sing }}, \ldots, L_{k}^{\text {sing }}$, Milnor
numbers $\mu_{1}, \ldots, \mu_{k}$, number of branches $c_{1}, \ldots, c_{k}$, and signatures $\sigma_{1}(\zeta), \ldots, \sigma_{k}(\zeta)$. Then

$$
\begin{align*}
-\sigma_{L}(\zeta)+n_{L}(\zeta)+\left(1-\chi\left(C_{\text {smooth }}\right)\right) & \geq \sum_{j=1}^{k}\left(-\sigma_{L_{j}^{\text {sing }}}(\zeta)+n_{j}(\zeta)+\mu_{j}\right) \\
\sigma_{L}(\zeta)+n_{L}(\zeta)+\left(1-\chi\left(C_{\text {smooth }}\right)\right) & \geq \sum_{j=1}^{k}\left(\sigma_{L_{j}^{\text {sing }}}(\zeta)+n_{j}(\zeta)+\mu_{j}\right) \tag{2.5.6}
\end{align*}
$$

Proof. We prove only the first part, in the second one we use $-u_{L}$ instead of $w_{L}$.
Let $r_{\text {min }}$ be minimal with $C \cap S(\xi, r)$ non-empty, and set $r_{0}:=r_{\min }+\varepsilon$ for $\varepsilon$ sufficiently small. Then $L_{0}$ is an unknot with $w_{L_{0}}(\zeta) \equiv 0, c_{0}=k_{0}=1$, thus (2.5.3) gives

$$
\begin{aligned}
-\sigma_{L}(\zeta)+n_{L}(\zeta) & +1-c_{L} \geq \\
& \geq \sum_{j=1}^{k}\left(-\sigma_{j}(\zeta)+n_{j}(\zeta)+\mu_{j}\right)-\sum_{j=1}^{k}\left(\mu_{j}+c_{j}-1\right)-2 g(C)-2 c_{L}+2 k_{1} .
\end{aligned}
$$

The proof is completed by applying the genus formula $2\left(g\left(C_{\text {smooth }}\right)-g(C)\right)=\sum_{j=1}^{k}\left(\mu_{j}+\right.$ $\left.c_{j}-1\right)$, the fact that $b_{1}\left(C_{\text {smooth }}\right)=2 g\left(C_{\text {smooth }}\right)+c_{L}-1$ and observing that $b_{0}\left(C_{\text {smooth }}\right) \leq 2 k_{1}$ (it is even bounded by $k_{1}$ alone).

Remark 2.5.7. The cited result (i.e. Proposition (2.5.2) does not really require Morse theoretical arguments, although they are very convenient. We could deduce it - with approximately the same amount of work - from the Murasugi inequality Kaw, Theorem 12.3.1] too. The argument is that $C_{01}=C \cap\left(B\left(\xi, r_{1}\right) \backslash B\left(\xi, r_{0}\right)\right)$ induces a cobordism between the links $L_{0}^{\prime}:=L_{0} \amalg L_{1}^{\text {sing }} \amalg \ldots \amalg L_{j}^{\text {sing }}$ and $L_{1}$. In this way we do not use anywhere that $C$ is a complex curve, only that its genus is the difference of the genera of the minimal Seifert surfaces of $L_{1}$ and $L_{0}^{\prime}$.

## 3. The Seifert form and the MHS of a polynomial at infinity

In this section we compare the Hodge-spectrum associated with the limit mixed Hodge structure of a polynomial map at infinity with the HVS-spectrum provided by its regular link at infinity. In this way we recover the Hodge-spectrum from the 'Seifert form at infinity'. For results concerning the limit mixed Hodge structure (MHS) and the Hodge spectrum at infinity the reader might consult [SSS, Di2, Br .
3.1. Basic definitions. Let $F: \mathbb{C}^{2} \rightarrow \mathbb{C}$ be a reduced polynomial with critical values $x_{1}, \ldots, x_{N}$. Since $\mathbb{C}^{2}$ is not compact, the topology of a fiber $F^{-1}(y)$ can vary even if $y$ changes in a set of regular values of $F$.

Definition 3.1.1. (Neu3]) The fiber $F^{-1}(c)$ is called regular at infinity if there exists a (small) disk $D \ni c$ in $\mathbb{C}$ and a (large) ball $B \subset \mathbb{C}^{2}$ such that $F$ restricted to $F^{-1}(D) \backslash B$ is a trivial fibration. The fiber is called irregular at infinity if it is not regular at infinity.

Consider all the values $y_{1}, \ldots, y_{M}$ such that $F^{-1}\left(y_{k}\right)$ is not regular at infinity. Set $\rho \in \mathbb{R}$ with $\rho>\max _{k, l}\left\{\left|x_{k}\right|,\left|y_{l}\right|\right\}$ and set $\gamma=\{z \in \mathbb{C}:|z|=\rho\}$. Then $F$ restricted to $F^{-1}(\gamma)$ is a locally trivial fibration, called the fibration of $F$ at infinity. It will be denoted $\mathrm{Fib}_{\infty}$. The fiber of $\mathrm{Fib}_{\infty}$ is the (generic) fiber $Y_{\infty}:=F^{-1}(\rho)$ of $F$. The induced algebraic monodromy over $\gamma$, called the monodromy of $F$ at infinity, will be denoted by

$$
\begin{equation*}
h_{\infty}: H_{1}\left(Y_{\infty}, \mathbb{Z}\right) \rightarrow H_{1}\left(Y_{\infty}, \mathbb{Z}\right) \tag{3.1.2}
\end{equation*}
$$

Furthermore, we consider on $H_{1}\left(Y_{\infty}, \mathbb{Z}\right)$ the intersection form $b_{\infty}$ too. Already this isometric structure $\left(H_{1}\left(Y_{\infty}\right) ; b_{\infty}, h_{\infty}\right)$ contains important information about the behavior of $F$ at
infinity, nevertheless, we will enhance it in two different ways. The first is topological: we investigate the possibility to extend the pair $\left(b_{\infty}, h_{\infty}\right)$ to a variation structure (this, strictly speaking, in general, only 'partially' is possible). The candidate for the variation operator is the inverse transpose of the Seifert matrix of the link at infinity. The second is algebraic: one lifts the pair $\left(b_{\infty}, h_{\infty}\right)$ to the level of a polarized mixed Hodge structure by considering the limit mixed Hodge structure of $F$ at infinity.

First we start with the topological part.
Fix a fiber $F^{-1}(c)$ which is regular at infinity. For sufficiently large $R$ the intersection $F^{-1}(c)$ with $\partial B(R)$ is transverse. This link $F^{-1}(c) \cap \partial B(R) \subset \partial B(R)$, denoted by $L_{\text {reg }}^{\infty}$, is independent (up to isotopy) of $R$ and $c$. It is called the regular link at infinity of $F$.

According to [Neu3, Theorem 5] we can associate with $L_{\text {reg }}^{\infty}$ the so-called fundamental multilink at infinity $L_{\text {fund }}$, which is fibered. This means the following: there exist a link $L_{\text {fund }}$ with components $\left\{L_{\text {fund }, i}\right\}_{i=1}^{\nu}$ and positive multiplicities $\mathbf{n}=\left\{n_{i}\right\}_{i=1}^{\nu}$ such that there is a fibration $\phi: S^{3} \backslash L_{\text {fund }} \rightarrow S^{1}$ with the following property: for any closed loop $\tau \in$ $S^{3} \backslash L_{\text {fund }}, \phi_{*}([\tau]) \in H_{1}\left(S^{1}\right)=\mathbb{Z}$ equals the linking number of $[\tau]$ with $\sum_{i} n_{i} L_{\text {fund }, i}$. Furthermore, the closure $\overline{Y_{t}}$ of the fiber $Y_{t}=\phi^{-1}\left(e^{2 \pi i t}\right)(t \in[0,1])$ is not a manifold with boundary, but homologically $\overline{Y_{t}} \backslash Y_{t}$ is the multilink $\sum_{i} n_{i} L_{\text {fund }, i}$.

Finally, the connection between the multilink $L_{\text {fund }}$ and the link at infinity $L_{\text {reg }}^{\infty}$ is the following. Let $T=T\left(L_{\text {fund }}\right)$ be a closed small tubular neighbourhood of $L_{\text {fund }}$. Then $L_{\text {reg }}^{\infty}$ is the intersection of a fiber $Y_{0}$ with $\partial T\left(L_{\text {fund }}\right)$.

Lemma 3.1.3. For any $i \in\{1, \ldots, \nu\}$, let $l_{i}$ be the linking number

$$
l_{i}=\operatorname{lk}\left(L_{\text {fund }, i}, \sum_{j \neq i} n_{j} L_{\text {fund }, j}\right)
$$

and $n_{i}^{\prime}$ the (positive) greatest common divisor of $n_{i}$ and $l_{i}$. Then the number of components of $Y_{0} \cap \partial T\left(L_{\text {fund }, i}\right)$ is exactly $n_{i}^{\prime}$. Hence, $L_{\text {reg }}^{\infty}$ has $\sum_{i=1}^{\nu} n_{i}^{\prime}$ components. Moreover, the components of $Y_{0} \cap \partial T\left(L_{\text {fund }, i}\right)$ are cyclically permuted by the geometric monodromy of $\phi$.
Proof. See [EN, §3 and 4].
Another important point about $L_{f \text { fund }}$ is that its fiber $Y_{0}$ can be identified with the generic fiber $Y_{\infty}$ of the polynomial $F$ [Neu3, Theorem 4]. In fact, by [AC, Theorem 1.1] one has

Lemma 3.1.4. The multilink fibration $S^{3} \backslash L_{\text {fund }} \rightarrow S^{1}$ associated with ( $L_{\text {fund }}, \mathbf{n}$ ) and the fibration $F i b_{\infty}$ of $F$ are isomorphic.

By [EN, page 37], $Y_{0}$ has $d=\operatorname{gcd}_{i}\left\{n_{i}\right\}$ connected components. In the sequel we will assume that $d=1$, that is, the generic fiber of $F$ is connected.
3.2. The multilink Seifert form of $L_{\text {fund }}$. The surface $Y_{0}$, the fiber of the multilink ( $L_{\text {fund }}, \mathbf{n}$ ), is a generalized Seifert surface of the multilink, cf. [EN, pages 28-29]. In the sequel we refer to it as the multilink Seifert surface. Using this surface, one can define multilink Seifert form associated with $Y_{0}$, cf. [EN, §15]. It is a bilinear form on $H_{1}\left(Y_{0}, \mathbb{Z}\right)$ defined similarly as the classical Seifert form, namely $S_{\text {fund }}(\alpha, \beta)$ for $\alpha, \beta \in H_{1}\left(Y_{0}, \mathbb{Z}\right)$ is the linking number $\operatorname{lk}\left(\alpha, \beta^{+}\right)$, where $\beta^{+}$is the push-forward of $\beta$ in the positive direction.

If all the multiplicities $\left\{n_{i}\right\}_{i}$ equal 1, then $L_{\text {fund }}$ is a fibred link, and $S_{\text {fund }}$ is its classical Seifert form, hence it has determinant $\pm 1$. In the case of general multiplicity system $\mathbf{n}$ this is not the case anymore. In fact, $S_{\text {fund }}$ can be even degenerate. Nevertheless, some parts of the classical theory survive.
Lemma 3.2.1. Let $H^{*}$ denote the dual of $H$, $T^{o}$ the interior of $T$, and $\bar{Y}_{[a, b]}:=\bigcup_{a \leq t \leq b} \overline{Y_{t}}$.
(a) The groups $H_{1}\left(\overline{Y_{0}}, \mathbb{Z}\right)$ and $H_{1}\left(Y_{0}, \mathbb{Z}\right)^{*}$ are isomorphic. In fact one has the following sequence of isomorphisms, denoted by $s$ :

$$
\begin{aligned}
& H_{1}\left(\overline{Y_{0}}\right) \xrightarrow{\partial^{-1}} H_{2}\left(S^{3}, \overline{Y_{0}}\right) \xrightarrow{(1)} H_{2}\left(S^{3}, \bar{Y}_{\left[0, \frac{1}{2}\right]}\right) \xrightarrow{(2)} H_{2}\left(\bar{Y}_{\left[\frac{1}{2}, 1\right]}, \overline{Y_{1 / 2}} \cup \overline{Y_{1}}\right) \xrightarrow{(3)} \\
& H_{2}\left(\bar{Y}_{\left[\frac{1}{2}, 1\right]}, \overline{Y_{1 / 2}} \cup \overline{Y_{1}} \cup\left(T \cap \bar{Y}_{\left[\frac{1}{2}, 1\right]}\right)\right) \xrightarrow{(4)} H_{2}\left(\bar{Y}_{\left[\frac{1}{2}, 1\right]} \backslash T^{o}, \partial\left(\bar{Y}_{\left[\frac{1}{2}, 1\right]} \backslash T^{o}\right)\right) \\
& \quad \xrightarrow{(5)} H_{1}\left(\bar{Y}_{\left[\frac{1}{2}, 1\right]} \backslash T^{o}\right)^{*} \xrightarrow{(6)} H_{1}\left(\overline{Y_{1}} \backslash T^{o}\right)^{*} \xrightarrow{(7)} H_{1}\left(Y_{1}\right)^{*}=H_{1}\left(Y_{0}\right)^{*} .
\end{aligned}
$$

(b) Let $j: H_{1}\left(Y_{0}, \mathbb{Z}\right) \rightarrow H_{1}\left(\overline{Y_{0}}, \mathbb{Z}\right)$ be induced by the inclusion. Then the composition

$$
H_{1}\left(Y_{0}, \mathbb{Z}\right) \xrightarrow{j} H_{1}\left(\overline{Y_{0}}, \mathbb{Z}\right) \xrightarrow{s} H_{1}\left(Y_{0}, \mathbb{Z}\right)^{*}
$$

can be identified with the multilink Seifert form $S_{\text {fund }}$.
(c) Identify the isometric structure $\left(b_{\infty}, h_{\infty}\right)$ with the intersection form and monodromy of $H_{1}\left(Y_{0}\right)$ (by 3.1.4). Then, in matrix notation,

$$
b_{\infty}=S_{f u n d}-S_{\text {fund }}^{T}, \quad \text { and } \quad S_{\text {fund }}^{T} h_{\infty}=S_{\text {fund }}
$$

In particular, $h_{\infty}$ is an automorphism of $S_{\text {fund }}$, that is $h_{\infty}^{T} S_{\text {fund }} h_{\infty}=S_{\text {fund }}$.
Proof. In the sequence of isomorphisms $\partial^{-1}$ comes from the exact sequence of the pair; (1), (3), (6) and (7) are induced by deformation retracts; (2) and (4) are excisions, while (5) is provided by duality of the manifold with boundary $\bar{Y}_{\left[\frac{1}{2}, 1\right]} \backslash T^{o}$. Part (b) and (c) follow by similar argument as in the classical case, see e.g. the survey [Nem5, (3.15)].

In the above composition, although $s$ is an isomorphism, $j$ in general is not. Since, by our assumption $\tilde{H}_{0}\left(Y_{0}, \mathbb{Z}\right)=0, j$ can be inserted in the following long exact sequence:

$$
\begin{equation*}
0 \rightarrow H_{2}\left(\overline{Y_{0}}\right) \rightarrow H_{2}\left(\overline{Y_{0}}, Y_{0}\right) \rightarrow H_{1}\left(Y_{0}\right) \xrightarrow{j} H_{1}\left(\overline{Y_{0}}\right) \rightarrow H_{1}\left(\overline{Y_{0}}, Y_{0}\right) \rightarrow 0 . \tag{3.2.2}
\end{equation*}
$$

Lemma 3.2.3.

$$
\begin{array}{ll}
(a) & H_{2}\left(\overline{Y_{0}}, Y_{0}, \mathbb{Z}\right)=\oplus_{i=1}^{\nu} \mathbb{Z}^{n_{i}^{\prime}-1}, \\
(b) & H_{1}\left(\overline{Y_{0}}, Y_{0}, \mathbb{Z}\right)=\oplus_{i=1}^{\nu}\left(\mathbb{Z}_{i}^{n_{i}^{\prime}-1} \oplus \mathbb{Z}_{n_{i} / n_{i}^{\prime}}\right) .
\end{array}
$$

In particular, $H_{2}\left(\overline{Y_{0}}, \mathbb{Z}\right)=0$ and

$$
\begin{equation*}
\operatorname{dim} \operatorname{ker} j=\sum_{i=1}^{\nu}\left(n_{i}^{\prime}-1\right) \tag{3.2.4}
\end{equation*}
$$

Proof. By excision and deformation retract argument $H_{q}\left(\overline{Y_{0}}, Y_{0}\right)=\oplus_{i} H_{q}\left(A_{i}, B_{i}\right)$, where

$$
\left(A_{i}, B_{i}\right):=\left(\overline{Y_{0}} \cap T\left(L_{f u n d, i}\right), \quad Y_{0} \cap \partial T\left(L_{\text {fund }, i}\right)\right) .
$$

Note that the homotopy type of $A_{i}$ is $L_{\text {fund }, i}$, while of $B_{i}$ is $n_{i}^{\prime}$ copies of $S^{1}$. Each of these copies maps (via the inclusion $B_{i} \hookrightarrow A_{i}$ ) onto $L_{f u n d, i}$ as the $n_{i} / n_{i}^{\prime}$-covering. Therefore, the inclusion $B_{i} \hookrightarrow A_{i}$ at $H_{1}$-level is $\mathbb{Z}^{n_{i}^{\prime}} \rightarrow \mathbb{Z},\left\{a_{1}, \ldots, a_{n_{i}^{\prime}}\right\} \mapsto \frac{n_{i}}{n_{i}^{\prime}} \cdot \sum_{k} a_{k}$. This gives (a) and (b). The rest follow by rank computation argument from (3.2.2).

Next, we will consider another compactification $\widetilde{Y}_{0}$ of $Y_{0}$. Denote $Y_{0}^{o}:=Y_{0} \backslash T\left(L_{\text {fund }}\right)^{o}$, the complement of the interior of the tube. $\partial Y_{0}^{o}$ consists of $\sum_{i} n_{i}^{\prime}$ copies of $S^{1}$. Let $\widetilde{Y}_{0}$ be obtained from $Y_{0}^{o}$ by gluing to each boundary circle a 2 -disc, in this way obtaining a compact smooth surface. In fact, the fibration at infinity $F$ over $\gamma$ can be compactified (even algebraically) to a fibration $\widetilde{F}$ over $\gamma$ with smooth compact fibers $\widetilde{Y}_{0}$, where in this language the compact fibers consists of $Y_{0}$ with additionally $\sum_{i} n_{i}^{\prime}$ 'points at infinity'. This point of view is used in Hodge theoretical computations, see e.g. [Di1] or [Di2, §3].

One has the following exact sequence:

$$
\begin{equation*}
0 \rightarrow \mathbb{Z} \rightarrow \oplus_{i} \mathbb{Z}^{n_{i}^{\prime}} \rightarrow H_{1}\left(Y_{0}, \mathbb{Z}\right) \rightarrow H_{1}\left(\widetilde{Y}_{0}, \mathbb{Z}\right) \rightarrow 0 \tag{3.2.5}
\end{equation*}
$$

Above, $\oplus_{i} \mathbb{Z}^{n_{i}^{\prime}}$ is generated by the discs, their images in $H_{1}\left(Y_{0}\right)$ are the classes of the circles $\partial Y_{0}^{o} . \mathbb{Z}$ from the left is $H_{2}\left(\widetilde{Y}_{0}\right)$; its image is generated by $\partial Y_{0}^{o}$. The monodromy extends to $H_{1}\left(\widetilde{Y}_{0}\right)$ (or to $\widetilde{F}$ ) (and will be denoted by $\widetilde{h}_{\infty}$ ), and also to the discs/points at infinity: it acts trivially on $\mathbb{Z}$, on $\mathbb{Z}^{n_{i}^{\prime}}$ acts by permutation of the base elements (denoted by $h_{p e r}$ ).

Let $\widetilde{b}_{\infty}$ be the intersection form on $H_{1}\left(\widetilde{Y}_{0}\right)$.

## Lemma 3.2.6.

(a) $\widetilde{h}_{\infty}$ has no eigenvalue 1, and all its Jordan blocks have size not larger than two.
(b) The exact sequence (3.2.5), together with the algebraic monodromy action on it, splits.

That is, $h_{\infty}$ has no Jordan block of size three, and the blocks of size two of $h_{\infty}$ and $\widetilde{h}_{\infty}$ agree. In other words, over $\mathbb{Q}$, one has a direct sum decomposition:

$$
\begin{equation*}
\left(H_{1}\left(Y_{0}\right) ; b_{\infty}, h_{\infty}\right)=\left(H_{1}\left(\widetilde{Y}_{0}\right) ; \widetilde{b}_{\infty}, \widetilde{h}_{\infty}\right) \oplus\left(\oplus_{i} \mathbb{Q}^{n_{i}^{\prime}} / \mathbb{Q} ; 0, h_{p e r}\right) \tag{3.2.7}
\end{equation*}
$$

Moreover, $\left(\widetilde{b}_{\infty}, \widetilde{h}_{\infty}\right)$ is a non-degenerate isometric structure.
(c) All roots of $h_{\infty}$ are roots of unity.

Proof. The statements follow from the mixed Hodge theory of the degeneration at infinity of $F$ and $\tilde{F}$. Part (a) is proved e.g. in (Di1, Di2]. Part (b) follows from the spectral pair computation of the mixed Hodge structure carried on $H^{1}\left(Y_{0}, \mathbb{C}\right)$. More precisely, there is a cohomological analogue of the sequence (3.2.5) which carries mixed Hodge structure compatible with the action of the monodromy, see again [Di2, §3]. The number of Jordan blocks of size two correspond to those spectral pairs $(\alpha, w)$ for which $w=0$. These are computed for both $H^{1}\left(Y_{0}, \mathbb{C}\right)$ and $H^{1}\left(\widetilde{Y}_{0}, \mathbb{C}\right)$ in $[\mathrm{Br}$, and their number agree. For (c) use e.g. the Monodromy Theorem for $\widetilde{F}$ at infinity.

Finally, we summarize the properties of $L_{\text {fund }}$ in the following proposition. As usual, if $h$ is an automorphism of the vector space $V$, then $V_{\lambda=1}$ denotes the generalized eigenspace corresponding to eigenvalue 1 , while $V_{\lambda \neq 1}$ is the direct sum of the other generalized eigenspaces.
Proposition 3.2.8. Set $U:=H_{1}\left(Y_{0}, \mathbb{Q}\right)$ and let $b_{\infty}$ and $h_{\infty}$ be the intersection form and the algebraic monodromy induced by the multilink fibration $\phi: S^{3} \backslash L_{\text {fund }} \rightarrow S^{1}$.

Then the following facts hold:
(a) $Y_{0}$ is the minimal multilink Seifert surface of the multilink $\left(L_{f u n d}, \mathbf{n}\right)$, and all minimal multilink Seifert surfaces of $\left(L_{f u n d}, \mathbf{n}\right)$ are isotopic to $Y_{0}$.
(b) One has a direct sum decomposition (Keef decomposition, cf. (2.2.1)):

$$
\begin{equation*}
\left(U, S_{\text {fund }}\right)=\left(U_{0} \oplus U_{\text {ndeg }}, S_{\text {fund }, 0} \oplus S_{\text {fund }, n d e g}\right) \tag{3.2.9}
\end{equation*}
$$

such that $S_{\text {fund }, 0}=0$ of size $\operatorname{Irr}=\sum_{i=1}^{\nu}\left(n_{i}^{\prime}-1\right)$, and $S_{\text {fund,ndeg }}$ is non-degenerate.
A possible free generator set for $U_{0}$ is the collection of the cycles $L_{\text {reg }, i, k}^{\infty}-L_{\text {reg, }, i, k+1}^{\infty}$ $\left(1 \leq i \leq \nu ; 1 \leq k<n_{i}^{\prime}\right)$, where $\left\{L_{\text {reg }, i, k}^{\infty}\right\}_{k=1}^{n_{i}^{\prime}}$ are the components of $Y_{0} \cap \partial T\left(L_{\text {fund }, i}\right)$.
(c) The compatibility of the decompositions (3.2.9) and (3.2.7) is the following:
(c.1) $\left(U_{\text {ndeg }}\right)_{\lambda \neq 1}=H_{\sim}^{1}\left(\widetilde{Y}_{0}\right)$. On this space $\left(S_{\text {fund,ndeg }}\right)_{\lambda \neq 1}$ completes the non-degenerate isometric structure $\left(\widetilde{b}_{\infty}, \widetilde{h}_{\infty}\right)$ to a simple $(-1)$-variation structure.
(c.2) $\left(\oplus_{i} \mathbb{Q}^{n_{i}^{\prime}} / \mathbb{Q} ; 0,\left[h_{\text {per }}\right]\right)=\left(\left(U_{\text {ndeg }}\right)_{\lambda=1} ; 0, \mathrm{Id}\right) \oplus\left(U_{0} ; 0,\left.h_{\infty}\right|_{U_{0}}\right)$ (and this is an eigenspace decomposition).
(c.3) $\left(U_{n d e g}\right)_{\lambda=1}$ has dimension $\nu-1$, on it the restriction of $b_{\infty}$ is trivial, the restriction of $h_{\infty}$ is the identity, and this degenerate isometric structure is completed by $\left(S_{f u n d, n d e g}\right)_{\lambda=1}$ to a simple variation structure.
(c.4) On $U_{0}$ the restrictions of $b_{\infty}$ and $S_{\text {fund }}$ are trivial (hence all the equivariant signature type invariants including the Tristram-Levine signatures of restrictions of $S_{\text {fund }}$ and $\left(b_{\infty}, h_{\infty}\right)$ are the same). Nevertheless, the restriction of $h_{\infty}$ is non-trivial (in fact, it has no eigenvalue 1), hence the isometric structure cannot be completed to a variation structure. The characteristic polynomial of the restriction of $h_{\infty}$ is

$$
\operatorname{det}\left(\left.h_{\infty}\right|_{U_{0}}-t \mathrm{Id}\right)=\prod_{i} \frac{t^{n_{i}^{\prime}}-1}{t-1} .
$$

Proof. (a) follows from [EN, (4.1)]. For (b) note that the generators listed are in the kernel of $j$. For this use e.g. the proof of (3.2.3), where $\left\{L_{r e g, i, k}^{\infty}\right\}_{k=1}^{n_{i}^{\prime}}$ are exactly the components of $B_{i}$. Another possibility is check directly that $S_{f u n d}\left(L_{r e g, i, k}^{\infty}-L_{r e g, i, k+1}^{\infty}, \beta\right)=S_{f u n d}\left(\beta, L_{r e g, i, k}^{\infty}-\right.$ $\left.L_{r e g, i, k+1}^{\infty}\right)=0$ for any $\beta$. Indeed, if $T$ is sufficiently small tubular neighborhood, then it does not intersect $\beta$, on the other hand inside of $T$ the circles $L_{r e g, i, k}^{\infty}$ and $L_{r e g, i, l}^{\infty}$ are homologous. For part (c) use Lemmas 3.2.1(c) and 3.2.6 for the characteristic polynomial use the fact that the components $\left\{L_{r e g, i, k}^{\infty}\right\}_{k}$ are cyclically permuted, cf. Lemma 3.1.3,

The multilink structure ( $L_{\text {fund }}, S_{\text {fund }}$ ) now will be used in two different aspects. First, it can be related with the link $L_{\text {reg }}^{\infty}$; in fact, one can recover it from $L_{\text {reg }}^{\infty}$, see (3.3). On the other hand, the multilink fibration of $L_{f u n d}$ can be identified with the fibration at infinity $F i b_{\infty}$ of $F$, cf. Lemma 3.1.4. In this way $L_{f u n d}$ creates the bridge between $L_{r e g}^{\infty}$ and $F i b_{\infty}$.
3.3. The Seifert form of $L_{\text {reg }}^{\infty}$. Set $Y_{0}^{o}:=Y_{0} \backslash T\left(L_{\text {fund }}\right)^{o}$ as above. Obviously, $Y_{0}^{o} \hookrightarrow Y_{0}$ admits a deformation retract, hence $H_{1}\left(Y_{0}^{o}, \mathbb{Z}\right)=H_{1}\left(Y_{0}, \mathbb{Z}\right)$ canonically.

Lemma 3.3.1. One has the following facts:
(a) $Y_{0}^{o}$ is the minimal Seifert surface of $L_{\text {reg }}^{\infty}$, and all minimal Seifert surfaces of $L_{\text {reg }}^{\infty}$ are isotopic to $Y_{0}^{o}$.
(b) The Seifert form $S_{\text {reg }}$ of $L_{\text {reg }}^{\infty}$ associated with $Y_{0}^{o}$ is identical with $S_{\text {fund }}$ (under the identification $H_{1}\left(Y_{0}^{o}, \mathbb{Z}\right)=H_{1}\left(Y_{0}, \mathbb{Z}\right)$ ). In particular, all the result listed in Proposition 3.2 .8 about $S_{\text {fund }}$ are valid for $S_{\text {reg }}$ too.
(c) Let $h_{\text {reg }}$ be the monodromy of the variation structure associated with $S_{\text {reg,ndeg }}=$ $S_{\text {fund,ndeg }}$ (as in (2.2). Then the higher Alexander polynomials $\Delta_{k}$ of $L_{\text {reg }}^{\infty}$ satisfies the following identities: $\Delta_{k} \equiv 0$ for $0 \leq k<\operatorname{Irr}, \Delta_{\operatorname{Irr}}(t)=\operatorname{det}\left(h_{\text {reg }}-t \mathrm{Id}\right)$.
(d) All the roots of the (higher) Alexander polynomial $\Delta_{\mathrm{Irr}}$ of $L_{\text {reg }}^{\infty}$ are roots of unity.

Proof. (a)-(b)-(c) follows from Proposition 3.2 .8 and from the construction of $Y_{0}^{o}$. For (d) use either Lemma 3.2.6(c) or note that the multilink fibration of $L_{\text {fund }}$ can be represented by a splice diagram [Neu3, Neu4, hence the characteristic polynomial of $h_{\text {fund }}$ is a product of cyclotomic polynomials by [EN, Theorem 13.6].

### 3.4. The HVS-spectrum of the regular link at infinity, $L_{\text {reg }}^{\infty}$.

For local isolated plane curve singularities we have the following classical result, which in the language of H -numbers $p_{\lambda}^{k}( \pm 1)$ and $q_{\lambda}^{k}$ of their local links can be formulated as follows (see e.g. [BN, Proposition 3.1.5, Lemma 3.1.6]).

Proposition 3.4.1 (Monodromy Theorem). Let L be an algebraic link and $p_{\lambda}^{k}( \pm 1), q_{\lambda}^{k}$ its H-numbers. Then
(a) $q_{\lambda}^{k}=0$ for all $k>0$ and $|\lambda|<1$; moreover $p_{\lambda}^{k}( \pm 1)=0$ unless $\lambda$ is a root of unity,
(b) $p_{\lambda}^{k}( \pm 1)=0$ for all $k>2$. Moreover $p_{1}^{2}( \pm 1)=0$,
(c) $p_{\lambda}^{2}(-1)=0$ and $p_{1}^{1}(-1)=0$.

The fundamental mulitlink at infinity $L_{\text {fund }}$, or the regular link at infinity $L_{\text {reg }}^{\infty}$, in general, cannot be realized by a local algebraic link. However, their H-numbers share similar properties as the H -numbers of local links.

Proposition 3.4.2. For the $H$-numbers of $L_{\text {reg }}^{\infty}$ the following facts hold.
(a) $q_{\lambda}^{k}=0$ for all $k>0$ and $|\lambda|<1$, moreover $p_{\lambda}^{k}( \pm 1)=0$ unless $\lambda$ is a root of unity,
(b) $p_{\lambda}^{k}( \pm 1)=0$ for all $k>2$, and $p_{1}^{2}( \pm 1)=0$,
(c) $p_{1}^{1}(-1)=0$,
(d) $p_{\lambda}^{2}(1)=0$ for $\lambda \neq 1$.

Proof. (a) and (b) follow from Lemma 3.2.6. Next we prove (c). First we recall that $\mathcal{W}_{1}^{1}( \pm 1)=(\mathbb{C} ; 0, \mathrm{Id}, \mp 1)$, hence we have to show that the restriction of $S_{\text {fund }}$ on $U_{\lambda=1}$ is negative definite. This follows from the more general result Proposition 3.7.2 of section 3.7,
(d) By Neu3, NeRu $L_{\text {fund }}$ can is represented by a splice diagram with all edges have negative determinants. Thus, $L_{\text {fund }}$ has uniform twists (all positive) (see [EN, Chapter 14]). Therefore, by the discussion in Neu1, Section 2] we have for each $x \in U_{\lambda}$

$$
S_{\text {fund }}\left(\lambda x,\left(h_{\infty}-\lambda\right) x\right) \geq 0
$$

This shows that $p_{\lambda}^{2}(+1)$ cannot occur.
Let $S p_{\text {HVS }}\left(L_{\text {reg }}^{\infty}\right)$ be the HVS-spectrum associated with the link $L_{\text {reg }}^{\infty}$.
Corollary 3.4.3. (a) The HVS-spectrum of $U_{\lambda=1}$ consists of $(\nu-1)$ copies of (1).
(b) All elements of $S p_{\mathrm{HVS}}\left(L_{\text {reg }}^{\infty}\right)$ are situated in $(0,2)$, and $S p_{\mathrm{HVS}}\left(L_{\text {reg }}^{\infty}\right)$ is symmetric with respect to 1 .

Remark 3.4.4. The proofs of Propositions 3.4.1 and 3.4.2 rely on some key properties of the splice diagrams of the corresponding links. The common properties (which imply the common $p_{1}^{1}(-1)=0$ ) is that in both cases the 'multiplicities of the nodes' and the '(near) weights' are positive. The crucial difference between the diagrams is that in the local case the edge determinants are positive, while for the diagram at infinity they are negative. This implies the sign difference in the $p_{\lambda}^{2}( \pm 1)$-vanishing. For more details, see subsection 3.7.

The next identity will be often used in the sequel.
Corollary 3.4.5. If $Y_{\infty}$ is a regular fiber of $F$, then $1-\chi\left(Y_{\infty}\right)=\operatorname{deg} \Delta_{\operatorname{Irr}}\left(L_{\text {reg }}^{\infty}\right)+\operatorname{Irr}$.
Proof. By Lemma 3.3.1(a) the size of $S_{\text {reg }}$ is equal to $1-\chi\left(Y_{\infty}\right)$. On the other hand, $\operatorname{deg} \Delta_{\text {Irr }}$ is equal to the size of $S_{n d e g}$ by Lemma 3.3.1(c). The difference of the sizes of the two matrices is equal to Irr by Proposition 3.2.8(b) (cf. also Lemma 3.3.1(b)).
3.5. The Hodge-spectrum of the fibration of $F$ at infinity. Let $S p_{\mathrm{MHS}, \infty}$ be the spectrum associated with the limit mixed Hodge structure of $F$ at infinity defined in a similar way as in subsection 2.3. The main result of this subsection shows that $S p_{\mathrm{MHS}, \infty}$ can be recovered from the rational Seifert form of $L_{\text {reg }}^{\infty}$ and from the integers $\left\{n_{i}^{\prime}\right\}_{i}$. Conversely, $S p_{\mathrm{HVS}}\left(L_{r e g}^{\infty}\right)$ is the maximal subset of $S p_{\mathrm{MHS}, \infty}$, which is symmetric with respect to 1 .

More precisely, in $\mathbb{Z}[\mathbb{Q}]$ one has

## Theorem 3.5.1.

$$
S p_{\mathrm{MHS}, \infty}=S p_{\mathrm{HVS}}\left(L_{r e g}^{\infty}\right)+\sum_{i=1}^{\nu}\left(\frac{1}{n_{i}^{\prime}}\right)+\cdots+\left(\frac{n_{i}^{\prime}-1}{n_{i}^{\prime}}\right)
$$

Proof. Let us consider the decomposition given in Proposition 3.2.8:

$$
\begin{equation*}
\left(U_{0} ; 0,\left.h_{\infty}\right|_{U_{0}}\right) \oplus\left(\left(U_{n d e g}\right)_{\lambda=1} ; 0, \mathrm{Id}\right) \oplus\left(\left(U_{n d e g}\right)_{\lambda \neq 1} ; \widetilde{b}_{\infty}, \widetilde{h}_{\infty}\right) \tag{3.5.2}
\end{equation*}
$$

The last component carries a limit mixed Hodge structure which is polarized by $\widetilde{b}_{\infty}$, and also it extends to a simple hermitian variation structure with $\left(S_{\text {fund,ndeg }}\right)_{\lambda \neq 1}$. In such a situation, the HVS-spectrum agrees with the Hodge spectrum. The proof is absolutely the same as in the local case, see Proposition 2.3.4, or the original source [Nem2, (6.5)] (or the affine polynomial case in [GN2]).

For the middle component both HVS and Hodge spectra are $(\nu-1)$ copies of (1): in the HVS case see Corollary 3.4.3 as a consequence of Proposition 3.4.2(c), while for the Hodge case see [Br] or Di2].

These two components provide the contribution from $S p_{\mathrm{HVS}}\left(L_{r e g}^{\infty}\right)$. The remaining part, provided by the first summand is computed in $[\mathrm{Br}$, and it is the sum in the right hand side of the identity of Theorem 3.5.1.

Example 3.5.3. Recall that $F$ is 'good at infinity' if and only if $L_{\text {reg }}^{\infty}$ is a fibred link, that is $n_{i}=1$ for all $i$, cf. [NeRu, Theorem 6.1]. By our result, in such a case one has $S p_{\mathrm{MHS}, \infty}=S p_{\mathrm{HVS}}\left(L_{\text {reg }}^{\infty}\right)$.
Corollary 3.5.4. For any $x \in[0,1]$ one has

$$
\left|S p_{\mathrm{HVS}}\left(L_{r e g}^{\infty}\right) \cap(x, x+1)\right| \leq\left|S p_{\mathrm{MHS}, \infty} \cap(x, x+1)\right| \leq\left|S p_{\mathrm{HVS}}\left(L_{r e g}^{\infty}\right) \cap(x, x+1)\right|+\operatorname{Irr} .
$$

and analogous inequality holds for $S p \backslash[x, x+1]$.
3.6. An example. The above discussion might have been technically quite involved. We want to illustrate the occuring phenomena by investigating one example, the Briançon polynomial, which appeared in [ACL, Br, Di2, DN] (we remark that in [AC, Exemple 4.14] there is a different polynomial called Briançon polynomial, it has different link at infinity and different irregular fibers).

The splice diagram of the fundamental link at infinity is as follows

(1) (1)

Here the numbers in parentheses are the multiplicities of the vertices and arrowheads (link components). The numbers not in parenthesis denote the weights of corresponding edges (those omitted equal 1 ). We have $n_{1}=4, n_{1}^{\prime}=\operatorname{gcd}(4,6)=2$, and $n_{2}=n_{2}^{\prime}=1$.

Computing the Euler characteristics of a minimal Seifert surface of $L_{\text {fund }}$ (as in [EN]) we get that this surface is a three times punctured torus ( 3 is the number of components of $\left.L_{\text {fund }}\right)$. The rank of $H_{1}\left(Y_{\infty}\right)$ is 4 , while the ranks of $U_{0}$ and $\left(U_{n d e g}\right)_{\lambda=1}$ are 1. The monodromy at infinity permutes $L_{1,1}, L_{1,2}$ and fixes $L_{2}$. The characteristic polynomial of the monodromy on boundary components is therefore $t^{2}-1$. The Alexander polynomial of $L_{\text {fund }}$, hence the characteristic polynomial of the monodromy at infinity, is $\left(t^{2}-1\right)\left(t^{2}+t+1\right)$.

The equivariant signatures (which correspond to jumps of the Tristram-Levine signatures) of $L_{\text {fund }}$ can be computed using [Neu2, Theorem 5.3 and Section 6]. Using [Neu2, Theorem 5.3] for the left-most splice component we compute that $\sigma_{e^{2 \pi i / 3}}^{-}=-1$
and $\sigma_{e^{-2 \pi i / 3}}^{-}=1$ so the jumps of the Tristram-Levine signature are respectively -2 and 2 , in other words $p_{e^{2 \pi i / 3}}^{1}(+1)=0, p_{e^{2 \pi i / 3}}^{1}(-1)=1, p_{e^{-2 \pi i / 3}}^{1}(+1)=1, p_{e^{-2 \pi i / 3}}^{1}(-1)=0$ (compare [Br, Sections 3.5, 3.6]). On the other hand, a straightforward computation shows that the right splice component does not contribute to the equivariant signature at all. Hence, the non-trivial H -numbers are $p_{e^{2 \pi i / 3}}^{1}(-1)=p_{e^{-2 \pi i / 3}}^{1}(+1)=p_{1}^{1}(1)=1$.

Concluding, the spectrum at infinity is equal to $\left\{\frac{2}{3}, \frac{4}{3}, \frac{1}{2}, 1\right\}$ (cf. [Di2, Example 3.6(ii)]), where $\left\{\frac{2}{3}, 1, \frac{4}{3}\right\}$ is the contribution from $L_{\text {reg }}^{\infty}$.
3.7. The definiteness of 'linking matrix'. The proof of (3.4.2)(c). We wish to prove that the restriction of $S_{\text {fund }}$ on $U_{\lambda=1}$ is negative definite. This follows from a more general combinatorial result which we now state.

Let $\Gamma$ be a rooted Eisenbud-Neumann diagram, cf. Neu3. For an edge we call the weight which is closer to the root vertex the near weight and the other one the far weight. For any two nodes $v$ and $w$, if the geodesics connecting $w$ and the root vertex contains $v$ then we say that $w$ is beyond $v$. We allow more than one near weight at each node to have weight different than 1 . The linking numbers and multiplicities are determined from the diagram as in [EN, $\S 10,11]$. The arrowhead vertices will be denoted by $L_{1}, \ldots, L_{\nu}$, their multiplicities are $n_{1}, \ldots, n_{\nu}$.

Let $\mathbb{Q}^{\nu}$ be the $\mathbb{Q}$-vector space generated by $\left\{L_{i}\right\}_{i}$, The linking matrix $\left\{\operatorname{lk}\left(L_{i}, L_{j}\right)\right\}_{i j}$ is defined as follows: for $i \neq j$ it is the standard linking pairing, while the self-linking $\operatorname{lk}\left(L_{i}, L_{i}\right)$ is defined via the identity $\operatorname{lk}\left(L_{i}, \sum_{j} n_{j} L_{j}\right)=0$. Equivalently,

$$
\begin{equation*}
\operatorname{lk}\left(n_{i} L_{i}, n_{i} L_{i}\right)=-\sum_{j \neq i} \operatorname{lk}\left(n_{i} L_{i}, n_{j} L_{j}\right) . \tag{3.7.1}
\end{equation*}
$$

In particular, the null-space of the linking matrix is at least 1-dimensional.
Proposition 3.7.2. Let $\Gamma$ be a rooted connected graph with the following properties
(a) all near weights are positive and no far weight is allowed to be zero.
(b) if the far weight at a node $v$ is negative, then all far weights of nodes beyond $v$ are also negative (this property is weaker than negativity of edge determinants);
(c) the multiplicities of all arrowhead and non-arrowhead vertices are positive.

Then, the linking matrix $\operatorname{lk}\left(L_{i}, L_{j}\right)$ is negative semi-definite with 1-dimensional null-space.
Proof. We begin with a following special case.
Lemma 3.7.3. The statement of Proposition 3.7.2 holds if $\operatorname{lk}\left(L_{i}, L_{j}\right)>0$ for all $i \neq j$.
Proof. The reasoning is exactly as in [Neu1, §3]: for $L=\sum \ell_{j} n_{j} L_{j}$ one has

$$
\operatorname{lk}(L, L)=\sum_{i<j} 2 \ell_{i} \ell_{j} \operatorname{lk}\left(n_{i} L_{i}, n_{j} L_{j}\right)+\sum_{i} \ell_{i}^{2} \operatorname{lk}\left(n_{i} L_{i}, n_{i} L_{i}\right) .
$$

Substituting (3.7.1), we get

$$
\begin{equation*}
\operatorname{lk}(L, L)=-\sum_{i<j}\left(\ell_{i}-\ell_{j}\right)^{2} \operatorname{lk}\left(n_{i} L_{i}, n_{j} L_{j}\right) . \tag{3.7.4}
\end{equation*}
$$

Hence $\operatorname{lk}(L, L)$ is zero if $\ell_{1}=\cdots=\ell_{n}$, and negative otherwise.
In general, if some far weights are negative, some of the linking numbers $\operatorname{lk}\left(L_{i}, L_{j}\right)$ might be negative too; in these cases the proof is more involved.

Lemma 3.7.5. If $\nu \geq 2$ then the self-linking number $1 \mathrm{k}\left(L_{i}, L_{i}\right)$ is negative for any $i$.

Proof. For each $i$, let $v_{i}$ be a node supporting $L_{i}, \alpha_{i}$ denotes the far weight at $v_{i}$ and $\beta_{i 1}, \ldots, \beta_{i k_{i}}$ the near weights at $v_{i}$, with $\beta_{i 1}$ the near weight on the edge supporting $L_{i}$.

If $\operatorname{lk}\left(L_{i}, L_{j}\right)>0$ for all $j \neq i$, then the statement follows from (3.7.1). Hence, assume that $\operatorname{lk}\left(L_{i}, L_{j}\right)<0$ for some $j$. Assume that $L_{i}$ and $L_{j}$ are supported by nodes $v_{i}$ and $v_{j}$ respectively (the case $v_{i}=v_{j}$ is also possible). Let $\gamma$ be a path in $\Gamma$ joining $L_{i}$ to $L_{j}$. Since $\operatorname{lk}\left(L_{i}, L_{j}\right)<0$, one of the vertices lying on $\gamma$, call it $v_{\gamma}$, must have a negative weight. This, by assumption (a), must be a far weight, hence there is a unique $v_{\gamma}$ along the path with this property. Now, if $v_{i}$ is beyond $v_{\gamma}$, then by (b) we have $\alpha_{i}<0$. Otherwise, $v_{i}=v_{\gamma}$ and $\alpha_{i}<0$ by the definition of $v_{\gamma}$. Next, let $M_{i}$ be the multiplicity of $v_{i}$, namely

$$
M_{i}=\sum_{j} \operatorname{lk}\left(v_{i}, n_{j} L_{j}\right)=\alpha_{i} \beta_{i 2} \ldots \beta_{i k_{i}} n_{i}+\sum_{j \neq i} \operatorname{lk}\left(v_{i}, n_{j} L_{j}\right) .
$$

But for $j \neq i$ one has $\operatorname{lk}\left(v_{i}, n_{j} L_{j}\right)=\beta_{i 1} \operatorname{lk}\left(L_{i}, n_{j} L_{j}\right)$, hence

$$
\begin{equation*}
-\operatorname{lk}\left(L_{i}, L_{i}\right)=\frac{1}{\beta_{i 1} n_{i}} \sum_{j \neq i} \operatorname{lk}\left(v_{i}, n_{j} L_{j}\right)=\frac{M_{i}}{n_{i} \beta_{i 1}}-\frac{\alpha_{i} \beta_{i 2} \ldots \beta_{i k_{i}}}{\beta_{i 1}}>0 \tag{3.7.6}
\end{equation*}
$$

as $M_{i}>0$.
Corollary 3.7.7. If the diagram has one or two arrowheads, then the statement of Proposition 3.7.2 holds.

Proof. Use Lemma 3.7.5 and the fcat that the null-space is not trivial.
The proof is based on induction via reduction of the diagram (via two operations).
Definition 3.7.8. Let $\Gamma$ be a rooted graph. Assume that the supporting node $v_{i}$ of the arrowhead vertex $L_{i}$ has the following properties: it is not the root vertex, there is no node beyond it, $L_{i}$ is the unique arrowhead supported by $v_{i}$. Hence, all its adjacent vertices except $L_{i}$ and another one (in the direction of the root) are leaves. As above, denote the valency of $v_{i}$ by $k_{i}+1$ (see the picture below).

A collapse of $v_{i}$ is a graph $\Gamma^{\prime}$ with $v_{i}$ replaced by an arrowhead vertex $L_{i}^{\prime}$ with multiplicity $n_{i} \beta$, where $\beta=\beta_{i 2} \cdots \cdots \cdots \beta_{i k_{i}}$ and all other weights and multiplicities are unchanged.


Lemma 3.7.9. The linking matrices of $\Gamma$ and $\Gamma^{\prime}$ are congruent. Moreover, if $\Gamma$ satisfies the assumptions (a), (b) and (c) of the proposition, then so does $\Gamma^{\prime}$.

Proof. We shall use the notation $\mathrm{l}_{\Gamma}$ and $\mathrm{l}_{\Gamma^{\prime}}$ for the linking forms on $\Gamma$ and $\Gamma^{\prime}$.
For any vertex $v$ (being node or arrowhead), different from the deleted ones, we have $\mathrm{lk}_{\Gamma}\left(v, L_{i}\right)=\mathrm{lk}_{\Gamma^{\prime}}\left(v, \beta L_{i}^{\prime}\right)$. We claim that $\mathrm{lk}_{\Gamma^{\prime}}\left(\beta L_{i}, \beta L_{i}\right)=\mathrm{l}_{\Gamma}\left(L_{i}, L_{i}\right)$. This follows from (3.7.1) applied for $\Gamma$ and $\Gamma^{\prime}$, and from the fact that in $\Gamma^{\prime}$ the relation $\beta n_{i} L_{i}+\sum_{j \neq i} n_{j} L_{j}=0$ holds. Thus, the linking matrix of $\Gamma$ written in a basis, $L_{1}, \ldots, L_{i}, \ldots, L_{\nu}$ is the same as the linking matrix of $\Gamma^{\prime}$ in the basis $L_{1}, \ldots, \beta L_{i}^{\prime}, \ldots, L_{\nu}$. This proves the first part. As for the
other part, the multiplicities of all vertices (besides the deleted ones) are preserved. This shows that if $\Gamma$ satisfies (c), then so does $\Gamma^{\prime}$, while (a) and (b) are obvious.

Definition 3.7.10. Let $v_{0}$ be a node with no other node beyond it. Let $L_{1}, \ldots, L_{k}$ be the arrowheads adjacent to $v_{0}(k \geq 2)$, denote their multiplicities by $n_{1}, \ldots, n_{k}$. $v_{0}$ might have several adjacent leaves as well, $\beta$ denotes the product of their near weights. Assume that the overall number of vertices of $\Gamma$ is at least three.

A squeeze of $\Gamma$ is a graph arising from $\Gamma$ by replacing two arrowheads supported by $v_{0}$ (say, $L_{1}$ and $L_{2}$ ) by a single one, denoted by $L_{s}$, which will gain multiplicity

$$
n_{s}:=n_{2} \beta_{1}+n_{1} \beta_{2}
$$

and the near weight $\beta_{s}:=\beta_{1} \beta_{2}$.


Lemma 3.7.11. Let $\Gamma^{\prime}$ be a squeeze of arrowhead $L_{1}$ and $L_{2}$ from $\Gamma$. If $\Gamma$ satisfies the assumptions (a), (b) and (c) of the proposition, then so does $\Gamma^{\prime}$. Moreover, the rational linking matrix of $\Gamma$ is a direct sum of the linking matrix of $\Gamma^{\prime}$ and a negative definite 1dimensional matrix.

Proof. As for the first part we observe that $\beta_{s}$ and $n_{s}$ were chosen in such a way that all multiplicities of vertices are preserved. Moreover, by construction we have

$$
\begin{array}{ll}
\mathrm{lk}_{\Gamma^{\prime}}\left(L_{i}, L_{j}\right)=\mathrm{k}_{\Gamma}\left(L_{i}, L_{j}\right) & \text { if }\{i, j\} \cap\{1,2\}=\emptyset \text { and } i \neq j,  \tag{3.7.12}\\
\mathrm{lk}_{\Gamma^{\prime}}\left(n_{s} L_{s}, L_{j}\right)=\mathrm{lk}_{\Gamma}\left(n_{1} L_{1}+n_{2} L_{2}, L_{j}\right) & \text { if } j \geq 3 .
\end{array}
$$

We claim $\mathrm{lk}_{\Gamma^{\prime}}\left(L_{j}, L_{j}\right)=\mathrm{lk}_{\Gamma}\left(L_{j}, L_{j}\right)$ for $j \geq 3$. Indeed, this follows from (3.7.12) and (3.7.1) applied for both graphs. Now, let us define

$$
\Lambda_{1}=\beta_{1} L_{1}-\beta_{2} L_{2} \quad \text { and } \quad \Lambda_{2}=x L_{1}+y L_{2}
$$

where the rational numbers $x$ and $y$ will be determined later. By definition,

$$
\mathrm{lk}_{\Gamma}\left(\Lambda_{1}, L_{j}\right)=0 \quad \text { for any } j \geq 3
$$

The self-linking of $\Lambda_{1}$ is equal to

$$
\mathrm{lk}_{\Gamma}\left(\Lambda_{1}, \Lambda_{1}\right)=\beta_{1}^{2} \mathrm{lk}_{\Gamma}\left(L_{1}, L_{1}\right)+\beta_{2}^{2} \mathrm{lk}_{\Gamma}\left(L_{2}, L_{2}\right)-2 \alpha \beta \beta_{1} \beta_{2} \ldots \beta_{k} .
$$

If $\alpha>0$, then the above expression is negative, because $\mathrm{lk}_{\Gamma}\left(L_{1}, L_{1}\right)$ and $\mathrm{lk}_{\Gamma}\left(L_{2}, L_{2}\right)$ are negative by Lemma 3.7.5. If $\alpha<0$, we use (3.7.6) to show that $\mathrm{lk}_{\Gamma}\left(\Lambda_{1}, \Lambda_{1}\right)=-M_{v_{0}}\left(\frac{\beta_{1}}{n_{1}}+\right.$ $\left.\frac{\beta_{2}}{n_{2}}\right)<0$, because the multiplicity of $M_{v_{0}}$ is positive. Hence $\mathrm{lk}_{\Gamma}\left(\Lambda_{1}, \Lambda_{1}\right)<0$ always.

Since $\mathrm{lk}_{\Gamma}\left(\Lambda_{1}, \Lambda_{1}\right)<0$ and $\mathrm{lk}_{\Gamma}\left(L_{1}, L_{2}\right) \neq 0$, there exist $x$ and $y$ such that $\mathrm{lk}_{\Gamma}\left(\Lambda_{2}, \Lambda_{1}\right)=0$ and $\Lambda_{1}, \Lambda_{2}$ are linearly independent. Such $x$ and $y$ are determined up to a multiplicative
constant. To choose it observe that

$$
\mathrm{lk}_{\Gamma}\left(\Lambda_{2}, L_{j}\right)=x \mathrm{lk}_{\Gamma}\left(L_{1}, L_{j}\right)+y \mathrm{lk}_{\Gamma}\left(L_{2}, L_{j}\right)=\left(x+y \frac{\beta_{1}}{\beta_{2}}\right) \mathrm{lk}_{\Gamma}\left(L_{1}, L_{j}\right)
$$

and $\mathrm{l}_{\Gamma^{\prime}}\left(L_{s}, L_{j}\right)=\frac{1}{\beta_{2}} \mathrm{l}_{\Gamma}\left(L_{1}, L_{j}\right)$. We chose the rational numbers $x$ and $y$ so that $x+y \frac{\beta_{1}}{\beta_{2}}=$ $\frac{1}{\beta_{2}}$. Then, we have for all $j \geq 3$

$$
\begin{equation*}
\mathrm{lk}_{\Gamma}\left(\Lambda_{2}, L_{j}\right)=\mathrm{lk}_{\Gamma^{\prime}}\left(L_{s}, L_{j}\right) \tag{3.7.13}
\end{equation*}
$$

Finally, we show that $\mathrm{l}_{\Gamma}\left(\Lambda_{2}, \Lambda_{2}\right)=\mathrm{lk}_{\Gamma^{\prime}}\left(L_{s}, L_{s}\right)$. This is done as follows. First, on $\Gamma$ we have the relation $n_{1} L_{1}+n_{2} L_{2}+\sum n_{j} L_{j}=0$, which can be rewritten as

$$
\lambda_{1} \Lambda_{1}+\lambda_{2} \Lambda_{2}+\sum_{j \geq 3} n_{j} L_{j}=0
$$

for some $\lambda_{1}$ and $\lambda_{2}$. On the other hand, on $\Gamma^{\prime}$ we have $n_{s} L_{s}+\sum_{j \geq 3} n_{j} L_{j}=0$. Now taking the linking numbers with $L_{r}$ for some $r \geq 3$ we obtain

$$
0=\sum_{j \geq 3} n_{r} \mathrm{l}_{\Gamma}\left(L_{r}, L_{j}\right)+\lambda_{2} \mathrm{lk}_{\Gamma}\left(L_{r}, \Lambda_{2}\right)=\sum_{j \geq 3} n_{r} \mathrm{lk}_{\Gamma^{\prime}}\left(L_{r}, L_{j}\right)+n_{s} \mathrm{lk}_{\Gamma^{\prime}}\left(L_{r}, L_{s}\right) .
$$

Now by (3.7.12), since $r \geq 3$ the above equation simplifies to

$$
\lambda_{2} \mathrm{lk}_{\Gamma}\left(L_{r}, \Lambda_{2}\right)=n_{s} \mathrm{lk}_{\Gamma^{\prime}}\left(L_{r}, L_{s}\right) .
$$

From (3.7.13) and $\mathrm{lk}_{\Gamma^{\prime}}\left(L_{r}, L_{s}\right) \neq 0$ it follows that $n_{s}=\lambda_{2}$. But then we have

$$
\mathrm{l}_{\Gamma}\left(\Lambda_{2}, \lambda_{2} \Lambda_{2}\right)=-\sum_{j \geq 3} \mathrm{lk}_{\Gamma}\left(\Lambda_{2}, n_{j} L_{j}\right)=-\sum_{j \geq 3} \mathrm{lk}_{\Gamma^{\prime}}\left(L_{s}, n_{j} L_{j}\right)=\mathrm{lk}_{\Gamma^{\prime}}\left(L_{s}, n_{s} L_{s}\right) .
$$

As $n_{s}=\lambda_{2}$ we conclude that $\operatorname{lk}\left(\Lambda_{2}, \Lambda_{2}\right)=\operatorname{lk}\left(L_{s}, L_{s}\right)$. Hence the linking form on $\Gamma$ restricted to $\Lambda_{2}, L_{3}, \ldots, L_{n}$ is the same as the linking form on $\Gamma^{\prime}$ written in basis $L_{s}, L_{3}, \ldots, L_{n}$, while the element $\Lambda_{1}$ splits out completely as an orthogonal summand.

Finishing the proof of Proposition 3.7.2. By applying collapses and squeezes to $\Gamma$ we end up with a diagram, for which no further collapse or squeeze is possible. This diagram has one or two arrowheads and we conclude the proof by Corollary 3.7.7.

Remark 3.7.14. If we assume that the multiplicities of nodes of $\Gamma$ are only non-negative (not just positive), we can still prove semidefiniteness of the linking matrix, possibly with higher dimensional null-space. We omit the details.

## 4. Semicontinuity results

Now we are ready to prove various semicontinuity results. In subsection 4.1 we recover (in a slightly weaker form) the classical semicontinuity results valid in the local case of algebraic plane curve singularities (classically proved by Varchenko in [Var, see also [St2]). Next, in 4.2, we analyse the behavious of spectra under a degeneration of affine plane curves in the spirit of [NS]. Finally, we consider an affine plane curve, and we relate the spectrum of a curve at infinity with the spectra of its singularities, see 4.3. This type of comparison is unknown in Hodge theory.
4.1. Semicontinuity of the local singularity spectrum. Recall that in the local case $S p_{\text {MHS }}=S p_{\text {HVS }}$ (cf. [2.3.4), which will be denoted just by $S p$.

Let us consider now the following situation. Let $f_{t}(x, y)$ be a smooth family of holomorphic functions in two local coordinates depending on a local parameter $t$. Assume that $f_{0}(x, y)=0$ has an isolated singularity at the origin. Let us introduce the following notation.

- $B$ is a small ball centered at the origin such that $f_{0}^{-1}(0)$ is transverse to $\partial B$ and $f_{0}(z) /\left|f_{0}(z)\right|: \partial B \backslash f_{0}^{-1}(0) \rightarrow S^{1}$ is a Milnor fibration;
- $L_{0}=f_{0}^{-1}(0) \cap \partial B$ is the link of $f_{0}$ at 0 , and $S p_{0}$ the spectrum of the link;
- $t \neq 0$ and $|t|$ is sufficiently small so that $f_{t}^{-1}(0) \cap \partial B$ is a transversal intersection, and this link is isotopic in $\partial B$ to $L_{0}$;
- $C=f_{t}^{-1}(0) \cap B$;
- $z_{1}, \ldots, z_{k}$ are singular points of $C, L_{1}^{\text {sing }}, \ldots, L_{k}^{\text {sing }}$ the corresponding local links of these singularities, and $S p_{1}, \ldots, S p_{k}$ are the spectra of $L_{1}^{\text {sing }}, \ldots, L_{k}^{\text {sing }}$.
Proposition 4.1.1. Fix $x \in[0,1]$ such that $e^{2 \pi i x}$ is not a root of the Alexander polynomial of $L_{0}$. Then

$$
\begin{align*}
\left|S p_{0} \cap(x, x+1)\right| & \geq \sum_{j}\left|S p_{j} \cap(x, x+1)\right|  \tag{4.1.2}\\
\left|S p_{0} \backslash[x, x+1]\right| & \geq \sum_{j}\left|S p_{j} \backslash[x, x+1]\right| .
\end{align*}
$$

Proof. Assume $x \neq 0,1$. We shall prove only the first inequality, the second is completely analogous (in Section 2.5 all inequalities are given in pairs, the first one we use to prove results about $S p \cap(x, x+1)$, the other one to prove results about $S p \backslash[x, x+1])$. As $L_{j}^{\text {sing }}$ is an algebraic links, $\mu_{j}$ is the degree of the Alexander polynomial of $L_{j}^{\text {sing }}$. Hence $\mu_{j}-\sigma_{L_{j}^{\text {sing }}}(\zeta)+n_{L_{j}^{\text {sing }}}(\zeta) \geq 2\left|S p_{j} \cap(x, x+1)\right|$ by Corollary 2.4.6.

By assumption $n_{L_{0}}(\zeta)=0$. Since $L_{0}$ is also an algebraic link, $1-\chi\left(C_{\text {smooth }}\right)$ is the degree of the Alexander polynomial of $L_{0}$. Thus, again by Corollary [2.4.6, one gets $-\sigma_{L_{0}}(\zeta)+(1-$ $\left.\chi\left(C_{\text {smooth }}\right)\right)=2\left|S p_{0} \cap(x, x+1)\right|$. Then we conclude by the inequality (2.5.6).

If $x \in\{0,1\}$ the assumption that $e^{2 \pi i x}$ is not a root of the Alexander polynomial means that $L_{0}$ is a knot and $\left|S p_{0} \cap(0,1)\right|=\left|S p_{0} \cap(1,2)\right|$ is the delta invariant $\delta_{0}$. For any singularity link, hence for $L_{j}^{\text {sing }}$ too, $\delta_{j}=|S p \cap(0,1]| \geq\left|S p_{j} \cap(0,1)\right|$. Hence the statement follows from $\delta_{0} \geq \sum \delta_{j}$.
4.2. Semicontinuity of spectrum at infinity of families of affine curves. The methods described in this paper allow us also to prove the results on semicontinuity of the spectrum at infinity in the sense of [NS].

Let $F_{t}: \mathbb{C}^{2} \rightarrow \mathbb{C}$ be a smooth family of polynomials with a local deformation parameter $t$. Let $S p_{t}$ be corresponding spectra of $M H S$ at infinity and $\operatorname{Irr}_{t}$ be the irregularity of the link at infinity $L_{t, r e g}^{\infty}$ associated with $F_{t}$.

Note that over a small punctured disc $D^{*} \ni t$ the spectrum $S p_{t}$ is constant.
Theorem 4.2.1. Fix $x \in[0,1]$ such that $\{x, x+1\} \cap S p_{t}=\emptyset$ for $t \in D^{*}$. Then

$$
\left|S p_{t} \cap(x, x+1)\right|+\operatorname{Irr}_{t} \geq\left|S p_{0} \cap(x, x+1)\right|
$$

and the same statement holds for $S p \backslash[x, x+1]$ instead of $S p \cap(x, x+1)$.
Proof. Again we shall assume that $x$ is not an integer, otherwise we use exactly the same reduction argument as at the end of the proof of 4.3.1. We write $\zeta:=e^{2 \pi i x} \in S^{1} \backslash\{1\}$.

Let us choose $c$ such that $C_{0}=F_{0}^{-1}(c)$ is smooth and regular at infinity. Furthermore, choose $\xi$ and $r_{0}$ such that $S^{3}\left(\xi, r_{0}\right) \cap C_{0}$ is the regular link of $F_{0}$ at infinity, denoted by
$L_{0}$. By openness of trasversality condition, there exist $D$, an open neighbourhood of 0 , and $W_{c}$, an open neighbourhood of $c$, such that for any $w \in W_{c}$ and $t \in D$, the intersection $F_{t}^{-1}(w) \cap S^{3}\left(\xi, r_{0}\right)$ is transverse and isotopic to $L_{0}$. Let us take any $t \in D^{*}$ and choose $w \in W_{c}$ such that $C_{t}=F_{t}^{-1}(w)$ is smooth and regular at infinity. Finally, choose $r_{t}$ such that $L_{t}:=S^{3}\left(\xi, r_{t}\right) \cap C_{t}$ is the regular link at infinity of $C_{t}$.

Since $C_{t} \cap B\left(\xi, r_{0}\right)$ is isotopic to $C_{0} \cap B\left(\xi, r_{0}\right)$, by Corollary 2.5.4 we get

$$
-\sigma_{L_{t}}(\zeta)+n_{L_{t}}(\zeta)+1-\chi\left(C_{t} \cap B\left(\xi, r_{t}\right)\right) \geq-\sigma_{L_{0}}(\zeta)+n_{L_{0}}(\zeta)+1-\chi\left(C_{0} \cap B\left(\xi, r_{0}\right)\right)
$$

By assumption, $\zeta$ is not a root of $\Delta_{\operatorname{Irr} t}\left(L_{t}\right)$. Hence, applying 3.4.5 for $F_{t}$ and $F_{0}$ we obtain

$$
-\sigma_{L_{t}}+\operatorname{deg} \Delta_{\operatorname{Irr}_{t}}\left(L_{t}\right)+2 \operatorname{Irr}_{t} \geq-\sigma_{L_{0}}(\zeta)+\tilde{n}_{L_{0}}(\zeta)+\operatorname{deg} \Delta_{\operatorname{Irr}_{0}}+2 \operatorname{Irr}_{0}
$$

Then Corollary 2.4.6 implies

$$
\left|S p_{\mathrm{HVS}}\left(L_{t}\right) \cap(x, x+1)\right|+\operatorname{Irr}_{t} \geq\left|S p_{\mathrm{HVS}}\left(L_{0}\right) \cap(x, x+1)\right|+\operatorname{Irr}_{0} .
$$

Finally, Corollary 3.5 .4 provides the result.
To show the statement for $S p \backslash[x, x+1]$ we use the same argument.
4.3. Spectrum at infinity of a singular curve. Let $C \subset \mathbb{C}^{2}$ be an irreducible plane algebraic curve given by zero set of a reduced polynomial $F$. Let $z_{1}, \ldots, z_{k}$ be its singular points and $S p_{1}, \ldots, S p_{k}$ their (Hodge or HVS) spectra. Let $S p_{\infty}$ be the Hodge-spectrum of $F$ at infinity. Similarly, let $L_{r e g}^{\infty}$ be the regular link of $F$ at infinity, $S p_{\mathrm{HVS}}\left(L_{r e g}^{\infty}\right)$ its HVS-spectrum and Irr be as defined in (3.2.4).

Theorem 4.3.1. With the above notations, for all $x \in[0,1]$ such that $e^{2 \pi i x}$ is not a root of the Alexander polynomial of $L_{\text {reg }}^{\infty}$ we have

$$
\begin{array}{r}
\left|S p_{\mathrm{HVS}}\left(L_{\text {reg }}^{\infty}\right) \cap(x, x+1)\right|+\operatorname{Irr} \geq \sum_{j}\left|S p_{j} \cap(x, x+1)\right| \\
\left|S p_{\infty} \cap(x, x+1)\right|+\operatorname{Irr} \geq \sum_{j}\left|S p_{j} \cap(x, x+1)\right| . \tag{4.3.2}
\end{array}
$$

Moreover, the analogous statement holds if we replace $S p \cap(x, x+1)$ by $S p \backslash[x, x+1]$.
In the good case, if the regular link at infinity is fibred (e.g. if it is a knot), then $\operatorname{Irr}=0$ and the second inequality of (4.3.2) takes the form $\left|S p_{\infty} \cap(x, x+1)\right| \geq \sum_{k=1}^{n}\left|S p_{k}(x, x+1)\right|$.

Proof. First we assume that $x \in(0,1)$. We focus on the case $S p \cap(x, x+1)$ the case $S p \backslash[x, x+1]$ is analogous.

If $C$ is regular at infinity, the inequality (2.5.6) reads as

$$
\begin{equation*}
-\sigma_{L_{r e g}^{\infty}}(\zeta)+n_{L_{\text {reg }}^{\infty}}(\zeta)+\left(1-\chi\left(C_{\text {smooth }}\right)\right) \geq \sum_{j}\left(-\sigma_{j}(\zeta)+n_{j}(\zeta)+\mu_{j}\right), \tag{4.3.3}
\end{equation*}
$$

where $C_{\text {smooth }}$ is the smoothing of $C$. Since each link $L_{j}^{\text {sing }}$ is algebraic, $-\sigma_{j}(\zeta)+n_{j}(\zeta)+\mu_{j} \geq$ $2\left|S p_{j} \cap(x, x+1)\right|$. On the other hand, by Proposition 3.4.5 we get

$$
\begin{equation*}
1-\chi\left(C_{\text {smooth }}\right)=\operatorname{Irr}+\operatorname{deg} \Delta_{\mathrm{Irr}} . \tag{4.3.4}
\end{equation*}
$$

By Lemma 3.3.1(d) $\Delta_{L_{\text {reg }}^{\infty}}^{h}$ has no roots outside the unit circle, hence Corollary 2.4.6 applies. Since $\zeta$ is not a root of $\Delta_{L_{\text {reg }}^{\infty}}^{h}, \widetilde{n}(\zeta)=0$, hence

$$
\begin{equation*}
-\sigma_{L_{\text {reg }}^{\infty}}(\zeta)+\operatorname{deg} \Delta_{L_{r e g}}^{h}=2\left|S p_{L_{\text {reg }}^{\infty}} \cap(x, x+1)\right|, \tag{4.3.5}
\end{equation*}
$$

and $n(\zeta)=\operatorname{Irr}($ see (3.2). Then (4.3.3), (4.3.4) and (4.3.5) prove the statement in this case.

If $C$ is not regular at infinity, we argue as follows. We take an $r_{0}$ such that $C \cap S^{3}\left(\xi, r_{0}\right)$ is the link of $C$ at infinity, denoted by $L_{C}$. Then (2.5.6) yields

$$
\begin{equation*}
-\sigma_{L_{C}}(\zeta)+n_{L_{C}}(\zeta)+\left(1-\chi\left(C_{\text {smooth }}^{r_{0}}\right)\right) \geq \sum_{j}\left(-\sigma_{j}(\zeta)+n_{j}(\zeta)+\mu_{j}\right), \tag{4.3.6}
\end{equation*}
$$

where $C_{s m o o t h}^{r_{0}}:=C_{\varepsilon} \cap B\left(\xi, r_{0}\right)$ is the smoothing of $C$ in $B\left(\xi, r_{0}\right)$. Here $C_{\varepsilon}:=F^{-1}(\varepsilon)$ is smooth and regular at infinity (for $\varepsilon$ non-zero and sufficiently small). Moreover, we can assume that the links $C \cap S^{3}\left(\xi, r_{0}\right)$ and $C_{\varepsilon} \cap S^{3}\left(\xi, r_{0}\right)$ are isotopic. Let $r_{1}$ be such that $C_{\varepsilon} \cap S^{3}\left(\xi, r_{1}\right)$ is the regular link of $F$ at infinity. Corollary 2.5.4 applied to $C_{\varepsilon}$ yields

$$
\begin{equation*}
-\sigma_{L_{r e g}^{\infty}}^{\infty}(\zeta)+n_{L_{r e g}^{\infty}}^{\infty}(\zeta)-\left(-\sigma_{L_{C}}(\zeta)+n_{L_{C}}(\zeta)\right) \geq \chi\left(C_{01}\right) \tag{4.3.7}
\end{equation*}
$$

where $C_{01}=C_{\varepsilon} \cap\left(B\left(\xi, r_{1}\right) \backslash B\left(\xi, r_{0}\right)\right)$. (4.3.6) and (4.3.7) combined yields

$$
-\sigma_{L_{r e g}^{\infty}}(\zeta)+n_{L_{r e g}^{\infty}}(\zeta)+\left(1-\chi\left(C_{\varepsilon}\right)\right) \geq \sum_{j}\left(-\sigma_{j}(\zeta)+n_{j}(\zeta)+\mu_{j}\right) .
$$

This inequality is identical to (4.3.3) and we proceed further as in the previous case.
Assume that $x=0$. Then, by the assumption, 1 is not a root of the Alexander polynomial of $L_{r e g}^{\infty}$, hence $L_{r e g}^{\infty}$ is a knot (because $U_{\lambda=1}$ is trivial, but its dimension is $\nu-1$ by Proposition 3.2.8). Therefore the link at infinity is good, $\operatorname{Irr}=0$ and $S p_{\mathrm{HVS}}\left(L_{r e g}^{\infty}\right)=S p_{\infty}$.

For $\theta>0$ sufficiently small $\left|S p_{\infty} \cap(0,1)\right|=\left|S p_{\infty} \cap(\theta, 1+\theta)\right|$ (because $1 \notin S p_{\text {HVS }}\left(L_{\text {reg }}^{\infty}\right)=$ $\left.S p_{\infty}\right)$. On the other hand, in the local case, $\left|S p_{j} \cap(0,1)\right| \leq\left|S p_{j} \cap(\theta, 1+\theta)\right|$, hence the statement follows from the case $x \in(0,1)$.

The case $x=1$ follows by the same argument with $\theta<0$.

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