1-quasi-hereditary algebras

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Abstract

Motivated by the structure of the algebras associated to the blocks of the BGG-category $\mathcal{O}$, we define a subclass of quasi-hereditary algebras called 1-quasi-hereditary. Many properties of these algebras only depend on the defining partial order. In particular, we can determine the quiver and the form of the relations. Moreover, if the Ringel dual of a 1-quasi-hereditary algebra is also 1-quasi-hereditary, then the structure of the characteristic tilting module can be computed.

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Introduction

The class of quasi-hereditary algebras, defined by Cline, Parshall and Scott [3], can be regarded as a generalization of the algebras associated to the blocks of the Bernstein-Gelfand-Gelfand category $\mathcal{O}(\mathfrak{g})$ of a complex semisimple Lie algebra $\mathfrak{g}$ (see [2]). Every block $\mathcal{B}(\mathfrak{g})$ is equivalent to the category of modules over a finite dimensional $\mathbb{C}$-algebra $\mathcal{A}_{\mathcal{B}}(\mathfrak{g})$.

The algebras $\mathcal{A}_{\mathcal{B}}(\mathfrak{g})$ are BGG-algebras as defined in [7] and in [12]. They are endowed with a duality functor on their module category which fixes the simple modules. Another important structural feature is the presence of exact Borel subalgebras and Δ-subalgebras introduced by König in [8]. These subalgebras provide a correspondence between Δ-good filtrations and Jordan-Hölder-filtrations. Moreover, Soergel has shown that $\mathcal{A}_{\mathcal{B}}(\mathfrak{g})$ is Morita equivalent to its Ringel dual $R(\mathcal{A}_{\mathcal{B}}(\mathfrak{g}))$ (see [11]).

Motivated by these results, in this paper we introduce a class of quasi-hereditary algebras, called 1-quasi-hereditary. Among other properties they are characterized by the fact that all possible non-zero filtration-multiplicities for Δ-good filtrations of indecomposable projectives and Jordan-Hölder filtrations of standard modules are equal to 1.

The class of 1-quasi-hereditary algebras is related to the aforementioned classes of quasi-hereditary algebras: Many factor algebras (related to saturated subsets) of an algebra of type $\mathcal{A}_{\mathcal{B}}(\mathfrak{g})$ are 1-quasi-hereditary. The understanding of 1-quasi-hereditary algebras gives some information on the relations, the structure of the characteristic tilting module etc. of $\mathcal{A}_{\mathcal{B}}(\mathfrak{g})$. Another class of examples is provided by the quasi-hereditary algebras considered by Dlab, Heath and Marko in [4]. These algebras are 1-quasi-hereditary BGG-algebras, however 1-quasi-hereditary algebras are in general not BGG-algebras. All known 1-quasi-hereditary algebras have exact Borel and Δ-subalgebras. Several examples, which show the complexity of such algebras and their additional properties are presented in [9].

Our first main result shows that many invariants of 1-quasi-hereditary algebras depend only on the given partial ordering:

Theorem A. *Let $A = (KQ/\mathcal{I}, \leqslant)$ be a (basic) 1-quasi-hereditary algebra. Then*

(1) Q is the double of the quiver of the incidence algebra corresponding to $\leqslant$, i.e. $Q_1 = \{\ i \rightleftarrows j \quad | \quad i$ and j are neighbours w.r.t. $\leqslant\ \}$.

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(2) $\mathcal{I}$ is generated by the relations of the form $p - \sum_{j,k \leqslant i} c_i \cdot p(j,i,k)$, where $p = (j \to \cdots \to k)$ and $p(j,i,k)$ are paths in Q of the form $(j = j_1 \to \cdots \to j_m \to i \to k_1 \to \cdots \to k_r = k)$ with $j_1 < \cdots < j_m < i > k_1 > \cdots > k_r$.

(3) The Δ-good filtrations of the projective indecomposable module at the vertex $i \in Q_0$ are in one-to-one correspondence with special sequences of vertices j with $j \geqslant i$.

An important feature in the representation theory of quasi-hereditary algebras is the concept of the Ringel dual: The algebra $R(A) := \mathrm{End}_A(T)^{op}$ is quasi-hereditary, where $T = \bigoplus_{i \in Q_0} T(i)$ is the characteristic tilting module. In view of Soergels work, this raises the question whether the class of 1-quasi-hereditary algebras is closed under Ringel-duality.

Theorem B. *Let $A = (KQ/\mathcal{I}, \leqslant)$ be a 1-quasi-hereditary algebra. Then*

$$R(A) \text{ is 1-quasi-hereditary } \text{ if and only if } \; T(i) \text{ is local for any } i \in Q_0.$$

Moreover, in this case we have a precise description of $T(i)$.

Our paper is organised as follows: In Section 1, we introduce some notations, recall some definitions and basic facts for later use.

In Section 2, we give several properties of 1-quasi-hereditary algebras, which can be derived from the definition using the general representation theory of bound quiver algebras. These properties are essential for the proof of Theorem A (1).

In Section 3, we present a particular basis of a 1-quasi-hereditary algebra A, which can be described combinatorially and only depends on the corresponding partial order (it consist the paths of the form $p(j,i,k)$). Consequently, we obtain a system of relations of A described in Theorem A (2).

In Section 4, we determine the set of Δ-good filtrations of all projective indecomposable modules over 1-quasi-hereditary algebras and establish their relationship with the Jordan-Hölder-filtrations of costandard modules. Using the result of Ringel [10], which says that the subcategory $\mathfrak{F}(\Delta)$ is resolving, we determine all local modules having Δ-good filtrations. We also record the dual results.

In Section 5, we consider factor algebras $A(i) := A/A(\sum_{j \not\leqslant i} e_j)A$ for $i \in Q_0$ of a 1-quasi-hereditary algebra A, where e_i is a primitive idempotent. If $A(i)$ is 1-quasi-hereditary, then we obtain an explicit expression of the direct summand $T(i)$ of the characteristic tilting module.

Using these results in Section 6, we turn to the question when the Ringel dual of a 1-quasi-hereditary algebra is 1-quasi-hereditary. We elaborate on Theorem B by establishing necessary and sufficient conditions involving the structure of tilting modules and projective indecomposable modules.

1. Preliminaries

Throughout the paper, $\mathcal{A}$ denotes a finite dimensional, basic K-algebra over an algebraically closed field K, which will be represented by a quiver and relations (Theorem of Gabriel) and $\mathrm{mod}\,\mathcal{A}$ is the category of finite dimensional left $\mathcal{A}$-modules. In the following part we will focus on some general facts from the representations theory of bound quiver algebras, which we will use in this paper.

The relevant material can be found in [1].

We consider algebras $\mathcal{A} = KQ/\mathcal{I}$ and by Q_0 (resp. Q_1) we denote the set of vertices (resp. the set of arrows) in Q. For any $i \in Q_0$ the corresponding trivial path will be denoted by e_i, the simple module, the projective indecomposable and the injective indecomposable $\mathcal{A}$-module, will be denoted by $S(i)$, $P(i)$ and $I(i)$ respectively. A path $p = (j \to \cdots \to i \to \cdots \to k)$ is the product of paths $p_1 = (i \to \cdots \to k)$ and $p_2 = (j \to \cdots \to i)$ written as $p = p_1 \cdot p_2$. The $\mathcal{A}$-map corresponding to p is given by $f_p : P(k) \to P(j)$ via $f_p(a \cdot e_k) = a \cdot p \cdot e_j$ for all $a \in \mathcal{A}$ and we have $f_p = f_{p_2} \circ f_{p_1}$.

For any $M \in \operatorname{mod} \mathcal{A}$ it is $M \cong \bigoplus_{i \in Q_0} M_i$, where M_i is the subspace of M corresponding to $i \in Q_0$. We denote by $[M : S(i)] = \dim_K M_i$ the Jordan-Hölder multiplicity of $S(i)$ in M. For any $m \in M$, we denote by $\langle m \rangle$ the submodule of M generated by m (i.e. $\langle m \rangle = \mathcal{A} \cdot m$). The set of all local submodules of M with top isomorphic to $S(i)$, we denote by $\mathrm{Loc}_i(M)$. It is clear that $\mathrm{Loc}_i(M) = \{\langle m \rangle \mid m \in M_i \setminus \{0\}\} = \{\mathrm{im}(f) \mid f \in \mathrm{Hom}_A(P(i), M),\ f \neq 0\}$.

The definition of quasi-hereditary algebras introduced by Cline-Parshall-Scott [3] implies in particular the presence of a partial order on the vertices of the corresponding quiver. The equivalent definition and relevant terminology is given by Dlab and Ringel in [5]. To recap briefly: For an algebra $\mathcal{A} \cong KQ/\mathcal{I}$ let $(Q_0, \leqslant)$ be a partially ordered set. For every $i \in Q_0$ the *standard* module $\Delta(i)$ is the largest factor module of $P(i)$ such that $[\Delta(i) : S(j)] = 0$ for all $j \in Q_0$ with $j \nleqslant i$ and the *costandard* module $\nabla(i)$ is the largest submodule of $I(i)$ such that $[\nabla(i) : S(j)] = 0$ for all $j \in Q_0$ with $j \nleqslant i$. We denote by $\mathfrak{F}(\Delta)$ the full subcategory of $\operatorname{mod} \mathcal{A}$ consisting of the modules having a filtration such that each subquotient is isomorphic to a standard module. The modules in $\mathfrak{F}(\Delta)$ are called Δ-*good* and the corresponding filtrations are Δ-*good filtrations* (resp. ∇-*good* modules have ∇-*good filtrations* and belong to $\mathfrak{F}(\nabla)$). For $M \in \mathfrak{F}(\Delta)$, we denote by $(M : \Delta(i))$ the (well-defined) number of subquotients isomorphic to $\Delta(i)$ in some Δ-good filtration of M (resp. $\nabla(i)$ appears $(M : \nabla(i))$ times in some ∇-good filtration of $M \in \mathfrak{F}(\nabla)$).

The algebra $\mathcal{A} = (KQ/\mathcal{I}, \leqslant)$ is *quasi-hereditary* if for all $i, j \in Q_0$ the following holds:

- $[\Delta(i) : S(i)] = 1$,
- $P(i)$ is a Δ-good module with $(P(i) : \Delta(j)) = 0$ for all $j \ngeqslant i$ and $(P(i) : \Delta(i)) = 1$.

1.1 Remark. If $(\mathcal{A}, \leqslant)$ is quasi-hereditary, then for any $i \in Q_0(\mathcal{A})$ the following holds:

$$\Delta(i) = P(i) / \left(\sum_{i<j} \sum_{f \in \mathrm{Hom}_A(P(j), P(i))} \mathrm{im}(f) \right) \quad \text{resp.} \quad \nabla(i) = \bigcap_{i<j} \bigcap_{f \in \mathrm{Hom}_A(I(i), I(j))} \ker(f)$$

Moreover, if $i \in Q_0$ is minimal with respect to $\leqslant$, then $\Delta(i) \cong \nabla(i) \cong S(i)$ and if $i \in Q_0$ is maximal then $P(i) \cong \Delta(i)$ as well as $I(i) \cong \nabla(i)$.

1.2 Definition. A quasi-hereditary algebra $A = (KQ/\mathcal{I}, \leqslant)$ is called *1-quasi-hereditary* if for all $i, j \in Q_0 = \{1, \ldots, n\}$ the following conditions are satisfied:

(1) There is a smallest and a largest element with respect to $\leqslant$, without loss of generality we will assume them to be 1 resp. n,

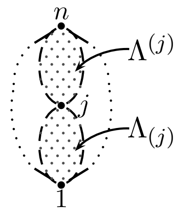
(2) $[\Delta(i) : S(j)] = \big(P(j) : \Delta(i)\big) = 1$ for $j \leqslant i$,

(3) $\operatorname{soc} P(j) \cong \operatorname{top} I(j) \cong S(1)$,

(4) $\Delta(i) \hookrightarrow \Delta(n)$ and $\nabla(n) \twoheadrightarrow \nabla(i)$.

The class of 1-quasi-hereditary algebras are related to several subclasses of quasi-hereditary algebras: Many factor algebras (related to a saturated subsets) of an algebra associated to a block of the category $\mathcal{O}(\mathfrak{g})$ of a semisimple $\mathbb{C}$-Lie algebra $\mathfrak{g}$ are 1-quasi-hereditary. If $\operatorname{rank}(\mathfrak{g}) \leq 2$, then an algebra corresponding to a block of $\mathcal{O}(\mathfrak{g})$ is 1-quasi-hereditary. This algebras are BGG-algebras in the sense of [12] and Ringel self-dual, however 1-quasi-hereditary algebras are not BGG-algebras in general and the class of 1-quasi-hereditary algebras is not closed under Ringel duality. Moreover all known 1-quasi-hereditary algebras have exact Borel and Δ-subalgebras in sense of König [8]. In [9] we give several examples to illustrate this specific properties.

Let $(Q_0, \leqslant)$ be the corresponding poset of a 1-quasi-hereditary algebra $KQ/\mathcal{I}$. For any $j \in Q_0$, we define



$$\Lambda_{(j)} := \{i \in Q_0 \mid i \leqslant j\} \quad \text{and} \quad \Lambda^{(j)} := \{i \in Q_0 \mid i \geqslant j\}$$

If $i < j$ (resp. $i > j$) and they are neighbours with respect to $\leqslant$, then we write $i \lhd j$ (resp $i \rhd j$). Obviously, $Q_0 = \Lambda^{(1)} = \Lambda_{(n)}$ and $i \in \Lambda^{(j)}$ if and only if $j \in \Lambda_{(i)}$.

According to the Brauer-Humphreys reciprocity formulas $(P(j) : \Delta(i)) = [\nabla(i) : S(j)]$ and $(I(j) : \nabla(i)) = [\Delta(i) : S(j)]$ (see [3]) the Axiom (2) in the Definition 1.2 is equivalent to the analog multiplicities axiom for injective indecomposable and costandard modules. For any 1-quasi-hereditary algebra $(A, \leqslant)$ and all $i, j \in Q_0(A)$ we thus have

$$(P(j) : \Delta(i)) = (I(j) : \nabla(i)) = [\Delta(i) : S(j)] = [\nabla(i) : S(j)] = \begin{cases} 1 & \text{if } i \in \Lambda^{(j)}, \\ 0 & \text{else.} \end{cases} \tag{$*$}$$

An algebra $\mathcal{A}$ is quasi-hereditary if and only if the opposite algebra $\mathcal{A}^{op}$ of $\mathcal{A}$ related to the same partial order $\leqslant$ on $Q_0(\mathcal{A}^{op}) = Q_0(\mathcal{A})$ is quasi-hereditary. There are the following relationships between the standard and costandard as well as between the Δ-good and ∇-good modules of $\mathcal{A}$ and $\mathcal{A}^{op}$ (we denote by $\mathcal{D}$ the standard K-duality): For all $i, j \in Q_0$, we have $\Delta_{\mathcal{A}}(i) \cong \mathcal{D}(\nabla_{\mathcal{A}^{op}}(i))$ and $[\Delta_{\mathcal{A}}(i) : S(j)] = [\nabla_{\mathcal{A}^{op}}(i) : S(j)]$. For $M \in \mathfrak{F}(\Delta_{\mathcal{A}})$, it is $\mathcal{D}(M) \in \mathfrak{F}(\nabla_{\mathcal{A}^{op}})$ and $(M : \Delta(i)) = (\mathcal{D}(M) : \nabla_{\mathcal{A}^{op}}(i))$. The corresponding dual properties hold for $\nabla_{\mathcal{A}}(i)$ and $M \in \mathfrak{F}(\nabla_{\mathcal{A}})$. The general properties of the standard duality imply that Axiom (3) and (4) in the Definition 1.2 are self-dual (see [1, Theorem 5.13]). This yields the following lemma.

1.3 Lemma. *An algebra A is 1-quasi-hereditary if and only if A^{op} is 1-quasi-hereditary.*

2. Projective indecomposables and the Ext-quiver

The structure of a 1-quasi-hereditary algebra A is related to properties of the projective indecomposable modules, which will be exhibited in this section. This implies that the structure of the standard A-modules, the quiver etc. is directly connected with the given partial order.

The relationship between the dimension vectors of an A-module M and of the subquotients of M as well as the equation $(*)$ shows that dimension vectors of modules $\Delta(j)$, $\nabla(j)$, $P(j)$, $I(j)$ and A only depend on the structure of the poset $(Q_0, \leqslant)$.

2.1 Lemma. *Let $A = (KQ/\mathcal{I}, \leqslant)$ be a 1-quasi-hereditary algebra and $j, k \in Q_0$. Then*

(1) $\dim_K \Delta(k) = \dim_K \nabla(k) = \left|\Lambda_{(k)}\right|$, $\quad \dim_K P(j) = \dim_K I(j) = \sum_{k\in\Lambda^{(j)}} \left|\Lambda_{(k)}\right|$ *and*

$$\dim_K A = \sum_{j\in Q_0} \left|\Lambda_{(j)}\right|^2.$$

(2) $[P(j) : S(k)] = [I(j) : S(k)] = \left|\Lambda^{(j)} \cap \Lambda^{(k)}\right|$.

(3) $P(1) \cong I(1)$*, where* $1 = \min\{Q_0, \leqslant\}$.

Proof. *(1)* The dimensions of $\Delta(i)$, $\nabla(i)$, $P(i)$, $I(i)$ and A we obtain directly from $(*)$.

(2) The equation $(*)$ implies $[P(j) : S(k)] = \sum_{i\in\Lambda^{(j)}}[\Delta(i) : S(k)] = \sum_{i\in\Lambda^{(j)}\cap\Lambda^{(k)}}[\Delta(i) : S(k)] + \sum_{i\in\Lambda^{(j)}\setminus\Lambda^{(k)}}[\Delta(i) : S(k)] = \left|\Lambda^{(j)} \cap \Lambda^{(k)}\right|$. Similarly, we have $[I(j) : S(k)] = \left|\Lambda^{(j)} \cap \Lambda^{(k)}\right|$.

(3) The Definition 1.2 (3) implies $P(1) \hookrightarrow I(1)$. Since $\dim_K P(1) \overset{(1)}{=} \dim_K I(1)$, we obtain $P(1) \cong I(1)$. $\square$

Any projective indecomposable module over a 1-quasi-hereditary algebra may be considered as a submodule of $P(1)$ because of Definition 1.2 (3) and Lemma 2.1 *(3)*.

2.2 Lemma. *Let $A = (KQ/\mathcal{I}, \leqslant)$ be a 1-quasi-hereditary algebra, $i, j \in Q_0$ and $M(i)$ be a submodule of $P(1)$ isomorphic to $P(i)$. Then*

(1) $\mathrm{Loc}_i(M(j)) \subseteq \mathrm{Loc}_i(M(i))$

(2) $\mathrm{Loc}_i(M(j)) = \mathrm{Loc}_i(M(i))$ *if and only if* $i \in \Lambda^{(j)}$.

In particular, $P(i) \hookrightarrow P(j)$ if and only if $i \in \Lambda^{(j)}$, and there exists a unique submodule of $P(j)$ which is isomorphic to $P(i)$.

Proof. *(1)* Since $\mathrm{Loc}_i(M(j)) = \{\langle m\rangle \mid m \in M(j)_i \setminus \{0\}\}$ for all $i, j \in Q_0$, it is enough to show $M(j)_i \subseteq M(i)_i$. Lemma 2.1 (2) implies $\dim_K P(1)_i = \dim_K M(i)_i = \left|\Lambda^{(i)}\right|$, thus $M(i) \subseteq P(1)$ yields $P(1)_i = M(i)_i$. Consequently, $M(j)_i \subseteq P(1)_i = M(i)_i$ for all $i, j \in Q_0$.

(2) Obviously, $i \in \Lambda^{(j)}$ if and only if $\left|\Lambda^{(i)} \cap \Lambda^{(j)}\right| = \left|\Lambda^{(i)}\right|$. In this case we have $\dim_K M(j)_i = \dim_K M(i)_i$, thus $M(j)_i \subseteq M(i)_i$ implies $M(j)_i = M(i)_i$.

Since $\mathrm{Loc}_i(P(1)) = \mathrm{Loc}_i(M(i))$, we obtain that for any submodule N of $P(1)$ with $N \cong P(i)$ it holds $N \subseteq M(i)$, thus $\dim_K N = \dim_K M(i)$ implies $N = M(i)$. Consequently, $M(i)$ is the unique submodule of $M(j)$ isomorphic to $P(i)$ if $i \in \Lambda^{(j)}$ because of *(2)*. $\square$

2.3 Remark. From now on, for $i, j \in Q_0$ with $i \in \Lambda^{(j)}$ we consider $P(i)$ as a submodule of $P(j)$. Since for every $F \in \mathrm{End}_A(P(j))$ with $F(P(i)) \neq 0$ the submodule $F(P(i))$ of $P(j)$ is local with top $F(P(i)) \cong S(i)$, Lemma 2.2 implies $F(P(i)) \subseteq P(i)$. The submodule $P(i)$ of $P(j)$ is an $\mathrm{End}_A(P(j))^{op}$-module for all $i \in \Lambda^{(j)}$.

2.4 Lemma. *Let $A = (KQ/\mathcal{I}, \leqslant)$ be a 1-quasi-hereditary algebra and $j \in Q_0$. Then*

$$\Delta(j) = P(j)/\left(\sum_{j \lhd i} P(i)\right) \quad \textit{and} \quad \nabla(j) = \bigcap_{j \lhd i} \ker(I(j) \twoheadrightarrow I(i)).$$

Proof. Since $\mathrm{Loc}_i(P(j)) = \{\mathrm{im}(f) \mid f \in \mathrm{Hom}_A(P(i), P(j)),\ f \neq 0\} \overset{2.2}{=} \mathrm{Loc}_i(P(i))$, we obtain $\sum_{f \in \mathrm{Hom}_A(P(i),P(j))} \mathrm{im}(f) = P(i)$ for every $i \in \Lambda^{(j)}$. Moreover, $\sum_{j<i} P(i) = \sum_{j \lhd i} P(i)$, since for every $k \in \Lambda^{(j)} \setminus \{j\}$ there exists $i \in Q_0$ with $j \lhd i \leqslant k$, thus $P(k) \subseteq P(i)$. We obtain $\Delta(j) \overset{1.1}{=} P(j)/\left(\sum_{j \lhd i} P(i)\right)$.

Using the standard duality we have $\nabla(j) = \bigcap_{j \lhd i} \ker(I(j) \twoheadrightarrow I(i))$. □

Definition 1.2 (4) shows that any standard module can be considered as a submodule of $\Delta(n)$. Thus we consider any submodule of $\Delta(j)$ as a submodule of $\Delta(n)$.

2.5 Lemma. *Let $A = (KQ/\mathcal{I}, \leqslant)$ be a 1-quasi-hereditary, $j \in Q_0$. Then M is a submodule of $\Delta(j)$ if and only if $M = \sum_{i \in \Lambda} \Delta(i)$ for some $\Lambda \subseteq \Lambda_{(j)}$. In particular,* $\mathrm{Loc}_i(\Delta(j)) = \{\Delta(i)\}$ *if $i \in \Lambda_{(j)}$ and* $\mathrm{Loc}_i(\Delta(j)) = \emptyset$ *if $i \notin \Lambda_{(j)}$. Moreover,* $\mathrm{rad}\,\Delta(j) = \sum_{j \rhd i} \Delta(i)$.

Proof. For every $i \in Q_0$ we have $\mathrm{Loc}_i(\Delta(n)) = \{\Delta(i)\}$, since $[\Delta(n) : S(i)] = 1$ (see Definition 1.2 (2)). If $i \in \Lambda_{(j)}$, then $[\Delta(j) : S(i)] = 1$, thus $\mathrm{Loc}_i(\Delta(j)) \neq \emptyset$. Since $\mathrm{Loc}_i(\Delta(j)) \subseteq \mathrm{Loc}_i(\Delta(n))$, we obtain $\mathrm{Loc}_i(\Delta(j)) = \{\Delta(i)\}$. If $i \notin \Lambda_{(j)}$, then $[\Delta(j) : S(i)] = 0$, thus $\mathrm{Loc}_i(\Delta(j)) = \emptyset$. Any submodule M of $\Delta(j)$ is a sum of some local submodules of $\Delta(j)$, thus $M = \sum_{i \in \Lambda} \Delta(i)$ for some $\Lambda \subseteq \Lambda_{(j)}$. In particularly, $\mathrm{rad}\,\Delta(j) = \sum_{i \in \Lambda_{(j)} \setminus \{j\}} = \sum_{i \lhd j} \Delta(i)$, since for any $k \in \Lambda_{(j)} \setminus \{j\}$ there exists $i \in Q_0$ with $k \leqslant i \lhd j$, thus $\Delta(k) \subseteq \Delta(i)$. □

2.6 Remark. Since for a 1-quasi-hereditary algebra A the algebra A^{op} is also 1-quasi-hereditary (see 1.3), every statement yields a corresponding dual statement. Lemma 2.5 and Lemma 2.2 implies that for all $j, l \in Q_0$ and all $i \in \Lambda_{(j)}$ and $k \in \Lambda^{(l)}$ we obtain

$$S(1) \hookrightarrow \Delta(i) \hookrightarrow \Delta(j) \hookrightarrow P(k) \hookrightarrow P(l) \hookrightarrow P(1) \cong I(1) \twoheadrightarrow I(l) \twoheadrightarrow I(k) \twoheadrightarrow \nabla(j) \twoheadrightarrow \nabla(i) \twoheadrightarrow S(1).$$

We are now going to determine the shape of the Ext-quiver of a 1-quasi-hereditary algebra (cf. Theorem A (1)).

2.7 Theorem. *Let $A = (KQ/\mathcal{I}, \leqslant)$ be a 1-quasi-hereditary algebra. In the* Ext*-quiver of A the vertices i and j are connected by an arrow if and only if they are neighbours with respect to $\leqslant$. Moreover, if $i \lhd j$ (or $i \rhd j$) then* $\left|\left\{\alpha \in Q_1 \mid i \xrightarrow{\alpha} j\right\}\right| = \left|\left\{\alpha \in Q_1 \mid j \xrightarrow{\alpha} i\right\}\right| = 1$.

Proof. Let $j, k \in Q_0$. The number of arrows from k to j is the number of $S(k)$ in the decomposition of $\mathrm{top}\,(\mathrm{rad}\,P(j))$ (see [1, Lemma 2.12]). We denote by $\mathtt{N}(j)$ the set $\{k \in Q_0 \mid k \lhd j\} \cup \{k \in Q_0 \mid k \rhd j\}$. We have to show $\mathrm{top}\,(\mathrm{rad}\,P(j)) \cong \bigoplus_{k \in \mathtt{N}(j)} S(k)$. In other words, for every $k \in \mathtt{N}(j)$ there exists $L(k) \in \mathrm{Loc}_k(P(j))$ with

$$\mathrm{rad}\,P(j) = \textstyle\sum_{k \in \mathtt{N}(j)} L(k) \quad \text{and} \quad L(t) \nsubseteq \sum_{\substack{k \in \mathtt{N}(j) \\ t \neq k}} L(k) \quad \text{for every } t \in \mathtt{N}(j).$$

We denote by $\mathrm{SM}(\Delta(j))$ the set of submodules of $\Delta(j)$ and by $\mathrm{SM}(P(j) \mid \sum_{j \lhd i} P(i))$ the set of submodules M of $P(j)$ with $\sum_{j \lhd i} P(i) \subseteq M$. The function $F : \mathrm{SM}\left(P(j) \mid \sum_{j \lhd i} P(i)\right) \to \mathrm{SM}(\Delta(j))$ with $F(M) = M/\left(\sum_{j \lhd i} P(i)\right)$ is bijective (see 2.4). By Lemma 2.5 for any $k \in \Lambda_{(j)}$ there exists $L(k) \in \mathrm{Loc}_k(P(j))$ such that $F\left(L(k) + \sum_{j \lhd i} P(i)\right) = \Delta(k)$ and $F\left(\sum_{k \in \Lambda} L(k) + \sum_{j \lhd i} P(i)\right) = \sum_{k \in \Lambda} \Delta(k)$ for any subset $\Lambda \subseteq \Lambda_{(j)}$, since F preserves and reflects inclusions. In particular, $F(\mathrm{rad}\, P(j)) = \mathrm{rad}\, \Delta(j) = \sum_{j > k} \Delta(k) \overset{2.5}{=} \sum_{j \rhd k} \Delta(k)$, thus

$$\mathrm{rad}\, P(j) = \textstyle\sum_{j \rhd k} L(k) + \sum_{j \lhd i} P(i).$$

Since $\Delta(t) \not\subseteq \sum_{\substack{j \rhd k \\ t \neq k}} \Delta(k)$, we obtain $L(t) \not\subseteq \sum_{\substack{j \rhd k \\ t \neq k}} L(k) + \sum_{j \lhd i} P(i)$ for every t with $j \rhd t$. In order to prove $P(t) \not\subseteq \sum_{j \rhd k} L(k) + \sum_{\substack{j \lhd i \\ t \neq i}} P(i)$ for t with $j \lhd t$, it is enough to show the following two statements: Let M, M' be some submodules of $P(j)$, then

① $P(t) \not\subseteq M$ and $P(t) \not\subseteq M'$ implies $P(t) \not\subseteq M + M'$,
② $P(t) \not\subseteq L(k)$ for every k with $j \rhd k$ and $P(t) \not\subseteq P(i)$ for every i with $j \lhd i \neq t$.

①: For all $m \in M_t$ and $m' \in M'_t$ we have $\langle m \rangle \neq P(t)$ and $\langle m' \rangle \neq P(t)$. Since $\langle m \rangle, \langle m' \rangle \in \mathrm{Loc}_t(P(j)) \overset{2.2}{\subseteq} \mathrm{Loc}_t(P(t))$ for $m, m' \neq 0$, we obtain $m, m' \in \mathrm{rad}\, P(t)$, thus $m + m' \in \mathrm{rad}\, P(t)$. Consequently, $\mathrm{Loc}_t(M+M') = \{\langle m + m' \rangle \mid m \in M_t \setminus \{0\}, m' \in M'_t \setminus \{0\}\} \subseteq \mathrm{Loc}_t(\mathrm{rad}\, P(t))$ and hence $P(t) \not\subseteq M + M'$.
②: Assume $P(t) \subseteq L(k)$ for some $j \rhd k$. Let $G : P(k) \twoheadrightarrow L(k)$, then $L(k) \cong P(k)/\ker(G)$ implies the existence of $N \in \mathrm{Loc}_t(P(k))$ with $\ker(G) \subseteq N$ such that $P(t) \cong N/\ker(G)$. Since $N \overset{2.2}{\subseteq} P(t)$, we have $\ker G = 0$ and $P(k) \cong L(k)$, a contradiction because for $j \rhd k$, it holds $P(k) \overset{2.2}{\not\subseteq} P(j)$. For all $i, t \in Q_0$ with $j \lhd i, t$ and $i \neq t$ we have $P(t) \not\subseteq P(i)$ by Lemma 2.2. $\square$

3. A basis of a 1-quasi-hereditary algebra

From now on $A = (KQ/\mathcal{I}, \leqslant)$ is a 1-quasi-hereditary algebra with $1 \leqslant i \leqslant n$ for all $i \in Q_0$. We use the same notations as in the previous section.

The structure of the quiver of a 1-quasi-hereditary algebra shows that for all $j, i, k \in Q_0$ with $i \in \Lambda^{(j)} \cap \Lambda^{(k)}$ there exists a path

$$\begin{array}{lll} j \to \lambda_1 \to \cdots \to \lambda_m \to i & \text{with} \;\; j \leqslant \lambda_1 \leqslant \cdots \leqslant \lambda_m \leqslant i & \text{resp.} \\ i \to \mu_1 \to \cdots \to \mu_r \to k & \text{with} \;\; i \geqslant \mu_1 \geqslant \cdots \geqslant \mu_r \geqslant k & \end{array}$$

called *increasing* path from j to i, resp. *decreasing* path from i to j. By concatenating these, we get a path from j to k passing through i, and we write $p(j, i, k)$ for the image in A of such path. When $i = j = k$, the path $p(j, j, j)$ is the trivial path e_j. All increasing resp. decreasing paths (as well all arrows) of the quiver occur in this way: A path of the form $p(j, i, i)$ is increasing resp. $p(i, i, k)$ is a decreasing path.

3.1 Remark. Recall that the module generated by $p(j,i,k)$ is the image of the A-map $f_{(j,i,k)} : P(k) \to P(j)$ via $f_{(j,i,k)}(e_k) = p(j,i,k)$, thus a submodule of $P(j)$ from $\mathrm{Loc}_k(P(j))$.

(a) Theorem 2.7 implies $\operatorname{rad} P(j) = \sum_{j \lhd i} \langle j \to i \rangle + \sum_{j \rhd i} \langle j \to i \rangle$ for any $j \in Q_0$. Since $\langle j \to i \rangle \in \mathrm{Loc}_i(P(j))$, we obtain that $\langle j \to i \rangle$ belongs to the submodule $P(i)$ of $P(j)$ for all i with $j \lhd i$ (see 2.2). It is easy to see that $\langle j \to i \rangle = P(i)$: Assume $\langle j \to i \rangle \subset P(i)$, then $\langle j \to i \rangle \not\subseteq \sum_{\substack{j \lhd i' \\ i \neq i'}} \langle j \to i' \rangle + \sum_{j \rhd i} \langle j \to i \rangle$ implies $P(i) \not\subseteq \operatorname{rad} P(j)$ (see ① in the proof of 2.7), a contradiction.

The A-map corresponding to $(j \to i)$ with $j \lhd i$ is therefore an inclusion. Consequently the A-map corresponding to an increasing path $p(j,i,i)$ provides a composition of the inclusions, thus $f_{(j,i,i)} : P(i) \hookrightarrow P(j)$. In particularly, for any two increasing paths p and q from j to i we have $\langle p \rangle = \langle q \rangle$, since $\operatorname{im}(f_p) = \operatorname{im}(f_q) = P(i)$ (see 2.2). Thus $\langle p(j,i,i) \rangle \cong \langle p(j',i,i) \rangle \cong P(i)$ for all $j, j' \in \Lambda_{(i)}$. Using our notations, we have $\operatorname{rad} P(j) = \sum_{j \lhd i} P(i) + \sum_{j \rhd k} \langle p(j,j,k) \rangle$.

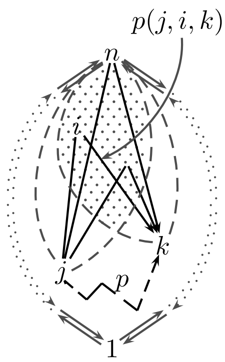
(b) A path $p(j,i,k)$ is the product of $p(i,i,k)$ and $p(j,i,i)$, therefore using (a) we have $f_{(j,i,k)} : P(k) \xrightarrow{f_{(i,i,k)}} P(i) \overset{f_{(j,i,i)}}{\hookrightarrow} P(j)$. Hence the module $\langle p(j,i,k) \rangle$ may be considered as a submodule of $P(i) (\subseteq P(j))$ from $\mathrm{Loc}_k(P(i))$. In particular, it is easy to see that for all $j, k \in Q_0$ we have $\langle p(j,n,k) \rangle \cong \Delta(k)$ because $\Delta(k)$ is the uniquely submodule of $P(n) = \Delta(n)$ from $\mathrm{Loc}_k(P(n))$ (see 2.5).

3.2 Theorem. *Let $A = (KQ/\mathcal{I}, \leqslant)$ be a 1-quasi-hereditary algebra and $j, k \in Q_0$. For any $i \in \Lambda^{(j)} \cap \Lambda^{(k)}$ we fix a path of the form $p(j,i,k)$. The set*

$$\left\{ p(j,i,k) \mid i \in \Lambda^{(j)} \cap \Lambda^{(k)} \right\} \quad \textit{is a } K\textit{-basis of } P(j)_k.$$

In particular, $\mathfrak{B}_j := \left\{ p(j,i,k) \mid i \in \Lambda^{(j)},\ k \in \Lambda_{(i)} \right\}$ is a K-basis of $P(j)$ for any $j \in Q_0$ and $\mathfrak{B} := \left\{ p(j,i,k) \mid j,k \in Q_0,\ i \in \Lambda^{(j)} \cap \Lambda^{(k)} \right\}$ is a K-basis of A.

The chosen paths $p(j,i,k)$ for all $i \in \Lambda^{(j)} \cap \Lambda^{(k)}$ are symbolically represented in the picture to the right (a path $p(j,i,k)$ is not uniquely determined). Theorem 3.2 shows that for any path p in A, which starts in j and ends in k there exist $c_i \in K$ such that $p = \sum_{i \in \Lambda^{(j)} \cap \Lambda^{(k)}} c_i \cdot p(j,i,k)$. In other words $p - \sum_{i \in \Lambda^{(j)} \cap \Lambda^{(k)}} c_i \cdot p(j,i,k) \in \mathcal{I}$ (here p and $p(j,i,k)$ are paths in Q). Using general methods any relation in $\mathcal{I}$ can by transform in this form (see for example [1, Section II.2, II.3]).



Part (2) of Theorem A follows directly from 3.2.

The proof of the Theorem 3.2 is based on the statements of the following lemma. Recall that for $i \in \Lambda^{(j)}$ we consider $P(i)$ as a submodule of $P(j)$ (see 2.3).

3.3 Lemma. *Let $A = (KQ/\mathcal{I}, \leqslant)$ be a 1-quasi-hereditary algebra and $j, k \in Q_0$. Let $0 \subset \cdots \subset D' \subset D \subset \cdots \subset P(j)$ be a Δ-good filtration of $P(j)$, where $D/D' \cong \Delta(i)$ for some $i \in \Lambda^{(j)} \cap \Lambda^{(k)}$. Then we have the following:*

(1) $D = P(i) + D'$.

(2) $D' \subset \langle p(j,i,k) \rangle + D' \subseteq D$ for any path of the form $p(j,i,k)$.

In particular, there exists a subset Λ *of* $\Lambda^{(j)}$ *with* $D = \sum\limits_{i\in\Lambda} P(i)$.

Proof. *(1)* Let $0 = D(r+1) \subset D(r) \subset \cdots \subset D(1) = P(j)$ be a Δ-good filtration with $D(l)/D(l+1) \cong \Delta(i_l)$ for all $r \leq l \leq 1$. There is some local submodule $L(l)$ of $P(j)$ with top isomorphic to $S(i_l)$ such that $D(l) = L(l) + D(l+1)$. Definition 1.2 yields $i_l \in \Lambda^{(j)}$ and therefore $L(l) \subseteq P(i_l) \subseteq P(j)$ (see 2.2). We obtain $D(l) = L(l) + D(l+1) \subseteq P(i_l) + D(l+1)$ for all $1 \leq l \leq r$. In order to show $D(l) = P(i_l) + D(l+1)$, we have to show $P(i_l) \subseteq D(l)$.

Assume $P(i_l) \not\subseteq D(l)$. There exists $t \in \{1, \ldots, l-1\}$ with $P(i_l) \subseteq D(t)$ and $P(i_l) \not\subseteq D(t+1)$ and hence $D(t+1) \subset P(i_l) + D(t+1) \subseteq D(t)$. We show now $P(i_l) + D(t+1) = D(t)$, this then implies $D(t)/D(t+1) \cong \Delta(i_t) \cong \Delta(i_l)$ and hence $(P(j) : \Delta(i_l)) \geq 2$, a contradiction (see Definition 1.2).

Since $0 \neq P(i_l)/\left(P(i_l) \cap D(t+1)\right) \hookrightarrow D(t)/D(t+1) \cong \Delta(i_t)$, the standard module $\Delta(i_t)$ has a local submodule with top isomorphic to $S(i_l)$. Thus $[\Delta(i_t) : S(i_l)] \neq 0$ and hence $i_t \overset{1.2}{\in} \Lambda^{(i_l)}$ and therefore $L(t) \subseteq P(i_t) \subseteq P(i_l)$ (see 2.2). Consequently, $D(t) = L(t) + D(t+1) \subseteq P(i_l) + D(t+1) \subseteq D(t)$. We have $P(i_l) + D(t+1) = D(t)$.

Via induction on $r-k$ we obtain $D(k) = \sum_{m=k}^{r} P(i_m)$ for any $0 \leq k \leq r$.

(2) By Lemma 2.4 and *(1)*, since $D/D' \cong P(i)/(P(i) \cap D') \cong \Delta(i)$, we obtain $P(i) \cap D' = \sum_{i\lhd l} P(l)$. Because $\langle p(j,i,k)\rangle$ is a submodule of $P(i) \subseteq P(j)$ (see 3.1(b)), it is enough to show $\langle p(j,i,k)\rangle \not\subseteq \sum_{i\lhd l} P(l)$. This implies $\langle p(j,i,k)\rangle \not\subseteq D'$ and consequently $D' \subset \langle p(j,i,k)\rangle + D' \subseteq P(i) + D' = D$.

Let $i \rhd k$, then $p(i,i,k) = (i \to k)$. We have $\langle p(i,i,k)\rangle \not\subseteq \sum_{i\lhd l} P(l)$, since $\operatorname{rad} P(i) = \sum_{i\rhd k} \langle p(i,i,k)\rangle + \sum_{i\lhd l} P(l)$ (see 3.1(a)). To deal with the general paths we consider maps. Because $\langle p(i,i,k)\rangle \overset{3.1}{=} \operatorname{im}(f_{(i,i,k)})$, we have $\operatorname{im}\left(P(k) \overset{f_{(i,i,k)}}{\to} P(i) \overset{\pi}{\twoheadrightarrow} P(i)/\left(\sum_{i\lhd l} P(l)\right)\right) \neq 0$. Since $\operatorname{im}\left(\pi \circ f_{(i,i,k)}\right) \subseteq P(i)/\left(\sum_{i\lhd l} P(l)\right) \overset{2.4}{=} \Delta(i)$ and $\operatorname{Loc}_k(\Delta(i)) \overset{2.5}{=} \{\Delta(k)\}$, we obtain $\operatorname{im}\left(\pi \circ f_{(i,i,k)}\right) = \Delta(k)$. Lemma 2.4 implies $\ker\left(\pi \circ f_{(i,i,k)}\right) = \sum_{k\lhd j} P(j)$. This implies a commutative diagram

$$\begin{tikzcd}
P(k) \arrow[r, "f_{(i,i,k)}"] \arrow[d] & P(i) \arrow[d, "\pi"] \\
\underbrace{P(k)/\left(\sum\limits_{k\lhd j} P(j)\right)}_{\Delta(k)} \arrow[r, "\overline{f_{(i,i,k)}}"] & \underbrace{P(i)/\left(\sum\limits_{i\lhd l} P(l)\right)}_{\Delta(i)}
\end{tikzcd}$$

The map $\overline{f_{(i,i,k)}}$ is an inclusion, since $\overline{f_{(i,i,k)}} \neq 0$.

Now let $i > k$ with $i \rhd l_1 \rhd \cdots \rhd l_m \rhd k$. Inductively we obtain the commutative diagrams for the path $p(i,i,k) = (i \to l_1 \to \cdots \to l_m \to k) = p(l_m, l_m, k) \cdot p(l_{m-1}, l_{m-1}, l_m) \cdots p(i,i,l_1)$

$$\begin{tikzcd}
P(k) \arrow[r, "f_{(l_m,l_m,k)}"] \arrow[d] & P(l_m) \arrow[r, "f_{(l_{m-1},l_{m-1},l_m)}"] \arrow[d] & \cdots \arrow[r, "f_{(l_1,l_1,l_2)}"] & P(l_1) \arrow[r, "f_{(i,i,l_1)}"] \arrow[d] & P(i) \arrow[d, "\pi"] \\
\Delta(k) \arrow[r, hook] & \Delta(l_m) \arrow[r, hook] & \cdots \arrow[r, hook] & \Delta(l_1) \arrow[r, hook] & \Delta(i)
\end{tikzcd}$$

For the maps $f_{(i,i,k)} = f_{(i,i,l_1)} \circ f_{(l_1,l_1,l_2)} \circ \cdots \circ f_{(l_m,l_m,k)}$ and $\pi : P(i) \twoheadrightarrow P(i)/\left(\sum_{i\lhd i'} P(i')\right) \cong \Delta(i)$ we have $\operatorname{im}\left(\pi \circ f_{(i,i,k)}\right) \neq 0$, thus $\operatorname{im}(f_{(i,i,k)}) = \langle p(i,i,k)\rangle \not\subseteq \sum_{i\lhd l} P(l)$. Therefore $f_{(j,i,k)} : P(k) \overset{f_{(i,i,k)}}{\longrightarrow} P(i) \overset{f_{(j,i,i)}}{\hookrightarrow} P(j)$ shows that the submodule $\operatorname{im}(f_{(j,i,k)}) = \langle p(j,i,k)\rangle$ of

$P(i) \subseteq P(j)$ is not the submodule of $\sum_{i \lhd l} P(l)$. $\square$

Proof of the theorem. Let $\mathfrak{F} : 0 = D(r+1) \subset D(r) \subset \cdots \subset D(1) = P(j)$ be Δ-good, then $\{D(l)/D(l+1) \mid 1 \leq l \leq r\} \overset{1.2}{\longleftrightarrow} \{\Delta(i) \mid i \in \Lambda^{(j)}\}$. Let $\{i_1, \ldots, i_m\} = \Lambda^{(j)} \cap \Lambda^{(k)}$ such that $\widetilde{\mathfrak{F}} : 0 \subseteq D(i_m+1) \subset D(i_m) \subseteq \cdots \subseteq D(i_2+1) \subset D(i_2) \subseteq D(i_1+1) \subset D(i_1) \subseteq P(j)$ is a subfiltration of $\mathfrak{F}$ with $D(i_t)/D(i_t+1) \cong \Delta(i_t)$ for $1 \leq t \leq m$. By Lemma 3.3 (2) the filtration $\widetilde{\mathfrak{F}}$ can be refined to

$$\begin{aligned} 0 \ &\subseteq D(i_m+1) \subset \langle p(j,i_m,k)\rangle + D(i_m+1) \subseteq D(i_m) \subseteq \cdots \\ &\qquad\qquad\vdots \\ &\subseteq D(i_2+1) \subset \langle p(j,i_2,k)\rangle + D(i_2+1) \subseteq D(i_2) \\ &\subseteq D(i_1+1) \subset \langle p(j,i_1,k)\rangle + D(i_1+1) \subseteq D(i_1) \subseteq P(j) \end{aligned}$$

Therefore $p(j,i_1,k), \ldots, p(j,i_m,k)$ are linear independent in $P(j)_k$. Since $m = \left|\Lambda^{(j)} \cap \Lambda^{(k)}\right| \overset{2.1(2)}{=} \dim_K P(j)_k$, the set $\left\{p(j,i,k) \mid i \in \Lambda^{(j)} \cap \Lambda^{(k)}\right\}$ is a K-basis of $P(j)_k$.

Because $\bigcup_{k \in Q_0} \left\{p(j,i,k) \mid i \in \Lambda^{(j)} \cap \Lambda^{(k)}\right\} = \left\{p(j,i,k) \mid i \in \Lambda^{(j)},\ k \in \Lambda_{(i)}\right\}$, the set $\mathfrak{B}_j$ is a K-basis of $P(j)$. $\square$

3.4 Remark. Let $j \in Q_0$ and $i, l \in \Lambda^{(j)}$ with $l \in \Lambda^{(i)}$, then $p(j,l,k) \overset{3.1(b)}{\in} P(l) \overset{2.2}{\subseteq} P(i) \overset{2.2}{\subseteq} P(j)$ for all $k \in \Lambda_{(l)}$. We obtain that the set

$$\mathfrak{B}_j(i) := \left\{p(j,l,k) \mid l \in \Lambda^{(i)},\ k \in \Lambda_{(l)}\right\} \ \text{ is a } K\text{-basis of the submodule } P(i) \text{ of } P(j),$$

since $\dim_K P(i) \overset{2.1}{=} \sum_{l \in \Lambda^{(i)}} \left|\Lambda_{(l)}\right| = |\mathfrak{B}_j(i)|$ and $\mathfrak{B}_j(i)$ is a subset of $\mathfrak{B}_j$ defined in 3.2. It is easy to check that for all subsets Γ_1, Γ_2 of $\Lambda^{(j)}$ and $\Gamma_{1,2} := \left(\bigcup_{i \in \Gamma_1} \Lambda^{(i)}\right) \cap \left(\bigcup_{i \in \Gamma_2} \Lambda^{(i)}\right)$ the set $\left(\bigcup_{i \in \Gamma_1} \mathfrak{B}_j(i)\right) \cap \left(\bigcup_{i \in \Gamma_2} \mathfrak{B}_j(i)\right) = \bigcup_{i \in \Gamma_{1,2}} \mathfrak{B}_j(i)$ is a K-basis of the submodule

$$\left(\sum_{i \in \Gamma_1} P(i)\right) \cap \left(\sum_{i \in \Gamma_2} P(i)\right) = \sum_{i \in \Gamma_{1,2}} P(i) \quad \text{of} \quad P(j).$$

4. Good filtrations

In this section, we show the relationship between the Jordan-Hölder filtrations of $\nabla(j)$ and Δ-good filtrations of $P(j)$ resp. the Jordan-Hölder filtrations of $\Delta(j)$ and ∇-good filtrations of $I(j)$ over a 1-quasi-hereditary algebra $(A, \leqslant)$. The sets of these Jordan-Hölder filtrations resp. good filtrations are finite and related to certain sequences of elements from $\Lambda_{(j)}$ resp. $\Lambda^{(j)}$ which depend on $\leqslant$.

For any $i \in \Lambda_{(j)}$ we can consider the standard module $\Delta(i)$ as a submodule of $\Delta(j)$ and $\nabla(i)$ as a factor module of $\nabla(n)$ (see 1.2(4)). We denote by $\mathfrak{K}(j)$ the kernel of the map $\nabla(n) \twoheadrightarrow \nabla(j)$. We have $\mathfrak{K}(j) \subseteq \mathfrak{K}(i)$ if and only if $i \in \Lambda_{(j)}$ (see 2.6).

4.1 Proposition. *Let $A = (KQ/\mathcal{I}, \leqslant)$ be 1-quasi-hereditary, $j \in Q_0$, $r = \left|\Lambda_{(j)}\right|$ and $\mathcal{T}(j) := \left\{\boldsymbol{i} = (i_1, i_2, \ldots, i_r) \mid i_m \in \Lambda_{(j)},\ i_k \ngeqslant i_t,\ 1 \leq k < t \leq r\right\}$. Then the following functions are bijective:*

(1) $\mathcal{S} : \mathcal{T}(j) \longrightarrow \{$*Jordan-Hölder-filtrations of* $\Delta(j)\}$ *with*

$$\mathcal{S}(\boldsymbol{i}):\ 0 = J(0) \subset J(1) \subset \cdots \subset J(t) \subset \cdots \subset J(r) \ \text{ such that } \ J(t) := \sum_{m=1}^{t} \Delta(i_m).$$

Moreover, $J(t)/J(t-1) \cong S(i_t)$ *for* $1 \leq t \leq r$.

(2) $\widetilde{\mathcal{S}} : \mathcal{T}(j) \longrightarrow \{$*Jordan-Hölder-filtrations of* $\nabla(j)\}$ *with*

$$\widetilde{\mathcal{S}}(\boldsymbol{i}):\ \mathfrak{J}(r) \subset \cdots \subset \mathfrak{J}(t) \subset \cdots \subset \mathfrak{J}(1) \subset \mathfrak{J}(0) = \nabla(j) \ \text{ such that } \ \mathfrak{J}(t) := \left(\bigcap_{m=1}^{t} \mathfrak{K}(i_m)\right)/\mathfrak{K}(j).$$

Moreover, $\mathfrak{J}(t-1)/\mathfrak{J}(t) \cong S(i_t)$ *for* $1 \leq t \leq r$.

Proof. (1) By definition of $\mathcal{T}(j)$ for $\boldsymbol{i} = (i_1, \ldots, i_t, \ldots, i_r) \in \mathcal{T}(j)$ we have $\Lambda_{(i_t)} \setminus \{i_t\} \subseteq \{i_1, \ldots, i_{t-1}\}$, thus $\operatorname{rad}\Delta(i_t) \overset{2.5}{=} \sum_{l<i_t} \Delta(l) \subseteq J(t-1)$ for all $1 \leq t \leq r$. For $l \in \{i_1, \ldots, i_{t-1}\}$ we have $[\Delta(l) : S(i_t)] \overset{1.2}{=} 0$, since $l \not\geqslant i_t$. Because $[\Delta(i_t) : S(i_t)] = 1$, we have $\Delta(i_t) \not\subseteq J(t-1)$. Hence $J(t)/J(t-1) \cong \Delta(i_t)/\left(\Delta(i_t) \cap J(t-1)\right) = \Delta(i_t)/\operatorname{rad}\Delta(i_t) \cong S(i_t)$ for every $1 \leq t \leq r$. The function $\mathcal{S}$ is well defined and injective.

Let $\mathcal{F}:\ 0 = M(0) \subset M(1) \subset \cdots \subset M(r') = \Delta(j)$ be a Jordan-Hölder-filtration of $\Delta(j)$ with $M(t)/M(t-1) \cong S(i_t)$ for all $1 \leq t \leq r'$. Then $i_t \in \Lambda_{(j)}$ and $r' \overset{1.2}{=} \left|\Lambda_{(j)}\right| = r$. There exists $\Lambda(t) \subseteq \Lambda_{(j)}$ with $M(t) \overset{2.5}{=} \sum_{i\in\Lambda(t)} \Delta(i)$ for any $1 \leq t \leq r$. By induction on t we can show $\Lambda(t) = \{i_1, \ldots, i_t\}$ with $i_k \not\geqslant i_v$ for $1 \leq k < v \leq t$: Let $t = 1$, then $\Delta(i_1) = \Delta(1) \overset{2.6}{=} \operatorname{soc}\Delta(j)$. Since $M(t+1)/M(t) \cong S(i_{t+1})$, we obtain $\Delta(i_{t+1}) \subseteq M(t+1)$ and $\Delta(i_{t+1}) \not\subseteq M(t)$, because $\operatorname{Loc}_{i_{t+1}}(\Delta(j)) \overset{2.5}{=} \{\Delta(i_{t+1})\}$. Thus $M(t+1) = M(t) + \Delta(i_{t+1})$ and $l \not\geqslant i_{t+1}$ for all $l \in \{i_1, \ldots, i_t\}$. This implies $\mathcal{F} = \mathcal{S}(i_1, \ldots, i_r)$, i.e. the function $\mathcal{S}$ is surjective.

(2) Since A^{op} is also 1-quasi-hereditary (see 1.3), by duality the function $\widetilde{\mathcal{S}}$ is bijective. □

In a similar way, we can determine all Δ-good filtrations of $P(j)$, resp. ∇-good filtrations of $I(j)$, for every $j \in Q_0$. For any $i \in \Lambda^{(j)}$ we continue denotind by $P(i)$ the projective submodule of $P(j)$ with top isomorphic to $S(i)$ and by $\mathcal{K}(j)$ we denote the kernel of the map $P(1) \twoheadrightarrow I(j)$ (see 2.6). Obviously, it is $\mathcal{K}(j) \subseteq \mathcal{K}(i)$ if and only if $i \in \Lambda^{(j)}$.

4.2 Proposition. *Let* $A = (KQ/\mathcal{I}, \leqslant)$ *be a 1-quasi-hereditary algebra,* $j \in Q_0$, $r = \left|\Lambda^{(j)}\right|$ *and* $\mathcal{L}(j) := \left\{\boldsymbol{i} = (i_1, i_2, \ldots, i_r) \mid i_m \in \Lambda^{(j)},\ i_k \not\geqslant i_t,\ 1 \leq k < t \leq r\right\}$. *Then the following functions are bijective:*

(1) $\mathscr{D} : \mathcal{L}(j) \longrightarrow \{\Delta$*-good filtrations of* $P(j)\}$ *with*

$$\mathscr{D}(\boldsymbol{i}):\ 0 = D(r+1) \subset D(r) \subset \cdots \subset D(t) \subset \cdots \subset D(1) \ \text{ such that } \ D(t) := \sum_{m=t}^{r} P(i_m).$$

Moreover, $D(t)/D(t+1) \cong \Delta(i_t)$ *for every* $1 \leq t \leq r$.

(2) $\mathscr{N} : \mathcal{L}(j) \longrightarrow \{\nabla$*-good filtrations of* $I(j)\}$ *with*

$$\mathscr{N}(\boldsymbol{i}):\ N(1) \subset \cdots \subset N(t) \subset \cdots \subset N(r) \subset I(j) \ \text{with}\ N(t) := \left(\bigcap_{m=t}^{r} \mathcal{K}(i_m)\right)/\mathcal{K}(j).$$

Moreover, $N(t+1)/N(t) \cong \nabla(i_t)$ *for every* $1 \leq t \leq r$.

Proof. *(1)* By definition of $\mathcal{L}(j)$ for $\boldsymbol{i} = (i_1, \ldots, i_t, \ldots, i_r) \in \mathcal{T}(j)$ we have $\Lambda^{(i_t)} \setminus \{i_t\} \subseteq \{i_{t+1}, \ldots, i_r\}$ for any $1 \leq t \leq r$. We obtain $\mathfrak{B}_j(i_t) \cap \left(\bigcup_{m=t+1}^{r} \mathfrak{B}_j(i_m)\right) = \bigcup_{i_t < i} \mathfrak{B}_j(i)$, using the notations from 3.4. Therefore $P(i_t) \cap D(t+1) = \sum_{i_t < i} P(i) = \sum_{i_t \lhd i} P(i)$ and consequently $D(t)/D(t+1) \cong P(i_t)/\left(\sum_{i_t \lhd i} P(i)\right) \overset{2.4}{=} \Delta(i_t)$. The filtration $\mathscr{D}(\boldsymbol{i})$ is Δ-good and $\mathscr{D}$ is injective.

Let $\mathcal{F} : 0 \subset D(r') \subset \cdots \subset D(1) = P(j)$ be a Δ-good filtration, with $D(t)/D(t+1) \cong \Delta(i_t)$, then $r' = r = \left|\Lambda^{(j)}\right|$ and $D(t) = \sum_{m=t}^{r} P(i_m)$ (see Lemma 3.3). The inclusion $D(t) \subset D(k)$ implies $P(i_k) \nsubseteq P(i_t)$ for $k < t$. Hence $i_k \not\geqslant i_t$ for all $1 \leq k < t \leq r$ (see 2.2)

(2) follows from the properies of the standard duality $\mathcal{D}$. □

The definitions of $\mathcal{T}(j)$ and $\mathcal{L}(j)$ yields $\mathcal{T}(n) = \mathcal{L}(1)$. Comparing the compositions factors of the filtrations corresponding to $\boldsymbol{i} \in \mathcal{T}(n)$, we obtain that the Jordan-Hölder filtration $\widetilde{\mathcal{S}}(\boldsymbol{i})$ of $\nabla(n)$ induces the Δ-good filtration $\mathscr{D}(\boldsymbol{i})$ of $P(1)$. Thus all Δ-good filtrations of $P(1)$ can be represented in a diagram whose shape coincides with the submodule diagram of $\nabla(n)$. Moreover, any sequence from $\mathcal{L}(j)$ can by completed to a sequence of $\mathcal{L}(1)$, thus all Δ-good filtrations of $P(j)$ are part of this diagram for every $i \in Q_0$. Analogously, the submodule diagram of $\Delta(n)$ and the diagram of all ∇-good filtrations of $I(1)$ has the same form (for the illustration of this see Example 4 of [9]): For $\boldsymbol{i} = (i_1, \ldots, i_m, \ldots, i_n) \in \mathcal{T}(n)$ we have the following relationship between the Jordan-Hölder filtrations of $\Delta(j)$ and $\Delta(n)$ (resp. $\nabla(j)$ and $\nabla(n)$) as well as Δ-good filtrations of $P(j)$ and $P(1)$ (resp. ∇-good filtrations of $I(j)$ and $I(1)$). The factors of the filtrations $\mathcal{S}(\boldsymbol{i})$ and $\mathscr{N}(\boldsymbol{i})$ (resp. $\widetilde{\mathcal{S}}(\boldsymbol{i})$ and $\mathscr{D}(\boldsymbol{i})$) are labeled by the same vertices (as indicated above the corresponding filtration):

$$\mathcal{S}(\boldsymbol{i}): \quad \underbrace{\overbrace{0 \subset J(1)}^{S(1)} \subset \cdots \subset \overbrace{J(t-1) \subset J(t)}^{S(i_t)}}_{\text{filtration of } \Delta(j)} \subset \cdots \subset \overbrace{J(n-1) \subset J(n)}^{S(n)} = \Delta(n) \qquad \text{for } (i_1, \ldots, i_t) \in \mathcal{T}(j),$$

$$\widetilde{\mathcal{S}}(\boldsymbol{i}): \quad \overbrace{\mathfrak{J}(n) \subset \mathfrak{J}(n-1)}^{S(n)} \subset \cdots \subset \underbrace{\overbrace{\mathfrak{J}(t) \subset \mathfrak{J}(t-1)}^{S(i_t)} \subset \cdots \subset \overbrace{\mathfrak{J}(1) \subset \mathfrak{J}(0)}^{S(1)}}_{\text{filtration of } \nabla(j)} = \nabla(n) \qquad \text{for } (i_1, \ldots, i_t) \in \mathcal{T}(j),$$

$$\mathscr{D}(\boldsymbol{i}): \quad \underbrace{\overbrace{0 \subset D(n)}^{\Delta(n)} \subset \cdots \subset \overbrace{D(t+1) \subset D(t)}^{\Delta(i_t)}}_{\Delta\text{-good filtration of } P(j)} \subset \cdots \subset \overbrace{D(2) \subset D(1)}^{\Delta(1)} = P(1) \qquad \text{for } (i_t, \ldots, i_n) \in \mathcal{L}(j),$$

$$\mathscr{N}(\boldsymbol{i}) \quad \overbrace{N(1) \subset N(2)}^{\nabla(1)} \subset \cdots \subset \underbrace{\overbrace{N(t) \subset N(t+1)}^{\nabla(i_t)} \subset \cdots \subset \overbrace{N(n) \subset N(n+1)}^{\nabla(n)}}_{\nabla\text{-good filtration of } I(j)} = I(1) \qquad \text{for } (i_t, \ldots, i_n) \in \mathcal{L}(j),$$

Let $\Lambda \subseteq Q_0$ and $\check{\Lambda} := \bigcup_{i \in \Lambda} \Lambda^{(i)}$. We can always construct a sequence $(i_1, \ldots, i_t, \ldots, i_n) \in \mathcal{L}(1)$ with $\{i_t, \ldots, i_n\} = \check{\Lambda}$. For any $k \in \check{\Lambda}$ there exists an $i \in \Lambda$ with $i \leqslant k$, thus $P(k) \subseteq P(i)$ and consequently $\sum_{l \in \Lambda} P(l) = \sum_{l \in \check{\Lambda}} P(j)$.

4.3 Corollary. *Let Λ_1 and Λ_2 be some subsets of Q_0 with $\check{\Lambda}_2 \subset \check{\Lambda}_1$. Then for the submodules $M_1 := \sum\limits_{l \in \Lambda_1} P(l)$ and $M_2 := \sum\limits_{l \in \Lambda_2} P(l)$ of $P(1)$, it is $M_2 \subset M_1$ and $M_1/M_2 \in \mathfrak{F}(\Delta)$ (resp. for $N_1 := \bigcap\limits_{l \in \Lambda_1} \mathcal{K}(l)$ and $N_2 := \bigcap\limits_{l \in \Lambda_2} \mathcal{K}(l)$ we have $N_1 \subset N_2$ and $N_2/N_1 \in \mathfrak{F}(\nabla)$) with*

$$(M_1/M_2 : \Delta(k)) = \begin{cases} 1 & \text{if } k \in \check{\Lambda}_1 \backslash \check{\Lambda}_2, \\ 0 & \text{else} \end{cases} \quad \text{and} \quad (N_2/N_1 : \nabla(k)) = \begin{cases} 1 & \text{if } k \in \check{\Lambda}_1 \backslash \check{\Lambda}_2, \\ 0 & \text{else.} \end{cases}$$

Proof. We can construct a sequence $\boldsymbol{i} = (i_1, \ldots, i_{t_1}, \ldots, i_{t_2}, \ldots, i_n) \in \mathcal{L}(1)$ such that $\{i_{t_v}, \ldots, i_n\} = \check{\Lambda}_v$ for $v = 1, 2$. In the Δ-good filtration $\mathscr{D}(\boldsymbol{i})$ of $P(1)$ we have $D(t_v) = \sum_{m=t_v}^{n} P(i_l) = M_v$ and a Δ-good filtrations of M_v for $v = 1, 2$

$$\mathscr{D}(\boldsymbol{i}) : \underbrace{\underbrace{0 \subset D(n) \subset \cdots \subset D(t_2)}_{\Delta\text{-good filtration of } M_2} \subset \cdots \subset D(t_1)}_{\Delta\text{-good filtration of } M_1} \subset \cdots \subset D(1) = P(1).$$

Since $\check{\Lambda}_2 \subset \check{\Lambda}_1$, we have $M_2 \subset M_1$ and $M_1/M_2 \in \mathfrak{F}(\Delta)$ because the induced filtration $D(t_2)/M_2 \subset D(t_2 - 1)/M_2 \subset \cdots \subset D(t_1)/M_2$ is Δ-good. The properties of the filtration $\mathscr{D}(\boldsymbol{i})$ implies $(M_v : \Delta(l)) = 1$ for all $l \in \check{\Lambda}_v$ and $(M_v : \Delta(l)) = 0$ for all $l \in Q_0 \backslash \check{\Lambda}_v$, here $v = 1, 2$. Thus $(M_1/M_2 : \Delta(k)) = (M_1 : \Delta(k)) - (M_2 : \Delta(k))$ implies the statement.

The dual statement follows by dual argumentation. □

For every quasi-hereditary algebra $\mathcal{A}$, the category $\mathfrak{F}(\Delta)$ is a *resolving* subcategory of $\operatorname{mod} \mathcal{A}$ (resp. $\mathfrak{F}(\nabla)$ is a *coresolving* subcategory of $\operatorname{mod} \mathcal{A}$), i.e. the category $\mathfrak{F}(\Delta)$ is closed under extensions, kernels of surjective maps and it contains all projective $\mathcal{A}$-modules (resp. $\mathfrak{F}(\nabla)$ is closed under extensions, cokernels of injective maps and contains all injective $\mathcal{A}$-modules) (see [10, Theorem 3 (resp. Theorem 3*)]).

Using this fact, when dealing with 1-quasi-hereditary algebras we can determine all local modules in $\mathfrak{F}(\Delta)$ resp. colocal modules in $\mathfrak{F}(\nabla)$.

4.4 Corollary. *Let $A = (KQ/\mathcal{I}, \leqslant)$ be a 1-quasi-hereditary algebra, $j \in Q_0$ and M, N be A-modules with* $\operatorname{top} M \cong S(j)$, $\operatorname{soc} N \cong S(j)$. *Then*

(1) $M \in \mathfrak{F}(\Delta)$ if and only if $M \cong P(j)/\left(\sum_{i \in \Lambda} P(i)\right)$ for some $\Lambda \subseteq \Lambda^{(j)} \backslash \{j\}$.

(2) $N \in \mathfrak{F}(\nabla)$ if and only if $N \cong \left(\bigcap_{i \in \Lambda} \mathcal{K}(i)\right)/\mathcal{K}(j)$ for some $\Lambda \subseteq \Lambda^{(j)} \backslash \{j\}$.

Proof. *(1)* The filtration $0 \subsetneq \ker(P(j) \twoheadrightarrow M) \subset P(j)$ can be refined to a Δ-good filtration $\mathscr{D}(\boldsymbol{i})$ for some $\boldsymbol{i} \in \mathcal{L}(j)$, thus $\ker(P(j) \twoheadrightarrow M) = \sum_{i \in \Lambda} P(i)$ for some $\Lambda \subseteq \Lambda^{(j)}$. Since $M \neq 0$, we have $j \notin \Lambda$. The other direction follows from Corollary 4.3.

(2) is the dual statement of *(1)*. □

4.5 Remark. If for all $j \in Q_0$ and all $i \in \Lambda^{(j)}$, and any two paths p, q of the form $p(j, i, i)$ it is $p = q$ and any two paths p', q' of the form $p(i, i, j)$ it is $p' = q'$, then the algebra $B(A) = KQ_{B(A)}/I_{B(A)}$ given by the quiver $Q_{B(A)} = (Q_0, \{(j \to i) \in Q_1 \mid j < i\})$ with all commutativity relations and the partially ordered set $(Q_0, \leqslant)$ is a (strong) exact Borel subalgebra of A and $C(A) := B(A)^{op}$ is a Δ-subalgebra of A in the sense of König (see [K]). The structure of the A-module $\Delta(j)$ corresponds to the structure of $P_{C(A)}(j)$ (this also holds for $\nabla(j)$ and $I_{B(A)}(j)$). In this case 4.2 is a consequence of [8, Proposition 2.5] and 4.1.

All known 1-quasi-hereditary algebras have exact Borel and Δ-subalgebras. We conjecture that this is in general the case.

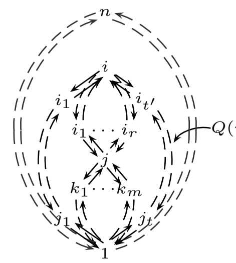
5. The characteristic tilting module

For any quasi-hereditary basic algebra $(\mathcal{A}, \leqslant)$ the full subcategory $\mathfrak{F}(\Delta) \cap \mathfrak{F}(\nabla)$ of $\operatorname{mod} \mathcal{A}$ consisting of all $\mathcal{A}$-modules which are Δ-good and ∇-good is determined by the so called *characteristic tilting module* $T_{\mathcal{A}}$ of A defined by Ringel in [10]: For any $i \in Q_0$ there exists an (up to isomorphism) uniquely determined indecomposable $\mathcal{A}$-module $T_{\mathcal{A}}(i)$ in $\mathfrak{F}(\Delta) \cap \mathfrak{F}(\nabla)$ with the following properties: For $j \nleqslant i$ it is $(T_{\mathcal{A}}(i) : \Delta(j)) = (T_{\mathcal{A}}(i) : \nabla(j)) = [T_{\mathcal{A}}(i) : S(j)] = 0$ and $(T_{\mathcal{A}}(i) : \Delta(i)) = (T_{\mathcal{A}}(i) : \nabla(i)) = [T_{\mathcal{A}}(i) : S(i)] = 1$. Moreover, there exists a submodule $Y_{\mathcal{A}}(i) \in \mathfrak{F}(\nabla)$ of $T_{\mathcal{A}}(i)$ with $T_{\mathcal{A}}(i)/Y_{\mathcal{A}}(i) \cong \nabla(i)$ (resp. a factor module $X_{\mathcal{A}}(i) \in \mathfrak{F}(\Delta)$ with $\ker(T_{\mathcal{A}}(i) \twoheadrightarrow X_{\mathcal{A}}(i)) \cong \Delta(i)$). The A-module $T_{\mathcal{A}}$ is isomorphic to $\bigoplus_{i \in Q_0} T_{\mathcal{A}}(i)$. Moreover, any module in $\mathfrak{F}(\Delta) \cap \mathfrak{F}(\nabla)$ is a direct sum of some copies of $T_{\mathcal{A}}(i)$.

We recall the notations and properties of some factor algebra of a quasi-hereditary algebra $\mathcal{A} = (KQ/\mathcal{I}, \leqslant)$, which will be used later: Let Λ be some saturated subset of Q_0 (i.e. if $v \in \Lambda$ and $k \in Q_0$ then $k < v$ implies $k \in \Lambda$), by $J(\Lambda)$ we denote the ideal $\mathcal{A}(\sum_{i \in Q_0 \setminus \Lambda} e_i)\mathcal{A}$ of $\mathcal{A}$. For the quiver $Q(\Lambda)$ of the factor algebra $\mathcal{A}(\Lambda) := \mathcal{A}/J(\Lambda)$ we have $Q_0(\Lambda) = \Lambda$ and $Q_1(\Lambda) = \{(i \to j) \in Q_1 \mid i, j \in \Lambda\}$. All paths $p = (k_1 \to k_2 \to \cdots \to k_m)$ in $\mathcal{A}$ with $k_t \notin \Lambda$ for some $1 \leq t \leq m$ span $J(\Lambda)$ as a K-space. Moreover, all $\mathcal{A}(\Lambda)$-modules can be considered as the $\mathcal{A}$-modules M with $[M : S(i)] = 0$ for all $i \in Q_0 \setminus \Lambda$. The projective $\mathcal{A}(\Lambda)$-module $P_{\mathcal{A}(\Lambda)}(i)$ is isomorphic to the $\mathcal{A}$-module $P(i)/J(\Lambda)P(i)$ for every $i \in Q_0(\Lambda)$. In particular, the algebra $(\mathcal{A}(\Lambda), \leqslant)$ is quasi-hereditary with $\Delta(i) \cong \Delta_{\mathcal{A}(\Lambda)}(i)$ and $\nabla(i) \cong \nabla_{\mathcal{A}(\Lambda)}(i)$ for all $i \in \Lambda$ (see [6]). We have $\mathfrak{F}(\Delta_{\mathcal{A}(\Lambda)}) \subseteq \mathfrak{F}(\Delta)$ (resp. $\mathfrak{F}(\nabla_{\mathcal{A}(\Lambda)}) \subseteq \mathfrak{F}(\nabla)$) and $T_{\mathcal{A}(\Lambda)}$ is a direct summand of $T_{\mathcal{A}}$ (more precisely $T_{\mathcal{A}(\Lambda)}(i) \cong T_{\mathcal{A}}(i)$).

Let $A = (KQ/\mathcal{I}, \leqslant)$ be a 1-quasi-hereditary algebra with $1 \leqslant i \leqslant n$ for all $i \in Q_0$. Since $P(1) \overset{2.1}{\cong} I(1)$ admits Δ-good and ∇-good filtrations with $X(n) = P(1)/P(n) \in \mathfrak{F}(\Delta)$ and $Y(n) = \ker(P(1) \twoheadrightarrow I(n)) \in \mathfrak{F}(\nabla)$ (see 4.4), we have $P(1) \cong T(n)$.

We fix $i \in Q_0$. The factor algebra $A(i)$ of A is defined as follows:



$$A(i) := A/J(i) \quad \text{where} \quad J(i) := A\left(\sum_{j \in Q_0 \setminus \Lambda_{(i)}} e_j\right)A.$$

For the quiver $Q(i)$ of $A(i)$ we have $Q_0(i) := \Lambda_{(i)}$ and $Q_1(i) := \{(j \to k) \in Q_1 \mid j, k \in \Lambda_{(i)}\}$.

5.1 Theorem. *Let $A = (KQ/\mathcal{I}, \leqslant)$ be a 1-quasi-hereditary algebra and $i \in Q_0$. The following statements are equivalent:*

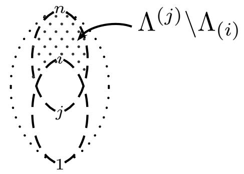
(i) *$A(i)$ is 1-quasi-hereditary,*

(ii) $T(i) \cong P(1)/\left(\sum_{l \in Q_0 \setminus \Lambda_{(i)}} P(l)\right)$, *(ii')* $T(i) \cong \bigcap_{l \in Q_0 \setminus \Lambda_{(i)}} \ker(I(1) \twoheadrightarrow I(l))$,

(iii) $\operatorname{soc} T(i)$ *is simple,* *(iii')* $\operatorname{top} T(i)$ *is simple.*

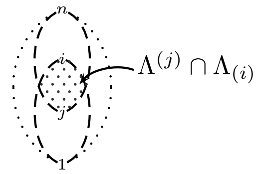
The subset $\Lambda_{(i)}$ of Q_0 is saturated, thus $(A(i), \leqslant)$ is a quasi-hereditary algebra. The proof of this theorem is based on some properties of projective $A(i)$-modules, which we consider in the next lemma. For $A(i)$-modules resp. paths we use the index (i). It should be noted that for any $l \in \Lambda_{(i)}$ a path $p(j,l,k)$ runs through some vertices from $\Lambda_{(i)}$ (see Sec.3).

5.2 Lemma. *Let $i \in Q_0$ and $(A(i), \leqslant)$ be defined as above. Then the following statements hold for any $j \in Q_0(i)$.*



(a) $P_{(i)}(j) \cong P(j) / \left(\sum_{l \in \Lambda^{(j)} \backslash \Lambda_{(i)}} P(l) \right)$ *and*

$\{p_{(i)}(j,l,k) \mid l \in \Lambda^{(j)} \cap \Lambda_{(i)},\ k \in \Lambda_{(l)}\}$ *is a K-basis of $P_{(i)}(j)$,*



(b) $P_{(i)}(j) \hookrightarrow P_{(i)}(1),\ \ I_{(i)}(1) \twoheadrightarrow I_{(i)}(j),$

(c) $\big(P_{(i)}(j) : \Delta_{(i)}(k)\big) = [\Delta_{(i)}(k) : S_{(i)}(j)] = 1$ *for all $k \in \Lambda^{(j)} \cap \Lambda_{(i)}$.*

Proof.(a) Since $P_{(i)}(j) \cong P(j)/(J(i)P(j))$, it is enough to show $J(i)P(j) = \sum_{l \in \Lambda^{(j)} \backslash \Lambda_{(i)}} P(l)$. The set $\underbrace{\{p(j,l,k) \mid l \in \Lambda^{(j)} \backslash \Lambda_{(i)},\ k \in \Lambda_{(l)}\}}_{\mathbf{B}_1 :=} \cup \underbrace{\{p(j,l,k) \mid l \in \Lambda^{(j)} \cap \Lambda_{(i)},\ k \in \Lambda_{(l)}\}}_{\mathbf{B}_2 :=}$ is a K-basis of $P(j)$ (see 3.2). Any path starting in j and passing through some $l \in Q_0 \backslash \Lambda_{(i)}$ belongs to $\operatorname{span}_K \mathbf{B}_1$. Thus $\mathbf{B}_1 = \bigcup_{l \in \Lambda^{(j)} \backslash \Lambda_{(i)}} \mathfrak{B}_j(l)$ is a K-basis of $J(i)P(j)$ and of the submodule $\sum_{l \in \Lambda^{(j)} \backslash \Lambda_{(i)}} P(l)$ of $P(j)$, in the notation of 3.4. We have $J(i)P(j) = \sum_{l \in \Lambda^{(j)} \backslash \Lambda_{(i)}} P(l)$ and $\{p_{(i)}(j,l,k) \mid p(j,l,k) \in \mathbf{B}_2\} = \{p_{(i)}(j,l,k) \mid l \in \Lambda^{(j)} \cap \Lambda_{(i)},\ k \in \Lambda_{(l)}\}$ is a K-basis of $P_{(i)}(j)$.

(b) We have $P(j) \cap \Big(\sum_{l \in Q_0 \backslash \Lambda_{(i)}} P(l) \Big) = \sum_{l \in \Lambda^{(j)} \backslash \Lambda_{(i)}} P(l)$, according to 3.4 for the subsets $\Gamma_1 = \Lambda^{(j)}$ and $\Gamma_2 = Q_0 \backslash \Lambda_{(i)}$ of $Q_0 = \Lambda^{(1)}$. Thus

$$\underbrace{P(j) / \left(\sum_{l \in \Lambda^{(j)} \backslash \Lambda_{(i)}} P(l) \right)}_{P_{(i)}(j)} \cong \left(P(j) + \sum_{l \in Q_0 \backslash \Lambda_{(i)}} P(l) \right) / \left(\sum_{l \in Q_0 \backslash \Lambda_{(i)}} P(l) \right) \subseteq \underbrace{P(1) / \left(\sum_{l \in Q_0 \backslash \Lambda_{(i)}} P(l) \right)}_{P_{(i)}(1)}.$$

Therefore $P_{(i)}(j)$ can be considered as a submodule of $P_{(i)}(1)$ for any $j \in \Lambda_{(i)}$.

Any projective indecomposable $A(i)^{op}$-module can by embedded in the projective indecomposable $A(i)^{op}$-module corresponding to the minimal vertex 1 because A^{op} is 1-quasi-hereditary and $A^{op}(i) \cong A(i)^{op}$. Using duality, we obtain $I_{(i)}(1) \twoheadrightarrow I_{(i)}(j)$.

(c) Since $\Delta(k) \cong \Delta_{(i)}(k)$, we have $[\Delta_{(i)}(k) : S_{(i)}(j)] \overset{1.2}{=} 1$ for all $k \in \Lambda^{(j)} \cap \Lambda_{(i)}$. For the sets $\Lambda_1 = \Lambda^{(j)} = \check{\Lambda}_1$ and $\Lambda_2 = \Lambda^{(j)} \backslash \Lambda_{(i)} = \check{\Lambda}_2$ (in the notation of 4.3) we have $M_1 = P(j)$ and $M_2 = \sum_{l \in \Lambda^{(j)} \backslash \Lambda_{(i)}} P(l)$. Thus $\big(M_1/M_2 : \Delta_{(i)}(k)\big) = 1$ for all $k \in \check{\Lambda}_1 \backslash \check{\Lambda}_2 = \Lambda^{(j)} \cap \Lambda_{(i)}$. $\square$

For all $j \in Q_0(i)$ it is $1 \leqslant j \leqslant i$ and $\Delta_{(i)}(j) \cong \Delta(j)$ as well as $\nabla_{(i)}(j) \cong \nabla(j)$, thus $\Delta_{(i)}(j) \hookrightarrow \Delta_{(i)}(i)$ and $\nabla_{(i)}(i) \twoheadrightarrow \nabla_{(i)}(j)$ (see 2.6). The foregoing lemma shows that the axioms of a 1-quasi-hereditary algebra are satisfied for $(A(i), \leqslant)$ if and only if if and only if $P_{(i)}(1) \cong I_{(i)}(1)$.

Proof of the theorem. Let $i \in Q_0$. Since $\operatorname{soc} \Delta(j) \overset{2.6}{\cong} S(1)$ for all $i \in Q_0$ and $T(i) \in \mathfrak{F}(\Delta)$, we obtain $\operatorname{soc} T(i) \cong S(1)^m$ for some $m \geq 1$.

$(i) \Rightarrow (ii)$ If $A(i)$ is 1-quasi-hereditary, then $P_{(i)}(1) \cong I_{(i)}(1)$ is isomorphic to $T_{(i)}(i)$, since i is maximal in $Q_0(i)$. The A-modules $T_{(i)}(i)$ and $T(i)$ are isomorphic. Lemma 5.2 *(a)* implies $T(i) \cong P_{(i)}(1) \cong P(1)/\left(\sum_{l\in Q_0\setminus\Lambda_{(i)}} P(l)\right)$.

$(ii) \Rightarrow (iii)$ Since $\operatorname{soc} T(i) \cong S(1)^m \cong \nabla(1)^m$ and $T(i) \in \mathfrak{F}(\nabla)$, the filtration $0 \subset \operatorname{soc} T(i) \subset T(i)$ can be refined to a ∇-good filtration of $T(i)$ since $\mathfrak{F}(\nabla)$ is coresolving. We have $\big(T(i) : \nabla(1)\big) = \big(\operatorname{soc} T(i) : \nabla(1)\big) + \big(T(i)/\operatorname{soc} T(i) : \nabla(1)\big)$. It is enough to show $\big(T(i) : \nabla(1)\big) = 1$, this implies $[\operatorname{soc} T(i) : S(1)] = (\operatorname{soc} T(i) : \nabla(1)) = m = 1$.

Since $T(i) \cong P(1)/\left(\sum_{l\in Q_0\setminus\Lambda_{(i)}} P(l)\right)$, $P(1) \in \mathfrak{F}(\Delta)$, the filtration $0 \subset \sum_{l\in Q_0\setminus\Lambda_{(i)}} P(l) \subset P(1)$ can be refined to a Δ-good filtration $\mathscr{D}(\boldsymbol{i})$ for some $\boldsymbol{i} = (i_1, \ldots, i_t, \ldots, i_n) \in \boldsymbol{\mathcal{L}}(1)$ (see 4.2). There exists $1 \leq t < n$ with $D(t+1) = \sum_{l\in Q_0\setminus\Lambda_{(i)}} P(l)$. Thus $(T(i) : \Delta(j)) = 1$ for $j \in \{i_1, \ldots, i_t\}$ and $(T(i) : \Delta(j)) = 0$ for $j \in \{i_{t-1}, \ldots, i_n\}$. In the notation of 4.3 for $\Lambda_1 = Q_0$ and $\Lambda_2 = Q_0\setminus\Lambda_{(i)}$ we obtain $T(i) \cong M_1/M_2$ and $(T(i) : \Delta(j)) = \begin{cases} 1 & \text{if } j \in \Lambda_{(i)}, \\ 0 & \text{else.} \end{cases}$
Hence $\Lambda_{(i)} = \{i_1, \ldots, i_t\}$. Since $i_k \not\geq i_v$ for $1 \leq k < v \leq t$, we have $(i_1, \ldots, i_t) \in \mathcal{T}(i)$ (see 4.1). Thus $i_1 = 1$ and $i_t = i$.

Let now $\mathscr{N} : 0 = N(r+1) \subset N(r) \subset \cdots \subset N(1) = T(i)$ be a ∇-good filtration with $N(v)/N(v+1) \cong \nabla(j_v)$ for every $1 \leq v \leq r$. We have to show $\{i_1, \ldots, i_t\} \subseteq \{j_1, \ldots, j_r\}$. Then the filtrations $\mathscr{D}(\boldsymbol{i})$ and $\mathscr{N}$ as well as $\dim_K \Delta(j) \overset{2.1}{=} \dim_K \nabla(j)$ implies

$$\dim T(i) = \sum_{j\in\{i_1,\ldots,i_t\}} \dim_K \Delta(j) = \sum_{j\in\{i_1,\ldots,i_t\}} \dim_K \nabla(j) + \underbrace{\sum_{j\in\{j_1,\ldots,j_r\}\setminus\{i_1,\ldots,i_t\}} \dim_K \nabla(j)}_{=0}.$$

In other words, this implies $\{i_1, \ldots, i_t\} = \{j_1, \ldots, j_r\}$ and $t = r$. Consequently, for all $j \in \{i_1, \ldots, i_t\}$ we obtain $(T(i) : \nabla(j)) = 1$ and therefore $(T(i) : \nabla(i_1)) = (T(i) : \nabla(1)) = 1$.

We show this by induction on $t - w$: If $w = 0$, then $i = i_t \in \{j_1, \ldots, j_r\}$, since $(T(i) : \nabla(i)) = 1$ by the properties of $T(i)$. Assume $i_{t-w}, i_{t-(w-1)}, \ldots, i_t \in \{j_1, \ldots, j_r\}$. For the k-th coordinate of the dimension vector of $T(i)$ we have

$$\begin{aligned}
[T(i) : S(k)] &= \sum_{l\in\{i_1,\ldots,i_{t-(w+1)}\}} [\Delta(l) : S(k)] + \sum_{j\in\{i_{t-w},\ldots,i_t\}} [\Delta(j) : S(k)] \quad (\Delta\text{-good filtration } \mathscr{D}(\boldsymbol{i})) \\
&= \sum_{\substack{j\in\{j_1,\ldots,j_r\} \\ j\notin\{i_{t-w},\ldots,i_t\}}} [\nabla(j) : S(k)] + \sum_{j\in\{i_{t-w},\ldots,i_t\}} [\nabla(j) : S(k)] \qquad (\nabla\text{-good filtration } \mathscr{N})
\end{aligned}$$

Let $X(k) := \sum_{l\in\{i_1,\ldots,i_{t-(w+1)}\}}[\Delta(l) : S(k)]$ and $Y(k) := \sum_{\substack{j\in\{j_1,\ldots,j_r\} \\ j\notin\{i_{t-w},\ldots,i_t\}}} [\nabla(j) : S(k)]$ for $k \in Q_0$. Since $[\Delta(j) : S(k)] = [\nabla(j) : S(k)]$ for all $j, k \in Q_0$ (see Sec.1 $(*)$), we obtain $X(k) = Y(k)$ for all $k \in Q_0$. By definition of $\mathcal{T}(i)$ for $(i_1, \ldots, i_{t-(w+1)}, \ldots, i_t) \in \mathcal{T}(i)$ we obtain $i_1, \ldots, i_{t-(w+1)} \notin \Lambda^{(i_{t-(w+1)})}\setminus\left\{i_{t-(w+1)}\right\}$. Thus $X(k) \overset{1.2}{=} 0 = Y(k)$ for all $k \in \Lambda^{(i_{t-(w+1)})}\setminus\left\{i_{t-(w+1)}\right\}$. We obtain $\{j_1, \ldots, j_r\}\setminus\{i_{t-w}, \ldots, i_t\} \not\subseteq \Lambda^{(i_{t-(w+1)})}\setminus\left\{i_{t-(w+1)}\right\}$. Moreover, for $k = i_{t-(w+1)}$ we have $X(k) \neq 0$ since $[\Delta(k) : S(k)] = 1$, therefore $Y(k) \neq 0$. There exists $j \in \{j_1, \ldots, j_r\}\setminus\{i_{t-w}, \ldots, i_t\}$ with $[\nabla(j) : S(i_{t-(w+1)})] = 1$, hence $j \in \Lambda^{(i_{t-(w+1)})}$. Thus $j \notin \Lambda^{(i_{t-(w+1)})}\setminus\left\{i_{t-(w+1)}\right\}$ and $j \in \Lambda^{(i_{t-(w+1)})}$ implies $j = i_{t-(w+1)} \in \{j_1, \ldots, j_r\}$.

$(iii) \Leftrightarrow (ii')$ The socle of $T(i)$ is simple if and only if $T(i)$ is a submodule of $I(1)$. The filtration $0 \subset T(i) \subset I(1)$ can be refined to a ∇-good filtration $\mathscr{N}(\boldsymbol{i})$ for some

$\boldsymbol{i} = (i_1, \ldots, i_t, \ldots, i_n) \in \mathcal{L}(1)$ (see 4.2) There exists $1 \leq t \leq n$ with $T(i) = N(t) = \bigcap_{m=t}^{n} \ker(I(1) \twoheadrightarrow I(i_m))$ and $(T(i) : \nabla(j)) = \begin{cases} 1 & \text{if } j \in \{i_1, \ldots, i_{t-1}\}, \\ 0 & \text{if } j \in \{i_t, \ldots, i_n\}. \end{cases}$ We know that $T(i)$ satisfies $(T(i) : \nabla(j)) = 0$ for all $j \in Q_0 \backslash \Lambda_{(i)}$ and $(T(i) : \nabla(j)) \neq 0$ implies $j \in \Lambda_{(i)}$. Since $i_1, \ldots, i_{t-1} \in \Lambda_{(i)}$ and $i \notin \{i_t, \ldots, i_n\}$, we obtain $\Lambda_{(i)} \cap \{i_t, \ldots, i_n\} = \emptyset$. Therefore $\{i_t, \ldots, i_n\} = Q_0 \backslash \Lambda_{(i)}$.

$(ii') \Rightarrow (iii')$ The dual argumentation of $(ii) \Rightarrow (iii)$.

$(iii') \Leftrightarrow (ii)$ The dual argumentation of $(iii) \Leftrightarrow (ii')$.

$(iii) \Rightarrow (i)$ If $\operatorname{soc} T(i) \cong S(1)$, then $(iii) \Rightarrow (ii') \Rightarrow (iii') \Rightarrow (ii)$ implies $T(i) \cong P(1)/\left(\sum_{l \in Q_0 \backslash \Lambda_{(i)}} P(l)\right) \overset{5.2(a)}{\cong} P_{(i)}(1)$. Since $\operatorname{soc} P_{(i)}(1) \cong S(1)$ and $\dim_K P_{(i)}(1) = \dim_K I_{(i)}(1)$ (see Brauer-Humphreys reciprocity formulas and Lemma 5.2 *(c)*), we obtain $P_{(i)}(1) \cong I_{(i)}(1)$. Therefore the algebra $A(i)$ is 1-quasi-hereditary. □

5.3 Remark. If $i \in Q_0$ is a neighbor of 1 (i.e. $1 \lhd i$), then for the $A(i)$-module $P_{(i)}(1)$ we have $\operatorname{rad} P_{(i)}(1) = P_{(i)}(i) \cong \Delta_{(i)}(i) \cong \Delta(i)$ because $0 \subset P_{(i)}(i) \subset P_{(i)}(1)$ is the uniquely determined Δ-good filtration of $P_{(i)}(1)$. Therefore $\operatorname{soc} P_{(i)}(1) \cong S(1)$ and consequently $A(i)$ is 1-quasi-hereditary. Theorem 5.1 implies that for any 1-quasi-hereditary algebra $A = (KQ/\mathcal{I}, \leqslant)$ with $1 \leqslant i \leqslant n$ it is:

- $T(1) \cong \Delta(1) \cong \nabla(1) \cong S(1)$,
- $T(n) \cong P(1) \cong I(1)$,
- $T(i) \cong P(1)/\left(\sum_{j \in Q_0 \backslash \{1,i\}} P(j)\right) \cong \bigcap_{j \in Q_0 \backslash \{1,i\}} \ker(P(1) \twoheadrightarrow I(i))$ for any $i \in Q_0$ with $1 \lhd i$.

An example of a 1-quasi-hereditary algebra A such that for some $i \in Q_0(A)$ the algebra $A(i)$ is not 1-quasi-hereditary can be found in [9].

6. The Ringel dual of a 1-quasi-hereditary algebra

The concept of Ringel duality is specific to the theory of quasi-hereditary algebras (see [10]): For any quasi-hereditary (basic) algebra $\mathcal{A}$ the endomorphism algebra of the characteristic tilting $\mathcal{A}$-module $T_{\mathcal{A}}$ is called the *Ringel dual* of $\mathcal{A}$, denoted by $R(\mathcal{A})$ [i.e. $R(\mathcal{A}) = \operatorname{End}_{\mathcal{A}}(T_{\mathcal{A}})^{op}$]. Since the direct summands of $T_{\mathcal{A}}$ are pairwise non isomorphic, $R(\mathcal{A})$ is a basic algebra. The vertices in the quiver $Q(R(\mathcal{A}))$ may be identified with the vertices of $Q(\mathcal{A})$ [$T(i) \leftrightsquigarrow i$]. The algebra $R(\mathcal{A})$ is quasi-hereditary with respect to the opposite order on $Q_0(\mathcal{A})$. Furthermore, $R(R(\mathcal{A}))$ and $\mathcal{A}$ are isomorphic as quasi-hereditary algebras. The functor $\mathscr{R}_{(\mathcal{A})}(-) := \operatorname{Hom}_{\mathcal{A}}(T_{\mathcal{A}}, -) : \operatorname{mod} \mathcal{A} \to \operatorname{mod} R(\mathcal{A})$ induces an equivalence between $\mathfrak{F}_{\mathcal{A}}(\nabla)$ and $\mathfrak{F}_{R(\mathcal{A})}(\Delta)$ and for any $i \in Q_0(\mathcal{A})$ hold

$$\mathscr{R}_{(\mathcal{A})}(\nabla_{\mathcal{A}}(i)) = \Delta_{R(\mathcal{A})}(i), \quad \mathscr{R}_{(\mathcal{A})}(T_{\mathcal{A}}(i)) = P_{R(\mathcal{A})}(i), \quad \mathscr{R}_{(\mathcal{A})}(I_{\mathcal{A}}(i)) = T_{R(\mathcal{A})}(i).$$

Applying $\mathscr{R}_{(\mathcal{A})}(-)$ to an exact sequence $0 \to M' \to M \to M'' \to 0$ in $\operatorname{mod} \mathcal{A}$ with $M', M, M'' \in \mathfrak{F}(\nabla)$ yields an exact sequence in $R(\mathcal{A})$ and $(M : \nabla(i)) = \left(\mathscr{R}_{(\mathcal{A})}(M) : \Delta_{R(\mathcal{A})}(i)\right)$ for all $i \in Q_0(\mathcal{A})$.

The next theorem shows that the class of 1-quasi-hereditary algebras is not closed under Ringel-duality (ct. Theorem B).

6.1 Theorem. *Let $A = (KQ/\mathcal{I}, \leqslant)$ be a 1-quasi-hereditary algebra with $1 \leqslant i \leqslant n$, then*

$$R(A) \text{ is 1-quasi-hereditary if and only if } T(i) = P(1)/\left(\sum_{l\in Q_0\setminus\Lambda_{(i)}} P(l)\right) \text{ for any } i \in Q_0.$$

Note that the Ringel dual of a 1-quasi-hereditary algebra is 1-quasi-hereditary if and only if the equivalent conditions of Theorem 5.1 are satisfied.

We now consider some properties of $R(A)$ for a 1-quasi-hereditary algebra A. The vertices in Q_0 and $Q_0(R)$ will be identified. By $\leqslant_{(R)}$ we denote the partial order on $Q_0(R)$, it means $i \leqslant j$ if and only if $j \leqslant_{(R)} i$. Obviously, $\{1\} = \max\left\{\left(Q_0(R), \leqslant_{(R)}\right)\right\}$ and $\{n\} = \min\left\{\left(Q_0(R), \leqslant_{(R)}\right)\right\}$. For the $R(A)$-modules we'll use the index (R).

6.2 Lemma. *Let $A = (KQ/\mathcal{I}, \leqslant)$ be a 1-quasi-hereditary algebra. Then, for the Ringel dual $R(A) = \left(KQ_{(R)}/\mathcal{I}_{(R)}, \leqslant_{(R)}\right)$ it is:*

(a) $P_{(R)}(n) \cong I_{(R)}(n) \cong T_{(R)}(1)$.

(b) $\Delta_{(R)}(j) \hookrightarrow \Delta_{(R)}(i)$ *if and only if* $\nabla_{(R)}(i) \twoheadrightarrow \nabla_{(R)}(j)$ *if and only if* $j \leqslant_{(R)} i$.

(c) $\operatorname{soc} P_{(R)}(i) \cong \operatorname{top} I_{(R)}(i) \cong S_{(R)}(n)$ *if and only if* $\operatorname{soc} T(i) \cong S(1)$.

(d) $\left[\Delta_{(R)}(j) : S_{(R)}(i)\right] = 1$ *for* $i \leqslant_{(R)} j$, *if* $\operatorname{top} T(i) \cong S(1)$.

Proof. (a) Using 1.3 and 5.3, we have $I(1) \cong T(n)$ resp. $I_{A^{op}}(1) \cong T_{A^{op}}(n)$. By applying $\mathscr{R}_{(A)}(-)$ resp. $\mathcal{D}\left(\mathscr{R}_{(A^{op})}(-)\right)$ we obtain $T_{(R)}(1) \cong P_{(R)}(n)$ resp. $T_{(R)}(1) \cong \mathcal{D}(T_{R(A^{op})}(1)) \cong \mathcal{D}(P_{R(A^{op})}(n)) \cong I_{(R)}(n)$ since $R(A^{op}) \cong R(A)^{op}$. Thus $P_{(R)}(n) \cong T_{(R)}(1) \cong I_{(R)}(n)$.

(b) There exist an exact sequence $\zeta : 0 \to \mathfrak{K} \to \nabla(j) \overset{\pi}{\twoheadrightarrow} \nabla(i) \to 0$, where $\mathfrak{K} = \ker \pi$ if and only if $i \leqslant j$ (see 2.6). By applying $\mathscr{R}_{(A)}(-)$ to ζ we obtain an exact sequence $0 \to \operatorname{Hom}_A(T, \mathfrak{K}) \to \Delta_{(R)}(j) \to \Delta_{(R)}(i)$. Since $\operatorname{top} \nabla(k) \overset{2.6}{\cong} S(1)$ and $[\nabla(k) : S(1)] = 1$ for all $k \in Q_0$ (see Sec.1 $(*)$), we obtain $\operatorname{top} T \in \operatorname{add} S(1)$ and $[\mathfrak{K} : S(1)] = 0$. This implies $\operatorname{Hom}_A(T, \mathfrak{K}) = 0$ and consequently $\Delta_{(R)}(j) \hookrightarrow \Delta_{(R)}(i)$ if and only if $i \leqslant j$ (i.e. $j \leqslant_{(R)} i$).

The algebra A^{op} is 1-quasi-hereditary, thus $\Delta_{(R(A^{op}))}(j) \hookrightarrow \Delta_{(R(A^{op}))}(i)$ if and only if $j \leqslant_{(R)} i$. Using duality, we have $\nabla_{(R)}(i) \twoheadrightarrow \nabla_{(R)}(j)$ if and only if $j \leqslant_{(R)} i$.

(c) "$\Leftarrow$" Since $\operatorname{soc} T(i) \cong S(1)$, we have $T(i) \hookrightarrow I(1) \cong T(n)$. Thus we have an exact sequence $\xi : 0 \to T(i) \to T(n) \to T(n)/T(i) \to 0$ with $T(i), T(n)$ and $T(n)/T(i) \in \mathfrak{F}(\nabla)$, since $\mathfrak{F}(\nabla)$ is coresolving. Applying $\mathscr{R}_{(A)}(-)$ to ξ yields an exact sequence $0 \to P_{(R)}(i) \to P_{(R)}(n) \to \mathscr{R}_{(A)}(T(n)/T(i)) \to 0$. Hence *(a)* implies $\operatorname{soc} P_{(R)}(i) \cong S_{(R)}(n)$.

According to Theorem 5.1 and $\operatorname{top} T(i) \in \operatorname{add}(S(1))$, we obtain that $\operatorname{soc} T(i) \cong S(1)$ imlies $\operatorname{top} T(i) \cong S(1)$. Thus $\operatorname{soc} \mathcal{D}(T(i)) \cong \operatorname{soc} T_{A^{op}}(i) \cong S(1)$ and therefore $\operatorname{soc} P_{R(A^{op})}(i) \cong S_{(R)}(n)$. Using duality, we obtain $\operatorname{top} I_{(R)}(i) \cong S_{(R)}(n)$.

"$\Rightarrow$" Let $\operatorname{soc} T(i) \cong S(1)^m$ [we know $\operatorname{soc} T(i) \in \operatorname{add}(S(1))$]. Since $T(i), I(1)^m \cong T(n)^m$ and $N := T(n)^m/T(i) \in \mathfrak{F}(\nabla)$, applying $\mathscr{R}_{(A)}(-)$ to the exact sequence $\xi : 0 \to T(i) \to T(n)^m \to N \to 0$ yields an exact sequence $0 \to P_{(R)}(i) \to P_{(R)}(n)^m \to \mathscr{R}_{(A)}(N) \to 0$. It is sufficient to show that $P_{(R)}(n)^m \overset{(a)}{\cong} I_{(R)}(n)^m$ is an injective envelope of $P_{(R)}(i)$. The assumption $\operatorname{soc} P_{(R)}(i) \cong S_{(R)}(n)$ implies then $m = 1$ and consequently $\operatorname{soc} T(i) \cong S(1)$:

Assume $P_{(R)}(n)^m$ is not an injective envelope of $P_{(R)}(i)$, then $P_{(R)}(n)$ is a direct summand of $\mathscr{R}_{(A)}(N)$. Since $P_{(R)}(n) \overset{(a)}{\cong} T_{(R)}(1)$ and $(T_{(R)}(1) : \Delta_{(R)}(1)) = 1$, we obtain $\big(\mathscr{R}_{(A)}(N) : \Delta_{(R)}(1)\big) \neq 0$. The properties of $\mathscr{R}_{(A)}(-)$ imply $(N : \nabla(1)) = (\mathscr{R}_{(A)}(N) : \Delta_{(R)}(1)) \neq 0$. Since $(T(n) : \nabla(1)) \overset{5.3}{=} 1$, the sequence ξ provides $m = (T(n)^m : \nabla(1)) = (T(i) : \nabla(1)) + (N : \nabla(1))$. Moreover, $(T(i) : \nabla(1)) \geq m$ because $\operatorname{soc} T(i) \cong \nabla(1)^m$ and the filtration $0 \subset \operatorname{soc} T(i) \subset T(i)$ can be refine to a ∇-good filtration of $T(i)$. We obtain $(T(i) : \nabla(1)) = m$ and therefore $(N : \nabla(1)) = 0$. We obtain a contradiction to our assumption.

(d) The structure of $\Delta_{(R)}(j)$ yields $[\Delta_{(R)}(j) : S_{(R)}(i)] = \dim_K \operatorname{Hom}_A(T(i), \nabla(j))$. If $\operatorname{top} T(i) \cong S(1)$ and an A-map $F : T(i) \to \nabla(j)$ is non zero, then F is surjective and $\dim_K \operatorname{Hom}_A(T(i), \nabla(j)) = 1$ because $\operatorname{top} \nabla(j) \cong S(1)$ and $[\nabla(j) : S(1)] = 1$. The properties of $T(i)$ yield $T(i) \twoheadrightarrow \nabla(i)$, thus we have a surjective map $F' : T(i) \twoheadrightarrow \nabla(i) \overset{2.6}{\twoheadrightarrow} \nabla(j)$ for all j with $j \leqslant i$. Thus $[\Delta_{(R)}(j) : S_{(R)}(i)] = \dim_K \operatorname{Hom}_A(T(i), \nabla(j)) = 1$ for all $i \leqslant_{(R)} j$. □

Proof of the theorem. " $\Rightarrow$ " If $R(A)$ is 1-quasi-hereditary, then for any $i \in Q_0(R)$ it is $\operatorname{soc} P_{(R)}(i) \cong S_{(R)}(n)$ (here $\{n\} = \min\big\{Q_0(R), \leqslant_{(R)}\big\}$). Lemma 6.2 *(d)* implies $\operatorname{soc} T(i) \cong S(1)$ and Theorem 5.1 provides $T(i) = P(1)/\left(\sum_{l \in Q_0 \setminus \Lambda_{(i)}} P(l)\right)$ for any $i \in Q_0$.

" $\Leftarrow$ " If $T(i) = P(1)/\left(\sum_{j \in Q_0 \setminus \Lambda_{(i)}} P(j)\right)$, then $\operatorname{soc} T(i) \cong \operatorname{top} T(i) \cong S(1)$ for any $i \in Q_0$ (see 5.1). Lemma 6.2 *(c)* and *(b)* provides (3) and (4) of Definition 1.2.

According to Theorem 5.1 and Lemma 5.2 *(a)* the A-module $T(i)$ can be considered as the module $P_{(i)}(1) \cong I_{(i)}(1)$ over a 1-quasi-hereditary algebra $A(i)$. Thus $(T(i) : \nabla(j)) = 1$ for every $j \in Q_0(i) = \Lambda_{(i)}$. We obtain $(P_{(R)}(i) : \Delta_{(R)}(j)) = 1$ for every $j \in Q_0(R)$ with $i \leqslant_{(R)} j$ and 6.2 *(e)* yields (2) of Definition 1.2. □

If for some 1-quasi-hereditary algebra A the algebra $R(A)$ is not 1-quasi-hereditary, then there exists $i \in Q_0$ such that $\operatorname{soc} T(i) \cong S(1)^m$ with $m \geq 2$, and consequently $(P_{(R)}(i) : \Delta_{(R)}(1)) \geq 2$. An example of a 1-quasi-hereditary algebra A such that $R(A)$ is not 1-quasi-hereditary can be found in [9].

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