# Harmonic Analysis on Heisenberg-Clifford Lie Supergroups 

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#### Abstract

We define a Fourier transform and a convolution product for functions and distributions on Heisenberg-Clifford Lie supergroups. The Fourier transform exchanges the convolution and a pointwise product, and is an intertwining operator for the left regular representation. We generalize various classical theorems, including the Paley-Wiener-Schwartz theorem, and define a convolution Banach algebra.


## 1 Introduction

Recently, several attempts have been made to extend the notion of a Fourier transform to a supersymmetric context. An obstacle one encounters is that supercommutative Lie supergroups with non-trivial odd part do not admit enough unitary representations to decompose reasonable spaces of superfunctions.

In this paper we associate a natural Fourier transform with purely odd Lie supergroups. The situation is analogous to geometric quantization of translation groups on vector spaces: The action on the cotangent bundle is Hamiltonian only after central extension. This leads to the canonical commutation relations and thus, to the definition of the Heisenberg group. Similarly, the central extension of the underlying supergroup of a super vector space produces the HeisenbergClifford supergroup which admits unitary representations in abundance. In fact, the representation theory of Heisenberg-Clifford algebras resembles the representation theory of Heisenberg groups, so that the harmonic analysis of phase space can be used as a guideline for the harmonic analysis of purely odd superspaces.

In the present paper, we will restrict ourselves to the case of a purely odd super vector space and its central extension. In a later paper, we will combine the purely odd with the classical, purely even Fourier analysis to provide a complete picture of the harmonic analysis for Heisenberg-Clifford supergroups.

To put our work in perspective, we mention some previous work on Fourier transforms of functions on linear supermanifolds. The earliest reference known to the authors is the article [15] by Rempel and Schmitt. More recent papers
include work by Brackx, De Schepper and Sommen [6, and by De Bie [8]. Investigation of Fourier transform on Heisenberg-Clifford Supergroups was started by Bieliavsky, de Goursac and Tuynman in their preprint [5]. These approaches have in common that the Fourier transform is defined in close formal analogy with the formula

$$
\hat{f}(\zeta)=\int_{\mathbb{R}} f(x) e^{-i \zeta x} \mathrm{~d} x
$$

and one of the crucial ideas is an appropriate generalization of the exponential $e^{i \zeta x}$. Our approach is somewhat different, in that we take as a starting point the formula

$$
\hat{f}(\pi)=\int_{G} f(x) \pi(x) \mathrm{d} x
$$

where $\pi$ is an irreducible representation of $G$, and $\mathrm{d} x$ is a left Haar measure on $G$. Of course, for $G=\mathbb{R}$ the two formulas agree if we take for $\pi$ the unitary character $x \mapsto e^{-i \zeta x}$. Our approach is naturally covariant and thus well-adapted to the supergroup structures. Indeed, we expect that it will generalize to arbitrary Lie supergroups (though, of course, these may not have any unitary representations in general). The irreducible unitary representations of the Heisenberg-Clifford Lie supergroup have been classified by Salmasian [17, and our approach depends heavily on his classification.

The Fourier transform $\mathcal{F}$ that we introduce takes values in a certain endomorphism algebra $\mathcal{H}$. We define a convolution product in analogy with

$$
(f * g)(x)=\int_{G} f(y) g\left(y^{-1} x\right) \mathrm{d} y
$$

which behaves well with respect to the Fourier transform in the sense that the Fourier transform $\mathcal{F}(F * G)$ agrees with the pointwise product $\mathcal{F}(F) \mathcal{F}(G)$ in $\mathcal{H}$. This convolution product seems to be new. (However, Bieliavsky et. al. [5] define a $\star$-product which is mapped to a pointwise product under a quantization map).

We give a brief summary of the paper and state the main results.
In Section 2 we define the Heisenberg-Clifford Lie algebra $\mathfrak{h c}=\mathfrak{h c}(V, \beta)$ associated with a symplectic super vector space $(V, \beta)$. We restrict our attention to $V$ purely odd and $\beta$ positive definite and non-degenerate. Then we define a Lie supergroup $\mathrm{HC}=\left(\mathrm{HC}_{0}, \mathcal{C}_{\mathrm{HC}}^{\infty}\right)$ with underlying Lie group $\mathrm{HC}_{0}=\mathbb{R}$, and Lie superalgebra $\mathfrak{h c}$. We introduce a left invariant integral $\int_{\mathrm{HC}} F$ for smooth compactly supported functions $F \in \mathcal{C}_{c}^{\infty}(\mathrm{HC})$ and define distributions on HC. This material is known, except possibly for our treatment of the invariant integral. If $\left(a_{i}\right)_{i=1}^{n}$ is an orthonormal basis of $(V, \beta)$, then $\gamma:=a_{1} \cdots a_{n} \in \mathfrak{U}(\mathfrak{h} \mathfrak{c})$ depends only on the orientation of the basis. We define $\int_{\mathrm{HC}} F:=\int_{\mathbb{R}} F(\gamma ; x) \mathrm{d} x$ - in our case, this is just the well-known Berezin integral, but an appropriate choice of $\gamma$ will yield an invariant integral on more general supergroups. The invariant integral gives rise to a non-degenerate invariant pairing $\langle\cdot, \cdot\rangle$ between $\mathcal{C}^{\infty}(\mathrm{HC})$ and $\mathcal{C}_{c}^{\infty}(\mathrm{HC})$. Lastly, we define spaces $\mathcal{D}^{\prime}(\mathrm{HC})$ and $\mathcal{E}^{\prime}(\mathrm{HC})$ of distributions and compactly supported distributions as topological dual spaces. The
non-degenerate pairing then allows us to identify a smooth function $F$ with the distribution $\Phi \mapsto\langle F, \Phi\rangle$.

The definition of a finite-dimensional unitary representation of HC is given in Section 3. It is known that all irreducible unitary representations of HC are finite-dimensional if $V$ is purely odd. Representations of the universal enveloping algebra $\mathfrak{U}(\mathfrak{h c})$ in which a central element $z$ acts by a scalar $i \zeta$ factor through a Clifford algebra $C l\left(V_{\mathbb{C}}, \zeta \beta\right)$. The spin module of this algebra is then used to define a representation $\left(\pi_{\zeta}, \mathcal{S}\right)$ whenever $\zeta \in \mathbb{C}$; real and positive values of $\zeta$ yield all unitary irreducible representations of HC, which follows from the results of Salmasian [17. It is well-known that the Clifford algebra $\mathrm{Cl}\left(V_{\mathbb{C}}, \zeta \beta\right)$ is isomorphic to an algebra $\mathcal{H}$ of endomorphisms of $\mathcal{S}$, and we define a trace $T$ and a sesquilinear form $\langle A \mid B\rangle=T\left(A B^{\dagger}\right)$ on $\mathcal{H}$.

In Section 4, we combine the invariant integral with the family $\left(\pi_{\zeta}, \mathcal{S}\right)$ of representations in order to define the Fourier transform of $F$ at $\zeta \in \mathbb{C}$ to be the $\mathcal{H}$-valued integral $\mathcal{F}(F)(\zeta):=\widehat{F}(\zeta):=\int_{\mathrm{HC}} F \cdot \pi_{-\zeta}$. The first main theorem is the following:

Theorem A. The Fourier transformation intertwines the left regular representation with $\pi_{-\zeta}$, that is,

$$
\left(L_{u ; x} F\right)^{\curlyvee}(\zeta)=\pi_{-\zeta}(u ; x) \widehat{F}(\zeta)
$$

for $\zeta \in \mathbb{C}, u \in \mathfrak{U}(\mathfrak{h c})$ and $x \in \mathbb{R}$.
We introduce the Schwartz space $\mathcal{S}(\mathrm{HC})$ as the space of functions $F \in$ $\mathcal{C}^{\infty}(\mathrm{HC})$ for which $F(u)$ is rapidly decreasing for all $u \in \mathfrak{U}(\mathfrak{h c})$. Then the preceding theorem suggests a definition of a space $\mathcal{S}(\mathbb{R}, \mathcal{H})$ of $\mathcal{H}$-valued Schwartz functions on $\mathbb{R}$. The main idea is to define the components $A(u ; \zeta)$ of a function $A: \mathbb{R} \rightarrow \mathcal{H}$ in such a way that if $A=\widehat{F}$, then $A(u)$ is the Fourier transform of $F(u)$. Then we can prove that the Fourier transform is an isomorphism of these topological vector spaces.

Theorem B. The Fourier transform restricts to an isomorphism of the topological vector spaces $\mathcal{S}(\mathrm{HC})$ and $\mathcal{S}(\mathbb{R}, \mathcal{H})$.

Next, we extend the definition of Fourier transform to compactly supported distributions. If $U \in \mathcal{D}^{\prime}(\mathrm{HC})$, then its Fourier transform $\widehat{U}(\zeta)$ extends to an entire holomorphic function on $\mathbb{C}$. Lastly, we prove a Paley-Wiener-Schwartz theorem, characterizing the image of the space $\mathcal{C}_{[-a, a]}^{\infty}(\mathrm{HC})$ of functions with support in the compact interval $[-a, a]$ under the Fourier transform.

Theorem C. The Fourier transform is a bijection between the space $\mathcal{C}_{[-a, a]}^{\infty}(\mathrm{HC})$ and the space of functions $A: \mathbb{C} \rightarrow \mathcal{H}$ whose components satisfy the following exponential growth condition:

For every $N \in \mathbb{N}$ there is a constant $C_{N}$ such that

$$
\left.\mid T\left(A(\zeta) d \pi_{-\zeta}(u)\right)\right) \mid \leq C_{N}(1+|\zeta|)^{-N} e^{a \operatorname{Im}(\zeta)} \quad \text { for all } u \in \mathfrak{U}(\mathfrak{h c}), \zeta \in \mathbb{C}
$$

Let $\left(m, m^{*}\right)$ and $\left(i, i^{*}\right)$ denote the multiplication and inversion morphism of the Lie supergroup HC. The last section begins with the definition of a convolution product

$$
(F * G)(u ; x):=(-1)^{|u|(|G|+|\gamma|)}\left\langle F, L_{u ; x} i^{*} G\right\rangle
$$

where $F$ and $G$ are smooth functions on HC , one of which is compactly supported. In the rather technical Proposition5.5 we prove that our formula indeed yields a smooth function on HC, and that

$$
\langle F * G, \Phi\rangle=\left\langle F \otimes G, m^{*} \Phi\right\rangle=\left\langle F, i^{*}\left(G * i^{*} \Phi\right)\right\rangle
$$

In Theorem 5.7 we prove the following property of the convolution product.
Theorem D. If $F$ and $G$ are smooth compactly supported functions on HC, then

$$
(F * G)^{\curlyvee}(\zeta)=\widehat{F}(\zeta) \widehat{G}(\zeta)
$$

The convolution product can be extended to include convolutions $U * F$ of a distribution $U$ and a smooth function $F$, if one of $F, U$ is compactly supported. Lastly, we define Sobolev-type spaces $\left(W^{k, p}(\mathrm{HC}),\|\cdot\|_{k, p}\right)$. If the order of differentiability $k$ is large enough, the convolution product can be extended to these Banach spaces, and we prove

Theorem E. If $n=\operatorname{dim} V$, the space $W^{n, 1}(\mathrm{HC})$ is a Banach algebra with respect to the convolution product.

We view the present set of results as a first step towards a systematic harmonic analysis on abelian Lie supergroups. We expect such a theory to have immediate applications to linear differential equations on superspaces. Moreover, it will be an important tool in a non-abelian harmonic analysis of homogeneous superspaces which is just evolving (see [2-4]).
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## 2 Preliminaries

In this section we provide the basic definitions necessary for our construction. The material in this section is mostly known, so we will omit proofs wherever possible. As general references, we mention [10, 14].

If $V=V_{0} \oplus V_{1}$ is a super vector space, we write $|v|$ for the parity of a homogeneous element $v \in V$. If $m=\operatorname{dim} V_{0}$ and $n=\operatorname{dim} V_{1}$, we say that $V$ is of graded dimension $(m, n)$. If $W$ is another super vector space, we equip $V \otimes W$ with a grading such that $|v \otimes w| \equiv|v|+|w|(2)$. The space of all linear maps from $V$ to $W$ is denoted $\operatorname{Hom}(V, W)$, and has grading defined in such a way that $|\phi(v)| \equiv|\phi|+|v|(2)$ for $\phi \in \underline{\operatorname{Hom}}(V, W), v \in V$. We let $\operatorname{Hom}(V, W)$ denote
the subspace of even linear maps. A bilinear form $\beta$ on $V$ is even if $|u|+|v|=1$ implies $\beta(u, v)=0$, and a non-degenerate even bilinear form $\beta$ is symplectic if

$$
\beta(u, v)=-(-1)^{|u||v|} \beta(v, u)
$$

for all homogeneous $u, v \in V$.
Definition 2.1. Given a finite-dimensional super vector space $V$ over $\mathbb{R}$ together with a symplectic form $\beta$ on $V$, we define the Heisenberg-Clifford Lie superalgebra by

$$
\mathfrak{h c}(V, \beta)=V \oplus \mathbb{R}
$$

with grading $\mathfrak{h c}(V, \beta)_{0}=V_{0} \oplus \mathbb{R}$ and $\mathfrak{h c}(V, \beta)_{1}=V_{1}$ and elements $u+x$ with $u \in V$ and $x \in \mathbb{R}$. We denote by $z$ the central element $0+1$. The bracket is given by

$$
\begin{equation*}
[u+\lambda z, v+\mu z]=2 \beta(u, v) z, u, v \in V, \lambda, \mu \in \mathbb{R} \tag{2.1}
\end{equation*}
$$

and the one-dimensional center $\mathfrak{z}(\mathfrak{h c}(V, \beta))$ of $\mathfrak{h c}(V, \beta)$ is spanned by $z$.
Remark 2.2. Throughout this article, we will assume that $V$ is purely odd super vector space of graded dimension $(0, n)$. Then, $\beta$ is simply a non-degenerate symmetric bilinear form, and we assume that $\beta$ is positive definite. We will write $\mathfrak{h c}$ or $\mathfrak{h c}(V)$ for $\mathfrak{h c}(V, \beta)$, and similarly $\mathfrak{z}$ for $\mathfrak{z}(\mathfrak{h c}(V, \beta))$, if no confusion is possible.
Remark 2.3. Let $\left(a_{i}\right)_{i=1}^{n}$ be an orthonormal basis of $(V, \beta)$. Then the universal enveloping algebra $\mathfrak{U}(\mathfrak{h c}(V, \beta))$ is generated by the elements $a_{i}$ and $z \in$ $\mathfrak{z}(\mathfrak{h c}(V, \beta))$, subject to the relations

$$
a_{i} a_{j}=\left\{\begin{array}{lll}
-a_{j} a_{i} & \text { if } \quad i \neq j  \tag{2.2}\\
z & \text { if } \quad i=j
\end{array}\right.
$$

and $z a_{i}=a_{i} z$ for all $i=1, \ldots, n$.
As in the ungraded case, there is a symmetrization map $\omega: S(\mathfrak{h c}) \rightarrow \mathfrak{U}(\mathfrak{h} \mathfrak{c})$. Here, $S(\mathfrak{h c}) \cong \mathbb{R}[z] \otimes \Lambda V$ is the symmetric algebra of the super vector space $\mathfrak{h c}$. The elements $a_{i} \in V$ pairwise anticommute, and therefore the map $\omega$ is simply given by $\omega\left(z^{k} \otimes\left(a_{i_{1}} \wedge \cdots \wedge a_{i_{k}}\right)\right)=z^{k} a_{i_{1}} \cdots a_{i_{k}}$.

Given a natural number $n \geq 1$, we let $\underline{n}=\{1,2, \ldots, n\}$. If $\left(a_{i}\right)_{i=1}^{n}$ is a basis of $V$, then the subsets of $\underline{n}$ parametrize a basis $\left(a_{I}\right)_{I \subset \underline{n}}$ of $\Lambda V$ in the usual way by

$$
a_{I}=a_{i_{1}} \wedge \cdots \wedge a_{i_{k}}
$$

where $I=\left\{i_{1}<\cdots<i_{k}\right\}$. We denote the images of $a_{I}$ under $\omega$ also by $a_{I}$. Then a Poincaré-Birkhoff-Witt basis of $\mathfrak{U}(\mathfrak{h c})$ is given by $\left\{z^{k} a_{I} \mid k \in \mathbb{N}, I \subset \underline{n}\right\}$.

We introduce a special element of $\mathfrak{h c}(V, \beta)$, which is up to a sign the chirality operator in the theory of Clifford algebras.
Definition 2.4. If $\left(a_{i}\right)_{i=1}^{n}$ is an orthonormal basis of $(V, \beta)$, we let

$$
\gamma:=a_{1} \cdots a_{n}=\omega\left(1 \otimes\left(a_{1} \wedge \cdots \wedge a_{n}\right)\right) \in \mathfrak{U}(\mathfrak{h} \mathfrak{c}(V, \beta)) .
$$

Since the volume element $a_{1} \wedge \cdots \wedge a_{n} \in \Lambda^{n} V$ only depends on the orientation of the orthonormal basis, the same is true for $\gamma$.

Recall that if $\mathfrak{g}$ is a Lie superalgebra, then $\mathfrak{U}(\mathfrak{g})$ carries the structure of a super Hopf algebra (see e.g. [12, Section 3]). The coproduct $\Delta$ and the antipode $S$ will be used below to define a Lie supergroup corresponding to $\mathfrak{h c}$. They are uniquely determined by the following properties: The coproduct $\Delta: \mathfrak{U}(\mathfrak{g}) \rightarrow$ $\mathfrak{U}(\mathfrak{g}) \otimes \mathfrak{U}(\mathfrak{g})$ is an even unital algebra homomorphism which satisfies $\Delta(x)=$ $x \otimes 1+1 \otimes x$ for $x \in \mathfrak{g}$. The antipode $S: \mathfrak{U}(\mathfrak{g}) \rightarrow \mathfrak{U}(\mathfrak{g})$ is an even unital superantiautomorphism, that is, $S(u v)=(-1)^{|u||v|} S(v) S(u)$, and on elements $x \in \mathfrak{g}$ it is given by $S(x)=-x$.
Remark 2.5. The element $\Delta(\gamma)$ will play an important role in this article. With respect to an orthonormal basis $\left(a_{i}\right)_{i=1}^{n}$, it is given by

$$
\begin{equation*}
\Delta(\gamma)=\prod_{i=1}^{n}\left(a_{i} \otimes 1+1 \otimes a_{i}\right)=\sum_{I \subset \underline{n}} a_{I} \otimes * a_{I} \tag{2.3}
\end{equation*}
$$

Here $* a_{I}= \pm a_{I^{c}}$, where the sign is such that $a_{I} \cdot\left(* a_{I}\right)=\gamma$ in $\mathfrak{U}(\mathfrak{h c})$, and $I^{c}=\underline{n} \backslash I$. Concretely, if $I=\left(i_{1}<\ldots<i_{k}\right)$, let $\sigma_{I}$ denote the permutation of $\underline{n}$ determined by $\sigma_{I}(j)=i_{j}$ for $1 \leq j \leq k$ and $\sigma_{I}(k+1)<\ldots<\sigma_{I}(n)$. Then

$$
* a_{I}=\operatorname{sgn}\left(\sigma_{I}\right) a_{I^{c}}
$$

Note that both $\left(a_{I}\right)_{I \subset \underline{n}}$ and $\left(* a_{I}\right)_{I \subset \underline{n}}$ form a bases of $\mathfrak{U}(\mathfrak{h c})$ as a $\mathfrak{z}$-module. We will use Sweedler's notation

$$
\begin{equation*}
\Delta(\gamma)=\sum_{i} \gamma_{i}^{(1)} \otimes \gamma_{i}^{(2)} \tag{2.4}
\end{equation*}
$$

This, however, requires some care, since the $\gamma_{i}^{(j)}$ are not uniquely determined by equation (2.4).

In [13, Koszul constructs a Lie supergroup associated with a Lie supergroup pair. We recall the definition of a Lie supergroup pair and the construction of the corresponding sheaf.

Definition 2.6. A Lie supergroup pair $G=\left(G_{0}, \mathfrak{g}\right)$ consists of a Lie group $G_{0}$, and a real Lie superalgebra $\mathfrak{g}$ whose even part $\mathfrak{g}_{0}$ is the Lie algebra of $G_{0}$, and a smooth linear action Ad of $G_{0}$ on $\mathfrak{g}$ by even linear automorphisms. We require that the action Ad extends the adjoint action of $G_{0}$ on $\mathfrak{g}_{0}$ and that its differential $d$ Ad : $\mathfrak{g}_{0} \times \mathfrak{g} \rightarrow \mathfrak{g}$ is the restriction of the bracket $[\cdot, \cdot]$. The subalgebra $\mathfrak{g}_{0}$ of $\mathfrak{g}$ acts on $\mathfrak{U}(\mathfrak{g})$ from the left, and if $U \subset G_{0}$ is open, then $\mathfrak{g}_{0}$ acts on $C^{\infty}(U)$ by left invariant differential operators. Consider $C^{\infty}(U)$ as a purely even $\mathfrak{g}_{0}$-module and define

$$
\mathcal{C}_{G}^{\infty}(U):=\underline{\operatorname{Hom}}_{\mathfrak{g}_{0}}\left(\mathfrak{U}(\mathfrak{g}), C^{\infty}(U)\right)
$$

If $F \in \mathcal{C}_{G}^{\infty}(U)$ we write $F(u ; x)$ for $F(u)(x)$ if $u \in \mathfrak{U}(\mathfrak{g})$ and $x \in U$.
Definition 2.7. We define the Heisenberg-Clifford Lie supergroup pair by $H C=(\mathbb{R}, \mathfrak{h c})$, where the action $\operatorname{Ad}$ of $\mathbb{R}$ on $\mathfrak{h c}$ is the trivial action. If $U \subset \mathbb{R}$ is open, we let $z \in \mathfrak{z}$ act on $C^{\infty}(U)$ by $z f=-f^{\prime}$.

Proposition 2.8. a) Let $U \subset \mathbb{R}$ be open, and denote by $\mu$ the pointwise multiplication of functions in $C^{\infty}(U)$. The assignment $U \mapsto \mathcal{C}_{\mathrm{HC}}^{\infty}(U)$ is a sheaf of supercommutative unital superalgebras on $\mathbb{R}$, if the algebra multiplication is defined by

$$
F \cdot G:=\mu \circ(F \otimes G) \circ \Delta .
$$

The pair $\left(\mathbb{R}, \mathcal{C}_{\mathrm{HC}}^{\infty}\right)$ is a supermanifold.
b) As a superalgebra, $\mathcal{C}_{\mathrm{HC}}^{\infty}(U)$ is isomorphic to

$$
\underline{\operatorname{Hom}}_{\mathbb{R}}\left(\Lambda V, C^{\infty}(U)\right) \cong C^{\infty}(U) \otimes \Lambda V^{*},
$$

where the algebra structure on the right hand side is the obvious one. The isomorphism is given by $\left.F \mapsto(F \circ \omega)\right|_{\Lambda V}$, where $\omega$ is the symmetrization map.

Remark 2.9. If $\left(a_{i}\right)_{i=1}^{n}$ is a basis of $V$, let $\left(\xi^{i}\right)_{i=1}^{n}$ denote the dual basis of $V^{*}$. We use superscripts $\xi^{I}=\xi^{i_{1}} \wedge \ldots \wedge \xi^{i_{k}}$ for the elements of $\Lambda V^{*}$, since this has become standard in the literature on supermanifolds. Due to the simple form of the symmetrization map $\omega$, the isomorphism in Proposition 2.8 b ) is given by

$$
F \mapsto \sum_{I \subset \underline{n}} f_{I} \otimes \xi^{I},
$$

where $f_{I}:=F\left(a_{I}\right)$. This coordinate-dependent notation for smooth functions is quite common in the literature. However, we will avoid using coordinates as far as possible.

We recall that a Lie supergroup is a supermanifold $\left(G, \mathcal{C}_{G}^{\infty}\right)$ together with morphisms $m=\left(m_{0}, m^{*}\right), i=\left(i_{0}, i^{*}\right)$ and $e: * \rightarrow\left(G, \mathcal{C}_{G}^{\infty}\right)$, satisfying the usual group axioms (here, $*$ is the ( 0,0 )-dimensional supermanifold).

Proposition 2.10. The supermanifold $\left(\mathbb{R}, \mathcal{C}_{\mathrm{HC}}^{\infty}\right)$ is a Lie supergroup with multiplication

$$
m=\left(m_{0}, m^{*}\right), \quad m_{0}(x, y):=x+y, \quad\left(m^{*} F\right)(u \otimes v ; x, y):=F(u v ; x+y)
$$

inversion

$$
i=\left(i_{0}, i^{*}\right), \quad i_{0}(x):=-x, \quad\left(i^{*} F\right)(u ; x):=F(S(u) ;-x),
$$

and identity element $e=\left(e_{0}, e^{*}\right)$ given by $e_{0}(*):=0$ and $e^{*} F:=F(1 ; 0)$.
Definition 2.11. We denote the algebra of global sections by $\mathcal{C}^{\infty}(\mathrm{HC}):=$ $\mathcal{C}_{\mathrm{HC}}^{\infty}(\mathbb{R})$ and refer to elements of $\mathcal{C}^{\infty}(\mathrm{HC})$ as smooth functions on HC .

The left regular action of HC on $\mathcal{C}^{\infty}(\mathrm{HC})$ is given by

$$
\left(L_{x} F\right)(u ; y):=F(u ; y-x), \quad\left(L_{u} F\right)(v, y):=(-1)^{|u||F|} F(S(u) v ; y)
$$

for $x \in \mathbb{R}$ and $u \in \mathfrak{U}(\mathfrak{h} \mathfrak{c})$. This defines linear maps $L_{x}, L_{u}: \mathcal{C}^{\infty}(\mathrm{HC}) \rightarrow \mathcal{C}^{\infty}(\mathrm{HC})$ of parity $\left|L_{x}\right|=0$ for $x \in \mathbb{R}$ and $\left|L_{u}\right|=|u|$ for $u \in \mathfrak{U}(\mathfrak{h c})$. We write $L_{u ; x}$ for $L_{u} \circ L_{x}=L_{x} \circ L_{u}$.

Lemma 2.12. a) The assignments $x \mapsto L_{x}$ and $u \mapsto L_{u}$ define representations of $\mathbb{R}$ and $\mathfrak{U}(\mathfrak{h c})$ on the vector space $\mathcal{C}^{\infty}(\mathrm{HC})$.
b) If $v \in V$, then $L_{v}$ is a super-derivation on $\mathcal{C}^{\infty}(\mathrm{HC})$, that is, $L_{v}(F \cdot G)=$ $L_{v} F \cdot G+(-1)^{|v||F|} F \cdot L_{v} G$.

Definition 2.13. By Proposition 2.8, $\mathcal{C}^{\infty}(\mathrm{HC})$ is isomorphic as a superalgebra to $C^{\infty}(\mathbb{R}) \otimes \Lambda V^{*}$. Since $C^{\infty}(\mathbb{R})$ carries a nuclear Fréchet topology, this tensor product also carries a nuclear Fréchet topology. For each compact $K \subset \mathbb{R}$ and $u \in \mathfrak{U}(\mathfrak{h} \mathfrak{c})$ we define a seminorm on $\mathcal{C}^{\infty}(\mathrm{HC})$ by

$$
p_{K, u}(F):=\max _{x \in K}\left|\left(L_{u} F\right)(1 ; x)\right|
$$

Given a basis $\left(a_{i}\right)_{i=1}^{n}$ of $V$ and a countable exhaustion $\left\{K_{j}\right\}_{j \in J}$ of $\mathbb{R}$ by compact sets, the Fréchet topology on $\mathcal{C}^{\infty}(\mathrm{HC})$ can be defined by the countable family $\left\{p_{K_{j}, z^{k} a_{I}}\right\}_{j, k, I}$ of seminorms.

We define vector valued and compactly supported functions as well as functions of Schwartz class.

Definition 2.14. If $K \subset \mathbb{R}$ is compact, we let

$$
\mathcal{C}_{K}^{\infty}(\mathrm{HC}):=\left\{F \in \mathcal{C}^{\infty}(\mathrm{HC}) \mid(\forall u \in \mathfrak{U}(\mathfrak{h} \mathfrak{c})): \operatorname{supp} F(u) \subset K\right\}
$$

be the space of smooth functions with support contained in $K$, and we give $\mathcal{C}_{K}^{\infty}(\mathrm{HC})$ the topology defined by the seminorms $p_{u}(F)=\max _{x \in K}\left|L_{u} F(1 ; x)\right|$. Then the union

$$
\mathcal{C}_{c}^{\infty}(\mathrm{HC}):=\cup_{i} \mathcal{C}_{K_{i}}^{\infty}(\mathrm{HC}),
$$

where $\left\{K_{i}\right\}$ is a countable exhaustion of $\mathbb{R}$ by compact sets, is the space of compactly supported smooth functions on HC, which is a countable strict inductive limit of Fréchet spaces, or an LF space.

If $W$ is a finite-dimensional super vector space over $\mathbb{R}$ or $\mathbb{C}$, we define the vector space of smooth $W$-valued functions on HC by $\mathcal{C}^{\infty}(\mathrm{HC}, W):=\left(\mathcal{C}^{\infty}(\mathrm{HC}) \otimes\right.$ $W)_{0}$.

Lastly, the Schwartz space $\mathcal{S}(\mathrm{HC})$ of rapidly decreasing functions is defined to be the space of $F \in \mathcal{C}^{\infty}(\mathrm{HC})$ for which

$$
s_{j, u}(F):=\sup _{x \in \mathbb{R}}\left|x^{j}\left(L_{u} F\right)(1 ; x)\right|<\infty
$$

for all $j \in \mathbb{N}$ and $u \in \mathfrak{U}(\mathfrak{h c})$.
Remark 2.15. a) The space $\mathcal{S}(\mathrm{HC})$ is simply the subspace of $\mathcal{C}^{\infty}(\mathrm{HC})$ of all $F$ which satisfy $F(u) \in \mathcal{S}(\mathbb{R})$ for all $u \in \mathfrak{U}(\mathfrak{h c})$.
b) The spaces of functions we have defined so far are isomorphic as vector spaces to $C^{\infty}(\mathbb{R}) \otimes W, C_{c}^{\infty}(\mathbb{R}) \otimes W$ and $\mathcal{S}(\mathbb{R}) \otimes W$, respectively, where $W=\Lambda V^{*}$ is finite-dimensional. Therefore, there is only one reasonable tensor product topology, and we will use this topology throughout.

Lemma 2.16. The linear maps $L_{u ; x}$ are continuous on $\mathcal{C}^{\infty}(\mathrm{HC}), \mathcal{C}_{c}^{\infty}(\mathrm{HC})$ and $\mathcal{S}(\mathrm{HC})$.

Proof. After choosing coordinates, the proof reduces to showing that the derivative $f \mapsto f^{\prime}$ is continuous on $C^{\infty}(\mathbb{R}), C_{c}^{\infty}(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$, which is trivial by definition.

## The Invariant Integral

Definition 2.17. Let $W$ be a finite-dimensional super vector space.
a) If $F \in \mathcal{C}_{c}^{\infty}(\mathrm{HC}, W)$, we define the integral of $F$ over HC as

$$
\int_{\mathrm{HC}} F:=\int_{\mathbb{R}} F(\gamma ; x) \mathrm{d} x
$$

where $\gamma$ is defined in 2.4
b) If $F \in \mathcal{C}^{\infty}(\mathrm{HC})$ and $G \in \mathcal{C}^{\infty}(\mathrm{HC}, W)$ are such that $F \cdot G$ has compact support, we let

$$
\langle F, G\rangle:=\int_{\mathrm{HC}} F \cdot G
$$

Remark 2.18. a) The integral and the pairing have parity $|\gamma|$, that is, if $|F|+$ $|\gamma| \equiv 1(2)$, then $\int_{\mathrm{HC}} F=0$, and if $|F|+|G|+|\gamma| \equiv 1(2)$, then $\langle F, G\rangle=0$.
b) The product in $\mathcal{C}^{\infty}(\mathrm{HC})$ is supercommutative, and therefore

$$
\langle F, G\rangle=(-1)^{|F||G|}\langle G, F\rangle .
$$

Lemma 2.19. The integral is left invariant in the sense that

$$
\int_{\mathrm{HC}} L_{x} F=\int_{\mathrm{HC}} F \quad \text { and } \quad \int_{\mathrm{HC}} L_{u} F=0
$$

for all $x \in \mathbb{R}$ and all $u \in \mathfrak{U}(\mathfrak{h c})$. The pairing $\langle\cdot, \cdot\rangle$ is invariant in the sense that

$$
\begin{equation*}
\left\langle L_{u ; x} F, G\right\rangle=(-1)^{|F||u|}\left\langle F, L_{S(u) ;-x} G\right\rangle \tag{2.5}
\end{equation*}
$$

for $x \in \mathbb{R}$ and $u \in \mathfrak{U}(\mathfrak{h c})$.
Proof. Invariance under $L_{x}, x \in \mathbb{R}$ follows from translation invariance of the Lebesgue measure, since $L_{x} F(\gamma ; y)=F(\gamma ; y-x)$. In order to check invariance under $\mathfrak{U}(\mathfrak{h c})$, choose an orthonormal basis $\left(a_{i}\right)_{i=1}^{n}$ of $V$. It then suffices to show that $\int L_{a_{i}} F=0$ for $1 \leq i \leq n$ and $\int L_{z} F=0$. We compute

$$
\begin{aligned}
\int_{\mathrm{HC}} L_{a_{i}} F & = \pm \int_{\mathbb{R}} F\left(a_{i} \gamma ; x\right) \mathrm{d} x \\
& = \pm \int_{\mathbb{R}} F\left(z a_{1} \ldots \hat{a}_{i} \ldots a_{n} ; x\right) \mathrm{d} x \\
& = \pm \int_{\mathbb{R}} F\left(a_{1} \ldots \hat{a}_{i} \ldots a_{n}\right)^{\prime}(x) \mathrm{d} x
\end{aligned}
$$

which is zero because $F\left(a_{1} \ldots \hat{a}_{i} \ldots a_{n}\right)$ is compactly supported. For the same reason, $\int L_{z} F=\int_{\mathbb{R}} F(\gamma)^{\prime}(x) \mathrm{d} x=0$. If $v \in \mathfrak{h c}$, we have $\int_{\mathrm{HC}} L_{v}(F \cdot G)=0$, and because $L_{v}$ is a super-derivation, this implies

$$
\left\langle L_{v} F, G\right\rangle=-(-1)^{|v||F|}\left\langle F, L_{v} G\right\rangle
$$

Let $u, v \in \mathfrak{h c}$. Then $-L_{u}=L_{-u}=L_{S(u)}$, and it follows that

$$
\left\langle L_{u v} F, G\right\rangle=(-1)^{|F|(|u v|)}\left\langle F, L_{S(u v)} G\right\rangle .
$$

This implies that equation (2.5) holds for arbitrary $u \in \mathfrak{U}(\mathfrak{h c})$ and $x \in \mathbb{R}$.
Lemma 2.20. If $F, G \in \mathcal{C}_{c}^{\infty}(\mathrm{HC})$, then

$$
\left|\int_{\mathrm{HC}} F\right| \leq \operatorname{vol}(\operatorname{supp} F) \cdot p_{\operatorname{supp} F, \gamma}(F)
$$

and

$$
p_{K, u}(F \cdot G) \leq \sum_{i} p_{K, u_{i}^{(1)}}(F) p_{K, u_{i}^{(2)}}(G)
$$

where $u \in \mathfrak{U}(\mathfrak{h c})$ and $\Delta(u)=\sum_{i} u_{i}^{(1)} \otimes u_{i}^{(2)}$.
In particular, the integral $\int_{\mathrm{HC}}$ is a continuous linear functional, and the algebra multiplication on $\mathcal{C}^{\infty}(\mathrm{HC})$ and the pairing $\langle\cdot, \cdot\rangle$ are continuous.

Lastly, we define distributions and compactly supported distributions.
Definition 2.21. We define the spaces $\mathcal{D}^{\prime}(\mathrm{HC})$ of distributions on HC and $\mathcal{E}^{\prime}(\mathrm{HC})$ of compactly supported distributions on HC to be the topological dual spaces of $\mathcal{C}_{c}^{\infty}(\mathrm{HC})$ and $\mathcal{C}^{\infty}(\mathrm{HC})$, respectively. We introduce the duality pairing $\langle\cdot, \cdot\rangle$ and write $\langle U, \Phi\rangle:=U(\Phi)$ if $U$ is a distribution and $\Phi$ is a smooth function.
Remark 2.22. a) By Lemma 2.20, every element $F \in \mathcal{C}^{\infty}(\mathrm{HC})$ defines a distribution via $\Phi \mapsto\langle F, \Phi\rangle$, and the corresponding map $\mathcal{C}^{\infty}(\mathrm{HC}) \rightarrow \mathcal{D}^{\prime}(\mathrm{HC})$ is injective. Similarly, there is an injection $\mathcal{C}_{c}^{\infty}(\mathrm{HC}) \rightarrow \mathcal{E}^{\prime}(\mathrm{HC})$.
b) The spaces $\mathcal{C}_{c}^{\infty}(\mathrm{HC})$ and $C_{c}^{\infty}(\mathbb{R}) \otimes \Lambda V^{*}$ are isomorphic as algebras and as topological vector spaces. Hence, the topological dual $\mathcal{D}^{\prime}(\mathrm{HC})$ can be identified with $\mathcal{D}^{\prime}(\mathbb{R}) \otimes \Lambda V$. For $U \in \mathcal{D}^{\prime}(\mathrm{HC})$ we denote $U\left(a_{I}\right) \in \mathcal{D}^{\prime}(\mathbb{R})$ the distribution determined by $U\left(f \otimes \xi^{I}\right)=U\left(a_{I}\right)(f)$. Similarly, we define $U\left(a_{I}\right) \in \mathcal{E}^{\prime}(\mathbb{R})$ if $U \in \mathcal{E}^{\prime}(\mathrm{HC})$.
Lemma 2.23. Let $F, G \in \mathcal{C}^{\infty}(\mathrm{HC})$, and assume that one of $F, G$ is compactly supported. Then

$$
\langle F, G\rangle=\sum_{i} \int_{\mathbb{R}}(-1)^{\left|\gamma_{i}^{(1)}\right|\left|\gamma_{i}^{(2)}\right|} F\left(\gamma_{i}^{(1)} ; x\right) G\left(\gamma_{i}^{(2)} ; x\right) \mathrm{d} x .
$$

Similarly, if $U \in \mathcal{D}^{\prime}(\mathrm{HC})$ or $U \in \mathcal{E}^{\prime}(\mathrm{HC})$, there are distributions $U\left(\gamma_{i}^{(1)}\right) \in$ $\mathcal{D}^{\prime}(\mathbb{R})$ or in $\mathcal{E}^{\prime}(\mathbb{R})$ such that

$$
\begin{equation*}
\langle U, \Phi\rangle=\sum_{i}(-1)^{\left|\gamma_{i}^{(1)}\right|\left|\gamma_{i}^{(2)}\right|}\left\langle U\left(\gamma_{i}^{(1)}\right), \Phi\left(\gamma_{i}^{(2)}\right)\right\rangle \tag{2.6}
\end{equation*}
$$

for all $\Phi \in \mathcal{C}_{c}^{\infty}(\mathrm{HC})$ or in $\mathcal{C}^{\infty}(\mathrm{HC})$, respectively.

Proof. By definition,

$$
\begin{aligned}
(F \cdot G)(\gamma ; x) & =\mu((F \otimes G)(\Delta(\gamma) ; x, x)) \\
& =\sum_{i} \mu\left((F \otimes G)\left(\gamma_{i}^{(1)} \otimes \gamma_{i}^{(2)} ; x, x\right)\right) \\
& =\sum_{i}(-1)^{\left|\gamma_{i}^{(1)}\right||G|} F\left(\gamma_{i}^{(1)} ; x\right) G\left(\gamma_{i}^{(2)} ; x\right)
\end{aligned}
$$

Now observe that $\left|G\left(\gamma_{i}^{(2)}\right)\right|=|G|+\left|\gamma_{i}^{(2)}\right|=0$, since $C^{\infty}(\mathbb{R})$ is purely even, and it follows that $|G|=\left|\gamma_{i}^{(2)}\right|$.

The fact that $\langle U, \Phi\rangle$ can be written as $\sum_{i}(-1)^{\left|\gamma_{i}^{(1)}\right|\left|\gamma_{i}^{(2)}\right|} U\left(\gamma_{i}^{(1)}\right)\left(\Phi\left(\gamma_{i}^{(2)}\right)\right)$ follows from Remark 2.22 b ).

## 3 Representations

We define representations and unitary representations of Lie supergroup pairs. In this, we follow Alldridge [4, Appendix B] and Carmeli et.al. [7]. Then we use spin modules to construct a family $\left(\pi_{\zeta}\right)_{\zeta \in \mathbb{C}}$ of representations of HC. Salmasian showed in [17] that all irreducible unitary representations of HC are, up to unitary equivalence, of the form $\pi_{\zeta}$ with $\zeta$ real and positive. Since the representations $\pi_{\zeta}$ are a crucial ingredient in our definition of the Fourier transform, we give a detailed description.

Definition 3.1. Let $V$ be a finite-dimensional super vector space over $\mathbb{K} \in$ $\{\mathbb{R}, \mathbb{C}\}$. A representation $\pi=\left(\pi_{0}, d \pi\right)$ of a Lie supergroup pair $G=\left(G_{0}, \mathfrak{g}\right)$ on $V$ consists of a representation $\pi_{0}$ of $G_{0}$ on $V$ by even $\mathbb{K}$-linear maps, and a Lie superalgebra representation $d \pi$ of $\mathfrak{h c}$ on $V$, such that $d\left(\pi_{0}\right)=\left.d \pi\right|_{\mathfrak{g}_{0}}$ and $d \pi(\operatorname{Ad}(g) x)=\pi_{0}(g) d \pi(x) \pi_{0}\left(g^{-1}\right)$ for all $g \in G_{0}$ and $x \in \mathfrak{g}$.

The global functions $\mathbb{A}:=\mathcal{C}_{G}^{\infty}\left(G_{0}\right)$ form a supercommutative $\mathbb{R}$-superalgebra. If $V$ is a finite-dimensional super vector space, then so is End $(V)$, and we define $\mathcal{C}^{\infty}\left(G_{0}, \underline{\operatorname{End}}(V)\right):=(\mathbb{A} \otimes \operatorname{End}(V))_{0}$. This space can be identified with the space $\operatorname{End}_{\mathbb{A}}(\mathbb{A} \otimes V)$ of even $\mathbb{A}$-linear endomorphisms of the left $\mathbb{A}$-module $\mathbb{A} \otimes V$. Consider the subset $G L(\mathbb{A} \otimes V)$ of invertible $\mathbb{A}$-linear endomorphisms. We have the following characterization of linear representations of HC on $V$.

Proposition 3.2. Linear representations $\pi$ of $G$ on a finite-dimensional super vector space $V$ are in bijective correspondence with elements $F \in \mathrm{GL}(\mathbb{A} \otimes V) \subset$ $\mathcal{C}^{\infty}\left(G_{0}, \underline{\text { End }}(V)\right)$ which satisfy

$$
\left(m^{*} \otimes \mathrm{id}_{V}\right) \circ F=\left(\mathrm{id}_{V} \otimes F\right) \circ F \quad \text { and } \quad\left(e^{*} \otimes \mathrm{id}_{V}\right) \circ F=\mathrm{id}_{V}
$$

Proof. See [4, Proposition B.19]. For later use, we just note that the element $F \in \mathcal{C}^{\infty}\left(G_{0}, \underline{\text { End }}(V)\right)$ corresponding to a representation $\pi=\left(\pi_{0}, d \pi\right)$ is given by

$$
\begin{equation*}
F(u ; x)=\pi_{0}(x) \circ d \pi(u) \in \underline{\operatorname{End}}(V) \tag{3.1}
\end{equation*}
$$

Definition 3.3. Let $(\mathcal{H},(\cdot, \cdot))$ be a $\mathbb{Z}_{2}$-graded Hilbert space over $\mathbb{C}$. We say that $(\mathcal{H},(\cdot, \cdot))$ is a super Hilbert space if the graded pieces are orthogonal with respect to $(\cdot, \cdot)$. If $(\mathcal{H},(\cdot, \cdot))$ is a super Hilbert space, we define the super inner product by $\langle u \mid v\rangle:=i^{|u||v|}(u, v)$, and the super adjoint $T^{\dagger}$ of a continuous linear operator by $T^{\dagger}:=(-1)^{|T|} T^{*}$, where $T^{*}$ is the usual adjoint.

Remark 3.4. The definitions of $\langle\cdot \mid \cdot\rangle$ and $T^{\dagger}$ are such that

$$
\langle u \mid v\rangle=(-1)^{|u||v|} \overline{\langle v \mid u\rangle} \quad \text { and } \quad\langle T u \mid v\rangle=(-1)^{|u||T|}\left\langle u \mid T^{\dagger} v\right\rangle .
$$

Definition 3.5. A representation $\pi=\left(\pi_{0}, d \pi\right)$ of a Lie supergroup pair $\left(G_{0}, \mathfrak{g}\right)$ on a finite-dimensional super Hilbert space is unitary if $\pi_{0}$ is a unitary representation of $G_{0}$ and $d \pi(u)^{\dagger}=-d \pi(u)$ for all $u \in \mathfrak{g}$.

Remark 3.6. a) Observe that if $\pi$ is a unitary representation, then

$$
d \pi(u)^{\dagger}=d \pi(S(u))
$$

for all $u \in \mathfrak{U}(\mathfrak{g})$.
b) We restrict our attention to finite-dimensional representations because all irreducible unitary representations of HC are finite-dimensional. In the general setting, there are technical subleties due to the fact that the operators $d \pi(x), x \in \mathfrak{g}_{1}$ are in general unbounded (see [7] Definition 2]).
c) Suppose that $\pi=\left(\pi_{0}, d \pi\right)$ is a unitary representation of $G=\left(G_{0}, \mathfrak{g}\right)$. If we let $\rho(x)=e^{-i \pi / 4} d \pi(x)$ for $x \in \mathfrak{g}_{1}$, then the $\rho(x)$ are self-adjoint and satisfy

$$
\rho(x) \rho(y)+\rho(y) \rho(x)=-i d \pi([x, y])
$$

(see [7, Section 2.3] for details).
We will use this observation in the next subsection by first constructing operators $c_{\zeta}(v)$ for $v \in \mathfrak{h c}_{1}=V$, which are self-adjoint if $\zeta$ is real and nonnegative, and then setting

$$
d \pi_{\zeta}(v)=e^{i \pi / 4} c_{\zeta}(v)
$$

for $v \in V$.

## Spin Modules

The construction by Carmeli et.al. [7] and Salmasian [17] of unitary representations of HC is based on the following idea. If $\mathcal{H}$ is an irreducible unitary representation of HC, then by a super version of Schur's lemma, the central element $z$ acts by a scalar $i \zeta$. This scalar $\zeta$ has to be positive, essentially because $z$ is the square of an odd element in $\mathfrak{h c}$. The operators $c_{\zeta}(v)=e^{-i \pi / 4} d \pi(v)$ for $v \in V$ are self-adjoint and satisfy

$$
\left.\left[c_{\zeta}(v), c_{\zeta} w\right)\right]=2 \zeta \beta(v, w) \mathrm{id}
$$

This means that $c_{\zeta}$ factors through a representation of the quotient of the complexified universal enveloping algebra $\mathfrak{U}(\mathfrak{h c})_{\mathbb{C}}$ by the ideal generated by $z-\zeta$. But this quotient is a Clifford algebra $C l\left(V_{\mathbb{C}}, \zeta \beta\right)$, and the irreducible representations of Clifford algebras are well-known.

We will need this construction also for general $\zeta \in \mathbb{C}$, in which case the corresponding representations are no longer unitary. Also, we need some refined information about the representation, and therefore we recall the construction in some detail. As additional references, we use the exposition by Deligne [9, Proposition 2.2], and the book by Rosenberg [16, Section 2.2.2].

Proposition 3.7. Consider the complex space $\left(V_{\mathbb{C}}, \zeta \beta\right)$, where $\zeta$ is any non-zero complex number.
a) If $\operatorname{dim} V=2 k>0$ is even, then $C l\left(V_{\mathbb{C}}, \zeta \beta\right)$ is isomorphic as complex superalgebra to $\mathcal{H}=\underline{\text { End }}(\mathcal{S})$, where $\mathcal{S}$ is the complex super vector space $\mathbb{C}^{N \mid N}, N=2^{k-1}$.
b) Let $D$ be the superalgebra $\mathbb{C}[\epsilon]$ with $\epsilon$ odd and $\epsilon^{2}=\zeta$. If $\operatorname{dim} V=2 k+1$, then $C l\left(V_{\mathbb{C}}, \zeta \beta\right)$ is isomorphic as complex superalgebra to $\mathcal{H}=$ End $_{D}(\mathcal{S})$, where $\mathcal{S}=D^{N}=D \otimes_{\mathbb{C}} \mathbb{C}^{N}, N=2^{k}$ is a left $D$-module.

Proof. We first consider the case $\operatorname{dim} V_{\mathbb{C}}=2 k>0$. The choice of an orthonormal basis in $V$ yields a tensor product decomposition

$$
C l\left(V_{\mathbb{C}}, \zeta \beta\right)=C l\left(\mathbb{C}^{2}\right) \otimes \cdots \otimes C l\left(\mathbb{C}^{2}\right)
$$

(see [9]), where the spaces $\mathbb{C}^{2}$ are equipped with the bilinear form $(u, v)=$ $\zeta\left(u_{1} v_{1}+u_{2} v_{2}\right)$. Therefore, it suffices to consider the case $k=1$. Let $a_{1}, a_{2}$ be an orthonormal basis of $V_{\mathbb{C}}$ and let

$$
c_{\zeta}\left(a_{1}\right):=\left(\begin{array}{cc}
0 & \zeta \\
1 & 0
\end{array}\right), \quad c_{\zeta}\left(a_{2}\right):=\left(\begin{array}{cc}
0 & i \zeta \\
-i & 0
\end{array}\right) \in \underline{\operatorname{End}}\left(\mathbb{C}^{1,1}\right) .
$$

This clearly defines a representation of $C l\left(\mathbb{C}^{2}\right)$ and the arguments in 9 show that $c_{\zeta}$ is an algebra isomorphism.

Now we turn to the case of $\operatorname{dim} V_{\mathbb{C}}=2 k+1$, and again we can reduce to $k=1$. It suffices to construct elements $c_{\zeta}\left(a_{i}\right) \in \operatorname{End}_{D}\left(D \otimes \mathbb{C}^{2}\right)$ for a basis $a_{0}, a_{1}, a_{2}$ of $V_{\mathbb{C}}$. To this end, we follow [7].

We let $x_{0}=\epsilon \otimes \mathrm{id}, x_{1}=1 \otimes c_{1}\left(a_{1}\right)$ and $x_{2}=1 \otimes c_{1}\left(a_{2}\right)$ in $D \otimes \operatorname{Mat}(2, \mathbb{C})$, where the $c_{1}\left(a_{i}\right)$ are defined as in the even case with $\zeta=1$. Now we let

$$
c_{\zeta}\left(a_{0}\right)=i x_{0} x_{1} x_{2}, c_{\zeta}\left(a_{1}\right)=-i c_{\zeta}\left(a_{0}\right) x_{1}, c_{\zeta}\left(a_{2}\right)=-i c_{\zeta}\left(a_{0}\right) x_{2}
$$

(see [7]). Then a simple computation shows that this defines a representation of $C l\left(V_{\mathbb{C}}, \zeta \beta\right)$, and an isomorphism of superalgebras.

Definition 3.8. We define the symbol $[n]$ for $n \in \mathbb{N}$ by $[n]=n / 2$ if $n$ is even and $[n]=(n+1) / 2$ if $n$ is odd. Let $\mathcal{H}$ be as in Proposition 3.7, and consider
the linear functional $T$ on $\mathcal{H}$ defined by

$$
T(A)= \begin{cases}\operatorname{STr}(A), & n \text { even } \\ \operatorname{Tr}\left(e^{i \pi / 4} \epsilon A\right), & n \text { odd }\end{cases}
$$

We define a sesquilinear form on $\mathcal{H}$ by $\langle A \mid B\rangle:=T\left(A B^{\dagger}\right)$.
Lemma 3.9. Let $\mathcal{S}$ and $\mathcal{H}$ be as in Proposition 3.7.
a) If $\zeta$ is real, then $\mathcal{S}$ carries a sesquilinear form $(\cdot, \cdot)_{\zeta}$ such that $c_{\zeta}(v)$ is selfadjoint for $v \in V$. The sesquilinear form is positive definite if $\zeta>0$. If $\zeta<0$ the form is positive definite on $\mathcal{S}_{0}$ and negative definite on $\mathcal{S}_{1}$.
b) Let $\left(a_{i}\right)_{i=1}^{n}$ be an orthonormal basis of $V$. Then

$$
T\left(e^{i n \pi / 4} c_{\zeta}(\gamma)\right)=(2 \zeta)^{[n]}
$$

and $T\left(c_{\zeta}\left(a_{I}\right)\right)=0$ if $I \subsetneq \underline{n}$. The sesquilinear form $\langle A \mid B\rangle=T\left(A B^{\dagger}\right)$ on $\mathcal{H}$ is non-degenerate, positive definite on $\mathcal{H}_{0}$ and negative definite on $\mathcal{H}_{1}$.

Proof. a) Assume that $\zeta$ is real. In the case of $n=2$ we define the sesquilinear form $(\cdot, \cdot)_{\zeta}$ on $\mathcal{S}=\mathbb{C}^{2}$ by $(u, v)_{\zeta}=u_{1} \bar{v}_{1}+\zeta u_{2} \bar{v}_{2}$. Clearly, the $c_{\zeta}\left(a_{i}\right)$ are selfadjoint. Moreover, the inner product is negative definite on $\mathcal{S}_{1}$ if $\zeta<0$. Taking tensor products yields the general result for the even case.

Now we consider the case when $n$ is odd. We identify $D \cong \mathbb{C}^{2}$, so that multiplication by $\epsilon$ has matrix $\left(\begin{array}{ll}0 & \zeta \\ 1 & 0\end{array}\right)$ with respect to the standard basis of $\mathbb{C}^{2}$. If we define $(\cdot, \cdot)_{\zeta}$ on $D$ by $(u, v)_{\zeta}=u_{1} \bar{v}_{1}+\zeta u_{2} \bar{v}_{2}$, then multiplication by $\epsilon$ is self-adjoint. By definition (see proof of Proposition 3.7), the operators $c_{\zeta}\left(a_{i}\right)$ are of the form $\epsilon \otimes A$ where $A$ is self-adjoint with respect to the standard inner product on $\mathbb{C}^{2^{k}}$. Extending these inner products to $\mathcal{S}=D \otimes \mathbb{C}^{2^{k}}$ makes all $c_{\zeta}(v), v \in V$ self-adjoint. Again, it is clear that if $\zeta<0$, then the inner product is negative definite on $\mathcal{S}_{1}$.
b) If $n=2$,

$$
c_{\zeta}\left(a_{1} a_{2}\right)=\left(\begin{array}{cc}
-i \zeta & 0 \\
0 & i \zeta
\end{array}\right)
$$

shows that $T\left(i c_{\zeta}(\gamma)\right)=2 \zeta$. If $n=3$, we have

$$
c_{\zeta}\left(a_{0}\right)=\epsilon \otimes\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right), c_{\zeta}\left(a_{1}\right)=\epsilon \otimes\left(\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right), c_{\zeta}\left(a_{2}\right)=\epsilon \otimes\left(\begin{array}{cc}
0 & 1 \\
1 & 0
\end{array}\right)
$$

and hence $c_{\zeta}(\gamma)=\zeta \epsilon \otimes\left(\begin{array}{cc}-i & 0 \\ 0 & -i\end{array}\right)$. If we multiply by $i \epsilon$, we obtain $i \epsilon c_{\zeta}(\gamma)=$ $\zeta^{2} \otimes \mathrm{id}$, and this endomorphism has trace $4 \zeta^{2}=(2 \zeta)^{[3]}$.

The fact that the trace of the tensor product of two linear maps is given by the product of the respective traces yields the statement for arbitrary $n$.

Remark 3.10. Note that the entries of $c_{\zeta}(u)$ are polynomial in $\zeta$. Hence for fixed $u \in \mathfrak{U}(\mathfrak{h c})$, the $\operatorname{map} \zeta \mapsto c_{\zeta}(u)$ is an entire holomorphic $\mathcal{H}$-valued function on $\mathbb{C}$.

Now we use the above construction to define representations of the HeisenbergClifford Lie supergroup.
Definition 3.11. Let $\mathcal{S}, \mathcal{H}$ and $c_{\zeta}$ be as in Proposition 3.7 and $(\cdot, \cdot)_{\zeta}$ as in Lemma 3.9 For every $\zeta \in \mathbb{C}$ we define a representation $\pi_{\zeta}=\left(\pi_{\zeta, 0}, d \pi_{\zeta}\right)$ of HC on $S$ by $\pi_{\zeta, 0}(x)=e^{i x \zeta}$ for $x \in \mathbb{R}$ and $d \pi_{\zeta}(v)=e^{i \pi / 4} c_{\zeta}(v)$ for $v \in \mathfrak{h c}$.

We also denote by $\pi_{\zeta}$ the element of $\mathcal{C}^{\infty}(\mathrm{HC}, \mathcal{H})$ corresponding to $\left(\pi_{\zeta, 0}, d \pi_{\zeta}\right)$ by Proposition 3.2. which is given by $\pi_{\zeta}(u ; x)=e^{i \zeta x} d \pi_{\zeta}(u)$ for $x \in \mathbb{R}$ and $u \in \mathfrak{U}(\mathfrak{h c})$. If $\zeta>0$ is real, then the representation $\pi_{\zeta}$ is unitary on the super Hilbert space $\left(\mathcal{S},(\cdot, \cdot)_{\zeta}\right)$.

Remark 3.12. The element $\gamma$ acts by $d \pi_{\zeta}(\gamma)=\left(e^{i \pi / 4}\right)^{n} c_{\zeta}(\gamma)$ Together with Lemma 3.9 this implies

$$
T\left(d \pi_{\zeta}(\gamma)\right)=(2 \zeta)^{[n]}
$$

and $T\left(d \pi_{\zeta}\left(a_{I}\right)\right)=0$ if $I \subsetneq \underline{n}$.

## 4 The Fourier Transform

Throughout the remainder of this article, we let $\mathcal{S}$ and $\mathcal{H}$ be as in Proposition 3.7, and for $\zeta \in \mathbb{C}$, we let $\left(\pi_{\zeta}, \mathcal{S}\right)$ be the representation of HC defined in 3.11 By Proposition 3.2 there is a unique element of $\mathcal{C}^{\infty}(\mathrm{HC}, \mathcal{H})$ corresponding to $\pi_{\zeta}$, which we denote by the same letter. We use the $\mathcal{H}$-valued function $\pi_{\zeta}$ to define the Fourier transform of a compactly supported smooth function on HC .

Definition 4.1. If $F \in \mathcal{C}_{c}^{\infty}(\mathrm{HC})$, we define the Fourier transform $\mathcal{F}(F)$ or $\widehat{F}$ by

$$
\mathcal{F}(F): \mathbb{C} \rightarrow \mathcal{H}, \quad \mathcal{F}(F)(\zeta):=\widehat{F}(\zeta):=\int_{\mathrm{HC}} F \cdot \pi_{-\zeta}
$$

This is well-defined since it is the integral of a compactly supported smooth function with values in the finite-dimensional complex vector space $\mathcal{H}$.
Remark 4.2. The Fourier transform of $F \in \mathcal{C}_{c}^{\infty}(\mathrm{HC})$ can be computed explicitly as

$$
\begin{aligned}
\left\langle F, \pi_{-\zeta}\right\rangle & =\sum_{i} \int_{\mathbb{R}} F\left(\gamma_{i}^{(1)} ; x\right) \pi_{-\zeta}\left(\gamma_{i}^{(2)} ; x\right) \mathrm{d} x \\
& =\sum_{i} \int_{\mathbb{R}} F\left(\gamma_{i}^{(1)} ; x\right) e^{-i \zeta x} d \pi_{-\zeta}\left(\gamma_{i}^{(2)}\right) \mathrm{d} x
\end{aligned}
$$

Here, we have used that representations have even parity, and that $\pi_{-\zeta}(u ; x)=$ $\pi_{-\zeta, 0}(x) d \pi_{-\zeta}(u)=e^{-i \zeta x} d \pi_{-\zeta}(u)$. We conclude that

$$
\begin{equation*}
\widehat{F}(\zeta)=\sum_{i} F\left(\gamma_{i}^{(1)}\right)^{\curlyvee}(\zeta) d \pi_{-\zeta}\left(\gamma_{i}^{(2)}\right) \tag{4.1}
\end{equation*}
$$

where $F\left(\gamma_{i}^{(1)}\right)^{\text {w }}$ is the classical Fourier transform of $F\left(\gamma_{i}^{(1)}\right) \in C_{c}^{\infty}(\mathbb{R})$. In particular, $\widehat{F}$ extends to an entire holomorphic $\mathcal{H}$-valued function.

An immediate consequence of this remark is the following Fourier inversion formula.

Proposition 4.3. For $F \in \mathcal{C}_{c}^{\infty}(\mathrm{HC})$ we have

$$
F(1 ; x)=\frac{1}{2 \pi} \int_{\mathbb{R}} T(\widehat{F}(\zeta))(-2 \zeta)^{-[n]} e^{i x \zeta} \mathrm{~d} \zeta
$$

Proof. We apply $T$ to the sum in equation (4.1). By Remark 3.12, only the summand with $\gamma_{i}^{(2)}=\gamma$ contributes. Then, $\gamma_{i}^{(1)}=1$ and we obtain

$$
T(\widehat{F}(\zeta))=F(1)^{\curlyvee}(\zeta) T\left(d \pi_{-\zeta}(\gamma)\right)=F(1)^{\Upsilon}(\zeta)(-2 \zeta)^{[n]}
$$

so that the claim follows from the classical Fourier inversion formula.
Theorem 4.4. The Fourier transform satisfies

$$
\left(L_{u ; x} F\right) \subsetneq(\zeta)=\pi_{-\zeta}(u ; x) \widehat{F}(\zeta)
$$

for all $x \in \mathbb{R}$ and $u \in \mathfrak{U}(\mathfrak{h c}(V))$.
Proof. Invariance of the integral implies

$$
\left(L_{u ; x} F\right)^{\Upsilon}(\zeta)=\left\langle L_{u ; x} F, \pi_{-\zeta}\right\rangle=(-1)^{|u||F|}\left\langle F, L_{S(u) ;-x} \pi_{-\zeta}\right\rangle .
$$

We have
$L_{S(u) ;-x} \pi_{-\zeta}(v, y)=\pi_{-\zeta}(u v ; x+y)=\pi_{-\zeta, 0}(x+y) d \pi_{-\zeta}(u v)=\pi_{-\zeta}(u ; x) \pi_{-\zeta}(v ; y)$, by equation (3.1) and because $\pi_{-\zeta, 0}(y)$ commutes with $d \pi_{-\zeta}(u)$. This implies

$$
\left\langle L_{u ; x} F, \pi_{-\zeta}\right\rangle=(-1)^{|u||F|}\left\langle F, \pi_{-\zeta}(u ; x) \pi_{-\zeta}\right\rangle=\pi_{-\zeta}(u ; x)\left\langle F, \pi_{-\zeta}\right\rangle,
$$

and hence the claim.
Remark 4.5. Theorem 4.4 together with Proposition 4.3, implies

$$
\begin{aligned}
F(u ; x) & =(-1)^{|u||F|}\left(L_{S(u)} F\right)(1 ; x) \\
& =(-1)^{|u||F|} \frac{1}{2 \pi} \int_{\mathbb{R}} T\left(\mathcal{F}\left(L_{S(u)} F\right)(\zeta)\right)(-2 \zeta)^{-[n]} e^{i x \zeta} \mathrm{~d} \zeta \\
& =(-1)^{|u||F|} \frac{1}{2 \pi} \int_{\mathbb{R}} T\left(d \pi_{-\zeta}(S(u)) \widehat{F}(\zeta)\right)(-2 \zeta)^{-[n]} e^{i x \zeta} \mathrm{~d} \zeta .
\end{aligned}
$$

Recall that $d \pi(S(u))=d \pi(u)^{\dagger}$ for unitary representations $\pi=\left(\pi_{0}, d \pi\right)$. Therefore, using $\langle A \mid B\rangle=T\left(A B^{\dagger}\right)$ we can write

$$
\begin{aligned}
F(u ; x) & =(-1)^{|u||F|} \frac{1}{2 \pi} \int_{\mathbb{R}} T\left(d \pi_{-\zeta}(u)^{\dagger} \widehat{F}(\zeta)\right)(-2 \zeta)^{-[n]} e^{i x \zeta} \mathrm{~d} \zeta \\
& =\frac{1}{2 \pi} \int_{\mathbb{R}} T\left(\widehat{F}(\zeta) d \pi_{-\zeta}(u)^{\dagger}\right)(-2 \zeta)^{-[n]} e^{i x \zeta} \mathrm{~d} \zeta \\
& =\frac{1}{2 \pi} \int_{\mathbb{R}}\left\langle\widehat{F}(\zeta) \mid d \pi_{-\zeta}(u)\right\rangle(-2 \zeta)^{-[n]} e^{i x \zeta} \mathrm{~d} \zeta .
\end{aligned}
$$

Definition 4.6. a) If $A: \mathbb{R} \rightarrow \mathcal{H}$ is any map, we define

$$
A(u ; \zeta):=\left\langle A(\zeta) \mid d \pi_{-\zeta}(u)\right\rangle(-2 \zeta)^{-[n]}
$$

for $u \in \mathfrak{U}(\mathfrak{h c})$ and non-zero $\zeta \in \mathbb{R}$. Note that

$$
A(z u ; \zeta)=\left\langle A(\zeta) \mid d \pi_{\zeta}(z u)\right\rangle(2 \zeta)^{-[n]}=(i \zeta) A(u ; \zeta)
$$

since $\langle\cdot \mid \cdot\rangle$ is conjugate linear in the second argument.
b) We say that $A: \mathbb{R} \rightarrow \mathcal{H}$ is of Schwartz class if for all $u \in \mathfrak{U}(\mathfrak{h c})$ and all $k \geq 0$, the function $A(u ; \zeta)$ is smooth on $\mathbb{R}$, and

$$
s_{k, u}(A):=\sup _{\zeta \in \mathbb{R}}\left|\frac{d^{k}}{d \zeta^{k}} A(u ; \zeta)\right|<\infty
$$

The space of $\mathcal{H}$-valued functions of Schwartz class is denoted $\mathcal{S}(\mathbb{R}, \mathcal{H})$.
c) If $A \in \mathcal{S}(\mathbb{R}, \mathcal{H})$, we define

$$
\begin{equation*}
\mathcal{F}^{-1}(A)(u ; x):=\frac{1}{2 \pi} \int_{\mathbb{R}} A(u ; \zeta) e^{i x \zeta} \mathrm{~d} \zeta \tag{4.2}
\end{equation*}
$$

If $\left(a_{i}\right)_{i=1}^{n}$ is a basis of $V$ and we let $s_{k, j, J}:=s_{k, z^{j} a_{J}}$, then the countable family $\left(s_{k, j, J}\right)$ defines a locally convex vector space topology on $\mathcal{S}(\mathbb{R}, \mathcal{H})$.
Remark 4.7. If $A \in \mathcal{S}(\mathbb{R}, \mathcal{H})$, then $\mathcal{F}^{-1}(A)$ is an element of $\mathcal{C}^{\infty}(\mathrm{HC})$, because

$$
\mathcal{F}^{-1}(A(z u))(x)=\mathcal{F}^{-1}(i \zeta A(u))(x)=\frac{\mathrm{d}}{\mathrm{~d} x} \mathcal{F}^{-1}(A(u))(x)
$$

Lemma 4.8. A function $A: \mathbb{R} \rightarrow \mathcal{H}$ is of Schwartz class if and only if for all $u \in \mathfrak{U}(\mathfrak{h c})$, the function $x \mapsto A(u ; x)$ is in $\mathcal{S}(\mathbb{R})$.
Proof. Clearly, if all $A(u)$ are in $\mathcal{S}(\mathbb{R})$, then $s_{k, u}(A)<\infty$ for all $u \in \mathfrak{U}(\mathfrak{h} \mathfrak{c}), k \in \mathbb{N}$. Conversely, suppose that $A \in \mathcal{S}(\mathbb{R}, \mathcal{H})$. Fix $u \in \mathfrak{U}(\mathfrak{h c})$. Then

$$
s_{k, z^{j} u}(A)=\sup _{\zeta \in \mathbb{R}}\left|\frac{d^{k}}{d \zeta^{k}} A\left(z^{j} u ; \zeta\right)\right|=\sup _{\zeta \in \mathbb{R}}\left|\frac{d^{k}}{d \zeta^{k}} \zeta^{j} A(u ; \zeta)\right|<\infty
$$

for all $k, j \in \mathbb{N}$, where we have used that $A\left(z^{j} u ; \zeta\right)=(i \zeta)^{j} A(u ; \zeta)$. It follows that $A(u) \in \mathcal{S}(\mathbb{R})$.

Theorem 4.9. The Fourier transform is an isomorphism of the topological vector spaces $\mathcal{S}(\mathrm{HC})$ and $\mathcal{S}(\mathbb{R}, \mathcal{H})$. Its inverse is given by (4.2).
Proof. Let $F \in \mathcal{S}(\mathrm{HC})$. Then

$$
\begin{aligned}
s_{k, j, J}(\widehat{F}) & =\sup _{\zeta \in \mathbb{R}}\left|\frac{d^{k}}{d \zeta^{k}} \widehat{F}\left(z^{j} a_{J} ; \zeta\right)\right|=\sup _{\zeta \in \mathbb{R}}\left|\frac{d^{k}}{d \zeta^{k}}\left\langle\widehat{F}(\zeta), d \pi_{-\zeta}\left(z^{j} a_{J}\right)\right\rangle\right| \\
& =\sup _{\zeta \in \mathbb{R}}\left|\frac{d^{k}}{d \zeta^{k}}\left(\zeta^{j} \widehat{F}\left(a_{J} ; \zeta\right)\right)\right|,
\end{aligned}
$$

which is finite because $\widehat{F}(u) \in \mathcal{S}(\mathbb{R})$ for all $u \in \mathfrak{U}(\mathfrak{h c})$. In particular, the last expression is continuous in $F$, which proves that $\mathcal{F}: \mathcal{S}(\mathrm{HC}) \rightarrow \mathcal{S}(\mathbb{R}, \mathcal{H})$ is continuous.

If $A \in \mathcal{S}(\mathbb{R}, \mathcal{H})$, then by Lemma 4.8, the components $A(u)$ are in $\mathcal{S}(\mathbb{R})$. It follows that $\mathcal{F}^{-1}(A) \in \mathcal{S}(\mathrm{HC})$. We apply the seminorms $s_{j, u}$, defined in 2.14, to obtain

$$
s_{j, u}\left(\mathcal{F}^{-1}(A)\right)=\sup _{x \in \mathbb{R}}\left|x^{j} \mathcal{F}^{-1}(A)(u ; x)\right|=\sup _{x \in \mathbb{R}}\left|\mathcal{F}^{-1}\left(\frac{\mathrm{~d}^{\mathrm{j}}}{\mathrm{~d} \zeta^{j}} A(u)\right)(x)\right| .
$$

The last expression involves the classical Fourier transform of the Schwartz function $\left(\mathrm{d}^{\mathrm{j}} / \mathrm{d} \zeta^{j}\right)(A(u))$ and is continuous in $A(u)$. This proves the continuity of $\mathcal{F}^{-1}$.

To conclude this section, we define a Fourier-Laplace transform for compactly supported distributions on HC and prove a theorem of Paley-WienerSchwartz type.

Definition 4.10. If $U \in \mathcal{E}^{\prime}(\mathrm{HC})$ is a compactly supported distribution, we let

$$
\widehat{U}(\zeta):=\left\langle U, \pi_{-\zeta}\right\rangle
$$

for $\zeta \in \mathbb{R}$.
Theorem 4.11. If $U \in \mathcal{E}^{\prime}(\mathrm{HC})$ we can extend the definition of $\widehat{U}(\zeta)$ to complex values of $\zeta$. The function $\widehat{U}$ is an entire holomorphic function with values in $\mathcal{H}$.
Proof. We write

$$
\langle U, F\rangle=\sum_{i}(-1)^{\left|\gamma_{i}^{(1)}\right||F|}\left\langle U\left(\gamma_{i}^{(1)}\right), F\left(\gamma_{i}^{(2)}\right)\right\rangle
$$

for appropriate distributions $U\left(\gamma_{i}^{(1)}\right) \in \mathcal{E}^{\prime}(\mathbb{R})$. Then

$$
\widehat{U}(\zeta)=U\left(\pi_{-\zeta}\right)=\sum_{i} U\left(\gamma_{i}^{(1)}\right)_{x}\left(\pi_{-\zeta}\left(\gamma_{i}^{(2)} ; x\right)\right)=\sum_{i} U\left(\gamma_{i}^{(1)}\right)_{x}\left(e^{-i \zeta x}\right) d \pi_{-\zeta}\left(\gamma_{i}^{(2)}\right) .
$$

Here $U\left(\gamma_{i}^{(1)}\right)_{x}\left(e^{-i \zeta x}\right)$ is the classical Fourier-Laplace transform of the compactly supported distribution $U\left(\gamma_{i}^{(1)}\right)$. In particular, $U\left(\gamma_{i}^{(1)}\right)_{x}\left(e^{-i \zeta x}\right)$ extends to an entire holomorphic function of $\zeta$ (see [11, Theorem 7.1.14]). The same is true for the matrices $d \pi_{-\zeta}\left(\gamma_{i}^{(2)}\right)$ by Remark 3.10, hence the claim follows.

Now we formulate a Paley-Wiener-Schwartz theorem, which characterizes the Fourier transformation of a compactly supported function by a growth condition on the 'components' $A(u)$ of $A=\mathcal{F}(F): \mathbb{C} \rightarrow \mathcal{H}$.

Definition 4.12. a) Let $\mathcal{C}_{[-a, a]}^{\infty}(\mathrm{HC})$ denote the space of functions $F \in \mathcal{C}_{c}^{\infty}(\mathrm{HC})$ for which the support of $F(u)$ is contained in the compact interval $[-a, a]$ for all $u \in \mathfrak{U}(\mathfrak{h c})$.
b) We say that an entire holomorphic function $A: \mathbb{C} \rightarrow \mathcal{H}$ is of exponential type $a$, where $a$ is a positive real number, if for every $N \in \mathbb{N}$ there is a constant $C_{N}$ such that

$$
\begin{equation*}
\left|\zeta^{-[n]} T\left(A(\zeta) d \pi_{-\zeta}(u)\right)\right| \leq C_{N}(1+|\zeta|)^{-N} e^{a \operatorname{Im}(\zeta)} \tag{4.3}
\end{equation*}
$$

for all $u \in \mathfrak{U}(\mathfrak{h} \mathfrak{c}), \zeta \in \mathbb{C}$.
Theorem 4.13. The Fourier-Laplace transform is a bijection between the space $\mathcal{C}_{[-a, a]}^{\infty}(\mathrm{HC})$ and the space of entire holomorphic functions $A: \mathbb{C} \rightarrow \mathcal{H}$ of exponential type $a$.

Proof. Let $F \in \mathcal{C}^{\infty}(\mathrm{HC})$. We use coordinates, so that

$$
\widehat{F}(\zeta)=\sum_{I \subset \underline{n}} F\left(a_{I}\right) \Upsilon(\zeta) d \pi_{-\zeta}\left(* a_{I}\right)
$$

and therefore

$$
\left.\mid \zeta^{-[n]} T\left(A(\zeta) d \pi_{-\zeta}\left(a_{I}\right)\right)\right)|=C| F\left(a_{I}\right)^{-}(\zeta) \mid
$$

for some positive constant $C$. By the corresponding classical Paley-WienerSchwartz theorem [11, Theorem 7.3.1], this shows that if $\operatorname{supp}\left(F\left(a_{I}\right)\right) \subset[-a, a]$, for all $I \subset \underline{n}$, then $\mathcal{F}(F)$ is of exponential type $a$.

Conversely, if $A: \mathbb{C} \rightarrow \mathcal{H}$ is of exponential type $a$, then the classical theorem implies that the components $A\left(a_{I}\right)$ are the Fourier transforms of smooth functions $F\left(a_{I}\right)$ with support in $[-a, a]$. The $F\left(a_{I}\right)$ now uniquely determine an element $F \in \mathcal{C}_{[-a, a]}^{\infty}(\mathrm{HC})$ with Fourier transform equal to $A$, and this concludes the proof.

## 5 The Convolution Product

In this section we study the convolution product for functions and distributions on HC. We prove basic properties of the convolution product, and in Theorem 5.7 we show that the Fourier transform interchanges the convolution product and the pointwise product of $\mathcal{H}$-valued functions. Lastly, we obtain a Banach convolution algebra as the completion of $\mathcal{C}_{c}^{\infty}(\mathrm{HC})$ with respect to a Sobolev norm.

Definition 5.1. Let $F, G \in \mathcal{C}^{\infty}(\mathrm{HC})$ and assume that one of $F, G$ is compactly supported. We define the convolution of $F$ and $G$ by

$$
(F * G)(u ; x):=(-1)^{|u|(|G|+|\gamma|)}\left\langle F, L_{u ; x} i^{*} G\right\rangle
$$

Proposition 5.2. a) Suppose that one of $F, G \in \mathcal{C}^{\infty}(\mathrm{HC})$ has compact support.
Then the convolution product $F * G$ is in $\mathcal{C}^{\infty}(\mathrm{HC})$.
b) $\operatorname{supp}(F * G) \subset \operatorname{supp} F+\operatorname{supp} G$
c) $L_{u ; x}(F * G)=\left(L_{u ; x} F\right) * G$ for all $u \in \mathfrak{U}(\mathfrak{h c}), x \in \mathbb{R}$.
d)

$$
(F * G)(u)=\sum_{i}(-1)^{\left|\gamma_{i}^{(1)}\right|\left|\gamma_{i}^{(2)}\right| F\left(\gamma_{i}^{(1)}\right) * G\left(S\left(\gamma_{i}^{(2)}\right) u\right) . . . . . . . .}
$$

Proof. To check that $F * G \in \mathcal{C}^{\infty}(\mathrm{HC})$ we need to verify $\mathfrak{z}$-linearity. This follows easily from

$$
F\left(\gamma_{i}^{(1)}\right) * G\left(S\left(\gamma_{i}^{(2)}\right) z u\right)=F\left(\gamma_{i}^{(1)}\right) * G^{\prime}\left(S\left(\gamma_{i}^{(2)}\right) u\right)=\left(F\left(\gamma_{i}^{(1)}\right) * G\left(S\left(\gamma_{i}^{(2)}\right) u\right)\right)^{\prime}
$$

The inclusion of supports in $b$ ) follows easily as in the classical case. Statement $c$ ) follows easily from the invariance of the pairing:

$$
\begin{aligned}
L_{u ; x}(F * G)(v ; y) & =(-1)^{|u|(|F|+|G|+|\gamma|)}(F * G)(S(u) v ; y-x) \\
& =(-1)^{|u||F|+|v|(|G|+|\gamma|)}\left\langle F, L_{S(u) v ; y-x} i^{*} G\right\rangle \\
& =(-1)^{|v|(|G|+|\gamma|)}\left\langle L_{u ; x} F, L_{v ; y} i^{*} G\right\rangle \\
& =\left(\left(L_{u ; x} F\right) * G\right)(v ; y) .
\end{aligned}
$$

We compute $F * G$ as follows.

$$
\begin{aligned}
(F * G) & (u ; x)= \\
& =(-1)^{|u|(|G|+|\gamma|)} \sum_{i}(-1)^{\left|\gamma_{i}^{(1)}\right|(|u|+|G|)} \int_{\mathbb{R}} F\left(\gamma_{i}^{(1)} ; y\right) L_{u ; x} i^{*} G\left(\gamma_{i}^{(2)} ; y\right) \mathrm{d} y \\
& =(-1)^{|u||\gamma|} \sum_{i}(-1)^{\left|\gamma_{i}^{(1)}\right|(|u|+|G|)} \int_{\mathbb{R}} F\left(\gamma_{i}^{(1)} ; y\right)\left(i^{*} G\right)\left(S(u) \gamma_{i}^{(2)} ; y-x\right) \mathrm{d} y \\
& =(-1)^{|u||\gamma|} \sum_{i}(-1)^{|u||\gamma|+\left|\gamma_{i}^{(1)}\right||G|} \int_{\mathbb{R}} F\left(\gamma_{i}^{(1)} ; y\right) G\left(S\left(\gamma_{i}^{(2)}\right) u ; x-y\right) \mathrm{d} y \\
& =\sum_{i}(-1)^{\left|\gamma_{i}^{(1)}\right||G|}\left(F\left(\gamma_{i}^{(1)}\right) * G\left(S\left(\gamma_{i}^{(2)}\right) u\right)\right)(x)
\end{aligned}
$$

Now the claim follows, since the summands are non-zero only if $|G|=\left|\gamma_{i}^{(2)}\right|$.
Remark 5.3. Choose coordinates and consider the elements $F=f \otimes \xi^{I}$ and $G=g \otimes \xi^{J}$ of $\mathcal{C}_{c}^{\infty}(\mathrm{HC})$. We compute $F * G$ using Proposition 5.2 b).

$$
\begin{aligned}
(F * G)\left(a_{K}\right) & =\sum_{L \subset \underline{n}}(-1)^{|L|\left|L^{c}\right|} F\left(a_{L}\right) * G\left(* a_{L} a_{K}\right) \\
& =(-1)^{|I|\left|I^{c}\right|} f * G\left(* a_{I} a_{K}\right) .
\end{aligned}
$$

This is non-zero only if $* a_{I} a_{k}$ is a multiple of $a_{J}$. Now $* a_{I}=\operatorname{sgn}\left(\sigma_{I}\right) a_{I^{c}}$, and in $\mathfrak{U}(\mathfrak{h c})$ we have the equality

$$
a_{I^{c}} a_{(I \Delta J)^{c}}= \pm z^{\left|I^{c} \cap J^{c}\right|} a_{J} \in \mathfrak{U}(\mathfrak{h c}),
$$

where $I \Delta J$ is the symmetric difference of the subsets $I, J \subset \underline{n}$. Hence, we choose $K=(I \Delta J)^{c}$ and obtain

$$
\left(f \otimes \xi^{I}\right) *\left(g \otimes \xi^{J}\right)= \pm(f * g)^{\left(\left|I^{c} \cap J^{c}\right|\right)} \otimes \xi^{(I \Delta J)^{c}}
$$

Proposition 5.4. The convolution product is an $\mathbb{R}$-bilinear, continuous map $*: \mathcal{C}^{\infty}(\mathrm{HC}) \times \mathcal{C}_{c}^{\infty}(\mathrm{HC}) \rightarrow \mathcal{C}^{\infty}(\mathrm{HC})$ with parity $|\gamma|$, that is, $|F * G|=|F|+|G|+|\gamma|$.

Proof. Bilinearity is clear, and continuity can be conveniently checked in coordinates. By Remark 5.3 we have

$$
\left(f \otimes \xi^{I}, g \otimes \xi^{J}\right) \mapsto \pm(f * g)^{\left(\left|I^{c} \cap J^{c}\right|\right)} \otimes \xi^{(I \Delta J)^{c}}
$$

hence continuity follows from the corresponding result for functions on $\mathbb{R}$, together with continuity of the derivative as a map from $C^{\infty}(\mathbb{R})$ to $C^{\infty}(\mathbb{R})$. The parity of the convolution product equals the parity of the pairing $\langle\cdot, \cdot\rangle$, which is $|\gamma|$ by Remark 2.18.
Proposition 5.5. Let $F, G$ and $\Phi$ be smooth functions on HC , at least two of which are compactly supported. Then

$$
\langle F * G, \Phi\rangle=\left\langle F \otimes G, m^{*} \Phi\right\rangle
$$

and

$$
\langle F * G, \Phi\rangle=\left\langle F, i^{*}\left(G * i^{*} \Phi\right)\right\rangle .
$$

Furthermore, if one of $F, G$ is compactly supported, then

$$
F * G=(-1)^{|F||G|} i^{*}\left(i^{*} G * i^{*} F\right) .
$$

Proof. We compute $\langle F * G, \Phi\rangle$ for $\Phi \in \mathcal{C}_{c}^{\infty}(\mathrm{HC})$ as

$$
\begin{aligned}
& \langle F * G, \Phi\rangle=\sum_{j}(-1)^{|\Phi|\left|\gamma_{j}^{(1)}\right|} \int_{\mathbb{R}}(F * G)\left(\gamma_{j}^{(1)} ; y\right) \Phi\left(\gamma_{j}^{(2)} ; y\right) \mathrm{d} y \\
& =\sum_{i, j}(-1)^{\left|\Phi \left\|\gamma _ { j } ^ { ( 1 ) } \left|+\left|G \| \gamma_{i}^{(1)}\right|\right.\right.\right.} \int_{\mathbb{R}}\left(F\left(\gamma_{i}^{(1)}\right) * G\left(S\left(\gamma_{i}^{(2)}\right) \gamma_{j}^{(1)}\right)\right)(y) \Phi\left(\gamma_{j}^{(2)} ; y\right) \mathrm{d} y \\
& =\sum_{i, j}(-1)^{\left|\Phi \left\|\gamma _ { j } ^ { ( 1 ) } \left|+\left|G \| \gamma_{i}^{(1)}\right|\right.\right.\right.} \int_{\mathbb{R}^{2}} F\left(\gamma_{i}^{(1)} ; x\right) G\left(S\left(\gamma_{i}^{(2)}\right) \gamma_{j}^{(1)} ; y-x\right) \Phi\left(\gamma_{j}^{(2)} ; y\right) \mathrm{d} x \mathrm{~d} y \\
& =\sum_{i, j}(-1)^{\left|\Phi \left\|\gamma_{j}^{(1)}|+|G \| \gamma|\right.\right.} \int_{\mathbb{R}^{2}} F\left(\gamma_{i}^{(1)} ; x\right)\left(L_{\gamma_{i}^{(2)} ; x} G\right)\left(\gamma_{j}^{(1)} ; y\right) \Phi\left(\gamma_{j}^{(2)} ; y\right) \mathrm{d} x \mathrm{~d} y \\
& =\sum_{i}(-1)^{|G \| \gamma|} \int_{\mathbb{R}} F\left(\gamma_{i}^{(1)} ; x\right)\left(L_{\gamma_{i}^{(2)} ; x} G, \Phi\right) \mathrm{d} x .
\end{aligned}
$$

Next, we use the invariance of the pairing $\langle\cdot, \cdot\rangle$ to obtain

$$
\begin{aligned}
\langle F * G, \Phi\rangle= & \sum_{i}(-1)^{|G|\left|\gamma_{i}^{(1)}\right|} \int_{\mathbb{R}} F\left(\gamma_{i}^{(1)} ; x\right)\left(G, L_{S\left(\gamma_{i}^{(2)}\right) ;-x} \Phi\right) \mathrm{d} x \\
= & \sum_{i, j}(-1)^{|G|\left|\gamma_{i}^{(1)}\right|+|\Phi|\left(\left|\gamma_{i}^{(1)}\right|+\left|\gamma_{j}^{(1)}\right|\right)+\left|\gamma_{j}^{(1)}\right|\left|\gamma_{i}^{(2)}\right|} \\
& \quad \int_{\mathbb{R}^{2}} F\left(\gamma_{i}^{(1)} ; x\right) G\left(\gamma_{j}^{(1)} ; x\right) \Phi\left(\gamma_{i}^{(2)} \gamma_{j}^{(2)} ; x+y\right) \mathrm{d} x \mathrm{~d} y \\
= & \left\langle F \otimes G, m^{*} \Phi\right\rangle .
\end{aligned}
$$

The function $i^{*}\left(G * i^{*} \Phi\right)$ takes values

$$
\begin{aligned}
i^{*}\left(G * i^{*} \Phi\right)(v ; y) & =\left(G * i^{*} \Phi\right)(S(v) ;-y) \\
& =\sum_{j}(-1)^{|\Phi|\left|\gamma_{j}^{(1)}\right|}\left(G\left(\gamma_{j}^{(1)}\right) *\left(i^{*} \Phi\right)\left(S\left(\gamma_{j}^{(2)}\right) S(v)\right)\right)(-y) \\
& =\sum_{j}(-1)^{|\Phi|\left|\gamma_{j}^{(1)}\right|} \int_{\mathbb{R}} G\left(\gamma_{j}^{(1)} ; x\right)\left(i^{*} \Phi\right)\left(S\left(\gamma_{j}^{(2)}\right) S(v)\right)(-x-y) \mathrm{d} x \\
& =\sum_{j}(-1)^{\left|\Phi \| \gamma_{j}^{(1)}\right|} \int_{\mathbb{R}} G\left(\gamma_{j}^{(1)} ; x\right) \Phi\left(\gamma_{j}^{(2)} v ; x+y\right) \mathrm{d} x .
\end{aligned}
$$

Therefore, comparing with the above computations yields

$$
\begin{aligned}
\langle F * G, \Phi\rangle & =\sum_{i}(-1)^{\left|\gamma_{i}^{(1)}\right|(|G|+|\Phi|)} \int_{\mathbb{R}} F\left(\gamma_{i}^{(1)} ; x\right) i^{*}\left(G * i^{*} \Phi\right)\left(\gamma_{i}^{(2)} ; x\right) \mathrm{d} x \\
& =\left\langle F, i^{*}\left(G * i^{*} \Phi\right)\right\rangle
\end{aligned}
$$

Lastly, we compute

$$
\begin{aligned}
(F * G)(u ; x) & =(-1)^{|u|(|G|+|\gamma|)}\left\langle F, L_{u ; x} i^{*} G\right\rangle \\
& =(-1)^{|u|(|G|+|\gamma|)+|F|(|u|+|G|)}\left\langle L_{u ; x} i^{*} G, F\right\rangle \\
& =(-1)^{|F||G|+|u|(|F|+|\gamma|)}\left\langle i^{*} G, L_{S(u) ;-x} F\right\rangle \\
& =(-1)^{|F||G|}\left(i^{*} G * i^{*} F\right)(S(u) ;-x),
\end{aligned}
$$

which proves that $F * G=(-1)^{|F||G|} i^{*}\left(i^{*} G * i^{*} F\right)$.
For later use we record the following corollary of our proof.
Corollary 5.6. a)

$$
\langle F * G, \Phi\rangle=\left\langle F, i^{*}\left(G * i^{*} \Phi\right)\right\rangle=\sum_{i}(-1)^{|G \||\gamma|} \int_{\mathbb{R}} F\left(\gamma_{i}^{(1)} ; x\right)\left(L_{\gamma_{i}^{(2)} ; x} G, \Phi\right) \mathrm{d} x
$$

b) The convolution product is associative.

Proof. Equality a) occurs in the course of the proof, and b) follows because

$$
(F, G) \mapsto\left\langle F \otimes G, m^{*} \Phi\right\rangle
$$

is associative.
Theorem 5.7. If $F, G \in \mathcal{C}_{c}^{\infty}(\mathrm{HC})$ and $\zeta \in \mathbb{R}$, then

$$
(F * G)^{\curlyvee}(\zeta)=\widehat{F}(\zeta) \widehat{G}(\zeta)
$$

in $\mathcal{H}$.

Proof. By definition of the Fourier transform and Proposition 5.5,

$$
(F * G)^{-}(\zeta)=\left\langle F * G, \pi_{-\zeta}\right\rangle=\left\langle F \otimes G,\left(m^{*} \otimes \operatorname{id}_{S}\right) \circ \pi_{-\zeta}\right\rangle
$$

By Proposition 3.2 we have $\left(m^{*} \otimes \mathrm{id}_{S}\right) \circ \pi_{-\zeta}=\left(\mathrm{id}_{S} \otimes \pi_{-\zeta}\right) \circ \pi_{-\zeta}$, and the latter is given by

$$
\left(\left(\operatorname{id}_{S} \otimes \pi_{-\zeta}\right) \circ \pi_{-\zeta}\right)(u \otimes v ; x, y)=\pi_{-\zeta}(u ; x) \circ \pi_{-\zeta}(v ; y) \in \mathcal{H}
$$

From this we conclude $\left\langle F \otimes G,\left(m^{*} \otimes \mathrm{id}_{S}\right) \circ \pi_{-\zeta}\right\rangle=\left\langle F, \pi_{-\zeta}\right\rangle\left\langle G, \pi_{-\zeta}\right\rangle$ and hence the claim.

For completeness, we also compute the Fourier transform of the pointwise product of compactly supported superfunctions. To this end, we use the notation

$$
((\Delta \otimes \mathrm{id}) \circ \Delta)(\gamma)=\sum_{i} \gamma_{i}^{(1)} \otimes \gamma_{i}^{(2)} \otimes \gamma_{i}^{(3)}
$$

Proposition 5.8. If $F, G \in \mathcal{C}_{c}^{\infty}(\mathrm{HC})$, then the Fourier transform of $F \cdot G$ is given by

$$
(F \cdot G)^{\Upsilon}(\zeta)=\frac{1}{2 \pi} \sum_{i}\left(F\left(\gamma_{i}^{(1)}\right)^{-} * G\left(\gamma_{i}^{(2)}\right)^{\curlyvee}\right)(\zeta) d \pi_{-\zeta}\left(\gamma_{i}^{(3)}\right) .
$$

Proof. We use the definition of Fourier transform and of the product in $\mathcal{C}^{\infty}(\mathrm{HC})$ to obtain

$$
\begin{aligned}
(F \cdot G)^{\prime}(\zeta) & =\left\langle F \cdot G, \pi_{-\zeta}\right\rangle=\left\langle\mu \circ F \otimes G \circ \Delta, \pi_{-\zeta}\right\rangle \\
& =\int_{\mathbb{R}}\left(F \otimes G \otimes \pi_{-\zeta}\right)(\Delta \circ(\mathrm{id} \otimes \Delta)(\gamma ; x) \mathrm{d} x \\
& =\sum_{i} \mathcal{F}\left(F\left(\gamma_{i}^{(1)}\right) * G\left(\gamma_{i}^{(2)}\right)\right)(\zeta) d \pi_{-\zeta}\left(\gamma_{i}^{(3)}\right) .
\end{aligned}
$$

Now the claim follows from the classical fact

$$
\widehat{f g}(\zeta)=\frac{1}{2 \pi}(\hat{f} * \hat{g})(\zeta)
$$

## Convolution with a Distribution

Our next goal is to define the convolution $U * F$ of a distribution $U \in \mathcal{D}^{\prime}(\mathrm{HC})$ and a compactly supported function $F$.

Definition 5.9. Let $U \in \mathcal{D}^{\prime}(\mathrm{HC})$ and $F \in \mathcal{C}_{c}^{\infty}(\mathrm{HC})$. We define a linear functional $U * F$ on $\mathcal{C}^{\infty}(\mathrm{HC})$ by

$$
\langle U * F, \Phi\rangle:=\left\langle U, i^{*}\left(F * i^{*} \Phi\right)\right\rangle .
$$

Remark 5.10. Note that by Proposition 5.5, for $U \in \mathcal{C}^{\infty}(\mathrm{HC})$ this definition agrees with Definition 5.1

Proposition 5.11. Given $U \in \mathcal{D}^{\prime}(\mathrm{HC})$ and $F \in \mathcal{C}_{c}^{\infty}(\mathrm{HC})$, the functional $U * F$ is a distribution on HC . The distribution $U * F$ is given by the smooth function $H \in \mathcal{C}^{\infty}(\mathrm{HC})$ defined by

$$
\begin{equation*}
H(u)=\sum_{i}(-1)^{\left|\gamma_{i}^{(1)}\right||F|} U\left(\gamma_{i}^{(1)}\right) * F\left(\gamma_{i}^{(2)} u\right) \tag{5.1}
\end{equation*}
$$

for $u \in \mathfrak{U}(\mathfrak{h c})$.
Proof. We need to show that

$$
\left\langle U, i^{*}\left(F * i^{*} \Phi\right)\right\rangle=\langle H, \Phi\rangle
$$

holds for all $\Phi \in \mathcal{C}_{c}^{\infty}(\mathrm{HC})$. First we compute $\langle H, \Phi\rangle$ as

$$
\begin{aligned}
\langle H, \Phi\rangle= & \sum_{j}(-1)^{\left|\gamma_{j}^{(1)}\right||\Phi|} \int_{\mathbb{R}} H\left(\gamma_{j}^{(1)} ; y\right) \Phi\left(\gamma_{j}^{(2)} ; y\right) \mathrm{d} y \\
= & \sum_{i, j}(-1)^{\left|\gamma_{j}^{(1)}\right||\Phi|+\left|\gamma_{i}^{(1)}\right||F|} \int_{\mathbb{R}}\left(U\left(\gamma_{i}^{(1)}\right) * F\left(S\left(\gamma_{i}^{(2)}\right) \gamma_{j}^{(1)}\right)\right)(y) \Phi\left(\gamma_{j}^{(2)} ; y\right) \mathrm{d} y \\
= & \sum_{i}(-1)^{\left|\gamma_{i}^{(1)}\right||F|} \\
& U\left(\gamma_{i}^{(1)}\right)\left(\sum_{j}(-1)^{\left|\gamma_{j}^{(1)}\right||\Phi|} \int_{\mathbb{R}} F\left(S\left(\gamma_{i}^{(2)}\right) \gamma_{j}^{(1)}\right)(y-x) \Phi\left(\gamma_{j}^{(2)} ; y\right) \mathrm{d} y\right) \\
= & \sum_{i}(-1)^{|F||\gamma|}\left\langle U\left(\gamma_{i}^{(1)}\right),\left\langle L_{\gamma_{i}^{(2)} ; x} F, \Phi\right\rangle\right\rangle .
\end{aligned}
$$

Here we have used that

$$
\int_{\mathbb{R}}(u * f)(y) \phi(y) \mathrm{d} y=u_{x}\left(\int_{\mathbb{R}} f(y-x) \phi(y) \mathrm{d} y\right)
$$

for $u \in \mathcal{D}^{\prime}(\mathbb{R})$ and $f, \phi \in C_{c}^{\infty}(\mathbb{R})$. Comparing with Corollary 5.6 we obtain

$$
\langle H, \Phi\rangle=\left\langle U, i^{*}\left(F * i^{*} \Phi\right)\right\rangle=\langle U * F, \Phi\rangle
$$

which completes the proof.
Example 5.12. Let $e=\left(e_{0}, e^{*}\right)$ be the identity of HC and define a distribution $U$ via $\langle U, \Phi\rangle:=e^{*} \Phi=\Phi(1 ; 0)$. Then $U \in \mathcal{E}^{\prime}(\mathrm{HC})$, which can be shown easily in coordinates. We compute

$$
\begin{aligned}
\langle U * F, \Phi\rangle & =\left\langle U, i^{*}\left(F * i^{*} \Phi\right)\right\rangle=\left(F * i^{*} \Phi\right)(1 ; 0) \\
& =\sum_{i}(-1)^{\left|\gamma_{i}^{(1)}\right||\Phi|} \int_{\mathbb{R}} F\left(\gamma_{i}^{(1)} ; x\right)\left(i^{*} \Phi\right)\left(S\left(\gamma_{i}^{(2)}\right) ;-x\right) \mathrm{d} x \\
& =\sum_{i}(-1)^{\left|\gamma_{i}^{(1)}\right||\Phi|} \int_{\mathbb{R}} F\left(\gamma_{i}^{(1)} ; x\right) \Phi\left(\gamma_{i}^{(2)} ; x\right) \mathrm{d} x=\langle F, \Phi\rangle
\end{aligned}
$$

for all $\Phi \in \mathcal{C}_{c}^{\infty}(\mathrm{HC})$. Therefore, $U * F=F$ for all $F \in \mathcal{C}^{\infty}(\mathrm{HC})$.

## The Convolution Algebra

We begin by recalling the definition of Sobolev spaces on $\mathbb{R}$. We refer to [1] for details. The Sobolev space $W^{k, p}(\mathbb{R})$ is the space of functions in $L^{p}(\mathbb{R})$ whose distributional derivatives up to order $k$ exist and are in $L^{p}(\mathbb{R})$. The space $W^{k, p}(\mathbb{R})$ is a Banach space with the norm

$$
\|f\|_{k, p}:=\left(\sum_{0 \leq j \leq k}\left\|f^{(j)}\right\|_{p}^{p}\right)^{1 / p}
$$

where $f^{(j)}$ denotes the $j$ th weak or distributional derivative of $f$.
For the rest of this section we fix an orthonormal basis $\left(a_{i}\right)_{i=1}^{n}$ of $V$. We define Sobolev norms on $\mathcal{C}_{c}^{\infty}(\mathrm{HC}(V))$ in analogy with the definition of $\|\cdot\|_{k, p}$ above, replacing the derivatives by the differential operators $L_{z^{j} a_{J}}$ for $z^{j} a_{J} \in \mathfrak{U}(\mathfrak{h c})$. As in the classical case, a different choice of basis $\left(a_{i}\right)_{i=1}^{n}$ will lead to an equivalent norm.

Definition 5.13. If $1 \leq p<\infty$ and $k \in \mathbb{N}$, we define a seminorm $\|\cdot\|_{k, p}$ on $\mathcal{C}_{c}^{\infty}(\mathrm{HC})$ by

$$
\|F\|_{k, p}=\left(\sum_{j+(\# I) \leq k}\left\|\left(L_{z^{j} a_{I}} F\right)(1)\right\|_{p}^{p}\right)^{1 / p}
$$

Here $\# I$ denotes the cardinality of $I \subset \underline{n}$.
Lemma 5.14. If $k \geq \operatorname{dim} V$, then $\|\cdot\|_{k, p}$ is a norm on $\mathcal{C}_{c}^{\infty}(\mathrm{HC})$. The completion of $\mathcal{C}_{c}^{\infty}(\mathrm{HC})$ with respect to $\|\cdot\|_{k, p}$ is a Banach space isomorphic to

$$
\bigoplus_{I \subset \underline{n}} W^{k-(\# I), p}(\mathbb{R})
$$

equipped with the norm

$$
\left\|\left(f_{I}\right)_{I \subset \underline{n}}\right\|=\left(\sum_{I \subset \underline{n}}\left\|f_{I}\right\|_{k-(\# I), p}^{p}\right)^{1 / p}
$$

Proof. Recall that after the choice of basis $\left(a_{i}\right)_{i=1}^{n}$ of $V$ we can identify

$$
\mathcal{C}_{c}^{\infty}(\mathrm{HC}) \cong \bigoplus_{I \subset \underline{n}} C_{c}^{\infty}(\mathbb{R})
$$

as vector spaces via $F \mapsto \sum_{I \subset \underline{n}} f_{I} \otimes \xi^{I}$, where $f_{I}=F\left(a_{I}\right)$. If $F=f \otimes \xi^{I}$ and $k \geq \# I$, then $\|F\|_{k, p}^{p}$ is given by

$$
\sum_{j+(\# J) \leq k}\left\|\left(L_{z^{j} a_{J}} F\right)(1)\right\|_{p}^{p}=\sum_{j+(\# J) \leq k}\left\|F\left(z^{j} a_{J}\right)\right\|_{p}^{p}=\sum_{j \leq k-(\# I)}\left\|f^{(j)}\right\|_{p}^{p},
$$

hence

$$
\left\|f \otimes \xi^{I}\right\|_{k, p}=\|f\|_{k-(\# I), p} .
$$

Clearly, if $k<|I|$, then $\left\|f \otimes \xi^{I}\right\|_{k, p}=0$, which shows that $\|\cdot\|_{k, p}$ is a norm on $\mathcal{C}_{c}^{\infty}(\mathrm{HC})$ only for $k \geq \operatorname{dim} V$. The completion of $\mathcal{C}_{c}^{\infty}(\mathrm{HC})$ is the direct sum of the completions of $\left(C_{c}^{\infty}(\mathbb{R}),\|\cdot\|_{k-(\# I), p}\right)$. But these completions are precisely the classical Sobolev spaces $\left(W^{k-(\# I), p}(\mathbb{R}),\|\cdot\|_{k-(\# I), p}\right)$, see [1, Theorem 3.23].

Definition 5.15. We denote the completion of $\mathcal{C}_{c}^{\infty}(\mathrm{HC})$ with respect to $\|\cdot\|_{k, p}$ by $W^{k, p}(\mathrm{HC})$.

Theorem 5.16. If $\mathrm{HC}=\mathrm{HC}(V)$ with $\operatorname{dim} V=n$, then $\left(W^{n, 1}(\mathrm{HC})\right.$,*) is a Banach algebra.

Proof. Let $\left(a_{i}\right)_{i=1}^{n}$ be the basis used to define the norm $\|\cdot\|_{n, 1}$, and let $F, G \in$ $W^{n, 1}(\mathrm{HC})$. Then

$$
\|F * G\|_{n, 1}=\sum_{j+(\# J) \leq n}\left\|F * G\left(z^{j} a_{J}\right)\right\|_{1} \leq \sum_{j+(\# J) \leq n} \sum_{I \subset \underline{n}}\left\|F\left(a_{I}\right) * G\left(z^{j} a_{I^{c}} a_{J}\right)\right\|_{1},
$$

by definition of the convolution product and the triangle inequality. Writing $C:=I \cap J$ and $A:=I \backslash C, B=J \backslash C$ we can express the last sum as a sum over pairwise disjoint subsets $A, B$ and $C$ of $\underline{n}$. In the following, the prime indicates that $A, B, C$ are pairwise disjoint, and we set $a=\# A, b=\# B$ and $c=\# C$.

$$
\begin{aligned}
\|F * G\|_{n, 1} & \leq \sum_{A, B, C}^{\prime} \sum_{j \leq n-b-c}\left\|\left(F\left(a_{A \cup C}\right) * G\left(a_{(A \cup C)^{c}} a_{B \cup C}\right)\right)^{(j)}\right\|_{1} \\
& =\sum_{A, B, C}^{\prime} \sum_{j \leq n-b-c}\left\|\left(F\left(a_{A \cup C}\right) * G\left(a_{(A \cup B)^{c}}\right)\right)^{(j+b)}\right\|_{1} .
\end{aligned}
$$

Now we need to 'distribute' the $(j+b)$-th derivative over the two factors in the convolution product. The first factor is in $W^{n-a-c, 1}(\mathbb{R})$ and the second is in $W^{a+b, 1}(\mathbb{R})$. Clearly, we can differentiate the second factor $b$ times. Then the factors are differentiable of order $n-a-c$ and $a$, respectively. But $n-a-c+a=$ $n-c \geq j+b \geq j$, since $j+b+c \leq n$. Hence we can distribute the remaining $j$ derivatives over the two factors. This shows that the last sum is less than or equal to

$$
\left(\sum_{i+(\# I) \leq n}\left\|F\left(z^{i} a_{I}\right)\right\|_{1}\right)\left(\sum_{j+(\# J) \leq n}\left\|G\left(z^{j} a_{J}\right)\right\|_{1}\right)=\|F\|_{n, 1}\|G\|_{n, 1}
$$

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