PICARD GROUPS OF CERTAIN STABLY PROJECTIONLESS C*-ALGEBRAS

NORIO NAWATA

ABSTRACT. We compute Picard groups of several nuclear and non-nuclear simple stably projectionless C*-algebras. In particular, the Picard group of the Razak-Jacelon algebra W_2 is isomorphic to a semidirect product of $Out(W_2)$ with \mathbb{R}^{\times}_+ . Moreover, for any separable simple nuclear stably projectionless C*-algebra with a finite dimensional lattice of densely defined lower semicontinuous traces, we show that \mathcal{Z} -stability and strict comparison are equivalent. (This is essentially based on the result of Matui and Sato, and Kirchberg's central sequence algebras.) This shows if A is a separable simple nuclear stably projectionless C*-algebra with a unique tracial state (and no unbounded trace) and has strict comparison, the following sequence is exact:

 $1 \longrightarrow \operatorname{Out}(A) \longrightarrow \operatorname{Pic}(A) \longrightarrow \mathcal{F}(A) \longrightarrow 1$ where $\mathcal{F}(A)$ is the fundamental group of A.

1. INTRODUCTION

Let A be a C*-algebra. Brown, Green and Rieffel introduced the Picard group Pic(A) of A in [5]. We say that an automorphism α of A is *inner* if there exists a unitary element u in the multiplier algebra M(A) of A such that $\alpha(a) = uau^*$ for any $a \in A$. Let Inn(A) denote the set of inner automorphisms of A, and let Out(A) = Aut(A)/Inn(A). They showed that if A is σ -unital, then Pic(A) is isomorphic to Out($A \otimes \mathbb{K}$). Kodaka computed Picard groups of several unital C*-algebras in [21], [22] and [23]. In particular he computed the Picard groups of the irrational rotation algebras A_{θ} . If θ is not quadratic irrational number, then Pic(A) is isomorphic to Out(A_{θ}) and if θ is a quadratic number, then Pic(A_{θ}) is isomorphic to Out(A_{θ}) $\rtimes \mathbb{Z}$. Kodaka considered the following set

 $FP/ \sim = \{[p] \mid p \text{ is a full projection in } A \otimes \mathbb{K} \text{ such that } p(A \otimes \mathbb{K})p \cong A\}$

where [p] is the Murray-von Neumann equivalence class of p and showed that if Out(A) is a normal subgroup of $Out(A \otimes \mathbb{K})$ and A is unital, then FP/ ~ has a suitable group structure and the following sequence is exact:

 $1 \longrightarrow \operatorname{Out}(A) \longrightarrow \operatorname{Pic}(A) \longrightarrow \operatorname{FP}/\sim \longrightarrow 1.$

Note that there exists a simple unital AF algebra B with a unique tracial state such that FP/ ~ of B does not have any suitable group structure. If A is unital, K-theoretical method enables us to show that Out(A) is a normal subgroup of $Out(A \otimes \mathbb{K})$ (see [21, Proposition 1.5]).

The set of FP/ \sim is similar to the fundamental group $\mathcal{F}(M)$ of a II₁ factor M introduced by Murray and von Neumann in [28]. Watatani and the author introduced the fundamental group $\mathcal{F}(A)$ of a simple unital C*-algebra A with a unique tracial state τ based on Kodaka's results. The fundamental group $\mathcal{F}(A)$

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is defined as the set of the numbers $\tau \otimes \text{Tr}(p)$ for some projection $p \in M_n(A)$ such that $pM_n(A)p$ is isomorphic to A. We showed that $\mathcal{F}(A)$ is a multiplicative subgroup of \mathbb{R}^{\times}_{+} and computed fundamental groups of several C*-algebras in [31]. Moreover we showed that any countable subgroup of \mathbb{R}^{\times}_{+} can be realized as the fundamental group of a separable simple unital C*-algebra with a unique tracial state in [32]. Note that the fundamental groups of separable simple unital C^* algebras are countable. Furthermore the author introduced the fundamental group of a simple stably projectionless C*-algebra with unique (up to scalar multiple) densely defined lower semicontinuous trace τ in [29]. If τ is a tracial state and A is σ -unital, then the fundamental group of $\mathcal{F}(A)$ of A is defined as the set of the numbers $d_{\tau}(h)$ for some positive element $h \in A \otimes \mathbb{K}$ such that $\overline{h(A \otimes \mathbb{K})h}$ is isomorphic to A where d_{τ} is the dimension function defined by τ . Note that if A is unital, then this definition coincides with the previous definition and there exist separable simple stably projectionless C*-algebras such that their fundamental groups are equal to \mathbb{R}_{+}^{\times} . The fundamental group of a II₁ factor M is equal to the set of trace-scaling constants for automorphisms of a II_{∞} factor $M \otimes B(\mathcal{H})$. This characterization shows that the fundamental groups of II_1 factors are related to the structure theorem for type III_{λ} factors where $0 < \lambda \leq 1$ (see [43] and [44]). We have a similar characterization, that is, if A is σ -unital, then the fundamental group of A is equal to the set of trace scaling constants for automorphisms of $A \otimes \mathbb{K}$.

We denote by Z the Jiang-Su algebra constructed in [14]. The Jiang-Su algebra Z is a unital separable simple infinite-dimensional nuclear C*-algebra whose Ktheoretic invariant is isomorphic to that of complex numbers. We may regard Z as the stably finite analogue of the Cuntz algebra \mathcal{O}_{∞} . We say that a C*-algebra A is Z-stable if A is isomorphic to $A \otimes Z$. It has recently become important to study regularity properties in Elliott's classification program for nuclear C*-algebras. In particular, Toms and Winter conjectured that for simple separable nuclear non-type I unital C*-algebras, the properties of (i) finite nuclear dimension, (ii) Z-stability and (iii) strict comparison of positive elements are equivalent (see, for example [46] and [49]). It is known that (i) implies (ii) and (ii) implies (iii) due to work of Winter [49] and Rørdam [40] respectively. Recently, Matui and Sato showed that (iii) implies (ii) in the case of finitely many extremal tracial states in [25].

In this paper we shall compute Picard groups of several nuclear and non-nuclear simple stably projectionless C^{*}-algebras. In the case of stably projectionless C^{*}-algebras, the theory of the Cuntz semigroup enables us to compute Picard groups of several examples. We shall show that if A is a separable simple exact Z-stable stably projectionless C^{*}-algebra with a unique tracial state τ and no unbounded trace, then the following sequence is exact:

$$1 \longrightarrow \operatorname{Out}(A) \longrightarrow \operatorname{Pic}(A) \longrightarrow \mathcal{F}(A) \longrightarrow 1$$

Since there exists a unital simple \mathbb{Z} -stable algebra A with a unique tracial state such that $\operatorname{Out}(A)$ is not a normal subgroup of $\operatorname{Pic}(A)$, \mathbb{Z} -stable stably projectionless \mathbb{C}^* -algebras are in this sense more well-behaved than unital stably finite \mathbb{Z} -stable \mathbb{C}^* -algebras. Let \mathcal{W}_2 be the Razak-Jacelon algebra studied in [13], [37], which has trivial K-groups and a unique tracial state and no unbounded trace. Then \mathcal{W}_2 is \mathbb{Z} -stable, and hence the sequence above is exact in this case. Moreover we shall show that the exact sequence above splits. Therefore $\operatorname{Pic}(\mathcal{W}_2)$ is isomorphic to $\operatorname{Out}(\mathcal{W}_2) \rtimes \mathbb{R}^*_+$.

Based on the result of Matui and Sato, and Kirchberg's central sequence algebras, for any separable simple infinite-dimensional non-type I nuclear C^{*}-algebra with a finite dimensional lattice of densely defined lower semicontinuous traces, we shall show that \mathcal{Z} -stability and strict comparison are equivalent. (It is important to consider property (SI).)

In particular, if A is a simple C*-algebra with a finite dimensional lattice of densely defined lower semicontinuous traces in the class of Robert's classification theorem ([37, Corollary 6.2.4]), then A is \mathbb{Z} -stable. Moreover we see that there are many examples that the sequence above is exact. But we do not know whether the exact sequence above splits in this case. This question is related to the existence of a one parameter trace scaling automorphism group of $A \otimes \mathbb{K}$. In the final part of this paper we shall give some remarks and a reason of the notation of \mathcal{W}_2 . Some results show every separable simple \mathbb{Z} -stable stably projectionless C*-algebra A with a unique tracial state has similar properties of (McDuff) II₁ factors.

2. The Picard group

In this section we shall review basic facts on the Picard groups of C^{*}-algebras introduced by Brown, Green and Rieffel in [5] and some results in [29].

Let A be a C*-algebra and \mathcal{X} a right Hilbert A-module, and let $\mathcal{H}(A)$ denote the set of isomorphic classes $[\mathcal{X}]$ of countably generated right Hilbert A-modules. We denote by $L_A(\mathcal{X})$ the algebra of the adjointable operators on \mathcal{X} . For $\xi, \eta \in \mathcal{X}$, a "rank one operator" $\Theta_{\xi,\eta}$ is defined by $\Theta_{\xi,\eta}(\zeta) = \xi\langle \eta, \zeta \rangle_A$ for $\zeta \in \mathcal{X}$. We denote by $K_A(\mathcal{X})$ the closure of the linear span of "rank one operators" $\Theta_{\xi,\eta}$ and by \mathbb{K} the C*-algebra of compact operators on an infinite-dimensional separable Hilbert space. Let \mathcal{X}_A be a right Hilbert A-module A with the obvious right Aaction and $\langle a, b \rangle_A = a^*b$ for $a, b \in A$. Then there exists a natural isomorphism of $K_A(\mathcal{X}_A)$ to A, where A acts on \mathcal{X}_A by left multiplication. Hence if A is unital, then $K_A(\mathcal{X}_A) = L_A(\mathcal{X}_A)$. A multiplier algebra, denote by M(A), of a C*-algebra A is the largest unital C*-algebra that contains A as an essential ideal. It is unique up to isomorphism over A and isomorphic to $L_A(\mathcal{X}_A)$. Let H_A denote the standard Hilbert module $\{(x_n)_{n\in\mathbb{N}} \mid x_n \in A, \sum x_n^* x_n$ converges in A} with an A-valued inner product $\langle (x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \rangle = \sum x_n^* y_n$. Then there exists a natural isomorphism of $A \otimes \mathbb{K}$ to $K_A(H_A)$.

Let A and B be C^{*}-algebras. An A-B-equivalence bimodule is an A-B-bimodule \mathcal{F} which is simultaneously a full left Hilbert A-module under a left A-valued inner product $_A\langle\cdot,\cdot\rangle$ and a full right Hilbert *B*-module under a right *B*-valued inner product $\langle \cdot, \cdot \rangle_B$, satisfying $_A\langle \xi, \eta \rangle \zeta = \xi \langle \eta, \zeta \rangle_B$ for any $\xi, \eta, \zeta \in \mathcal{F}$. We say that A is Morita equivalent to B if there exists an A-B-equivalence bimodule. There exists an isomorphism φ of A to $K_B(\mathcal{F})$ such that $\varphi(A(\xi, \eta)) = \Theta_{\xi, \eta}$ for any $\xi, \eta \in$ \mathcal{F} . The standard Hilbert module H_A can be regard as an $A \otimes \mathbb{K}$ -A-equivalence bimodule. A dual module \mathcal{F}^* of an A-B-equivalence bimodule \mathcal{F} is a set $\{\xi^*; \xi \in \mathcal{F}\}$ with the operations such that $\xi^* + \eta^* = (\xi + \eta)^*$, $\lambda \xi^* = (\overline{\lambda}\xi)^*$, $b\xi^* a = (a^*\xi b^*)^*$, $_B\langle\xi^*,\eta^*\rangle = \langle\eta,\xi\rangle_B$ and $\langle\xi^*,\eta^*\rangle_A = _A\langle\eta,\xi\rangle$. The bimodule \mathcal{F}^* is a *B*-*A*-equivalence bimodule. We refer the reader to [35] and [36] for the basic facts on equivalence bimodules and Morita equivalence. For A-A-equivalence bimodules \mathcal{E}_1 and \mathcal{E}_2 , we say that \mathcal{E}_1 is isomorphic to \mathcal{E}_2 as an equivalence bimodule if there exists a \mathbb{C} -linear one-to-one map Φ of \mathcal{E}_1 onto \mathcal{E}_2 with the properties such that $\Phi(a\xi b) = a\Phi(\xi)b$, $_{A}\langle \Phi(\xi), \Phi(\eta) \rangle = _{A}\langle \xi, \eta \rangle$ and $\langle \Phi(\xi), \Phi(\eta) \rangle_{A} = \langle \xi, \eta \rangle_{A}$ for $a, b \in A, \xi, \eta \in \mathcal{E}_{1}$. The set of isomorphic classes $[\mathcal{E}]$ of the A-A-equivalence bimodules \mathcal{E} forms a group under the product defined by $[\mathcal{E}_1][\mathcal{E}_2] = [\mathcal{E}_1 \otimes_A \mathcal{E}_2]$. We call it the *Picard group* of A and denote it by Pic(A). The identity of Pic(A) is given by the A-A-bimodule $\mathcal{E} := A$ with $_A\langle a_1, a_2 \rangle = a_1 a_2^*$ and $\langle a_1, a_2 \rangle_A = a_1^* a_2$ for $a_1, a_2 \in A$. The inverse element of $[\mathcal{E}]$ in the Picard group of A is the dual module $[\mathcal{E}^*]$. Let α be an automorphism of A, and let $\mathcal{E}^{A}_{\alpha} = A$ with the obvious left A-action and the obvious A-valued inner product. We define the right A-action on \mathcal{E}^A_{α} by $\xi \cdot a = \xi \alpha(a)$ for

any $\xi \in \mathcal{E}^A_{\alpha}$ and $a \in A$, and the right A-valued inner product by $\langle \xi, \eta \rangle_A = \alpha^{-1}(\xi^*\eta)$ for any $\xi, \eta \in \mathcal{E}^A_{\alpha}$. Then \mathcal{E}^A_{α} is an A-A-equivalence bimodule. For $\alpha, \beta \in \operatorname{Aut}(A)$, \mathcal{E}^A_{α} is isomorphic to \mathcal{E}^A_{β} if and only if there exists a unitary $u \in M(A)$ such that $\alpha = ad \ u \circ \beta$. Moreover, $\mathcal{E}^A_{\alpha} \otimes \mathcal{E}^A_{\beta}$ is isomorphic to $\mathcal{E}^A_{\alpha\circ\beta}$. Hence we obtain an homomorphism ρ_A of $\operatorname{Out}(A)$ to $\operatorname{Pic}(A)$. Note that for any $\alpha \in \operatorname{Aut}(A), \mathcal{E}^A_{\alpha}$ is isomorphic to \mathcal{X}_A as a right Hilbert A-module. Conversely we have the following proposition.

Proposition 2.1. Let \mathcal{E} be an A-A-equivalence bimodule such that \mathcal{E} is isomorphic to \mathcal{X}_A as a right Hilbert A-module. Then there exists an automorphism α of A such that \mathcal{E} is isomorphic to \mathcal{E}^A_{α} as an A-A-equivalence bimodule.

Proof. Let Φ be a right Hilbert A-module isomorphism of \mathcal{X}_A to \mathcal{E} , and let ψ be an isomorphism of $K_A(\mathcal{E})$ to $K_A(\mathcal{X}_A)$ induced by Φ . Since $K_A(\mathcal{X}_A)$ is naturally isomorphic to A, we may regard ψ as an isomorphism of $K_A(\mathcal{E})$ to A. There exists an isomorphism φ of A to $K_A(\mathcal{E})$ such that $\varphi(_A\langle \xi, \eta \rangle) = \Theta_{\xi,\eta}$ for any $\xi, \eta \in \mathcal{E}$ because \mathcal{E} is an A-A-equivalence bimodule.

Put $\alpha := (\psi \circ \varphi)^{-1}$, and define a map F of \mathcal{E} to \mathcal{E}^A_{α} by $F(\Phi(a)) := \alpha(a)$ for any $a \in A$. Note that we have

$${}_A\langle \Phi(a), \Phi(b) \rangle = \varphi^{-1}(\Theta_{\Phi(a), \Phi(b)}) = \varphi^{-1} \circ \psi^{-1}(ab^*) = \alpha(ab^*)$$

and

$$a \cdot \Phi(b) = \varphi(a)\Phi(b) = \Phi(\psi \circ \varphi(a)b) = \Phi(\alpha^{-1}(a)b)$$

for any $a, b \in A$. Therefore it can easily be checked that F is an A-A-equivalence bimodule isomorphism.

An A-B-equivalence bimodule \mathcal{F} induces an isomorphism Ψ of $\operatorname{Pic}(A)$ to $\operatorname{Pic}(B)$ by $\Psi([\mathcal{E}]) = [\mathcal{F}^* \otimes \mathcal{E} \otimes \mathcal{F}]$ for $[\mathcal{E}] \in \operatorname{Pic}(A)$. Therefore if A is Morita equivalent to B, then $\operatorname{Pic}(A)$ is isomorphic to $\operatorname{Pic}(B)$. Brown, Green and Rieffel showed that if A is σ -unital, then $\operatorname{Pic}(A)$ is isomorphic to $\operatorname{Out}(A \otimes \mathbb{K})$ (see [5, Theorem 3.4 and Corollary 3.5]). Indeed a homomorphism $\rho_{A \otimes \mathbb{K}}$ of $\operatorname{Aut}(A \otimes \mathbb{K})$ to $\operatorname{Pic}(A \otimes \mathbb{K})$ induces an isomorphism of $\operatorname{Out}(A \otimes \mathbb{K})$ onto $\operatorname{Pic}(A \otimes \mathbb{K})$.

A sequence $\{\xi_i\}_{i\mathbb{N}}$ of a right Hilbert A-module \mathcal{X} is called *countable basis* of \mathcal{X} if $\eta = \sum_{i=1}^{\infty} \xi_i \langle \xi_i, \eta \rangle_A$ in norm for any $\eta \in \mathcal{X}$. If $K_A(\mathcal{X})$ is σ -unital, then \mathcal{X} has a countable basis. A sequence $\{\xi_i\}_{i\mathbb{N}}$ is a countable basis if and only if $\{\sum_{i=1}^{N} \Theta_{\xi_i,\xi_i}\}_{N\in\mathbb{N}}$ is an approximate unit for $K_A(\mathcal{X})$. See [15], [16], [29] and [50] for details of bases of Hilbert modules. We denote by T(A) the set of densely defined lower semicontinuous traces on A and $T_1(A)$ the set of tracial states on A. We have the following proposition.

Proposition 2.2. ([29, Proposition 2.4])

Let A be a simple σ -unital C^{*}-algebra and \mathcal{X} a countably generated Hilbert Amodule, and let τ be a densely defined lower semicontinuous trace on A. For $x \in K_A(\mathcal{X})_+$, define

$$Tr_{\tau}^{\mathcal{X}}(x) := \sum_{i=1}^{\infty} \tau(\langle \xi_i, x\xi_i \rangle_A)$$

where $\{\xi_i\}_{i=1}^{\infty}$ is a countable basis of \mathcal{X} . Then $Tr_{\tau}^{\mathcal{X}}$ does not depend on the choice of basis and is a densely defined (resp. strictly densely defined) lower semicontinuous trace on $K_A(\mathcal{X})$ (resp. $L_A(\mathcal{X})$).

The following proposition is [29, Remark 2.5]. Moreover it is well-known (see for example [6]). But we include the proof for completeness.

Proposition 2.3. Let A be a simple σ -unital C*-algebra and \mathcal{X} a countably generated Hilbert A-module. Then there exists a bijective correspondence between T(A)and $T(K_A(\mathcal{X}))$.

Proof. Since a right Hilbert A-module \mathcal{X} is a $K_A(\mathcal{X})$ -A-equivalence bimodule, \mathcal{X}^* is an A- $K_A(\mathcal{X})$ -equivalence bimodule. Let $\{\xi_j\}_{j\in\mathbb{N}}$ be a countable basis of \mathcal{X} and $\{\eta_i^*\}_{i\in\mathbb{N}}$ a countable basis of \mathcal{X}^* . For any $a \in A_+$ and $\tau \in T(A)$, we have

$$Tr_{Tr_{\tau}^{\mathcal{X}}}^{\mathcal{X}^{*}}(a) = \lim_{n \to \infty} \sum_{i=1}^{n} Tr_{\tau}^{\mathcal{X}}(\langle \eta_{i}^{*}, a\eta_{i}^{*} \rangle_{K_{A}(\mathcal{X})})$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} Tr_{\tau}^{\mathcal{X}}(_{K_{A}(\mathcal{X})}\langle \eta_{i}a^{\frac{1}{2}}, \eta_{i}a^{\frac{1}{2}} \rangle)$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} Tr_{\tau}^{\mathcal{X}}(\Theta_{\eta_{i}a^{\frac{1}{2}}, \eta_{i}a^{\frac{1}{2}}})$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{\infty} \tau(\langle \xi_{j}, \eta_{i}a^{\frac{1}{2}} \rangle_{A}\langle \eta_{i}a^{\frac{1}{2}}, \xi_{j} \rangle_{A}\rangle)$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{\infty} \tau(\langle \eta_{i}a^{\frac{1}{2}}, \xi_{j} \rangle_{A}\langle \xi_{j}, \eta_{i}a^{\frac{1}{2}} \rangle_{A})$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{\infty} \tau(\langle \eta_{i}a^{\frac{1}{2}}, \xi_{j} \rangle_{A}\langle \xi_{j}, \eta_{i}a^{\frac{1}{2}} \rangle_{A}).$$

Since $\{\xi_j\}_{j\in\mathbb{N}}$ is a countable basis of \mathcal{X} , we see that

$$\langle \eta_i a^{\frac{1}{2}}, \sum_{j=1}^m \xi_j \langle \xi_j, \eta_i a^{\frac{1}{2}} \rangle_A \rangle_A \nearrow \langle \eta_i a^{\frac{1}{2}}, \eta_i a^{\frac{1}{2}} \rangle_A \ (m \to \infty).$$

Hence

$$\lim_{n \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{\infty} \tau(\langle \eta_i a^{\frac{1}{2}}, \xi_j \langle \xi_j, \eta_i a^{\frac{1}{2}} \rangle_A \rangle_A) = \lim_{n \to \infty} \sum_{i=1}^{n} \tau(\langle \eta_i a^{\frac{1}{2}}, \eta_i a^{\frac{1}{2}} \rangle_A)$$

by the lower semicontinuity of τ . Since $\Theta_{\eta_i^*,\eta_i^*}$ is corresponding to $\langle \eta_i, \eta_i \rangle_A$, we see that $\{\sum_{i=1}^n \langle \eta_i, \eta_i \rangle_A\}_{n \in \mathbb{N}}$ is an approximate unit for A. Therefore we have

$$\lim_{n \to \infty} \sum_{i=1}^{n} \tau(\langle \eta_i a^{\frac{1}{2}}, \eta_i a^{\frac{1}{2}} \rangle_A) = \lim_{n \to \infty} \tau(a^{\frac{1}{2}} (\sum_{i=1}^{n} \langle \eta_i, \eta_i \rangle_A) a^{\frac{1}{2}}) = \tau(a)$$

by the lower semicontinuity of τ . Consequently $Tr_{Tr_{\tau}^{\mathcal{X}}}^{\mathcal{X}^*} = \tau$.

Since $(\mathcal{X}^*)^*$ is naturally isomorphic to \mathcal{X} as a $K_A(\mathcal{X})$ -A-equivalence bimodule, we see that $Tr_{Tr_{\tau}^{\mathcal{X}^*}}^{\mathcal{X}} = \tau$ for any $\tau \in T(K_A(\mathcal{X}))$ as above. Hence we obtain the conclusion.

The following Corollary is folklore.

Corollary 2.4. Let A be a simple σ -unital C^{*}-algebra and h be a non-zero positive element in A. Then every densely defined lower semicontinuous trace on \overline{hAh} is a restriction of some densely defined lower semicontinuous trace on A.

Proof. Put $\mathcal{X} = \overline{hA}$. Since $K_A(\overline{hA})$ is naturally isomorphic to \overline{hAh} , it is enough to show that $Tr_{\tau}^{\mathcal{X}}(\tau) = \tau|_{\overline{hAh}}$ for any $\tau \in T(A)$ by Proposition 2.3. Let $\{a_n\}_{n \in \mathbb{N}}$ be

a countable basis of \overline{hA} . (Note that the norm of Hilbert A-module \overline{hA} is equal to the norm of C*-algebra A.) Since $\tau(a_n^*xa_n) = \tau(x^{\frac{1}{2}}a_na_n^*x^{\frac{1}{2}})$ for any $x \in \overline{hAh}_+$ and $\tau \in T(A)$, we have $\sum_{n=1}^N \tau(a_n^*xa_n) \leq \sum_{n=1}^{N+1} \tau(a_n^*xa_n)$ for any $N \in \mathbb{N}$. Therefore we see that $Tr_{\tau}^{\mathcal{X}}(x) = \tau(x)$ by the lower semicontinuity of τ . \Box

For $\tau \in T(A)$, define a map \hat{T}_{τ} of $\mathcal{H}(A)$ to $[0, \infty]$ by

$$\hat{T}_{\tau}([\mathcal{X}]) := \sum_{n=1}^{\infty} \tau(\langle \xi_n, \xi_n \rangle_A)$$

where $\{\xi_n\}_{n\in\mathbb{N}}$ is a countable basis of \mathcal{X} . This map is well-defined map and does not depend on the choice of basis. Moreover we have $\hat{T}_{\tau}(\mathcal{X}) = Tr_{\tau}^{\mathcal{X}}(1_{L_A(\mathcal{X})}) = ||Tr_{\tau}^{\mathcal{X}}||$.

Put $d_{\tau}(h) = \lim_{n \to \infty} \tau \otimes \operatorname{Tr}(h^{\frac{1}{n}})$ for $h \in (A \otimes \mathbb{K})_+$. Then d_{τ} is a dimension function. We have the following proposition.

Proposition 2.5. [29, Proposition 3.1]

Let A be a simple σ -unital C^{*}-algebra with unique (up to scalar multiple) densely defined lower semicontinuous trace τ and h a positive element in $A \otimes \mathbb{K}$. Then $\hat{T}_{\tau}(\overline{hH_A}) = d_{\tau}(h)$.

The following proposition is an immediate corollary of [29, Proposition 3.3]

Proposition 2.6. Let A be a simple σ -unital C*-algebra with a unique tracial state τ and no unbounded trace. Then for every right Hilbert A-module \mathcal{X} and every A-A-equivalence bimodule \mathcal{E} ,

$$\hat{T}_{\tau}([\mathcal{X} \otimes \mathcal{E}]) = \hat{T}_{\tau}([\mathcal{X}])\hat{T}_{\tau}([\mathcal{E}]).$$

If A is σ -unital, then for any A-A-equivalence bimodule \mathcal{E} there exists a positive element h in $A \otimes \mathbb{K}$ such that \mathcal{E} is isomorphic to $\overline{hH_A}$ as a right Hilbert A-module. Note that $\overline{h(A \otimes \mathbb{K})h}$ is isomorphic to A and $\overline{hH_A}$ has a suitable structure as an A-A-equivalence bimodule in this case. (See, for example, [29, Proposition 2.3].) The following proposition is a key proposition in this paper.

Proposition 2.7. Let A be a simple σ -unital C*-algebra with a unique tracial state τ and no unbounded trace. Define a map T of $\operatorname{Pic}(A)$ to \mathbb{R}_+^{\times} by $T([\overline{hH_A}]) = d_{\tau}(h)$. Then T is a well-defined multiplicative map and $T([\mathcal{E}_{\alpha}^A]) = 1$ for any $\alpha \in \operatorname{Aut}(A)$. Moreover $\operatorname{Im}(T)$ is equal to the set

 $\{d_{\tau}(h) \in \mathbb{R}^{\times}_{+} \mid h \text{ is a positive element in } A \otimes \mathbb{K} \text{ such that } A \cong \overline{h(A \otimes \mathbb{K})h} \}.$

Proof. Let $[\overline{hH_A}] \in \operatorname{Pic}(A)$. Then $d_{\tau}(h) = \hat{T}_{\tau}(\overline{hH_A}) = ||Tr_{\tau}^{\overline{hH_A}}||$ by Proposition 2.5. Since $K_A(\overline{hH_A}) \cong A$ has no unbounded trace, $d_{\tau}(h) = ||Tr_{\tau}^{\overline{hH_A}}|| < \infty$. Hence we see that T is well-defined map and $\operatorname{Im}(T)$ is equal to the set

 $\{d_{\tau}(h) \in \mathbb{R}^{\times}_{+} \mid h \text{ is a positive element in } A \otimes \mathbb{K} \text{ such that } A \cong \overline{h(A \otimes \mathbb{K})h} \}$

by an argument above. Proposition 2.6 implies that T is a multiplicative map. It is easy to see that \mathcal{E}^A_{α} is isomorphic to $\overline{sA} = A$ as a right Hilbert A-module where s is a strictly positive element in A. Since τ is a tracial state, $T([\mathcal{E}^A_{\alpha}]) = d_{\tau}(s) = 1$. \Box

Put $\mathcal{F}(A) = \text{Im}(T)$. We call $\mathcal{F}(A)$ the fundamental group of A, which is a multiplicative subgroup of \mathbb{R}^{\times}_{+} by the proposition above.

Let A a simple C*-algebra with unique (up to scalar multiple) densely defined lower semicontinuous trace τ . For any $\alpha \in \operatorname{Aut}(A \otimes \mathbb{K}), \tau \otimes \operatorname{Tr} \circ \alpha$ is a densely defined lower semicontinuous trace on $A \otimes \mathbb{K}$. Hence there exists a positive number λ such that $\tau \otimes \operatorname{Tr} \circ \alpha = \lambda \tau \otimes \operatorname{Tr}$. Define a map S of $\operatorname{Out}(A \otimes \mathbb{K})$ to \mathbb{R}^+_+ by $S([\alpha]) = \lambda$ where $\tau \otimes \text{Tr} \circ \alpha = \lambda \tau \otimes \text{Tr}$. Then S is a well-defined multiplicative map and Im(S) is equal to the set

$$\mathfrak{S}(A) := \{ \lambda \in \mathbb{R}_+^{\times} \mid \tau \otimes \operatorname{Tr} \circ \alpha = \lambda \tau \otimes \operatorname{Tr} \text{ for some } \alpha \in \operatorname{Aut}(A \otimes \mathbb{K}) \}.$$

The following proposition is a strengthened version of [29, Proposition 4.20].

Proposition 2.8. Let A be a simple σ -unital C*-algebra with a unique tracial state τ and no unbounded trace. Then there exists an isomorphism Ψ of $\operatorname{Out}(A \otimes \mathbb{K})$ to $\operatorname{Pic}(A)$ such that $S([\alpha])^{-1} = T \circ \Psi([\alpha])$ for any $[\alpha] \in \operatorname{Out}(A \otimes \mathbb{K})$, and hence $\mathcal{F}(A) = \mathfrak{S}(A)$.

Proof. Define $\Psi([\alpha]) := [(H_A)^* \otimes \mathcal{E}_{\alpha}^{A \otimes \mathbb{K}} \otimes H_A]$. Since H_A is an $A \otimes \mathbb{K}$ -A-equivalence bimodule and $\rho_{A \otimes \mathbb{K}}$ induces an isomorphism of $\operatorname{Out}(A \otimes \mathbb{K})$ to $\operatorname{Pic}(A \otimes \mathbb{K})$ by [5, Corollary 3.5], Ψ is an isomorphism of $\operatorname{Out}(A \otimes \mathbb{K})$ to $\operatorname{Pic}(A)$. Note that $(H_A)^*$ is naturally isomorphic to $\overline{(s \otimes e_{11})(A \otimes \mathbb{K})}$ as an A- $A \otimes \mathbb{K}$ -equivalence bimodule where s is a strictly positive element in A and e_{11} is a rank one projection in \mathbb{K} . It is easy to see that for any element ζ in an algebraic tensor product $(s \otimes e_{11})(A \otimes \mathbb{K}) \otimes \mathcal{E}_{\alpha}^{A \otimes \mathbb{K}} \odot H_A$, there exists an element ξ in H_A such that

$$\zeta = (s^{\frac{1}{2}} \otimes e_{11}) \otimes (s^{\frac{1}{4}} \otimes e_{11}) \otimes \alpha^{-1} (s^{\frac{1}{4}} \otimes e_{11}) \xi.$$

Therefore it can easily be checked that $(H_A)^* \otimes \mathcal{E}_{\alpha}^{A \otimes \mathbb{K}} \otimes H_A$ is isomorphic to $\overline{\alpha^{-1}(s \otimes e_{11})H_A}$ as a right Hilbert A-module. We have

$$d_{\tau}(\alpha^{-1}(s \otimes e_{11})) = \lim_{n \to \infty} \tau \otimes \operatorname{Tr}(\alpha^{-1}((s \otimes e_{11})^{\frac{1}{n}})) = S([\alpha])^{-1}$$

since τ is a tracial state on A. Hence we obtain the conclusion.

3. The Cuntz semigroup

In this section we shall review basic facts of the Cuntz semigroup and some results in [7], [10], [38] and [40]. See, for example, [2] for details of the Cuntz semigroup. Let A be a C*-algebra. For positive elements $a, b \in A$ we say that a is Cuntz smaller than b, written $a \preceq b$, if there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ of A such that $||x_n^*bx_n - a|| \to 0$. Positive elements a and b are said to be Cuntz equivalent, written $a \preceq b$, if $a \preceq b$ and $b \preceq a$. Define the Cuntz semigroup Cu(A) as the set of Cuntz equivalence classes of positive elements in $A \otimes \mathbb{K}$ endowed with the order $[a] \leq [b]$ if a is Cuntz smaller than b, and the addition [a] + [b] = [a' + b'] where $a \sim a', b \sim b'$ and a'b' = 0. Note that this definition is different from the original definition W(A) in [8]. (We have Cu(A) = W(A \otimes \mathbb{K}).) The Cuntz semigroup Cu(A) is also defined using right Hilbert A-modules (see [7]). For positive elements $a, b \in A \otimes \mathbb{K}$ we say that a is compactly contained in b, written $a \ll b$ if whenever $[b] \leq \sup_{n \in \mathbb{N}} [b_n]$ for an increasing sequence $\{[b_n]\}_{n \in \mathbb{N}}$, then there exists a natural number n such that $[a] \leq [b_n]$. Coward, Elliott and Ivanescu [7] showed that Cu(A) has the following properties:

(1) every increasing sequence in Cu(A) has a supremum,

(2) for any element [a] in Cu(A) there exists an increasing sequence $\{[a_n]\}_{n\in\mathbb{N}}$ of Cu(A) such that $[a_n] \ll [a_{n+1}]$ for any $n \in \mathbb{N}$ and $[a] = \sup[a_n]$,

(3) the operation of passing to the supremum of an increasing sequence and the relation \ll are compatible with addition.

Moreover they showed that $\operatorname{Cu}(A)$ is a functor which is continuous with respect to inductive limits ([7, Theorem 2]). For a positive element $a \in A \otimes \mathbb{K}$ and $\epsilon > 0$ we denote by $(a - \epsilon)_+$ the element f(a) in $A \otimes \mathbb{K}$ where $f(t) = \max\{0, t - \epsilon\}, t \in \sigma(a)$. Then we have $(a - \epsilon)_+ \ll a$.

Following the definition in [40], the Cuntz semigroup $\operatorname{Cu}(A)$ is said to be *almost* unperforated if $(k+1)[a] \leq k[b]$ for some $k \in \mathbb{N}$ implies that $[a] \leq [b]$. Rørdam

showed that if A is \mathbb{Z} -stable, then Cu(A) is almost unperforated (see [40, Theorem 4.5]). If A is a simple exact C^{*}-algebra with traces, then Cu(A) is almost unperforated if and only if A has strict comparison, that is, if $a, b \in (A \otimes \mathbb{K})_+$ with $d_{\tau}(a) < d_{\tau}(b) < \infty$ for any $\tau \in T(A)$, then $[a] \leq [b]$. (See [10, Proposition 4.2, Remark 4.3 and Proposition 6.2] and [40, Proposition 3.2 and Corollary 4.6].)

Lemma 3.1. Let A be a simple C^{*}-algebra and a non-zero positive element in $A \otimes \mathbb{K}$. Then for any positive element b in $A \otimes \mathbb{K}$, $[b] \leq \sup_{n \in \mathbb{N}} n[a]$.

Proof. Let $B := (a + b)(A \otimes \mathbb{K})(a + b)$. Then B is a σ -unital hereditary subalgebra of $A \otimes \mathbb{K}$. By a variant of Brown's theorem (see for example [27, Theorem 1.9]), $\overline{aH_B}^{\infty}$ is isomorphic to H_B as a right Hilbert module. Since $\overline{bH_B} \subseteq H_B$, we see that $[b] \leq \sup_{n \in \mathbb{N}} n[a]$ in Cu(B). We obtain the conclusion because B is a hereditary subalgebra of $A \otimes \mathbb{K}$.

The following proposition is an immediate corollary of [10, Theorem 6.6]. (Note that they considered the more general case.) But we shall give a self-contained proof based on their arguments (see also [10, Proposition 6.4]).

Proposition 3.2. Let A be a simple exact C*-algebra, and let a and b be positive elements in $A \otimes \mathbb{K}$. Assume that $\operatorname{Cu}(A)$ is almost unperforated and 0 is an accumulation point of the spectrum $\sigma(a)$ of a. Then if $d_{\tau}(a) \leq d_{\tau}(b) < \infty$ for any $\tau \in T(A)$, then a is Cuntz smaller than b.

Proof. Let a and b be positive elements in $A \otimes \mathbb{K}$ such that $d_{\tau}(a) \leq d_{\tau}(b)$ for any $\tau \in T(A)$. We may assume that ||a|| = ||b|| = 1. For any $k \in \mathbb{N}$ we have

$$d_{\tau}(\operatorname{diag}(\overbrace{a,..,a}^{k})) = kd_{\tau}(a) \le kd_{\tau}(b) < (k+1)d_{\tau}(b) = d_{\tau}(\operatorname{diag}(\overbrace{b,..,b}^{k+1})).$$

Hence $k[a] \leq (k+1)[b]$ for any $k \in \mathbb{N}$ because A has strict comparison. Let $\epsilon > 0$, and choose a positive function c_{ϵ} on $\sigma(a)$ such that $c_{\epsilon}(t) > 0$ on $t \in (0, \epsilon)$ and $c_{\epsilon}(t) = 0$ on $\sigma(a) \setminus (0, \epsilon)$. Then we have $[c_{\epsilon}(a)] + [(a - \epsilon)_{+}] \leq [a]$. Note that for any $\epsilon > 0$, $c_{\epsilon}(a)$ is a nonzero positive element because 0 is an accumulation point of $\sigma(a)$. Hence we have $2[a] \leq \sup_{n \in \mathbb{N}} n[c_{\epsilon}]$ by Lemma 3.1. There exists a natural number m such that $2[(a - \epsilon)_{+}] \leq m[c_{\epsilon}(a)]$ since $2[(a - \epsilon)_{+}] \ll 2[a]$. Therefore we have

$$(m+2)[(a-\epsilon)_+] \le m[(a-\epsilon)_+] + m[c_{\epsilon}(a)] \le m[a] \le (m+1)[b].$$

By the assumption that $\operatorname{Cu}(A)$ is almost unperforated, we see that $[(a - \epsilon)_+] \leq [b]$ for any $\epsilon > 0$, and hence we have $[a] \leq [b]$.

Corollary 3.3. Let A be a simple exact stably projectionless C*-algebra, and let a and b be positive elements in $A \otimes \mathbb{K}$. Assume that Cu(A) is almost unperforated. Then if $d_{\tau}(a) = d_{\tau}(b) < \infty$ for any $\tau \in T(A)$, then a is Cuntz equivalent to b.

Proof. For any nonzero positive element a in $A \otimes \mathbb{K}$, 0 is an accumulation point of $\sigma(a)$ because A is a stably projectionless C*-algebra. Hence we obtain the conclusion by Proposition 3.2.

Based on the result in [38], we say that a C*-algebra A has almost stable rank one if for every σ -unital hereditary subalgebra $B \subseteq A \otimes \mathbb{K}$ we have $B \subseteq \overline{\operatorname{GL}(\widetilde{B})}$. Robert showed that if A is a simple \mathbb{Z} -stable stably projectionless C*-algebra, then A has almost stable rank one (see [38, Corollary 4.5] and [40]). The following proposition is [38, Proposition 4.7]. See [7, Theorem 3] for the proof.

Proposition 3.4. Let A be a simple σ -unital C*-algebra such that A has almost stable rank one and a and b positive elements in $A \otimes \mathbb{K}$. Then a is Cuntz smaller than b if and only if there exists a right Hilbert A-module $\mathcal{X} \subseteq \overline{bH_A}$ such that \mathcal{X}

is isomorphic to $\overline{aH_A}$ as a right Hilbert A-module, and a is Cuntz equivalent to b if and only if $\overline{aH_A}$ is isomorphic to $\overline{bH_A}$ as a right Hilbert A-module.

Corollary 3.3 and Proposition 3.4 are important in the proof of our main result. These propositions show that every separable simple \mathcal{Z} -stable stably projectionless C*-algebra A with a unique tracial state has similar properties of II₁ factors (Murray-von Neumann comparison theory). Moreover we have the following proposition.

Proposition 3.5. Let A be a simple exact σ -unital stably projectionless C*-algebra with a unique tracial state τ and no unbounded trace. Assume that $\operatorname{Cu}(A)$ is almost unperforated, A has almost stable rank one and $\mathcal{F}(A) = \mathbb{R}_+^{\times}$. Then every nonzero hereditary subalgebra of A is isomorphic to A.

<u>Proof.</u> Let B be a non-zero hereditary subalgebra of A. Then B is isomorphic to $\overline{h_0Ah_0}$ for some non-zero positive element h_0 in A. Since $d_{\tau}(h_0) \in \mathbb{R}_+^{\times} = \mathcal{F}(A)$, there exists a positive element h in $A \otimes \mathbb{K}$ such that $d_{\tau}(h) = d_{\tau}(h_0)$ and $\overline{h(A \otimes \mathbb{K})h}$ is isomorphic to A. But then $h \sim h_0$ by Corollary 3.3 and so $\overline{hH_A}$ is isomorphic to $(\overline{h_0 \otimes e_{11}})H_A$ by Proposition 3.4. Hence $A \cong K_A(\overline{hH_A}) \cong K_A((\overline{h_0 \otimes e_{11}})H_A) \cong B$.

4. MAIN RESULT

The following theorem is the main result in this paper. See [21, Corollary 4.8] and [31, Proposition 3.26] for the unital case.

Theorem 4.1. Let A be a simple exact σ -unital stably projectionless C*-algebra with a unique traical state τ and no unbounded trace. Assume that $\operatorname{Cu}(A)$ is almost unperforated and A has almost stable rank one. Then the following sequence is exact:

$$1 \longrightarrow \operatorname{Out}(A) \xrightarrow{\rho_A} \operatorname{Pic}(A) \xrightarrow{T} \mathcal{F}(A) \longrightarrow 1$$

Proof. It is clear that T is onto by definition of $\mathcal{F}(A)$. We see that ρ_A is one-to-one and $\operatorname{Im}(\rho_A) \subseteq \operatorname{Ker}(T)$ by [5, Corollary 3.2] and Proposition 2.7 respectively. We shall show that $\operatorname{Ker}(T) \subseteq \operatorname{Im}(\rho_A)$. Let $[\mathcal{E}] \in \operatorname{Ker}(T)$. Then Corollary 3.3 and Proposition 3.4 imply \mathcal{E} is isomorphic to $(s \otimes e_{11})H_A$ as a right Hilbert A-module where s is a strict positive element in A and e_{11} is a rank one projection in \mathbb{K} because we have $d_{\tau}(s \otimes e_{11}) = 1$ by $\|\tau\| = 1$. Since $(s \otimes e_{11})H_A$ is isomorphic to \mathcal{X}_A as a right Hilbert A-module, there exists some automorphism α such that $[\mathcal{E}] = [\mathcal{E}^A_{\alpha}]$ by Proposition 2.1. Hence $[\mathcal{E}] \in \operatorname{Im}(\rho_A)$.

Corollary 4.2. Let A be a simple exact separable \mathcal{Z} -stable stably projectionless C*-algebra with a unique tracial state τ and no unbounded trace. Then the following sequence is exact:

 $1 \longrightarrow \operatorname{Out}(A) \xrightarrow{\rho_A} \operatorname{Pic}(A) \xrightarrow{T} \mathcal{F}(A) \longrightarrow 1$

Proof. This is an immediate consequence of [40, Theorem 4.5], [38, Corollary 4.5] and Theorem 4.1. $\hfill \Box$

Remark 4.3. There exists a unital simple AF algebra A with a unique tracial state such that Out(A) is not a normal subgroup of Pic(A). (See [30].) Of course A is a unital stably finite \mathcal{Z} -stable C*-algebra. Therefore the corollary above shows that \mathcal{Z} -stable stably projectionless C*-algebras are in this sense more well-behaved than unital stably finite \mathcal{Z} -stable C*-algebras.

We shall show some examples.

Let W_2 be the Razak-Jacelon algebra studied in [13], [37] and [38], which has trivial K-groups and a unique tracial state and no unbounded trace. The Razak-Jacelon algebra W_2 is constructed as an inductive limit C*-algebra of Razak's building block in [34], that is,

$$A(n,m) = \left\{ f \in C([0,1]) \otimes M_m(\mathbb{C}) \mid f(0) = \operatorname{diag}(\overbrace{c,..,c}^k, 0_n), f(1) = \operatorname{diag}(\overbrace{c,..,c}^{k+1}), \\ c \in M_n(\mathbb{C}) \right\}$$

where *n* and *m* are natural numbers with n|m and $k := \frac{m}{n} - 1$. Let \mathcal{O}_2 denote the Cuntz algebra generated by 2 isometries S_1 and S_2 . For every $\lambda_1, \lambda_2 \in \mathbb{R}$ there exists by universality a one-parameter automorphism group α of \mathcal{O}_2 given by $\alpha_t(S_j) = e^{it\lambda_j}S_j$. Kishimoto and Kumjian showed that if λ_1 and λ_2 are all nonzero of the same sign and λ_1 and λ_2 generate \mathbb{R} as a closed subgroup, then $\mathcal{O}_2 \rtimes_{\alpha} \mathbb{R}$ is a simple stable projectionless C*-algebra with unique (up to scalar multiple) densely defined lower semicontinuous trace in [19] and [20]. Moreover Robert [37] showed that $\mathcal{W}_2 \otimes \mathbb{K}$ is isomorphic to $\mathcal{O}_2 \rtimes_{\alpha} \mathbb{R}$ for some λ_1 and λ_2 . (See also [9].) In particular, $\mathcal{W}_2 \otimes \mathbb{K}$ has a one parameter trace scaling automorphism group σ (see [19]).

Theorem 4.4. The Picard group of Razak-Jacelon algebra \mathcal{W}_2 is isomorphic to a semidirect product of $\operatorname{Out}(\mathcal{W}_2)$ with \mathbb{R}_+^{\times} . Moreover if A is a simple exact σ -unital C*-algebra with a unique tracial state τ and no unbounded trace, then the Picard group of $A \otimes \mathcal{W}_2$ is isomorphic to a semidirect product of $\operatorname{Out}(A \otimes \mathcal{W}_2)$ with \mathbb{R}_+^{\times} .

Proof. Note that we see that $A \otimes W_2$ is stably projectionless C*-algebra because $A \otimes W_2 \otimes \mathbb{K}$ has a one parameter trace scaling automorphism group id $\otimes \sigma$. Since W_2 is \mathcal{Z} -stable, we have the following exact sequence:

$$1 \longrightarrow \operatorname{Out}(A \otimes \mathcal{W}_2) \xrightarrow{\rho_A} \operatorname{Pic}(A \otimes \mathcal{W}_2) \xrightarrow{T} \mathcal{F}(A \otimes \mathcal{W}_2) \longrightarrow 1$$

by Corollary 4.2. By Proposition 2.8, we see that $\mathcal{F}(A \otimes \mathcal{W}_2) = \mathbb{R}_+^{\times}$ and the exact sequence above splits because $A \otimes \mathcal{W}_2 \otimes \mathbb{K}$ has a one parameter trace scaling automorphism group. Consequently $\operatorname{Pic}(A \otimes \mathcal{W}_2)$ is isomorphic to $\operatorname{Out}(A \otimes \mathcal{W}_2) \rtimes \mathbb{R}_+^{\times}$.

Remark 4.5. (i) Note that we have

$$\operatorname{Out}(\mathcal{W}_2 \otimes \mathbb{K}) \cong \operatorname{Out}(\mathcal{W}_2) \rtimes \mathbb{R}_+^{\times}$$

(ii) We do not assume that A is nuclear in the theorem above. Hence we have

$$\operatorname{Pic}(\mathcal{W}_2 \otimes C_r^*(\mathbb{F}_n)) \cong \operatorname{Out}(\mathcal{W}_2 \otimes C_r^*(\mathbb{F}_n)) \rtimes \mathbb{R}_+^{\times}$$

where \mathbb{F}_n is a non-amenable free group with n generators. Moreover Proposition 3.5 shows that every nonzero hereditary subalgebra of $\mathcal{W}_2 \otimes C_r^*(\mathbb{F}_n)$ is isomorphic to $\mathcal{W}_2 \otimes C_r^*(\mathbb{F}_n)$.

(iii) Let *B* be a simple unital AF algebra with two extremal tracial states. Then $\mathcal{W}_2 \otimes B$ is a simple stably projectionless C*-algebra with two extremal tracial states and in the class of Robert's classification theorem [37]. It can be checked that $\operatorname{Out}(\mathcal{W}_2 \otimes B)$ is not a normal subgroup of $\operatorname{Pic}(\mathcal{W}_2 \otimes B)$ by Robert's classification theorem and a similar proposition as [21, Proposition 1.5]. (We need to replace the K₀-groups with the trace spaces.)

5. Z-stability of stably projectionless C*-algebras

In this section we shall generalize the result of Matui and Sato in [25] to stably projectionless C^* -algebras. Note that our arguments are essentially based on their arguments.

We shall review some results of Kirchberg's central sequence algebra in [17]. We denote by \tilde{A} the unitization algebra of A. Note that we consider $A = \tilde{A}$ when A is unital. For a separable C*-algebra A, set

$$c_0(A) := \{ (a_n)_{n \in \mathbb{N}} \in \ell^{\infty}(\mathbb{N}, A) \mid \lim_{n \to \infty} ||a_n|| = 0 \}, \ A^{\infty} := \ell^{\infty}(\mathbb{N}, A) / c_0(A).$$

Let B be a C^{*}-subalgebra of A. We identify A and B with the C^{*}-subalgebras of A^{∞} consisting of equivalence classes of constant sequences. Put

 $A_{\infty} := A^{\infty} \cap A', \text{ Ann}(B, A^{\infty}) := \{(a_n)_n \in A^{\infty} \cap B' \mid (a_n)_n b = 0 \text{ for any } b \in B\}.$ Then Ann (B, A^{∞}) is an closed two-sided ideal of $A^{\infty} \cap B'$, and define

$$F(A) := A_{\infty} / \operatorname{Ann}(A, A^{\infty}).$$

We call F(A) the central sequence algebra of A. A sequence $(a_n)_n$ is said to be central if $\lim_{n\to\infty} ||a_n a - aa_n|| = 0$ for all $a \in A$. A central sequence is a representative of an element in A_{∞} . Since A is separable, A has a countable approximate unit $\{h_n\}_{n\in\mathbb{N}}$. It is easy to see that $[(h_n)_n]$ is a unit in F(A). If A is unital, then $F(A) = A_{\infty}$. Moreover we see that F(A) is isomorphic to $M(A)^{\infty} \cap A'/\operatorname{Ann}(A, M(A)^{\infty})$ since for any $(y_n)_n \in M(A)^{\infty} \cap A', (y_nh_n)_n$ is a central sequence in A and $[(y_n)_n] = [(y_nh_n)_n]$ in $M(A)^{\infty} \cap A'/\operatorname{Ann}(A, M(A)^{\infty})$. Let $\{e_{ij}\}_{i,j\in\mathbb{N}}$ be the standard matrix units of K. Define a map φ of F(A) to $F(A \otimes \mathbb{K})$ by $\varphi([(x_n)_n]) = [(x_n \otimes \sum_{i=1}^n e_{ii})_n]$. Then it is easily seen that φ is a welldefined injective homomorphism. A similar argument as above shows any element in $F(A \otimes \mathbb{K})$ is equal to $[(\sum_{i,j=1}^{n} x_{n,i,j} \otimes e_{i,j})_n]$ for some sequence $\{x_{n,i,j}\}_{n \in \mathbb{N}}$ in A. Using matrix units and the centrality of sequence, we can show that if $i \neq j$, then $\lim_{n\to\infty} x_{n,i,j}a = 0$ for any $a \in A$ and $\lim_{n\to\infty} (x_{n,i,i} - x_{n,j,j})a = 0$ for any $i, j \in \mathbb{N}$ and $a \in A$. Since $M_{\infty}(A)$ is dense in $A \otimes \mathbb{K}$, it can be checked that φ is surjective. Hence F(A) is isomorphic to $F(A \otimes \mathbb{K})$. (See [17, Proposition 1.9] for more general cases.)

We denote by I(k, k+1) the prime dimension drop algebra

 $\{f \in C([0,1]) \otimes M_k(\mathbb{C}) \otimes M_{k+1}(\mathbb{C}) \mid f(0) \in M_k(\mathbb{C}) \otimes \mathrm{id}_{k+1}, f(1) \in \mathrm{id}_k \otimes M_{k+1}(\mathbb{C})\}$

for $k \in \mathbb{N}$. The Jiang-Su algebra \mathcal{Z} is constructed as an inductive limit C*-algebra of prime dimension drop algebras in [14]. We shall show the following proposition (which is based on [48, Proposition 2.2]) by a similar way as in [39, Theorem 7.2.2]. See [17, Proposition 4.11] for more general cases.

Proposition 5.1. Let A be a separable C*-algebra. There exist a unital homomorphism of the prime dimension drop algebra I(k, k + 1) to F(A) for any $k \in \mathbb{N}$ if and only if A is \mathcal{Z} -stable.

Proof. Assume that there exist a unital homomorphism of the prime dimension drop algebra I(k, k + 1) to F(A) for any $k \in \mathbb{N}$. By a similar argument as in [48, Proposition 2.2] and the construction of \mathcal{Z} in [14], we see that there exists a unital homomorphism α of \mathcal{Z} to F(A).

Let φ be an injective homomorphism of A to $A \otimes \mathbb{Z}$ defined by $\varphi(a) = a \otimes 1_{\mathbb{Z}}$, and put $C := M(A \otimes \mathbb{Z})^{\infty} \cap \varphi(A)' / \operatorname{Ann}(\varphi(A), M(A \otimes \mathbb{Z})^{\infty})$. Then we can regard α as a unital homomorphism of \mathbb{Z} to C since F(A) is isomorphic to $M(A)^{\infty} \cap$ $A' / \operatorname{Ann}(A, M(A)^{\infty})$. Define a unital homomorphism of β of \mathbb{Z} to $M(A \otimes \mathbb{Z})^{\infty} \cap$ $\varphi(A)'$ by $\beta(x) = (1_{M(A)} \otimes x)_n$, and let $[\beta] : \mathbb{Z} \to C$ be the quotient homomorphism of β . Then we see that $C^*(\alpha(\mathbb{Z}), [\beta](\mathbb{Z}))$ in C is isomorphic to $\mathbb{Z} \otimes \mathbb{Z}$. Since \mathbb{Z} has

approximately inner flip and is K₁-injective (see [47, Proposition 1.13]), there exists a sequence $\{w_m\}_{m\in\mathbb{N}}$ of unitary elements in C such that $\lim_{m\to\infty} w_m^*[\beta](x)w_m = \alpha(x)$ for any $x \in \mathbb{Z}$ and w_m is in the connected component of 1_C in U(C) for any $m \in \mathbb{N}$. Since w_m is in the connected component of 1_C in U(C), there exists a unitary element u_m in $M(A \otimes \mathbb{Z})^{\infty} \cap \varphi(A)'$ such that $[u_m] = w_m$ for any $m \in \mathbb{N}$. For any $a \in A, x \in \mathbb{Z}$ and all $y \in M(A \otimes \mathbb{Z})^{\infty} \cap \varphi(A)'$ such that $[y] = \alpha(x)$, we have

$$y\varphi(a) = \lim_{m \to \infty} u_m^*\beta(x)u_m\varphi(a) = \lim_{m \to \infty} u_m^*\beta(x)\varphi(a)u_m = \lim_{m \to \infty} u_m^*(a \otimes x)u_m$$

by $[y] = \lim_{m\to\infty} [u_m^*\beta(x)u_m]$ and the definition of $\operatorname{Ann}(\varphi(A), M(A \otimes \mathbb{Z})^{\infty})$. Since $[y] = \alpha(x)$, we can take $y \in M(\varphi(A))^{\infty} \cap \varphi(A)' \subseteq M(A \otimes \mathbb{Z})^{\infty} \cap \varphi(A)'$. Hence we see that $\lim_{m\to\infty} u_m^*(a \otimes x)u_m$ is an element in $\varphi(A)^{\infty}$. Therefore for any $z \in A \otimes \mathbb{Z}$, $\lim_{m\to\infty} d(u_m^*zu_m, \varphi(A)^{\infty}) = 0$. We see that A is \mathbb{Z} -stable by a similar argument as in [39, Proposition 2.3.5 and Proposition 7.2.1].

Conversely assume that A is \mathbb{Z} -stable. Then A is isomorphic to $A \otimes (\bigotimes_{k=1}^{\infty} \mathbb{Z})$. Since M(A) is the largest unital C*-algebra that contains A as an essential ideal, $\tilde{A} \otimes (\bigotimes_{k=1}^{\infty} \mathbb{Z})$ is a unital subalgebra of $M(A \otimes (\bigotimes_{k=1}^{\infty} \mathbb{Z}))$. Hence there exists a unital homomorphism of \mathbb{Z} to $M(A)^{\infty} \cap A'$. Therefore we see that there exists a unital homomorphism of the prime dimension drop algebra I(k, k+1) to F(A) for any $k \in \mathbb{N}$ because F(A) is isomorphic to $M(A)^{\infty} \cap A'/\operatorname{Ann}(A, M(A)^{\infty})$.

We denote by $\operatorname{Ped}(A)$ the Pedersen ideal of A. The Pedersen ideal $\operatorname{Ped}(A)$ is a minimal dense two-sided ideal of A. Hence every densely defined lower semicontinuous trace τ on A is finite on $\operatorname{Ped}(A)$ because τ is finite on a dense two-sided ideal. Moreover for any positive element h in $\operatorname{Ped}(A)$, \overline{hAh} is contained in $\operatorname{Ped}(A)$. We refer the reader to [3, II 5.2.4] and [33, Section 5.6] for details of the Pedersen ideal. If A is unital, every densely defined lower semicontinuous trace on A is bounded. Hence if A is simple and $A \otimes \mathbb{K}$ has a nonzero projection, then there exists a full hereditary subalgebra B of A such that every densely defined lower semicontinuous trace on B is bounded. In general, we have the following proposition.

Proposition 5.2. Let A be a σ -unital simple C*-algebra. Then there exists a full hereditary subalgebra B of A such that every densely defined lower semicontinuous trace on B is bounded.

Proof. Let h be a nonzero positive element in Ped(A). Then any $\tau \in T(A)$ restricts to a bounded trace on \overline{hAh} because every positive liner functional is automatically bounded. We obtain the conclusion by Corollary 2.4.

If A is separable, then A is \mathbb{Z} -stable if and only if some full hereditary subalgebra is \mathbb{Z} -stable by Proposition 5.1 and Brown's theorem in [4] since F(A) is isomorphic to $F(A \otimes \mathbb{K})$. (See also [47].) Therefore we may assume that A has no unbounded trace by the proposition above. Note that if A has strict comparison and no unbounded trace, then for any $a, b \in A_+$ satisfying $d_{\tau}(a) < d_{\tau}(b)$ for all $\tau \in T_1(A)$, we have $a \preceq b$.

Proposition 5.3. Let A be a separable C*-algebra such that $T_1(A)$ is a non-empty compact set, and let $\{h_m\}_{m\in\mathbb{N}}$ be a countable approximate unit for A and $\epsilon > 0$. Then there exists a natural number N such that

$$\max_{\tau \in T_1(A)} |\tau(f_n) - \tau(h_m f_n)| < \epsilon$$

for any $m \geq N$ and for any sequence $(f_n)_{n \in \mathbb{N}}$ of positive contractions in A. In particular, we have

$$\lim_{n \to \infty} \max_{\tau \in T_1(A)} |\tau(h_n f_n) - \tau(f_n)| = 0.$$

Proof. For any $\tau \in T_1(A)$, we have $\tau(h_m) \leq \tau(h_{m+1})$ and $\lim \tau(h_m) = 1$. By Dini's theorem, there exists a natural number N such that

$$\max_{\tau \in T_1(A)} |1 - \tau(h_m)| < \epsilon$$

for any $m \geq N$. For any sequence $(f_n)_{n \in \mathbb{N}}$ of positive contractions in A,

$$\max_{\tau \in T_1(A)} |\tau(f_n) - \tau(h_m f_n)| = \max_{\tau \in T_1(A)} |\tau((1 - h_m)^{1/2} f_n (1 - h_m)^{1/2})|$$

$$\leq \max_{\tau \in T_1(A)} |1 - \tau(h_m)| < \epsilon$$

for $m \geq N$.

We recall some definitions.

Definition 5.4. Let A be a separable C*-algebra with no unbounded trace. Assume that $T_1(A)$ is a non-empty compact set. We say that A has *property* (SI) if for any central sequences $(e_n)_n$ and $(f_n)_n$ of positive contractions in A satisfying

$$\lim_{n \to \infty} \max_{\tau \in T_1(A)} \tau(e_n) = 0, \ \lim_{m \to \infty} \liminf_{n \to \infty} \min_{\tau \in T_1(A)} \tau(f_n^m) > 0,$$

there exists a central sequence $(s_n)_n$ in A such that

$$\lim_{n \to \infty} \|s_n^* s_n - e_n\| = 0, \ \lim_{n \to \infty} \|f_n s_n - s_n\| = 0.$$

For a completely positive map φ of \tilde{A} to \tilde{A} , we say that φ can be excised in small central sequences in A if for any central sequences $(e_n)_n$ and $(f_n)_n$ of positive contractions in A satisfying the property above, there exists a sequence $(s_n)_{n \in \mathbb{N}}$ in A such that

$$\lim_{n \to \infty} \|s_n^* a s_n - \varphi(a) e_n\| = 0 \text{ for any } a \in \tilde{A}, \ \lim_{n \to \infty} \|f_n s_n - s_n\| = 0.$$

Remark 5.5. In the definition above, it is important that e_n and f_n are elements in A. We see that if $id_{\tilde{A}}$ can be excised in small central sequences in A, then A has property (SI) (see [25, Proof of (iii) \Rightarrow (iv) of Theorem1.1]).

We shall generalize [24, Lemma 4.6] and [25, Lemma 2.4] to non-unital C^{*}-algebras.

Lemma 5.6. Let c be a positive element in a separable C*-algebra A such that $T_1(A)$ is a non-empty compact set, and let $\theta \in \mathbb{R}$. For any central sequence $(f_n)_n$ of positive contractions in A, we have

$$\limsup_{n \to \infty} \max_{\tau \in T_1(A)} |\tau(cf_n) - \theta \tau(f_n)| \le 2 \max_{\tau \in T_1(A)} |\tau(c) - \theta|.$$

Proof. Let $\{h_m\}_{m\in\mathbb{N}}$ be a countable approximate unit for A. Replacing f_n and θ in [24, Lemma 4.6] with $h_m f_n$ and θh_m respectively, the same argument in the proof of [24, Lemma 4.6] shows that

$$\limsup_{n \to \infty} \max_{\tau \in T_1(A)} |\tau(cf_n) - \theta \tau(h_m f_n)| \le 2 \max_{\tau \in T_1(A)} |\tau(c) - \theta \tau(h_m)|$$

for any $m \in \mathbb{N}$. By Proposition 5.3, we have

$$\limsup_{n \to \infty} \max_{\tau \in T_1(A)} |\tau(cf_n) - \theta \tau(f_n)| \le 2 \max_{\tau \in T_1(A)} |\tau(c) - \theta|.$$

Lemma 5.7. Let A be a separable simple C*-algebra such that $T_1(A)$ is a nonempty compact set, and let a be a nonzero positive element in \tilde{A} . If $(f_n)_n$ is a central sequence of positive contractions in A such that

$$\lim_{m \to \infty} \liminf_{n \to \infty} \min_{\tau \in T_1(A)} \tau(f_n^m) > 0,$$

then

$$\lim_{m \to \infty} \liminf_{n \to \infty} \min_{\tau \in T_1(A)} \tau(f_n^{m/2} a f_n^{m/2}) > 0.$$

Proof. Put $R := a^{1/2}A$. Since A is simple, R is a right ideal of A such that $R^*R = AaA$ is a dense ideal of A. Therefore there exists a sequence $\{v_j\}_{j\in\mathbb{N}}$ in A such that $\{\sum_{j=1}^n v_j^*av_j\}_{n\in\mathbb{N}}$ is an approximate unit for A by a similar argument as in [4, Lemma 2.3]. By Proposition 5.3, there exists a natural number N such that

$$\lim_{m \to \infty} \liminf_{n \to \infty} \min_{\tau \in T_1(A)} \tau(\sum_{j=1}^N v_j^* a v_j f_n^m) > 0.$$

We have

$$\lim_{m \to \infty} \liminf_{n \to \infty} \min_{\tau} \tau(\sum_{j=1}^{N} v_{j}^{*} a v_{j} f_{n}^{m}) = \lim_{m \to \infty} \liminf_{n \to \infty} \min_{\tau} \sum_{j=1}^{N} \tau(v_{j}^{*} a^{1/2} f_{n}^{m} a^{1/2} v_{j})$$
$$= \lim_{m \to \infty} \liminf_{n \to \infty} \min_{\tau} \sum_{j=1}^{N} \tau(f_{n}^{m/2} a^{1/2} v_{j} v_{j}^{*} a^{1/2} f_{n}^{m/2})$$
$$\leq \sum_{j=1}^{N} \|v_{j}\|^{2} \lim_{m \to \infty} \liminf_{n \to \infty} \min_{\tau} \tau(f_{n}^{m/2} a f_{n}^{m/2}).$$

Hence we obtain the conclusion.

Let A be a separable simple C*-algebra, and let τ be a tracial state on A. Consider the GNS representation $(\pi_{\tau}, H_{\tau}, \xi_{\tau})$ associated with τ . Then $\pi_{\tau}(A)''$ is a finite von Neumann algebra and $\pi_{\tau}(A)$ is strongly dense subalgebra of $\pi_{\tau}(A)''$ in general. (Indeed, every approximate unit for $\pi_{\tau}(A)$ is strongly convergent to $1_{H_{\tau}}$.) In particular, Kaplansky density theorem shows that for any positive contraction $H \in \pi_{\tau}(A)''$ there exists a sequence $\{a_n\}_{n \in \mathbb{N}}$ of positive contractions in A such that $\pi(a_n)$ is strongly converge to H. We can identify $C^*(\pi_{\tau}(A), 1_{H_{\tau}})$ in $B(H_{\tau})$ with its unitization algebra \tilde{A} . Therefore the same proof as [42, Lemma 2.1] shows the following lemma. See also [26, Proposition 3.5 and Theorem 4.3].

Lemma 5.8. ([42, Lemma 2.1])

Let A be a separable simple nuclear C*-algebra, and let τ be a tracial state on A. For any sequence $\{H_n\}_{n\in\mathbb{N}}$ of positive contractions in $\pi_{\tau}(A)''$ such that $\|[H_n, x]\|_{\tau} \to 0$ for all $x \in \pi_{\tau}(A)''$, there exists a central sequence $(c_n)_n$ of positive contractions in A such that $\|c_n - H_n\|_{\tau} \to 0$.

Maybe someone considers that [42, Lemma 2.1] depends on a unit for the application of Haagerup's theorem ([12, Theorem 3.1]); see for example [11, Theorem 2.1] for details. But we can check that the same proof of [42, Lemma 2.1] works for non-unital C^{*}-algebras because A is a two-sided ideal of \tilde{A} and for any positive contraction $H \in \pi_{\tau}(A)''$ there exists a sequence $\{a_n\}_{n \in \mathbb{N}}$ of positive contractions in A such that $\pi(a_n)$ is strongly converge to H.

If τ is an extremal tracial state on a separable simple infinite-dimensional nuclear C^{*}-algebra A, then $\pi_{\tau}(A)''$ is the AFD II₁ factor in general. Therefore Lemma 5.8 and the same proof as [25, Lemma 3.3] show the following lemma.

Lemma 5.9. ([25, Lemma 3.3])

Let A be a separable simple infinite-dimensional nuclear C*-algebra with finitely many extremal tracial states. For any $k \in \mathbb{N}$, there exist central sequences $(c_{i,n})_n$ in A, i = 1, 2, ..., k such that $c_{1,n}$ is a positive contraction for any $n \in \mathbb{N}$, $(c_{i,n}c_{i,n}^*)_n =$

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 $\delta_{i,j}(c_{1,n}^2)_n$ and

$$\lim_{n \to \infty} \max_{\tau \in T_1(A)} |\tau(c_{1,n}^m) - \frac{1}{k}| = 0$$

for any $m \in \mathbb{N}$.

Note that we need to consider unitaries in \tilde{A} in the proof above. But it is also no problem because A is a two-sided ideal of \tilde{A} .

Let ω be a pure state on A. Then we can uniquely extend ω to a pure state $\tilde{\omega}$ on \tilde{A} . Moreover if A is a separable simple non-type I C*-algebra, then $\pi_{\omega}(A) \cap K(H_{\omega}) = \{0\}$. Therefore the same proof as [25, Lemma 3.1] shows that every completely positive map of \tilde{A} to \tilde{A} can be approximated in the pointwise norm topology by completely positive map φ of the form

$$\varphi(x) = \sum_{l=1}^{N} \sum_{i,j=1}^{N} \tilde{\omega}(d_i^* x d_j) c_{l,i}^* c_{l,j}, \ x \in \tilde{A}$$

where $c_{l,i}, d_i \in \tilde{A}$. For $1 \leq l \leq N$, let $\varphi_l(x) = \sum_{i,j=1}^N \tilde{\omega}(d_i^* x d_j) c_{l,i}^* c_{l,j}$. Then $\varphi = \varphi_1 + \ldots + \varphi_N$. Using Lemma 5.7 instead of [25, Lemma 2.4], we can prove a version of [25, Proposition 2.2], i.e. that each φ_l can be excised in small central sequences in A. (See the proof of [25, Lemma 2.5], which is where [25, Lemma 2.4] gets used; note also this where we need strict comparison.) We can check that [25, Lemma 3.4] holds without the assumption of a unit by using Lemma 5.6 and Lemma 5.9 instead of [24, Lemma 4.6] and [25, Lemma 3.3] respectively. By this lemma, we see that a sum of completely positive maps $\tilde{A} \to \tilde{A}$, each of which can be excised in small central sequences in A. Therefore we obtain the following theorem.

Theorem 5.10. Let A be a separable simple infinite-dimensional nuclear C^{*}algebra with finitely many extremal tracial states and no unbounded trace. If A has a strict comparison, then any completely positive map of \tilde{A} to \tilde{A} can be excised in small central sequences in A.

The following theorem is the main theorem in this section.

Theorem 5.11. Let A be a separable simple infinite-dimensional non-type I nuclear C^{*}-algebra with a finite dimensional lattice of densely defined lower semicontinuous traces. Then A has strict comparison if and only if A is \mathcal{Z} -stable.

Proof. Rørdam showed that if A is \mathbb{Z} -stable, then A has strict comparison (see [40, Corollary 4.6]). We shall show the only if part. By Proposition 5.2, we may assume that A has no unbounded trace. Hence A has property (SI) by Remark 5.5 and Theorem 5.10. For any $k \in \mathbb{N}$, there exist central sequences $(c_{i,n})_n$ in A, i = 1, 2, ..., k such that $c_{1,n}$ is a positive contraction, $(c_{i,n}c_{i,n}^*)_n = \delta_{i,j}(c_{1,n}^2)_n$ and

$$\lim_{n \to \infty} \max_{\tau \in T_1(A)} |\tau(c_{1,n}^m) - \frac{1}{k}| = 0$$

for any $m \in \mathbb{N}$ by Lemma 5.9. Let $\{h_n\}_{n\in\mathbb{N}}$ be an approximate unit for A. Taking a suitable subsequence of $\{h_n\}_{n\in\mathbb{N}}$, we may assume that $(h_n)_n(c_{1,n})_n = (c_{1,n})_n(h_n)_n$. Define central sequences $(f_{i,n})_n$ in A, i = 1, ..., k by $(f_{i,n})_n := (c_{i,n}h_n^{1/2})_n$, and put $(e_n)_n := (h_n - \sum_{i=1}^k f_{i,n}^* f_{i,n})_n$. Then we may assume that $(e_n)_n$ is a central sequence of positive contractions in A. Proposition 5.3 implies $\lim_{n\to\infty} \max_{\tau} |\tau(f_{i,n}^* f_{i,n} - c_{i,n}^* c_{i,n})| = 0$ for any $1 \le i \le k$, and hence we have

$$\lim_{n \to \infty} \max_{\tau \in T_1(A)} \tau(e_n) = 0.$$

Note that $(f_{1,n})_n$ is a central sequence of positive contractions in A by the assumption of $(h_n)_n$. Because $\{h_n^{1/2}\}_{n \in \mathbb{N}}$ is also an approximate unit for A, we have

$$\lim_{n \to \infty} \sup_{\tau} \max_{\tau} \|c_{1,n} - c_{1,n}^{1/2} h_n^{1/2} c_{1,n}^{1/2} \|_{\tau}^2 = \limsup_{n \to \infty} \max_{\tau} \tau((c_{1,n} - c_{1,n}^{1/2} h_n^{1/2} c_{1,n}^{1/2})^2)$$
$$\leq \limsup_{n \to \infty} \max_{\tau} \tau(c_{1,n} - c_{1,n}^{1/2} h_n^{1/2} c_{1,n}^{1/2})$$
$$= 0$$

by Proposition 5.3. Hence

$$\begin{split} \limsup_{n \to \infty} \max_{\tau} |\tau(c_{1,n}^m) - \tau(f_{1,n}^m)| &= \limsup_{n \to \infty} \max_{\tau} |\tau(c_{1,n}^m - (c_{1,n}h_n^{1/2})^m)| \\ &= \limsup_{n \to \infty} \max_{\tau} |\tau(c_{1,n}^m - (c_{1,n}^{1/2}h_n^{1/2}c_{1,n}^{1/2})^m)| \\ &\leq \limsup_{n \to \infty} \max_{\tau} \|c_{1,n}^m - (c_{1,n}^{1/2}h_n^{1/2}c_{1,n}^{1/2})^m)\|_{\tau} = 0 \end{split}$$

for any $m \in \mathbb{N}$. Therefore we have

$$\lim_{m \to \infty} \liminf_{n \to \infty} \min_{\tau \in T_1(A)} \tau(f_{1,n}^m) = 1/k > 0.$$

Since A has property (SI), there exists a central sequence $(s_n)_n$ in A such that $(s_n^*s_n + \sum_{i=1}^k f_{i,n}^*f_{i,n})_n = (h_n)_n$ and $(f_{1,n}s_n)_n = (s_n)_n$. We have $[(f_{i,n}f_{j,n}^*)_n] = \delta_{i,j}[(f_{1,n}^2)_n]$ and $[(s_n^*s_n + \sum_{i=1}^k f_{i,n}^*f_{i,n})_n] = 1$ in F(A) because $[(h_n^{1/2})_n]$ is a unit in F(A). It follows from [41, Proposition 2.1] that there exists a unital homomorphism of I(k, k+1) to F(A). Consequently A is \mathcal{Z} -stable by Proposition 5.1.

Remark 5.12. Let A be a separable simple infinite-dimensional non-type I nuclear C*-algebra with a finite dimensional lattice of densely defined lower semicontinuous traces, that has strict comparison. Since A is \mathcal{Z} -stable by the theorem above, there exists a unital homomorphism of \mathcal{Z} to $M(A)^{\infty} \cap A'$. But we do not know that we could show this fact directly without using Kirchberg's central sequence algebras. Note that if A is non-unital, then there exists no unital homomorphism of \mathcal{Z} to $(\tilde{A})^{\infty} \cap A'$ because \tilde{A} is not \mathcal{Z} -stable.

The following corollary is an immediate consequence of the theorem above and Corollary 4.2.

Corollary 5.13. Let A be a separable simple nuclear stably projectionless C^* -algebra with a unique tracial state and no unbounded trace. Assume that A has strict comparison. Then we have the following exact sequence:

 $1 \longrightarrow \operatorname{Out}(A) \xrightarrow{\rho_A} \operatorname{Pic}(A) \xrightarrow{T} \mathcal{F}(A) \longrightarrow 1$

We shall consider some examples. We refer the reader to [45] for details of slow dimensional growth for nonunital C^{*}-algebras. Tikuisis showed that if A is a simple separable approximately subhomogeneous C^{*}-algebra with slow dimension growth, then Cu(A) is almost unperforated in [45, Corollary 5.9]. The following immediate corollary of this result and Theorem 5.11 is suggested by the referee.

Corollary 5.14. Let A be a simple separable non-type I approximately subhomogeneous C*-algebra with slow dimension growth and a finite dimensional lattice of densely defined lower semicontinuous traces. Then A is \mathcal{Z} -stable.

We say that A is a 1-dimensional NCCW complex if A is a pullback C*-algebra of the form

$$\begin{array}{ccc} A & \stackrel{\pi_2}{\longrightarrow} & E \\ & \downarrow^{\pi_1} & \downarrow^{\rho} \\ C([0,1]) \otimes F \xrightarrow{\delta_0 \oplus \delta_1} & F \oplus F \end{array}$$

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where E and F are finite-dimensional C*-algebras and δ_i is the evaluation map at i. Razak's building block A(n,m) is a 1-dimensional NCCW complex. The Cuntz semigroup of a 1-dimensional NCCW complex was computed in [1]. Every simple inductive limit C*-algebras of 1-dimensional NCCW complexes is approximately subhomogeneous C*-algebra with slow dimension growth. The following example is also suggested by the referee.

Example 5.15. For $n \geq 2$, let \mathcal{O}_n denote the Cuntz algebra generated by n isometries $S_1, ..., S_n$. Given $\lambda_1, ..., \lambda_n \in \mathbb{R}$, there exists by universality a one-parameter automorphism group α of \mathcal{O}_n given by $\alpha_t(S_j) = e^{it\lambda_j}S_j$. Kishimoto and Kumjian showed that if λ_j are all nonzero of the same sign and $\{\lambda_1, ..., \lambda_n\}$ generates \mathbb{R} as a closed subgroup, then $\mathcal{O}_n \rtimes \mathbb{R}$ is a simple stable projectionless C*-algebra with unique (up to scalar multiple) densely defined lower semicontinuous trace in [19] and [20]. In particular, $\mathcal{O}_n \rtimes_\alpha \mathbb{R}$ has a one parameter trace scaling automorphism group. Dean showed that there exist many sets of numbers $\{\lambda_1, ..., \lambda_n\}$ such that $\mathcal{O}_n \rtimes_\alpha \mathbb{R}$ can be expressed as an inductive limit C*-algebra of 1-dimensional NCCW-complexes in [9, Theorem 5.1]. Therefore for α defined by such a set of numbers $\{\lambda_1, ..., \lambda_n\}$, $\mathcal{O}_n \rtimes_\alpha \mathbb{R}$ is \mathcal{Z} -stable. Moreover it can be checked that for any positive element h in $\operatorname{Ped}(\mathcal{O}_n \rtimes_\alpha \mathbb{R})$, $\operatorname{Pic}(\overline{h(\mathcal{O}_n \rtimes_\alpha \mathbb{R})h})$ is isomorphic to a semidirect product of $\operatorname{Out}(\overline{h(\mathcal{O}_n \rtimes_\alpha \mathbb{R})h})$ with \mathbb{R}^{\times}_+ by the same argument of the proof in Theorem 4.4.

Robert classified inductive limit C*-algebras of 1-dimensional NCCW complexes with trivial K_1 -groups in [37].

Corollary 5.16. Let A be a simple stably projectionless C*-algebra with a unique tracial state and no unbounded trace, that is expressible as an inductive limit C*-algebra of 1-dimensional NCCW-complexes with trivial K_1 -groups and B a separable simple C*-algebra with a unique tracial state and no unbounded trace. Then we have the following exact sequence:

 $1 \longrightarrow \operatorname{Out}(A \otimes B) \xrightarrow{\rho_{A \otimes B}} \operatorname{Pic}(A \otimes B) \xrightarrow{T} \mathbb{R}_{+}^{\times} \longrightarrow 1$

Proof. For any $r \in (0, 1)$ there exists a positive element h in A such that $d_{\tau}(h) = r$ because A has a positive element with a continuous spectrum. Note that the class of C*-algebras covered by Robert's classification theorem in [37] is closed under stable isomorphism (see [37, Theorem 1.0.1]). By [37, Proposition 3.1.7], we see that a classifying invariant of the class of C*-algebras which contains A and \overline{hAh} is equal to that of [37, Corollary 6.2.4]. Hence we see that A is isomorphic to \overline{hAh} . Therefore $\mathcal{F}(A) = \mathbb{R}_+^{\times}$. Since $\mathcal{F}(A \otimes B) = \mathbb{R}_+^{\times}$ and $A \otimes B$ is separable, $A \otimes B$ is a stably projectionless C*-algebra by [29, Corollary 4.10]. Therefore we obtain the conclusion by Corollary 4.2 and Corollary 5.14.

We do not know whether the exact sequence above splits. This question is related to the existence of a one parameter trace scaling automorphism group of $A \otimes \mathbb{K}$. For any countable abelian groups G_1 and G_2 , Kishimoto showed that there exists a stable projectionless simple separable nuclear C*-algebra A with unique (up to scalar multiple) densely defined lower semicontinuous trace with $K_0(A) = G_1$ and $K_1(A) = G_2$ in [18]. These stably projectionless C*-algebras are constructed as

the crossed products $\mathcal{O} \rtimes_{\alpha} \mathbb{R}$ by certain one parameter automorphism groups α of Kirchberg algebras \mathcal{O} and the dual actions of α are trace scaling actions of $\mathcal{O} \rtimes_{\alpha} \mathbb{R}$. Hence it is natural to believe that there exists a kind of duality between \mathcal{Z} -stable stably projectionless C*-algebras (with unique trace) and \mathcal{O}_{∞} -stable C*-algebras. From this view point, it seems to be possible to introduce the stably projectionless C*-algebra \mathcal{W}_n for any $n \geq 3$. Hence we denote by \mathcal{W}_2 the Razak-Jacelon algebra. On the other hand, Tikuisis [45] constructed a simple separable nuclear stably projectionless C*-algebra whose Cuntz semigroup is not almost unperforated.

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References

- R. Antoine, F. Perera and L. Santiago, Pullbacks, C(X)-algebras, and their Cuntz semigroup, J. Funct. Anal. 260 (2011), no. 10, 2844–2880.
- [2] P. Ara, F. Perera and A. Toms, K-theory for operator algebras. Classification of C^{*}-algebras, Aspects of operator algebras and applications, 1–71, Contemp. Math., 534, Amer. Math. Soc., Providence, RI, 2011.
- [3] B. Blackadar, Operator Algebras: Theory of C*-Algebras and von Neumann Algebras, Encyclopaedia of Mathematical Sciences, 122, Springer, 2006.
- [4] L. G. Brown, Stable isomorphism of hereditary subalgebras of C*-algebras, Pacific J. Math. 71 (1977), 335–348.
- [5] L. G. Brown, P.Green and M. A. Rieffel, Stable isomorphism and strong Morita equivalence of C^{*}-algebras, Pacific J. Math. **71** (1977), no. 2, 349–363.
- [6] F. Combes and H. Zettl, Order structures, traces and weights on Morita equivalent C^{*}algebras, Math. Ann. 265 (1983), no. 1, 67–81.
- [7] K. Coward, G. A. Elliott and C. Ivanescu, The Cuntz semigroup as an invariant for C^{*}algebras, J. Reine Angew. Math. 623 (2008), 161–193.
- [8] J. Cuntz, Dimension functions on simple C*-algebras, Math. Ann. 233 (1978), no. 2, 145– 153.
- [9] A. Dean, A continuous field of projectionless C*-algebras, Canad. J. Math. 53 (2001), no. 1, 51–72.
- [10] G. A. Elliott, L. Robert and L. Santiago, The cone of lower semicontinuous traces on a C^{*}-algebra, Amer. J. Math. 133 (2011), no. 4, 969–1005.
- [11] H. Futamura, N. Kataoka and A. Kishimoto, Type III representations and automorphisms of some separable nuclear C^{*}-algebras, J. Funct. Anal. 197 (2003), no. 2, 560–575.
- [12] U. Haagerup, All nuclear C*-algebras are amenable, Invent. Math. 74 (1983), no. 2, 305–319.
- [13] B. Jacelon, A simple, monotracial, stably projectionless C*-algebra, preprint, arXiv:1006.5397v3 [math.OA], to appear in J. London Math. Soc.
- [14] X. Jiang and H. Su, On a simple unital projectionless C*-algebra, Amer. J. Math. 121 (1999), no. 2, 359–413.
- [15] T. Kajiwata, C. Pinzari and Y. Watatani, Jones index theory for Hilbert C^{*}-bimodules and its equivalence with conjugation theory, J. Funct. Anal. 215 (2004), no. 1, 1–49.
- [16] T. Kajiwara and Y. Watatani, Jones index theory by Hilbert C*-bimodules and K-theory, Trans. Amer. Math. Soc. 352 (2000), no. 8, 3429–3472.
- [17] E. Kirchberg, Central sequences in C*-algebras and strongly purely infinite algebras, Operator Algebras: The Abel Symposium 2004, 175–231, Abel Symp., 1, Springer, Berlin, 2006.
- [18] A. Kishimoto, Pairs of simple dimension groups, Internat. J. Math. 10 (1999), no. 6, 739–761.
 [19] A. Kishimoto and A. Kumjian, Simple stably projectionless C^{*}-algebras arising as crossed
- products, Canad. J. Math. 48 (1996), no. 5, 980–996.
- [20] A. Kishimoto and A. Kumjian, Crossed products of Cuntz algebras by quasi-free automorphisms, in Operator algebras and their applications (Waterloo, ON, 1994/1995), 173–192, Fields Inst. Commun., 13, Amer. Math. Soc., Providence, RI, 1997.

- [21] K. Kodaka, Full projections, equivalence bimodules and automorphisms of stable algebras of unital C*-algebras, J. Operator Theory, 37 (1997), no. 2, 357–369.
- [22] K. Kodaka, Picard groups of irrational rotation C*-algebras, J. London Math. Soc. (2) 56 (1997), no. 1, 179–188.
- [23] K. Kodaka, Projections inducing automorphisms of stable UHF-algebras, Glasg. Math. J. 41 (1999), no. 3, 345–354.
- [24] H. Matui and Y. Sato, Z-stability of crossed products by strongly outer actions, Comm. Math. Phys. 314 (2012), 193–228.
- [25] H. Matui and Y. Sato, Strict comparison and Z-absorption of nuclear C^{*}-algebras, Acta Math. 209 (2012), no. 1, 179–196. arXiv:1111.1637v1 [math.OA].
- [26] H. Matui and Y. Sato, Z-stability of crossed products by strongly outer actions II, preprint, arXiv:1205.1590v1 [math.OA].
- [27] J. A. Mingo and W. J. Phillips. Equivariant triviality theorems for Hilbert C^{*}-modules, Proc. Amer. Math. Soc. 91 (1984), no. 2, 225–230.
- [28] F. Murray and J. von Neumann, On rings of operators IV, Ann. Math. (2) 44, (1943), 716–808.
- [29] N. Nawata, Fundamental group of simple C*-algebras with unique trace III, Canad. J. Math. 64, (2012), no. 3, 573–587.
- [30] N. Nawata, A note on trace scaling actions and fundamental groups of C^{*}-algebras, preprint:arXiv:1009.1722 [math.OA].
- [31] N. Nawata and Y. Watatani, Fundamental group of simple C*-algebras with unique trace, Adv. Math. 225 (2010), no. 1, 307–318.
- [32] N. Nawata and Y. Watatani, Fundamental group of simple C*-algebras with unique trace II, J. Funct. Anal. 260 (2011), no. 2, 428–435.
- [33] G. K. Pedersen, C*-Algebras and Their Automorphism Groups, Academic Press, London-New York-San Francisco, 1979.
- [34] S. Razak, On the classification of simple stably projectionless C*-algebras, Canad. J. Math. 54 (2002), no. 1, 138–224.
- [35] I. Raeburn and D. P. Williams, Morita Equivalence and Continuous-Trace C*-Algebras, Mathematical Surveys and Monographs, 60, American Mathematical Society, Providence, RI, 1998.
- [36] M. A. Rieffel, Morita equivalence for operator algebras, Operator algebras and applications, Part I (Kingston, Ont., 1980), pp. 285–298, Proc. Sympos. Pure Math., 38, Amer. Math. Soc., Providence, R.I., 1982.
- [37] L. Robert, Classification of inductive limits of 1-dimensional NCCW complexes, Adv. Math. 231 (2012), no. 5, 2802–2836. arXiv:1007.1964v2 [math.OA].
- [38] L. Robert, *Remarks on R*, preprint, arXiv:1112.6069v1 [math.OA].
- [39] M. Rørdam, Classification of nuclear, simple C*-algebras, in:Classification of nuclear C*algebras. Entropy in operator algebras, 1–145, Encyclopaedia of Mathematical Sciences, 126, Springer, 2002.
- [40] M. Rørdam, The stable and the real rank of Z-absorbing C*-algebras, Internat. J. Math. 15 (2004), no. 10, 1065–1084.
- [41] Y. Sato, The Rohlin property for automorphisms of the Jiang-Su algebra, J. Funct. Anal. 259 (2010), no. 2, 453–476.
- [42] Y. Sato, Discrete amenable group actions on von Neumann algebras and invariant nuclear C*-subalgebras, preprint, arXiv:1104.4339v1 [math.OA].
- [43] M. Takesaki, Theory of operator algebras. II, Encyclopaedia of Mathematical Sciences, 125, Springer, 2003.
- [44] M. Takesaki, Theory of operator algebras. III, Encyclopaedia of Mathematical Sciences, 127, Springer, 2003.
- [45] A. Tikuisis, Regularity for stably projectionless, simple C*-algebras, J. Funct. Anal. 263 (2012), no. 5, 1382–1407.
- [46] A. Toms, Characterizing classifiable AH algebras, C. R. Math. Acad. Sci. Soc. R. Can. 33 (2011), no. 4, 123–126.
- [47] A. S. Toms and W. Winter, Strongly self-absorbing C*-algebras, Trans. Amer. Math. Soc. 359 (2007), no. 8, 3999–4029.
- [48] A. S. Toms and W. Winter, *Z-stable ASH algebras*, Canad. J. Math. **60** (2008), no. 3, 703– 720.
- [49] W. Winter, Nuclear dimension and Z-stability of pure C*-algebras, Invent. Math. 187 (2012), no. 2, 259–342.
- [50] Y. Watatani, Index for C^{*}-subalgebras, Memoir AMS **424** (1990).

DEPARTMENT OF MATHEMATICS AND INFORMATICS, GRADUATE SCHOOL OF SCIENCE, CHIBA UNIVERSITY,1-33 YAYOI-CHO, INAGE, CHIBA, 263-8522, JAPAN *E-mail address:* nawata@math.s.chiba-u.ac.jp