UNIVERSALLY REVERSIBLE JC^*-TRIPLES AND OPERATOR SPACES

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Abstract. We prove that the vast majority of JC^*-triples satisfy the condition of universal reversibility. Our characterisation is that a JC^*-triple is universally reversible if and only if it has no triple homomorphisms onto Hilbert spaces of dimension greater than two nor onto spin factors of dimension greater than four. We establish corresponding characterisations in the cases of JW^*-triples and of TROs (regarded as JC^*-triples). We show that the distinct natural operator space structures on a universally reversible JC^*-triple E are in bijective correspondence with a distinguished class of ideals in its universal TRO, identify the Shilov boundaries of these operator spaces and prove that E has a unique natural operator space structure precisely when E contains no ideal isometric to a nonabelian TRO. We deduce some decomposition and completely contractive properties of triple homomorphisms on TROs.

1. Introduction

The norm closed subspaces of $\mathcal{B}(H)$ invariant under the ternary product $[a, b, c] = ab^*c$ and known as TROs (ternary rings of operators) have a well-documented significance in the category of operator spaces in which, up to complete isometry, they occur as the noncommutative Shilov boundaries, injective envelopes and as Hilbert C^*-modules [2, Chapters 4, 8]. TROs have a natural operator space structure since the algebraic isomorphisms between them are exactly the surjective complete isometries. This is a state generally not possessed by the norm closed subspaces of TROs invariant under the symmetrised triple product $\{a, b, c\} = (1/2)([a, b, c] + [c, b, a])$ which arose in a different tradition and were named J^*-algebras by Harris [17, 18] who showed that the open unit ball of each such space is a bounded symmetric domain, that is, the group of biholomorphic automorphisms of the open unit ball is transitive. Kaup's extension [23] is a formidable conjunction of algebra and analysis: the open unit ball of a complex Banach space E is a bounded symmetric domain if and only if there is a continuous

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map $\{\cdot,\cdot,\cdot\}\colon E^3 \to E$ such that the operator $D(a,b)$ on E given by $D(a,b)(c) = \{a,b,c\}$ is sesquilinear, satisfies the *Jordan triple identity*, $[D(a,b), D(x,y)] = D(\{a,b,x\},y) - D(x,\{y,a,b\})$ and that $D(x,x)$ is a positive hermitian operator on E with norm $\|x\|^2$ (for all a,b,c,x and y in E). This class of Banach spaces, the *JB^*-triples*, is invariant under linear isometries and a key feature is that the *Jordan triple product* $\{a,b,c\}$ on a JB^*-triple is unique: the triple isomorphisms between JB^*-triples are the surjective linear isometries.

The linear isometric copies of J^*-algebras, hereafter referred to as JC^*-triples, are principal examples of JB^*-triples. In these terms J^*-algebras are precisely the JC^*-subtriples of $\mathcal{B}(H)$, or *concrete* JC^*-triples, and by the triple Gelfand-Naimark theorem of Friedman and Russo [13] JC^*-triples are the JB^*-triples with vanishing exceptional ideal. The JC^*-triples with a predual are called *JW^*-triples*. Hilbert spaces are JC^*-triples as are all (linear isometric copies of) Cartan factors, C^*-algebras, TROs and Jordan operator algebras. Friedman and Russo [12] have also shown that the range of a contractive projection on a C^*-algebra is a JC^*-triple, though not necessarily a subtriple of the ambient C^*-algebra (one by-product of this paper is the passing observation that all simple JC^*-triples arise in this way up to linear isometry).

Operator space structure of reflexive JC^*-triples has been investigated in important articles by Neal, Ricard and Russo [25] and by Neal and Russo [26, 28], who also proved [27] that an operator space X is completely isometric to a TRO if and only if $M_n(X)$ is a JC^*-triple (in the abstract sense defined above) for all $n \geq 2$. In [3] B. Feely and the authors began a general study of *JC-operator spaces* (the operator spaces induced by linear isometries onto concrete JC^*-triples) which was continued in [4, 5]. The JC-operator spaces of all Cartan factors (see §2) were described and enumerated in the process via instrumental use of the notions (inaugurated in [3]) of the universal TRO of a JC^*-triple and of a universally reversible JC^*-triple, conceived by analogy with companion notions in Jordan operator algebras [15, 16] which they precisely generalise [3, §4].

Universally reversible JC^*-triples form a class of Banach spaces preserved by linear isometries. The extent and influence of this class is the subject of this paper.

The setting is that given $a_1, \ldots, a_{2n+1}$ in a JC^*-subtriple E of $\mathcal{B}(H)$ with $n \geq 2$, the *reversible* element

$$a_1 a_2^* a_3 \cdots a_{2n}^* a_{2n+1} + a_{2n+1} a_{2n}^* \cdots a_3 a_2^* a_1$$

(lying in the TRO generated by E) is not a Jordan-theoretic product and has no compelling algebraic claim to membership of E: if E contains all reversible elements arising in this way it is said to be *reversible* in $\mathcal{B}(H)$. A JC^*-triple E is defined to be *universally reversible* if $\pi(E)$ is reversible in $\mathcal{B}(H)$ for every triple homomorphism $\pi\colon E \to \mathcal{B}(H)$.

By [3] Cartan factors are universally reversible with the exceptions of Hilbert spaces of dimension ≥ 3 and spin factors of dimension ≥ 5. We shall prove that the latter are essentially the only obstacles to universal reversibility thus showing that most JC^*-triples satisfy the condition and that failure to do so is confined to a sharply delineated isolated class.

Our characterisation is that a JC^*-triple is not universally reversible when, and only when, it has a triple homomorphism onto a Hilbert space of dimension ≥ 3 or a spin factor of dimension ≥ 5 (Theorem 4.7). Further results in §4 give the structures of universally reversible JW^*-triples (in Theorem 4.14) and W^*-TROs (Corollary 4.15), and that section contains other results of independent interest. The prior section, §3, deals with important special cases such as one-sided weak*-closed ideals in von Neumann algebras. Having shown the prevalence of universally reversible JC^*-triples we study their JC-operator spaces in §5, distilling the significance (as it turns out) of nonabelian TROs in the process (Theorem 5.12). For a given universally reversible JC^*-triple E we prove that its JC-operator spaces are in bijective correspondence with a distinguished family of ideals in its universal TRO (Theorem 5.10), identify the Shilov boundaries of each such operator space (Corollary 5.11) and severally characterise those E with a unique JC-operator space structure (Theorem 5.12). Modulo a mild restriction, we apply our results to show that triple homomorphisms on a TRO routinely decompose into the sum of a TRO homomorphism and a TRO antihomomorphism (Proposition 5.14) and deduce that a triple automorphism on a W^*-TRO factor not isometric to a von Neumann algebra must be a TRO automorphism (Theorem 5.18).

2. Preliminaries

We refer to surveys [29, 30] and articles [7, 13, 17, 18, 24] for general background on JC^*-triples and to [2, 10] for the theory of operator spaces and also for TROs, further information about which may be found in [3, 5, 9, 14, 35]. Related to JC^*-triples are the JC^*-algebras, the complexifications of the JC-algebras studied thoroughly in [16]. JC^*-algebras are the norm closed subspaces of C^*-algebras invariant under the involution and Jordan product $a \circ b = (ab + ba)/2$, which are all JC^*-triples because $\{a, b, c\} = (a \circ b^*) \circ c + a \circ (b^* \circ c) - (a \circ c) \circ b^*$. A JW^*-algebra is a weakly closed JC^*-subalgebra of a von Neumann algebra. Given elements $a_1, \ldots, a_{2n+1}$ of a TRO T, we often write $[a_1, \cdots, a_{2n+1}]$ for the element $a_1 a_2^* a_3 \cdots a_{2n}^* a_{2n+1}$ of T.

Throughout this paper the term ideal shall mean norm closed ideal. In each of the four above mentioned categories ideals have the obvious algebraic definition with respect to the relevant product and, by a result of Harris [18, Proposition 5.8], the definitions coincide whenever the categories overlap. We write $\mathrm{TRO}(X)$ for the TRO generated by subset X in a C^*-algebra and recall [5, Proposition 3.2] that $\mathrm{TRO}(I)$ is an ideal of $\mathrm{TRO}(E)$ with $I = E \cap \mathrm{TRO}(I)$ whenever I is an ideal of a concrete JC^*-triple E.

Let E be a JC^*-triple. Elements x and y in E are said to be orthogonal (denoted by $x \perp y$) if $\{x, x, y\} = 0$, which is equivalent to $x^*y = xy^* = 0$ when E is a JC^*-subtriple of $\mathcal{B}(H)$, implying that $\|x+y\| = \max(\|x\|, \|y\|)$ [17, p. 18]. Given $x \in E$ and $S \subseteq E$ we write $x \perp S$ if $x \perp y$ for all y in S and denote $\{x \in E : x \perp S\}$ by $S^\perp$. Ideals I and J of E are orthogonal, that is, $I \subseteq J^\perp$ (equivalently, $J \subseteq I^\perp$) if and only if $I \cap J = \{0\}$, in which case, $I + J = I \oplus_\infty J$. If E is a JW^*-triple with a weak*-closed ideal I then E is the sum of I and its complementary weak*-closed ideal $I^\perp$. A JW^*-triple E is a factor if it has no nontrivial weak*-closed ideals. Given x in E, Q_x denotes the conjugate linear operator $y \mapsto \{x, y, x\}$ on E.

An element u of a JC^*-triple E is a *tripotent* if $\{u, u, u\} = u$. The *Peirce projections* $P_k(u)$ ($k = 0, 1, 2$, u a tripotent) on E associated with u are given by

$$P_2(u) = Q_u^2, P_1(u) = 2(D(u, u) - P_2(u)) \text{ and } P_0(u) = I - P_2(u) - P_0(u)$$

and satisfy $P_i(u)P_j(u) = 0$ whenever $i \neq j$. Also $E = E_2(u) \oplus E_1(u) \oplus E_0(u)$ [linear direct sum] where $E_k(u) = P_k(u)E = \{x \in E : 2\{u, u, x\} = kx\}$ is the *Peirce k-space* for u and each $E_k(u)$ is a JC^*-subtriple of E. In addition, $E_2(u)$ is a JC^*-algebra with product $x \circ y = \{x, u, y\}$, identity element u and involution $x^\# = \{u, x, u\}$. If E is a JW^*-triple then $E_2(u)$ is a JW^*-algebra.

A tripotent $u \in E$ is said to be *abelian* if $E_2(u)$ is an abelian JC^*-algebra (equivalently, an abelian C^*-algebra) and to be *minimal* if $E_2(u) = \mathbb{C}u$. If $P_0(u) = 0$, then u is called a *complete* tripotent of E and is called a *unitary* tripotent if $P_0(u) = P_1(u) = 0$. The completeness of a tripotent u is equivalent to $u^\perp = \{0\}$ (by [17, p. 18]). and to u being an extreme point of the closed unit ball of E (by [17, Theorem 11]).

We note the following.

Lemma 2.1. *Let u be a tripotent in a JC^*-triple E, M a JW^*-triple, and let $\pi\colon E \to M$ be a nonzero triple homomorphism with weak*-dense range. If u is a complete (respectively, a unitary, an abelian, a minimal) tripotent of E, then $\pi(u)$ is a complete (respectively, a unitary, an abelian, a minimal) tripotent of M. In particular $\pi(u) \neq 0$ if u is a complete tripotent of E.*

By a *Cartan factor* we shall mean a JC^*-triple linearly isometric to one of the following *canonical forms* (i) $\mathcal{B}(H)e$; (ii) $\{x \in \mathcal{B}(H) : x^t = x\}$, $\dim H \geq 2$; (iii) $\{x \in \mathcal{B}(H) : x^t = -x\}$, $\dim H \geq 4$; (iv) a spin factor, where in (i) – (iii) H is a Hilbert space, $e \in \mathcal{B}(H)$ is a projection and $x \mapsto x^t$ is a transposition. Up to linear isometry, the forms (ii), (iii) when H has even or infinite dimension, and (iv), are type I JW^*-triples and those of form (i) where e is a minimal projection are Hilbert spaces. The spin factor of finite dimension $n+1$ is denoted by V_n.

We denote the norm closed linear span of the minimal tripotents in a Cartan factor C by K_C, an ideal of C (which consists of the compact operators contained in C when C is of type (i), (ii) or (iii) above). The Cartan factors may be characterised as the JW^*-triple factors possessing minimal tripotents. The *rank* of a Cartan factor C is the cardinality of a maximal family of orthogonal minimal tripotents in C.

The following three conditions are equivalent for a Cartan factor C:

$$\text{(a) } C \text{ has finite rank; (b) } K_C = C\text{; (c) } C \text{ is reflexive.} \tag{2.1}$$

A *Cartan factor representation* of a JC^*-triple E is a triple homomorphism $\pi\colon E \to C$ where C is a Cartan factor with $\pi(E)$ weak*-dense in C (so that $\pi(E) = C$ if C has finite rank). If $\pi\colon E \to C$ is a a Cartan factor representation, the weak*-continuous extension $\tilde{\pi}\colon E^{**} \to C$ is a surjective triple homomorphism so that the orthogonal complement of $\ker\tilde{\pi}$ is a weak*-closed ideal of E^{**} linearly isometric to C. Conversely, if J is a weak*-closed ideal of E^{**} linearly isometric to C, then the restriction to E of the natural projection from E^{**} onto J induces a Cartan factor representation $\pi\colon E \to C$. Every JC^*-triple E has a separating family of Cartan factor representations [13].

The type of a JW^*-algebra is the type of its self-adjoint part, a JW-algebra, and the corresponding decomposition theory of JW^*-algebras may be read directly from [16, 31, 32, 34]. In particular, a JW^*-algebra is of type I if it is the norm closed linear span of abelian projections and is continuous if it has no nonzero abelian projections. We briefly recall the extension to JW^*-triples due to Horn and Neher [19, 20, 21].

Following Horn [20] we use $B \,\overline{\otimes}\, N$ to denote the weak* closure of the algebraic tensor product $B \otimes N$ in the von Neumann algebra tensor product $\mathcal{B}(H) \,\overline{\otimes}\, \mathcal{B}(K)$ when B is an abelian von Neumann subalgebra of $\mathcal{B}(H)$ and N is a JW^*-subtriple of $\mathcal{B}(K)$.

A JW^*-triple is defined to be of *type I* if it is the weak*-closed linear span of its abelian tripotents, and to be *continuous* if it contains no nonzero abelian tripotents [19, 21]. By Horn's theorem [20], a JW^*-triple M is type I if and only if M is linearly isometric to an ℓ^∞ sum

$$\bigoplus_j B_j \,\overline{\otimes}\, C_j \tag{2.2}$$

where each B_j is an abelian von Neumann algebra and each C_j is a Cartan factor in canonical form. When all B_j are nonzero and the C_j are mutually distinct we shall refer to M as being $\{C_j\}$-*homogeneous.*

We further say that a JW^*-triple M is of *type* I_{finite} if each C_j has finite rank in the decomposition (2.2). Specialising, we refer to M as *type I_1* if each C_j is (isometric to) a Hilbert space, and as *type I_{spin}* if each C_j is a spin factor. Equivalently (by [20, §2]) M is type I_1 if and only if it contains a complete abelian tripotent and (by [16, Theorem 6.3.14]) M is type I_{spin} if and only if it is linearly isometric to a JW^*-algebra with self-adjoint part of type I_2 in the sense of [16, 5.3.3]. If each C_j has infinite rank we say that M has type I_∞, in which case, collecting terms, with $\cong$ meaning 'linearly isometric to', we have

$$M \cong N \oplus_\infty We, \tag{2.3}$$

where N is a type I_∞ JW^*-algebra and e is a properly infinite projection in a type I_∞ von Neumann algebra W.

In general, by [21, (1.17)] every JW^*-triple M has a decomposition

$$M \cong M_1 \oplus_\infty M_c, \tag{2.4}$$

where the first summand is type I and the second is continuous, referred to as the type I and continuous parts of M, respectively. In addition (see [21, §4])

$$M_c \cong N \oplus_\infty We, \tag{2.5}$$

where N is a continuous JW^*-algebra and e is a projection in a continuous von Neumann algebra W.

We remark that a JW^*-algebra is continuous as a JW^*-algebra if and only if it is continuous as a JW^*-triple, the 'if' part being clear since abelian projections are abelian tripotents. Conversely, suppose M is a continuous JW^*-algebra. If M is not a continuous JW^*-triple, then the decomposition theory implies a surjective linear isometry $\pi\colon N \to B \mathbin{\overline{\otimes}} C$, for some weak*-closed ideal N (a continuous JW^*-algebra) of M, where B is an abelian von Neumann algebra and C is a Cartan factor in canonical form possessing a unitary tripotent, u, say. In which case, with $v = 1 \otimes u$ and $w = \pi^{-1}(v)$, we arrive at the contradiction that the type I JW^*-algebra $B \mathbin{\overline{\otimes}} C_2(u)$ is Jordan*-isomorphic to $N_2(w)$ which, by [21, (5.2) Lemma], is a continuous JW^*-algebra. It follows that the two forms of 'type I' for JW^*-algebras also coincide.

We have defined universally reversible JC^*-triples in the Introduction. By [3, §4], when A is a JC^*-algebra, A is universally reversible if and only if $\pi(A)_{sa}$ is a reversible JC-algebra (see [16, 2.3.2]) for each Jordan *-homomorphism $\pi\colon A \to \mathcal{B}(H)$.

It follows from these remarks together with the coordinatisation theorem [16, 2.8.9] that a unital JC^*-algebra is a universally reversible JC^*-triple if it contains, for $3 \le n < \infty$, n orthogonal projections with sum 1 and pairwise exchanged by symmetries. The following is recorded for later use.

Proposition 2.2. *Let M be a JW^*-algebra such that M is continuous or of type I_∞ or of type I_n with $3 \le n < \infty$. Then M is a universally reversible JC^*-triple.*

Proof. M contains m orthogonal projections with sum 1 pairwise exchanged by symmetries with $m = 4$ if M is continuous or of type I_∞ (see the proof of [16, Theorem 5.3.9]), and with $m = n$ if M has type I_n for $3 \leq n < \infty$. □

The *universal TRO* [3] of a JC^*-triple E is a pair $(T^*(E), \alpha_E)$ consisting of an injective triple homomorphism

$$\alpha_E \colon E \to T^*(E)$$

where $T^*(E)$ is a TRO generated by $\alpha_E(E)$ (as a TRO) and possessing the universal property that for each triple homomorphism $\pi \colon E \to \mathcal{B}(H)$ there is a (unique) TRO homomorphism $\tilde{\pi} \colon T^*(E) \to \mathcal{B}(H)$ with $\tilde{\pi} \circ \alpha_E = \pi$. The *canonical involution* $\Phi \colon T^*(E) \to T^*(E)$, which has order 2, is the unique antiautomorphism of $T^*(E)$ fixing each element of $\alpha_E(E)$. By [3], a JC^*-triple E is universally reversible if and only if $\alpha_E(E) = \{x \in T^*(E) : \Phi(x) = x\}$. When A is a JC^*-algebra $(T^*(A), \alpha_A)$ coincides with the universal C^*-algebra $(C^*_{\mathrm{J}}(A), \beta_A)$ of A [3, Proposition 3.7].

Proposition 2.3. *Let A be a JC^*-algebra and T a TRO and let $\pi \colon A \to T$ be a surjective linear isometry. Then there is a C^*-algebra B, a Jordan *-isomorphism $\phi \colon A \to B$ and a TRO isomorphism $\psi \colon B \to T$ such that $\pi = \psi \circ \phi$.*

Proof. We may suppose that $A \subseteq C^*_{\mathrm{J}}(A)$. By [3, Proposition 3.7] there is a TRO homomorphism $\tilde{\pi} \colon C^*_{\mathrm{J}}(A) \to T$ extending π. Putting $I = \ker \tilde{\pi}$ we have the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\pi"] \arrow[d, hook] & T \\
C^*_{\mathrm{J}}(A) \arrow[r, "q"] \arrow[ur, "\tilde{\pi}"] & C^*_{\mathrm{J}}(A)/I \arrow[u, "\tilde{\pi}_I"']
\end{tikzcd}$$

where q is the quotient map and $\tilde{\pi}_I$ is the map sending $x + I$ to $\pi(x)$ for each x in $C^*_{\mathrm{J}}(A)$. Since $\pi(A) = T$, $\tilde{\pi}_I$ is a TRO isomorphism and since $\tilde{\pi}_I \circ q$ agrees with π on A we have $q(A) = C^*_{\mathrm{J}}(A)/I$. The assertion follows upon setting $B = C^*_{\mathrm{J}}(A)/I$, $\psi = \tilde{\pi}_I$ and ϕ to be the restriction of q to A. □

In outline an *operator space* is a complex Banach space E together with a linear isometric embedding into $\mathcal{B}(H)$ and the resulting operator space structure on E is determined by the matrix norms on $M_n(E)$ conferred thus by the C^*-algebras $M_n(\mathcal{B}(H))$, for all $n \in \mathtt{N}$. A linear map $\pi \colon E \to F$ between operator spaces is said to be *completely contractive* if, for each n, $\|\pi_n\| \leq 1$ where π_n is the tensored map $\pi \otimes I_n \colon M_n(E) \to M_n(F)$, and to be *completely isometric* if each π_n is a surjective isometry. Complete isometries are the isomorphisms in the category of operator spaces. By a JC-operator space structure on a Banach space E we mean an operator space structure determined by a linear isometry from E onto a JC^*-subtriple of $\mathcal{B}(H)$ (in which case, E is a JC^*-triple), and by a JC-operator space we mean a JC^*-triple E together with a prescribed JC-operator space structure on E. By [3, Proposition 6.2] the possible JC-operator space structures on a JC^*-triple E arise from (norm closed) ideals $\mathcal{I}$ of $T^*(E)$ for which $\alpha_E(E) \cap \mathcal{I} = \{0\}$, called *operator space ideals* of $T^*(E)$. For each operator space ideal $\mathcal{I}$ of $T^*(E)$ we have the JC-operator space structure, $E_{\mathcal{I}}$, on E determined by the isometric embedding $E \to T^*(E)/\mathcal{I}$ $(x \mapsto \alpha_E(x) + \mathcal{I})$.

Given an ideal I of a JC^*-triple E we recall [5, Theorem 3.3] that $\mathrm{TRO}(\alpha_E(I))$ is an ideal of $T^*(E)$, $(T^*(I), \alpha_I) = (\mathrm{TRO}(\alpha_E(I)), \alpha_E|_I)$ and $T^*(E/I) = T^*(E)/T^*(I)$ with $\alpha_{E/I}(x + I) = \alpha_E(x) + T^*(I)$ for $x \in E$.

Lemma 2.4. *Let I be an ideal of a JC^*-triple E and $\mathcal{J}$ an ideal of $T^*(E)$. Then $\mathcal{J}$ is an operator space ideal of $T^*(I)$ if and only if $\mathcal{J}$ is an operator space ideal of $T^*(E)$ contained in $T^*(I)$; in which case $I_{\mathcal{J}}$ is an operator subspace of $E_{\mathcal{J}}$.*

Thus, E has at least as many distinct JC-operator space structures as does I.

Proof. We have

$$\mathcal{J} \cap T^*(I) \cap \alpha_E(I) = \mathcal{J} \cap \alpha_E(I) = \mathcal{J} \cap \alpha_I(I)$$

and an ideal of $T^*(I)$ is an ideal of $T^*(E)$ (see [5, Proposition 3.2] for example), from which the assertions follow. □

3. Reversibility for JW^*-triples

The aim of this section is to show that all continuous and type I_∞ JW^*-triples are universally reversible (Theorem 3.10) and to use this to establish reversibility criteria for JW^*-triples. In view of the Horn-Neher structure theory (2.3) and (2.5), and Proposition 2.2 (and [3, Proposition 3.6] which allows passage to direct sums of finitely many terms) we may confine our attention to JW^*-triples of the form eW where e is a projection in a von Neumann algebra W. Moreover it suffices to deal with the cases where (a) W is continuous (Propositions 3.6 and 3.9) and (b) eWe is type I_∞ (Proposition 3.9).

Given projections e and f in a von Neumann algebra W we write $e \sim f$ if and only if $e = uu^*$ and $f = u^*u$ for some (partial isometry) $u \in W$, and $e \precsim f$ if and only if there is a projection q in W with $e \sim q \leq f$. In addition for $n \in \mathbb{N}$ we use the notation $n \cdot e \precsim f$ to mean the existence of orthogonal projections $e_1, \ldots, e_n \in W$ with $\sum_{i=1}^n e_i \precsim f$ and $e \sim e_i$ for $i = 1, \ldots, n$. We note that if e, f, p are projections in W with $e, f \leq p$ and $e \sim f$ in W then $e \sim f$ in pWp since if $u \in W$ with $e = uu^*$ and $f = u^*u$ the $u = euf \in pWp$.

We shall make frequent use of the properties [33, p. 291] that, in a von Neumann algebra, if e and f are projections with $e \precsim f$ and $f \precsim e$ then $e \sim f$, and that if $(e_i)_{i\in I}$ and $(f_i)_{i\in I}$ are two families of orthogonal projections with $e_i \sim f_i$ for each $i \in I$, then $\sum_{i\in I} e_i \sim \sum_{i\in I} f_i$. We refer to [33, Chapter V] for any undefined terms used below.

Lemma 3.1. *Let e and f be projections in a von Neumann algebra W such that $e \sim f$. Then eWf is a universally reversible JC^*-triple.*

Proof. Given $u \in W$ with $e = u^*u$ and $f = uu^*$, the map $x \mapsto ux$ is a TRO isomorphism from eWf onto the von Neumann algebra fWf, which is universally reversible as a JC^*-triple by [1, Lemma 3.4 and Theorem 4.6] (together with [3, §4]). □

The following folklore is included for want of an exact reference.

Lemma 3.2. *Let e and f be projections in a von-Neumann algebra with $e \precsim f$. Then there is a projection $p \in W$ with $e \le p \sim f$.*

Proof. Let q be a projection in W such that $e \sim q \le f$. By the comparability theorem ([33, p. 293] there is a central projection z of W such that $(1-e)z \precsim (f-q)z$ and $(f-q)(1-z) \precsim (1-e)(1-z)$. Thus, using $ez \sim qz$,

$$z = (1-e)z + ez \precsim (f-q)z + qz = fz \le z,$$

giving $z \sim fz$.

Since there is a projection r in W with $(f-q)(1-z) \sim r \le (1-e)(1-z)$, so that

$$f(1-z) = (f-q)(1-z) + q(1-z) \sim r + e(1-z)$$

we have

$$f = fz + f(1-z) \sim z + (r + e(1-z)) = e + (1-e)z + r \ge e,$$

as required. □

Lemma 3.3. *Let e be a projection in a continuous von Neumann algebra W, let u be a tripotent in eW, and let $n \in \mathbb{N}$. Then there exist orthogonal tripotents $u_1, \ldots, u_n \in eW$ with $u = \sum_{i=1}^n u_i$ and $n\cdot(u_i^*u_i) \precsim e$ for $i = 1, \ldots, n$.*

Proof. Let $f = uu^*$. We may choose orthogonal projections $f_1, \ldots, f_n$ in W with $f = \sum_{i=1}^n f_i$ with $f_i \sim f$ for $i = 1, \ldots, n$ (see [22, 6.5.6], for instance). Now letting $u_i = f_i u$ for $i = 1, \ldots, n$, we have that $u_1, \ldots, u_n$ are orthogonal tripotents with $u = u_1 + \cdots + u_n$ and $u_i^*u_i \sim u_iu_i^* = f_i \le e$ for $i = 1, \ldots, n$ giving $n \cdot (u_i^*u_i) \precsim e$. □

Lemma 3.4. *Let e be a finite projection in a continuous von Neumann algebra W and let $f_1, f_2, \ldots, f_n$ be projections in W with $n \cdot f_i \precsim e$ for $i = 1, \ldots, n$. Then $\bigvee_{i=1}^n f_i \precsim e$.*

Proof. Let $f = \bigvee_{i=1}^n f_i$ and $g = e \vee f$ (finite by [33, Theorem V.1.37]) and let τ be the centre-valued trace on the finite von Neumann algebra gWg. Using [33, Corollary V.2.8] we have $\tau(f_i) \leq (1/n)\tau(e)$ for $i = 1, \ldots, n$ so that $\tau(f) \leq \sum_{i=1}^n \tau(f_i) \leq \tau(e)$ and hence $f \precsim e$. □

Lemma 3.5. *Let $u_1, \ldots, u_n$ be tripotents in eW where e is a finite projection in a continuous von Neumann algebra W such that $n \cdot (u_i^* u_i) \precsim e$ for $1 \leq i \leq n$. Then there is a universally reversible JC^*-subtriple of eW containing $u_1, \ldots, u_n$.*

Proof. By Lemma 3.4 together with Lemma 3.2 there is a projection f in W with $\bigvee_{i=1}^n u_i^* u_i \leq f \sim e$ so that each $u_i \in eWf$ which, by Lemma 3.1, is a universally reversible JC^*-triple. □

Proposition 3.6. *Let e be a finite projection in a continuous von Neumann algebra W. Then eW is a universally reversible JC^*-triple.*

Proof. Let $\pi \colon eW \to \mathcal{B}(H)$ be a triple homomorphism and let $a_1, \ldots, a_n \in \pi(eW)$ for an odd integer $n \geq 3$. We must show that

$$[a_1, \cdots, a_n] + [a_n, \cdots, a_1] \in \pi(eW). \tag{3.1}$$

Choose $x_i \in eW$ such that $\pi(x_i) = a_i$, for $1 \leq i \leq n$. In order to establish (3.1), since eW is the norm closed linear span of its tripotents, we may suppose without loss that each x_i is a tripotent. In which case, by Lemma 3.3, we may write $x_i = \sum_{j=1}^n u_{i,j}$ where the $u_{i,j}$ are tripotents in eW with $n \cdot (u_{i,j}^* u_{i,j}) \precsim e$ for $1 \leq i, j \leq n$. Put $a_{i,j} = \pi(u_{i,j})$ for $1 \leq i, j \leq n$. By Lemma 3.1, for each choice of $j_1, \ldots, j_n$ in $\{1, \ldots, n\}$ there is a universally reversible JC^*-subtriple M of eW containing $u_{1,j_1}, \ldots, u_{n,j_n}$ from which it follows that

$$[a_{1,j_1}, \cdots, a_{n,j_n}] + [a_{n,j_n}, \cdots, a_{1,j_1}] \in \pi(M) \subseteq \pi(eW).$$

By summing over all such choices of $j_1, \ldots, j_n$ we obtain (3.1). □

We next turn to the more pliable properly infinite projections e in a von Neumann algebra W. In this case we recall [33, Proposition V.1.36] there are projections p and q in W with $e = p + q \sim p \sim q$, so that if f_1 and f_2 are projections with $f_1 \precsim e$ and $f_2 \precsim e$ then $f_1 \vee f_1 - f_2 \sim f_1 - f_1 \wedge f_2 \precsim p$ and $f_2 \precsim q$ giving $f_1 \vee f_2 \precsim p + q = e$. The next assertion is an immediate consequence.

Lemma 3.7. *If $f_1, \ldots, f_n$ and e are projections in a von-Neumann algebra W where $f_i \precsim e$ for $1 \leq i \leq n$ and e is properly infinite then $\bigvee_{i=1}^n f_i \precsim e$.*

For an element x in a von-Neumann algebra W the range projection $r(x^*x)$ of x^*x is the least projection p in W with $xp = x$, and $r(x^*x) \sim r(xx^*)$ [33, Proposition V.1.5].

Lemma 3.8. *Let S be a finite subset of eW where e is a properly infinite projection in the von-Neumann algebra W. Then S is contained in a universally reversible JC^*-subtriple of eW.*

Proof. Let $x_1, \dots, x_n$ be the elements of S. For $1 \le i \le n$ we have $r(x_i^* x_i) \sim r(x_i x_i^*) \le e$ so that by Lemmas 3.2 and 3.7 there is a projection f in W with $\bigvee_{i=1}^n r(x_i^* x_i) \le f \sim e$. Hence for $1 \le i \le n$, $x_i \in eW \cap Wf = eWf$, whence the result by Lemma 3.1. □

Proposition 3.9. *Let e be a projection in a von-Neumann algebra W such that (a) e is properly infinite or (b) W is continuous. Then eW is a universally reversible JC^*-triple.*

Proof. If e is properly infinite and S is a finite subset of $\pi(eW)$, where $\pi\colon eW \to \mathcal{B}(H)$ is a triple homomorphism, then Lemma 3.8 implies the existence of a universally reversible JC^*-subtriple M of eW such that $S \subset \pi(M)$, implying that the latter is reversible in $\mathcal{B}(H)$ and proving the result in case (a). The remaining case follows from this together with Proposition 3.6. □

We now state and prove the main result of this section.

Theorem 3.10. *Every JW^*-triple with zero type $\mathrm{I}_{\text{finite}}$ part is a universally reversible JC^*-triple.*

Proof. The JW^*-triples in question are those of the form

$$(\text{type } I_\infty) \oplus_\infty (\text{continuous})$$

(it being understood that one or both summands can be zero). Recalling the Horn-Neher structure theory related in (2.3) and (2.5), the assertion follows from Proposition 3.9 together with Proposition 2.2. □

Definition 3.11. We say that a JW^*-triple M is *weakly universally reversible* if for every weak*-continuous triple homomorphism $\pi\colon M \to \mathcal{B}(H)$, $\pi(M)$ is reversible in $\mathcal{B}(H)$.

Remarks 3.12. By [3, Theorem 5.6] a Cartan factor is universally reversible if and only if it is not a Hilbert space of dimension ≥ 3 nor a spin factor of dimension ≥ 5. If B is an abelian von Neumann algebra and C is a Cartan factor, then [5, Corollary 4.10] (for the case $\dim C < \infty$) together with Proposition 3.9 (for the case $C = \mathcal{B}(H)e$) and Proposition 2.2 (for the remaining cases) imply that $B \mathbin{\overline{\otimes}} C$ is universally reversible if and only if C is universally reversible. We note also that the class of JW^*-triples satisfying Definition 3.11 is stable under arbitrary ℓ_∞ direct sums.

Combining these remarks with [5, Theorem 4.12] and Theorem 3.10, and using the notation V_n for a spin factor of dimension $n+1$, we have the following characterisation.

Proposition 3.13. *The following are equivalent for a JW^*-triple M.*

(a) M is weakly universally reversible.

(b) The type I_1 and type I_{spin} parts of M are linearly isometric to $B_0 \oplus_\infty B_1 \otimes \ell_2^2$ and to $B_2 \otimes V_2 \oplus_\infty B_3 \otimes V_3$ (respectively), where the B_i are abelian von Neumann algebras (or zero).

In particular M is weakly universally reversible if it has zero type I_1 and type I_{spin} parts.

In the next section we shall exploit Proposition 3.13 to show amongst other things that weakly universally reversible JW^*-triples are universally reversible. For the time being we note the following.

Proposition 3.14. *Let E be a JC^*-triple. Then E is universally reversible if and only if its bidual E^{**} is weakly universally reversible.*

Proof. If E is universally reversible and $\pi\colon E^{**} \to \mathcal{B}(H)$ is a weak*-continuous triple homomorphism then, since $\pi(E)$ is reversible in $\mathcal{B}(H)$, separate weak*-continuity of multiplication together with a simple induction shows that $\pi(E^{**}) = \overline{\pi(E)}$ is reversible in $\mathcal{B}(H)$.

Conversely, let E^{**} be weakly universally reversible. Identifying E with its image in $T^*(E)$ (and $T^*(E)$ canonically with its embedding in its bidual, E^{**} with the weak*-closure of E in $T^*(E)^{**}$) we have the following commutative diagram

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d, hook] & T^*(E) \arrow[d, hook] \\ E^{**} \arrow[r, hook] & T^*(E)^{**} \end{tikzcd}$$

Let $x_1, \ldots, x_{2n+1} \in E$. Then, since E^{**} is reversible in $T^*(E)^{**}$, by assumption, $[x_1, \cdots, x_{2n+1}] + [x_{2n+1}, \cdots, x_1] \in E^{**} \cap T^*(E) = E$ (since elements of $T^*(E) \setminus E$ are not in the weak*-closure of E). Therefore E is reversible in $T^*(E)$ and so is universally reversible. $\square$

4. Characterising Universal Reversibility

We establish the prevalence of universally reversible JC^*-triples by proving that the property fails only when a JC^*-triple possesses a quotient (by an ideal) linearly isometric to a Hilbert space of dimension ≥ 3 or a spin factor of dimension ≥ 5 (Theorem 4.7). In this sense we prove that 'almost all' JC^*-triples are universally reversible. In like vein we characterise universally reversible TROs and JW^*-triples, in the latter case showing that universal reversibility is equivalent to the formally weaker version of Definition 3.11. We conclude this section by showing that the number of terms in the definition of universal reversibility can be reduced to the minimum possible of five.

Lemma 4.1. *Let u be a nonzero tripotent in a JC^*-subtriple E of a spin factor V. Then*

(a) u is unitary if it is not minimal;
(b) E is linearly isometric to a spin factor, a Hilbert space or to $\mathbb{C} \oplus_\infty \mathbb{C}$.

Proof. (a) See [8, Lemma 5.4], for example.

(b) Since V is reflexive, E is a JW^*-subtriple. If E is a factor and not a Hilbert space we may choose nonzero orthogonal tripotents u and v in E which, by (a), are minimal in V with $u+v$ unitary, so that $E = E_2(u+v)$ is a spin factor. If E is not a factor, then $E = I \oplus_\infty J$ for certain nonzero ideals I and J of E and now we may choose minimal tripotents $u \in I$ and $v \in J$ giving $E = E_2(u+v) = \mathbb{C}u \oplus_\infty \mathbb{C}v$.

□

If B is an abelian C^*-algebra and C a finite dimensional Cartan factor, we note that the JC^*-triple $B \otimes C$ has a separating family of Cartan factor representations onto C given by $\rho \otimes \mathrm{id}_C$ as ρ ranges over the pure states of B. Recall that we introduced following (2.2) the notion of a JW^*-triple being $\{C_j : j \in J\}$-homogeneous (for distinct Cartan factors C_j).

Lemma 4.2. *Let E be a JC^*-triple with a separating family of Cartan factor representations $\{\pi_\lambda \colon E \to C_\lambda\}_{\lambda\in\Lambda}$ where C_λ has finite rank $\forall \lambda \in \Lambda$. (Hence $\pi_\lambda(E) = C_\lambda$ for all $\lambda \in \Lambda$.)*

(a) If D is a Cartan subfactor of E, then D is linearly isometric to a subfactor of C_λ for some $\lambda \in \Lambda$.
(b) If E is a JW^*-triple and F a JW^*-subtriple of E, then F is $\{D_i : i \in I\}$-homogeneous where each D_i is contained in C_λ for some λ (depending on i).
(c) If Λ is finite and each C_λ is finite-dimensional, then E^{**} is $\{D_1, \ldots, D_n\}$-homogeneous, where each D_i is contained in C_λ for some λ (depending on i).

Proof. (a) Let D be a Cartan subfactor of E and choose $\lambda \in \Lambda$ such that π_λ does not vanish on the elementary ideal K_D of D. Then (since K_D is has no nontrivial ideals) $K_D \cong \pi_\lambda(K_D) \subseteq C_\lambda$, implying that D has finite rank, giving $K_D = D$ and proving (a).
(b) This follows from (a) and Horn's type I classification (2.2) (since the existence of a weak*-continuous triple homomorphism $\pi \colon F \to D$ onto a Cartan factor D implies that D is isomorphic to a summand of F).
(c) In this case, for appropriate sets X_λ, E may be represented as a subtriple of a finite ℓ_∞-sum $\bigoplus_{\lambda\in\Lambda} \ell_\infty(X_\lambda) \otimes C_\lambda$ and , in turn, E^{**} may be mapped to a JW^*-subtriple of $\bigoplus_{\lambda\in\Lambda} B_\lambda \otimes C_\lambda$ (with B_λ abelian von Neumann algebras). The assertion now follows from (b).

□

Lemma 4.3. *Let M be a type I_1 JW^*-triple. Then every JW^*-subtriple of M is type I, M^{**} is type I, and all Cartan factor representations of M are onto Hilbert spaces.*

Proof. The first assertion follows from Lemma 4.2 (b). Since M contains an abelian complete tripotent u, which is automatically complete and abelian in M^{**} (Lemma 2.1), the latter is type I. Finally, for any Cartan factor representation $\pi\colon M \to C$, $\pi(u)$ is a minimal complete tripotent in C, so that C is a Hilbert space. ☐

Proposition 4.4. *The following are equivalent for a JC^*-triple E.*

(a) All Cartan factor representations of E are onto Hilbert spaces.
(b) E has a separating family of Hilbert space representations.
(c) E^{**} is type I_1.

Proof. (a) $\Rightarrow$ (b): This is clear.

(b) $\Rightarrow$ (c): Given (b), E may be realised as a subtriple of an ℓ_∞-sum, M (say), of Hilbert spaces. Since M is a type I_1 JW^*-triple and $E^{**} \subset M^{**}$, E^{**} is type I_1 by Lemma 4.3.

(c) $\Rightarrow$ (a): The implication follows from Lemma 4.3 (since any Cartan factor representation $\pi\colon E \to C$ extends to a triple homomorphism from E^{**} onto C). ☐

Proposition 4.5. *Let E be a JC^*-triple. Let J be the intersection of the kernels of all nonzero Hilbert space representations of E (it being understood that $J = E$ if there are no nonzero Hilbert space representations). Then*

(a) E^{**} has nonzero type I_1 part if and only if E has a nonzero Hilbert space representation.
(b) $(E/J)^{**}$ is type I_1 and J^{**} has zero type I_1 part.
(c) E^{**} has nonzero type I_{spin} part if and only if E has a spin factor representation.

Proof. (a) Suppose the type I_1 part, M, of E^{**} is nonzero and consider the (weak*-continuous) canonical projection $P\colon E^{**} \to M$. Since $P(E)$ is a nonzero JC^*-subtriple of M, by Lemma 4.3 there is a Hilbert space representation π of M which does not vanish on $P(E)$, implying that $\pi \circ P$ is a Hilbert space representation of E. The converse is clear.

(b) By construction E/J has a separating family of Hilbert space representations so that $(E/J)^{**} \equiv E^{**}/J^{**}$ is type I_1 by Proposition 4.4. Further, since any Hilbert space representation of J will extend to one of E, J^{**} has zero type I_1 part by (a).

(c) The converse being clear, suppose that the type $\mathrm{I}_{\mathrm{spin}}$ part, N, of E^{**} is nonzero. By (b), J^{**} is the orthogonal complement of the type I_1 part of E^{**} and so contains N as a weak*-closed ideal. Let $Q\colon E^{**} \to N$ be the canonical projection. Since N has a separating family of spin factor representations [32, Lemma 2], we may choose a spin factor representation $\pi\colon N \to V$ with $\pi(Q(J)) \neq \{0\}$. By the definition of J together with Lemma 4.1, $\pi \circ Q$ induces a spin factor representation of J, which extends to a spin factor representation of E ([5, Remarks 2.5 (b)]).

□

Lemma 4.6. *Let E be a JC^*-triple. Each of the following two conditions separately implies that E is universally reversible.*

(a) E has separating family of representations onto Cartan factors $C \in \{\mathsf{C}, \ell_2^2, V_2, V_3\}$
(b) E has no nonzero Hilbert space representations and no spin factor representations.

Proof. By Proposition 3.14, in either case it is enough to show that E^{**} is weakly universally reversible.

(a) In this case by Lemma 4.2 (c) and Lemma 4.1 E^{**} is linearly isometric to an ℓ_∞-sum of JW^*-triples of the form $B \overline{\otimes} C$ where $C \in \{\mathsf{C}, \ell_2^2, V_2, V_3\}$ and B is an abelian von Neumann algebra, and so is universally reversible (see Remarks 3.12).
(b) By Proposition 4.5 (a), (c) E^{**} has zero type I_1 and type $\mathrm{I}_{\mathrm{spin}}$ parts and thus is weakly universally reversible by Proposition 3.13.

□

Our characterisation of universally reversible JC^*-triples below shows that Hilbert spaces of dimension ≥ 3 and spin factors of dimension ≥ 5 are essentially the only obstacles to the property.

Theorem 4.7. *Let E be a JC^*-triple. Then E is universally reversible if and only if E satisfies both of the following conditions.*

(a) E has no representation onto a Hilbert space of dimension ≥ 3.
(b) E has no representation onto a spin factor of dimension ≥ 5.

Proof. Let E satisfy (a) and (b). Let J be as in Proposition 4.5. Note that the hypotheses pass to ideals (since Cartan factor representations of ideals extend) and we rely on the fact that universal reversibility of both an ideal J of a JC^*-triple E and the quotient E/J imply universal reversibility of E [5, Corollary 3.4].

Now either $J = E$ (in which case E/J is trivially reversible) or E/J has a separating family of representations onto Hilbert spaces of dimensions one or two and thus is universally reversible by Lemma 4.6 (a). We show now that J is universally reversible. Suppose that $J \neq \{0\}$ and let K be the intersection of the kernels of spin factor representations of J. Then J/K is zero or has a separating family of representations onto spin factors of dimensions 3 or 4 so that J/K is universally reversible, again by Lemma 4.6 (a). The ideal K is universally reversible because it satisfies the condition of Lemma 4.6 (b). Hence J is universally reversible, as therefore is E.

The converse is clear. □

Remark 4.8. It is immediate from Theorem 4.7 that the only simple JC^*-triples failing to be universally reversible are the Hilbert spaces of dimension ≥ 3 and the spin factors of dimension ≥ 5.

Since realisable as a corner of a C*-algebra (see [9, p. 493], for example), up to complete isometry every TRO is the range of a completely contractive projection on a C^*-algebra. If E is a JC^*-triple then the set of points in $T^*(E)$ fixed by its canonical involution Φ is the range of the bicontractive projection $\frac{1}{2}(\mathrm{id} + \Phi)$ on $T^*(E)$. It follows that every universally reversible JC^*-triple linearly isometric to the range of a contractive projection on a C^*-algebra. Since Hilbert spaces and spin factors also possess this property, by [11] in the spin case, we note that every simple JC^*-triple has the same property.

We recall from [5, Theorem 3.5] that every JC^*-triple has a largest universally reversible ideal. We can reprove its existence and characterise it as follows.

Corollary 4.9. *Let E be a JC^*-triple and let I denote the intersections of the kernels of representations onto Hilbert space of dimension ≥ 3 and spin factors of dimension ≥ 5 (understood as E if there are no such representations). Then I is the largest universally reversible ideal of E.*

Proof. If $\pi\colon E \to C$ is a Cartan factor representation such that C fails to be universally reversible, then C has finite rank and so $\pi(E) = C$. Hence if J is an ideal in E we have $\pi(J)$ an ideal in C, giving $\pi(J)$ equal to $\{0\}$ or C. If J is universally reversible we must have $J \subseteq \ker \pi$ for all such π and so $J \subseteq I$.

On the other hand, by the extension property of Cartan factor representations from ideals and the construction of I, Theorem 4.7 implies that I is universally reversible. □

Corollary 4.10. *Let E be a JC^*-triple with a complete tripotent u. If $E_2(u)$ is universally reversible with no one dimensional representations, then E is universally reversible.*

Proof. Let $\pi\colon E \to C$ be a Hilbert space or spin factor representation and put $v = \pi(u)$. Then v is a complete tripotent of C and the induced map $\pi_2\colon M_2(u) \to C_2(v)$ is surjective (since $\pi(E) = C$). If C is a spin factor, then $C = C_2(v)$ is universally reversible and so has dimension 3 or 4. If C is a (nonzero) Hilbert space then $C_2(v) = \mathbb{C}v$, a contradiction. Hence, E is universally reversible by Theorem 4.7. □

By passing to a quotient, if a spin factor is a triple homomorphic image of a TRO, then it is linearly isometric to a TRO and thus, via Proposition 2.3, Jordan *-isomorphic to a type I_2 von Neumann algebra, and hence to $M_2(\mathbb{C})$.

Theorem 4.11. *Let T be a TRO. Then the following are equivalent.*

(a) T is universally reversible (as a JC^*-triple).
(b) T has no triple homomorphisms onto a Hilbert space of dimension ≥ 3.
(c) T has no TRO homomorphisms onto a Hilbert space of dimension ≥ 3.

Proof. The equivalence of (a) and (b) follows from Theorem 4.7 and the above remark. That (c) $\Rightarrow$ (b) is seen on passing to quotients (recalling that triple ideals in a TRO are TRO ideals ([18, Proposition 5.8]), and (b) $\Rightarrow$ (c) is clear. □

Lemma 4.12. *Let M be an ℓ_∞-sum $\bigoplus_{i\in I} B_i \overline{\otimes} \mathcal{B}(H_i)e_i$ where B_i is an abelian von Neumann algebra and e_i is a rank 2 projection in $\mathcal{B}(H_i)$, for each $i \in I$. Then M is universally reversible.*

Proof. With $W = \bigoplus_{i\in I} B_i \overline{\otimes} \mathcal{B}(H_i)$ and $e = \bigoplus_{i\in I} 1 \otimes e_i$, we have e a complete tripotent in $M = We$ and $M_2(e) = eWe$ is a type I_2 von Neumann algebra. Thus $M_2(e)$ is universally reversible and has no *-homomorphism onto $\mathbb{C}$. Hence M is universally reversible by Corollary 4.10. □

If e and f are projections in a JW^*-algebra M we use $e \underset{1}{\sim} f$ to denote that e and f are exchanged by a symmetry and we recall [16, Lemma 5.2.9] that if $(e_i)_{i\in I}$ and $(f_i)_{i\in I}$ are two families of orthogonal projections in M with $e_i \underset{1}{\sim} f_i$ for all $i \in I$ and $\sum_{i\in I} e_i \perp \sum_{i\in I} f_i$, then $\sum_{i\in I} e_i \underset{1}{\sim} \sum_{i\in I} f_i$.

Lemma 4.13. *Let M be a type I finite JW^*-algebra with no nonzero summands of type I_1 nor of type $\mathrm{I}_{\mathrm{spin}}$. Then M is a universally reversible JC^*-triple with no one dimensional representations.*

Proof. We may suppose that there is a sequence $(z_i)_i$ of orthogonal central projections in M with (weak) sum 1 such that each Mz_i is a type I_{n_i} JW^*-algebra with n_i strictly increasing and $3 \leq n_i < \infty$, for all i.

Fixing i, we have

$$z_i = e_{i,1} + \cdots e_{i,n_i} \tag{4.1}$$

where the $e_{i,j}$ $(1 \leq j \leq n_i)$ are orthogonal and exchanged by symmetries. Taking equivalence classes modulo 3, write $n_i = 3k + r$ where $r = 0, 1$ or 3. Letting $f_{i,j} = \sum_{s=1}^{k} e_{i,3(s-1)+j}$ $(1 \leq j \leq 3)$ we then have

$$z_i = f_{i,1} + f_{i,2} + f_{i,3} + g_i$$

where $f_{i,1}$, $f_{i,2}$ and $f_{i,3}$ are exchanged by symmetries and $g_i \underset{1}{\sim} p_i \leq f_{i,1} + f_{i,2}$ for some projection p_i in M. Repeating this process for each i and putting $f_j = \sum_i f_{i,j}$ for $1 \leq j \leq 3$, $g = \sum_i g_i$ and $p = \sum_i p_i$ we have

$$f_1 + f_2 + f_3 + g = 1,\ f_j \underset{1}{\sim} f_m \ (1 \leq j, m \leq 3) \text{ and } g \sim p \leq f_1 + f_2. \tag{4.2}$$

The condition (4.2) continues to hold in M^{**}, showing that M^{**} has zero type $\mathrm{I}_{\mathrm{spin}}$ part since, for any nonzero central projection z in M^{**}, zf_1, zf_2 and zf_3 are nonzero orthogonal projections. Therefore, by Proposition 3.13, M^{**} is weakly universally reversible. Hence M is a universally reversible JC^*-triple by Proposition 3.14.

By a similar argument, any quotient of M by an ideal must have dimension at least 3. □

Theorem 4.14. *The following are equivalent for a JW^*-triple M.*
(a) M is universally reversible.
(b) M is weakly universally reversible.
(c) The type $I_1 \oplus_\infty \mathrm{I}_{\mathrm{spin}}$ part of M is linearly isometric to

$$B_0 \oplus_\infty B_1 \otimes \ell_2^2 \oplus_\infty B_2 \otimes V_2 \oplus_\infty B_3 \otimes V_3,$$

where the B_i are abelian von Neumann algebras (or zero).

Proof. (c) ⇒ (a): Assume (c). Let J denote the type $\mathrm{I}_1 \oplus_\infty \mathrm{I}_{\mathrm{spin}}$ part of M. By Remarks 3.12, J is universally reversible and, passing to the orthogonal complement of J in M, we may suppose $J = \{0\}$. By Theorem 3.10, we may assume further that $M = \bigoplus_i B_i \overline{\otimes} C_i$, where the B_i are abelian von Neumann algebras and the C_i are Cartan factors with $3 \le \mathrm{rank}(C_i) < \infty$ for all i. For each i, choose a complete tripotent $u_i \in B_i \overline{\otimes} C_i$ and put $u = \bigoplus_i u_i$. Then u is a complete tripotent of M with $M_2(u)$ a JW^*-algebra of the form considered in Lemma 4.13, and hence $M_2(u)$ is universally reversible with no one dimensional representations. By Corollary 4.10, M is universally reversible.

The implication (a) ⇒ (b) is clear, and (b) ⇒ (c) is given by Proposition 3.13. □

Corollary 4.15. *A W^*-TRO is universally reversible (as a JC^*-triple) if and only if its type I_1 part is linearly isometric to $B_0 \oplus_\infty B_1 \overline{\otimes} \ell_2^2$, where B_0 and B_1 are abelian von Neumann algebras.*

Given a JC^*-triple E and a Cartan factor C, we have that C is linearly isometric to a weak*-closed ideal of E^{**} if and only if there is a Cartan factor representation $\pi\colon E \to C$. Recall (see [13, Proposition 2 and Corollary 4]) that the weak*-closed linear space of the minimal tripotents in E^{**} is called the *atomic part* E^{**}_{at} of E^{**} and E^{**}_{at} is the ℓ_∞ sum of the distinct Cartan factor ideals of E^{**}. Combining this with Theorems 4.7 and 4.14 we have the following.

Corollary 4.16. *The following are equivalent for a JC^*-triple E.*
(a) E is universally reversible.
(b) E^{**}_{at} is universally reversible.
(c) E^{**} is universally reversible.

We shall conclude this section by showing that the number of terms in the definition of universally reversible JC^*-triples can be reduced to the smallest possible. We recall that the universal TRO $(T^*(A), \alpha_A)$ coincides with the universal C^*-algebra $(C^*_{\mathrm{J}}(A), \beta_A)$ when A is a JC^*-algebra [3, Proposition 3.7].

Theorem 4.17. *The following are equivalent for a JC^*-triple E.*
(a) E is universally reversible.
(b) $[x_1, \cdots, x_5] + [x_5, \cdots, x_1] \in \alpha_E(E)$ whenever $x_1, \ldots, x_5 \in \alpha_E(E)$.
(c) $[y_1, \cdots, y_5] + [y_5, \cdots, y_1] \in \pi(E)$ whenever $y_1, \ldots, y_5 \in \pi(E)$ and $\pi\colon E \to \mathcal{B}(H)$ is a triple homomorphism.

Proof. (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (b) are clear.

(b) $\Rightarrow$ (c): Assume (c). Let $\pi\colon E \to \mathcal{B}(H)$ be a triple homomorphism, $y_i \in \pi(E)$ and $x_i \in E$ with $\pi(x_i) = y_i$ for $1 \le i \le 5$. Let $\tilde{\pi}\colon T^*(E) \to \mathcal{B}(H)$ be the TRO homomorphism with $\pi = \tilde{\pi} \circ \alpha_E$. Then with $z_i = \alpha_E(x_i)$, applying $\tilde{\pi}$ to $[z_1, \cdots, z_5] + [z_5, \cdots, z_1] \in \alpha_E(E)$ gives $[y_1, \cdots, y_5] + [y_5, \cdots, y_1] \in \tilde{\pi}(\alpha_E(E)) = \pi(E)$.

(c) $\Rightarrow$ (a): Assume (c). Let $\psi\colon E \to F \subset \mathcal{B}(H)$ be a surjective triple homomorphism and consider $\pi\colon E \to \alpha_F(F) \subset T^*(F)$ where π denotes $\alpha_F \circ \psi$. By [3, Theorem 5.1 and proof], if F is a Hilbert space of dimension ≥ 3 then $\alpha_F\colon F \to T^*(F)$ fails condition (c), as therefore does π. If F is a spin factor of dimension ≥ 5 then (using $(T^*(F), \alpha_F) = (C^*_{\mathrm{J}}(F), \beta_F)$) $\alpha_F(F_{sa}) = \alpha_F(F)_{sa}$ is not reversible in $T^*(F)$ so that by [6] (or see [16, p. 142]) there exist $y_1, y_2, y_3, y_4 \in \alpha_F(F)_{sa}$ with $y_1y_2y_3y_4 + y_4y_3y_2y_1 \notin \alpha_F(F)_{sa}$. Thus, in this case π fails condition (c) for y_1, y_2, y_3, y_4, y_5 with $y_5 = 1$.

Therefore E is universally reversible by Theorem 4.7. □

5. JC-operator spaces

We recall from §2 that the JC-operator space structures of a JC^*-triple E, the operator space structures determined by triple embeddings $\pi\colon E \to \mathcal{B}(H)$, are the $E_{\mathcal{I}}$ induced by the maps $E \to T^*(E)/\mathcal{I}$ $(x \mapsto \alpha_E(x) + \mathcal{I})$ as $\mathcal{I}$ ranges over the ideals of $T^*(E)$ having vanishing intersection with $\alpha_E(E)$ (the operator space ideals of $T^*(E)$).

Let T be a subTRO of $\mathcal{B}(H)$. We use T^{op} to denote the identical image of T in the opposite C^*-algebra $\mathcal{B}(H)^{\mathrm{op}}$. Given a transposition $x \mapsto x^t$ on $\mathcal{B}(H)$ we write $T^t = \{x^t : x \in \mathcal{B}(H)\}$. Thus $\mathrm{id}\colon T \to T^{\mathrm{op}}$ and the map $T^{\mathrm{op}} \to T^t$ $(x \mapsto x^t)$ are a TRO anti-isomorphism and a TRO isomorphism, respectively.

By definition, T is an *abelian TRO* if $[x, y, z] = [z, y, x]$ for all $x, y, z \in T$. Evidently, T is an abelian TRO if and only if $\mathrm{id}\colon T \to T^{\mathrm{op}}$ is a TRO isomorphism (equivalently, a complete isometry).

Lemma 5.1. *Let T be a TRO. Then*
(a) $\mathrm{id}\colon T \to T^{\mathrm{op}}$ *is completely contractive if and only if T is abelian;*
(b) a TRO antihomomorphism $\pi\colon T \to \mathcal{B}(H)$ is completely contractive if and only if $\pi(T)$ is abelian.

Proof. (a) If $\mathrm{id}\colon T \to T^{\mathrm{op}}$ is completely contractive, then so is $\mathrm{id}\colon T^{\mathrm{op}} \to (T^{\mathrm{op}})^{\mathrm{op}} = T$ by [2, 1.1.25], implying that it is a complete isometry, and hence that T is abelian. The converse is clear.

(b) Let $\pi\colon T \to \mathcal{B}(H)$ be a completely contractive TRO antihomomorphism. Let S denote $T/\ker\pi$ and let $\tilde{\pi}\colon S \to \pi(T)$ denote the induced TRO antihomomorphism, also completely contractive (by [2, 1.1.15]). Since $\tilde{\pi}^{-1}\colon \pi(T) \to S^{\mathrm{op}}$ is a TRO isomorphism, $\tilde{\pi}^{-1}\circ\pi\colon S \to S^{\mathrm{op}}$, which is the identity map, is completely contractive, so that S is abelian by (a). Conversely, a TRO antihomomorphism into an abelian TRO is a TRO homomorphism and so is completely contractive.

□

A JC^*-triple E is said to be *abelian* if

$$\{x, y, \{a, b, c\}\} = \{\{x, y, a\}, b, c\}$$

for all $x, y, a, b, c \in E$. An abelian TRO is an abelian JC^*-triple. On the other hand, given an abelian JC^*-subtriple E of $\mathcal{B}(H)$, the abelian JW^*-triple $\bar{E}$ (the weak*-closure of E) is linearly isometric to an abelian von Neumann algebra by [19, (3.11)] and we may choose a unitary tripotent $u \in \bar{E}$. Then $\bar{E}_2(u)$ is an abelian von Neumann subalgebra of $\mathcal{B}(H)_2(u)$, the binary product and involution of the latter being given by $x \bullet y = [x, u, y]$ and $x^\# = [u, x, u]$. In particular, with $a, b, c \in E$ (since $E \subseteq \bar{E} = \bar{E}_2(u)$) we have $[a, b, c] = ab^*c = a \bullet b^\# \bullet c = c \bullet b^\# \bullet a = cb^*a$. Thus E is a TRO and the inclusion $E \hookrightarrow \bar{E}_2(u)$ is a TRO isomorphism onto a subTRO of $\bar{E}_2(u)$.

Proposition 5.2. *The following are equivalent for a JC^*-subtriple E of $\mathcal{B}(H)$.*

(a) E is an abelian JC^*-triple.
(b) E is an abelian TRO.
(c) E is completely isometric to a subTRO of an abelian C^*-algebra.
(d) E has a separating family of representations onto $\mathbb{C}$.
(e) $T^*(E) = E$

Proof. The equivalence of (a), (b) and (c) is verified by the preamble, and (a) $\iff$ (d) is shown in [24, Proposition 6.2]. If (b) holds then every triple homomorphism from E into a C^*-algebra is a TRO homomorphism, giving (e). Conversely (e) implies that E is a TRO and the universal property of $T^*(E)$ implies that $\mathrm{id}\colon E \to E^{\mathrm{op}}$ is a TRO isomorphism, giving (b). □

Corollary 5.3. *Every abelian JC^*-triple has a unique JC-operator space structure.*

Theorem 5.4. *Let $T \subset \mathcal{B}(H)$ be a TRO with no nonzero representations onto a Hilbert space of any dimension other (possibly) than two. Suppose $x \mapsto x^t$ is a transposition of $\mathcal{B}(H)$. Then*

(a) $T^*(T) = T \oplus T^t$ *with* $\alpha_T(x) = x \oplus x^t$ *and canonical involution* $\Phi(x \oplus y^t) = y \oplus x^t$ *(for* $x, y \in T$*);*
(b) T *has at least three distinct* JC*-operator space structures.*

Proof. (a) By the assumptions on T and Theorem 4.11, T is universally reversible with no ideals of codimension one. Moreover $x \mapsto x^t$ is a TRO antiautomorphism of $\mathcal{B}(H)$. Therefore (a) follows from [3, Corollary 4.5].
(b) Regarding T and T^t as operator subspaces of $\mathcal{B}(H)$ and $T \oplus T^t$ as an operator subspace of $\mathcal{B}(H) \oplus \mathcal{B}(H)$, neither the map $T \to T^t$ $(x \mapsto x^t)$ nor its inverse can be completely contractive, by Lemma 5.1 (a), since T is not abelian. Further, since the natural TRO projection $T \oplus T^t \to T^t$ is completely contractive, $\alpha_T \colon T \to T \oplus T^t$ cannot be completely contractive. Thus if $\mathcal{I}_0$ denotes the zero ideal and $\mathcal{I}_1 = T \oplus \{0\}$, $\mathcal{I}_2 = \{0\} \oplus T^t$ (which are nonzero operator space ideals of $T^*(T)$), then the JC-operator spaces $T_{\mathcal{I}_j}$ $(j = 0, 1, 2)$ are mutually distinct.

□

Given an ideal I of a JC^*-triple E we recall [5, Theorem 3.3] that the canonical involution Φ of $T^*(E)$ restricts to that of $T^*(I)$ and the canonical involution of $T^*(E/I) = T^*(E)/T^*(I)$ is given by $x + T^*(I) \mapsto \Phi(x) + T^*(I)$ $(x \in T^*(E))$. In each of the following three technical results Φ denotes the canonical involution of $T^*(E)$ for the JC^*-triple E in question.

Lemma 5.5. *Let* E *be a universally reversible* JC^**-triple and let* $\mathcal{I}$ *be an ideal of* $T^*(E)$*. Then*
(a) if $\mathcal{I}$ *is an operator space ideal with* $\Phi(\mathcal{I}) = \mathcal{I}$ *then* $\mathcal{I} = \{0\}$*;*
(b) $\mathcal{I} \cap \Phi(\mathcal{I}) = T^*(J)$ *where* J *is the ideal of* E *given by* $\alpha_E(J) = \alpha_E(E) \cap \mathcal{I}$*;*
(c) $(\mathcal{I} + \Phi(\mathcal{I})) \cap \alpha_E(E) = \{x + \Phi(x) : x \in \mathcal{I}\}$*.*

Proof. (a) See [3, Lemma 4.3].
(b) Letting $\tilde{\Phi}$ be the canonical involution of $T^*(E/J) = T^*(E)/T^*(J)$ and $\mathcal{K} = \mathcal{I} \cap \Phi(\mathcal{I})$, $\mathcal{K}/T^*(J)$ is a $\tilde{\Phi}$-invariant operator space ideal of $T^*(E/J)$. Since E/J is universally reversible, $\mathcal{K} = T^*(J)$ by (a).
(c) Given $x, y \in \mathcal{I}$ with $x + \Phi(y) \in \alpha_E(E)$, we have $x + \Phi(y) = a + \Phi(a)$ where $a = (1/2)(x + y)$, so that the left hand side is contained in the right. On the other hand, since E is universally reversible, the right hand side is contained in $\alpha_E(E)$, establishing the assertion.

□

Proposition 5.6. *Let* E *be a universally reversible* JC^**-triple. Then the following are equivalent for an ideal* $\mathcal{I}$ *of* $T^*(E)$
(a) $\mathcal{I}$ *is an operator space ideal of* $T^*(E)$*;*
(b) $\mathcal{I} \cap \Phi(\mathcal{I}) = \{0\}$*;*

(c) the map $\mathcal{I} \to (\mathcal{I} + \Phi(\mathcal{I})) \cap \alpha_E(E)$ $(x \mapsto x + \Phi(x))$ is a triple isomorphism (onto an ideal of $\alpha_E(E)$).

Proof. (a) $\Rightarrow$ (b) is immediate from Lemma 5.5 (a) since $\mathcal{I} \cap \Phi(\mathcal{I})$ is a Φ-invariant ideal of $T^*(E)$.

(b) $\Rightarrow$ (c). Given (b) we have $\mathcal{I} + \Phi(\mathcal{I}) = \mathcal{I} \oplus_\infty \Phi(\mathcal{I})$ so that the stated map is a linear isometry, surjective by Lemma 5.5 (c), and thus a triple isomorphism.

(c) $\Rightarrow$ (a). If (c) holds and $x \in \mathcal{I} \cap \alpha_E(E)$, then $\Phi(x) = x$ and $\|x\| = \|2x\|$, giving $x = 0$. □

Lemma 5.7. *Let $\mathcal{I}$, $\mathcal{J}$ be operator space ideals of $T^*(E)$ with $\mathcal{J} \subseteq \mathcal{I}$, where E is a universally reversible JC^*-triple. Put $\mathcal{K} = \mathcal{J} + \Phi(\mathcal{J})$ and let L be the ideal of E such that $\alpha_E(L) = \alpha_E(E) \cap \mathcal{K}$. Then*

(a) $\mathcal{I} \cap (\alpha_E(E) + \mathcal{K}) = \mathcal{J} = \mathcal{I} \cap \mathcal{K}$;

(b) $(\mathcal{I} + \mathcal{K}) \cap (\alpha_E(E) + \mathcal{K}) = \mathcal{K}$;

(c) $\mathcal{I}/\mathcal{J}$ is TRO isomorphic to an operator space ideal of $T^*(E/L) = T^*(E)/\mathcal{K}$.

Proof. (a) That $\mathcal{J}$ is contained in the two stated sets is clear. Conversely given $x \in \mathcal{I} \cap (\alpha_E(E) + \mathcal{K})$ we have $x = y + a + \Phi(b)$ for some y $(= \Phi(y))$ in $\alpha_E(E)$ and $a, b \in \mathcal{J}$. Putting $c = a - b$ we have $\Phi(x - c) = x - c$ and so, since E is universally reversible, $x - c \in \alpha_E(E) \cap \mathcal{I} = \{0\}$ giving $x = c \in \mathcal{I}$.

(b) Given z in the left hand set we have $x \in \mathcal{I}$, $y \in \alpha_e(E)$ and $a, b \in \mathcal{K}$ such that $z = x + a = y + b$. Then $x = y + b - a \in \mathcal{I} \cap (\alpha_e(E) + \mathcal{K}) = \mathcal{J}$, by (a), so that $z \in \mathcal{K}$.

(c) The quotients $\mathcal{I}/\mathcal{J}$ and $(\mathcal{I} + \mathcal{K})/\mathcal{K}$ are TRO isomorphic by (a). By Lemma 5.5 (b), $T^*(L) = \mathcal{K}$, so that $T^*(E/L) = T^*(E)/\mathcal{K}$. Since $(\mathcal{I} + \mathcal{K})/\mathcal{K}$ has vanishing intersection with $(\alpha_E(E) + \mathcal{K})/\mathcal{K}$, by (b), it is an operator space ideal of $T^*(E/L)$.

□

Lemma 5.8. *Let E be a JC^*-triple and let E_1 denote the intersection of the kernels of all the one dimensional representations of E (with $E_1 = E$ if E has no representations onto $\mathbb{C}$). Then*

(a) E_1 has no (nonzero) one dimensional representations and E/E_1 is abelian;

(b) $T^*(E_1)$ has no one dimensional representations and $T^*(E/E_1) = T^*(E)/T^*(E_1)$ is abelian;

(c) an ideal $\mathcal{I}$ of $T^*(E)$ is abelian if and only if $\mathcal{I} \cap T^*(E_1) = \{0\}$.

Proof. (a) and (b) are immediate by Proposition 5.2 and the properties of the universal TRO. To see (c), if $\mathcal{I}$ has trivial intersection with $T^*(E_1)$ then it may be realised as a subtriple of $T^*(E)/T^*(E_1)$ and so is abelian by the second part of (b). On the other hand the first part of (b) implies that $T^*(E_1)$ has no nonzero abelian ideal. □

Proposition 5.9. *Let E be a universally reversible JC^*-triple and let E_1 be as in Lemma 5.8. Then*
(a) $T^*(E)$ has no nonzero abelian operator space ideals;
(b) all operator space ideals of $T^*(E)$ are contained in $T^*(E_1)$.

Proof. (a) Let $\mathcal{I}$ be an abelian operator space ideal of $T^*(E)$ and let J be the ideal of E with $\alpha_E(J) = \alpha_E(E) \cap (\mathcal{I} + \Phi(\mathcal{I}))$. Since $\mathcal{I}$ and $\Phi(\mathcal{I})$ are orthogonal, by Proposition 5.6 (b), $\mathcal{I} + \Phi(\mathcal{I})$ is abelian, as therefore is J. Hence, using Proposition 5.2 (e) in the first equality and Lemma 5.5 (b) in the second, we have $\alpha_E(J) = T^*(J) = \mathcal{I} + \Phi(\mathcal{I})$, so that $\mathcal{I}$ is contained in $\alpha_E(E)$. Hence $\mathcal{I} = \{0\}$.
(b) Let $\mathcal{I}$ be a nonzero operator space ideal of $T^*(E)$. Put $\mathcal{J} = \mathcal{I} \cap T^*(E_1)$, $\mathcal{K} = \mathcal{J} + \Phi(\mathcal{J})$. [Note that $\mathcal{J}$ is nonzero by (a) together with Lemma 5.8 (c).] By construction $\mathcal{I}/\mathcal{J}$ is TRO isomorphic to $(T^*(E_1) + \mathcal{J})/T^*(E_1)$ and so is abelian by Lemma 5.8 (b). In addition, by Lemma 5.7 (c), there is a JC^*-triple quotient F of E such that $\mathcal{I}/\mathcal{J}$ is TRO isomorphic to an operator space ideal of $T^*(F)$ and therefore vanishes, by (a), since F is universally reversible. Hence, $\mathcal{I} = \mathcal{J}$.

□

Given a JC^*-triple E and an operator space ideal $\mathcal{I}$ of $T^*(E)$, in the proof below we denote the triple homomorphic image $\{\alpha_E(x) + \mathcal{I} : x \in E\}$ of E in $T^*(E)/\mathcal{I}$ by $\tilde{E}_{\mathcal{I}}$, noting that the latter is a completely isometric copy of $E_{\mathcal{I}}$.

Theorem 5.10. *Let E be a universally reversible JC^*-triple. Then $\mathcal{I} \mapsto E_{\mathcal{I}}$ is a bijective correspondence between the operator space ideals of $T^*(E)$ and the JC-operator space structures of E.*

Proof. We know that the stated correspondence is surjective. By [3, Theorem 6.5] given an operator space ideal $\mathcal{J}$ of $T^*(E)$ there is a largest operator space ideal $\mathcal{I}$ of $T^*(E)$ with $E_{\mathcal{I}} = E_{\mathcal{J}}$. To establish injectivity it is enough to show that this largest $\mathcal{I}$ coincides with $\mathcal{J}$. Suppose on the contrary that $\mathcal{I}$, $\mathcal{J}$ are operator space ideal of $T^*(E)$ with $\mathcal{J}$ strictly contained in $\mathcal{I}$ but with $E_{\mathcal{I}} = E_{\mathcal{J}}$. Simplifying notation we shall regard E as a subtriple of $T^*(E)$. Put $K = E \cap (\mathcal{I} + \Phi(\mathcal{I}))$. Then, by Lemma 5.5 and Proposition 5.6, we have $K = \{x + \Phi(x) : x \in \mathcal{I}\}$ and $T^*(K) = \mathcal{I} + \Phi(\mathcal{I}) = \mathcal{I} \oplus_\infty \Phi(\mathcal{I})$. In particular, $\mathcal{I}$ and $\mathcal{J}$ are operator space ideals of $T^*(K)$ and we may regard $\tilde{K}_{\mathcal{I}}$ and $\tilde{K}_{\mathcal{J}}$ as operator subspaces of $T^*(K)/\mathcal{I}$ and $T^*(K)/\mathcal{J}$, respectively. We shall show that the linear isometry $\pi : \tilde{K}_{\mathcal{I}} \to \tilde{K}_{\mathcal{J}}$ given by $\pi(x + \mathcal{I}) = x + \mathcal{J}$, for all x in K, is not a complete contraction, implying $K_{\mathcal{I}} \neq K_{\mathcal{J}}$ and hence $E_{\mathcal{I}} \neq E_{\mathcal{J}}$. We have $\tilde{K}_{\mathcal{I}} = (K + \mathcal{I})/\mathcal{I} = \{\Phi(x) + \mathcal{I} : x \in \mathcal{I}\} = T^*(K)/\mathcal{I} \cong \Phi(\mathcal{I})$ (as TROs). The map $\beta \colon \tilde{K}_{\mathcal{I}} \to \Phi(\mathcal{I})$ $(\Phi(x) + \mathcal{I} \mapsto \Phi(x))$ is a TRO isomorphism and hence a complete isometry. Define $\psi \colon T^*(K)/\mathcal{J} \to \mathcal{I}/\mathcal{J}$ by $\psi(a + \Phi(b) + \mathcal{J}) = a + \mathcal{J}$, whenever $a, b \in \mathcal{I}$.

Then ψ is a TRO homomorphism restricting to a surjective complete contraction $\Upsilon\colon \tilde{K}_{\mathcal{J}} \to \mathcal{I}/\mathcal{J}$. We note that $\mathcal{I}/\mathcal{J}$ cannot be abelian, by Lemma 5.7 (c) together with Proposition 5.9 (a). On the other hand the composition $\Upsilon \circ \pi \circ \beta^{-1}$

$$\Phi(\mathcal{I}) \to \tilde{K}_{\mathcal{I}} \to \tilde{K}_{\mathcal{J}} \to \mathcal{I}/\mathcal{J}$$

which sends $\Phi(x)$ to $x + \mathcal{J}$ for all x in $\mathcal{I}$, is a TRO antihomomorphism and so, by Lemma 5.1 (b) cannot be completely contractive. Hence, π is not a complete contraction. $\square$

For the definition and properties of the triple envelope $(\mathcal{T}(X), j)$ of an an operator space X we refer to [14] and [2, Chapter 8], and again to the latter for a justification that it be regarded as the noncommutative Shilov boundary of X. We should point out a clash of usage: the terms "triple system" and "triple homomorphism" used in [2, 14] should be read as "TRO" and "TRO homomorphism". In particular, $\mathcal{T}(X)$ is a TRO. The following is immediate from Theorem 5.10 and [4, Proposition 1.2].

Corollary 5.11. *(a) If E is a universally reversible JC^*-triple and $\mathcal{I}$ is an operator space ideal of $T^*(E)$ then the triple envelope of the operator space $E_{\mathcal{I}}$ is $(T^*(E)/\mathcal{I}, j)$, where $j\colon E_{\mathcal{I}} \to T^*(E)/\mathcal{I}$ $(x \mapsto \alpha_E(x) + \mathcal{I})$.*

(b) If E is a universally reversible JC^*-subtriple of a C^*-algebra A (regarded as an operator subspace of A) then
   - *(i) the triple envelope of E is* $(\mathrm{TRO}(E), \text{inclusion})$*;*
   - *(ii) every complete isometry from E onto a JC^*-subtriple F of $\mathcal{B}(H)$ extends to a TRO isomorphism from* $\mathrm{TRO}(E)$ *onto* $\mathrm{TRO}(F)$.

Theorem 5.12. *The following are equivalent for a universally reversible JC^*-triple E.*

(a) E has a unique JC-operator space structure.
(b) $T^*(E)$ has no nonzero operator space ideal.
(c) E has no ideal linearly isometric to a nonabelian TRO.
(d) If $\pi : E \to \mathcal{B}(H)$ is an injective triple homomorphism, then $(\mathrm{TRO}(\pi(E)), \pi)$ *is the universal TRO of E.*

Proof. The equivalences (a) $\iff$ (b) and (b) $\iff$ (d) are immediate from Theorem 5.10 and properties of $T^*(\cdot)$, respectively.

(a) $\Rightarrow$ (c). Suppose that E has an ideal I linearly isometric to a nonabelian TRO. Via Lemma 5.8 I has a nonzero ideal I_1 linearly isometric to a universally reversible TRO without one dimensional representations. By Theorem 4.11 the latter TRO satisfies the conditions of Theorem 5.4 so that I_1, and hence E, has at least three JC-operator structures.

(c) $\Rightarrow$ (b). Suppose $\mathcal{I}$ is a nonzero operator space ideal of $T^*(E)$. Then $\mathcal{I}$ is nonabelian by Proposition 5.9 and is linearly isometric to an ideal of E by Proposition 5.6, proving the implication. $\square$

Remarks 5.13. Since the nonzero operator space ideals occur in pairs (Lemma 5.5 (a)), it follows from Theorem 5.10 that the number of distinct JC-operator spaces of a universally reversible JC*-triple must be odd or infinite. In [3, Remark 6.9] we proved that for a projection e of rank ≥ 2 in $\mathcal{B}(H)$, the universally reversible Cartan factor $\mathcal{B}(H)e$ has three or $2|\alpha|+5$ JC-operator space structures according to whether the rank of e is finite or an infinite cardinal $\aleph_\alpha$ respectively, with $|\alpha|$ denoting the cardinality of the ordinal segment $[0, \alpha)$ so that, as $|\alpha|$ varies over finite cardinalities all odd numbers ≥ 3 arise (the case $\alpha > 0$ was mis-stated in [3, Remark 6.9 (iii)]).

On the other hand, if E is a universally reversible JC^*-triple with no Cartan factor representations $\pi\colon E \to C$ such that C has the form $\mathcal{B}(H)e$, then E must satisfy condition (c) of Theorem 5.12 and so must have a unique JC-operator space structure.

Another corollary of Theorem 5.12 is that a simple universally reversible JC^*-triple E either has a unique JC-operator space structure or it has exactly three. By Remark 4.8, this together with [4, Proposition 2.4 and Theorem 3.7] and [5, §6], accounts for the enumeration of the JC-operator space structures of all simple JC^*-triples.

Consider a triple homomorphism $\pi\colon E \to \mathcal{B}(K)$ where E is a universally reversible JC^*-subtriple of $\mathcal{B}(H)$. If E has no ideals isometric to a nonabelian TRO, then by Theorem 5.12 ((c) $\Rightarrow$ (d)) π extends to a TRO homomorphism on TRO$(E)(= T^*(E))$ and so is completely contractive, and further is completely isometric (onto its range) if π is injective. By contrast, for TROs with few nonzero Hilbert space representations we have the following consequence of Theorem 5.4 (which should be compared with the C^*-algebra result [15, Corollary 4.6]).

Proposition 5.14. *Let T be a TRO with no nonzero Hilbert space representations other (possibly) than of dimension two. Let $\pi\colon T \to \mathcal{B}(H)$ be a triple homomorphism. Then there exist $\pi_1, \pi_2\colon T \to \mathcal{B}(H)$ where π_1 is a TRO homomorphism and π_2 is a TRO antihomomorphism such that $\pi = \pi_1 + \pi_2$ and $\pi_1(T) \perp \pi_2(T)$.*

Proof. Supposing $T \subseteq \mathcal{B}(K)$ and $x \mapsto x^t$ to be a transposition of $\mathcal{B}(K)$ Theorem 5.4 implies that there is a TRO homomorphism $\tilde{\pi}\colon T \oplus T^t \to \mathcal{B}(H)$ such that $\pi(x) = \tilde{\pi}(x \oplus x^t)$ for all $x \in T$. Defining $\pi_1, \pi_2\colon T \to \mathcal{B}(H)$ by $\pi_1(x) = \tilde{\pi}(x \oplus 0)$ and $\pi_2(x) = \tilde{\pi}(0 \oplus x^t)$, the result follows. $\square$

We remark that by the results of [5, §3 and §4], with the notation of Proposition 5.14, there is a central projection z in the right von Neumann algebra of $\pi(T)$ such that $\pi_1(x) = \pi(x)z$ and $\pi_2(z) = \pi(x)(1-z)$ for all $x \in T$.

Corollary 5.15. *Let T be as in Proposition 5.14 and let $\pi\colon T \to \mathcal{B}(H)$ be a triple homomorphism such that the weak*-closure $\overline{\mathrm{TRO}(\pi(T))}$ of $\mathrm{TRO}(\pi(T))$ is a W^*-TRO factor. Then π is either a TRO homomorphism or a TRO antihomomorphism.*

Proof. The factor condition implies that $\mathrm{TRO}(\pi(T))$ cannot contain a nontrivial pair of orthogonal ideals. Thus, in the notation of Proposition 5.14, either π_1 is trivial or π_2 is, whence the assertion. □

Corollary 5.16. *Let $\pi\colon T \to \mathcal{B}(H)$ be a completely contractive triple homomorphism where T is as in Proposition 5.14. Then π is a TRO homomorphism.*

Proof. By Proposition 5.14 and its proof, $\pi_1(T)$ and $\pi_2(T)$ are triple ideals of $\pi(T)$ so that $\mathrm{TRO}(\pi(T)) = \mathrm{TRO}(\pi_1(T)) \oplus_\infty \mathrm{TRO}(\pi_2(T)) = \pi_1(T) \oplus_\infty \pi_2(T)$, and so π_2 must be completely contractive. Hence $\pi_2(T)$ is abelian by Proposition 5.1 (b), Hence $\pi_2(T) = \{0\}$ as otherwise T would have a one dimensional (Hilbert space) representation, proving the result. □

Lemma 5.17. *If T is a a W^*-TRO, there exist centrally orthogonal projections e and f in a von Neumann algebra W such that T is TRO isomorphic to the direct sum $eW + Wf$. If T is a factor, then T is TRO isomorphic to a weak*-closed one-sided ideal in a W^*-algebra factor.*

Proof. Up to TRO isomorphism, a W^*-TRO T has the form $T = gMh$ where g and h are projections in a von Neumann algebra M [9, §2]. By comparison theory, there exist projections e, f and z in M, with z central, such that

$$gz \sim e \leq hz \text{ and } h(1-z) \sim f \leq g(1-z)$$

For $W_1 = zhMh$ and $W_2 = (1-z)gMg$, centrality of z implies that $Tz = (gz)Mh$ is TRO isomorphic to $pMh = e(hMh)z = eW_1$ (see proof of Lemma 3.1) and similarly that $T(1-z)$ is TRO isomorphic to $W_2 f$. The result follows from putting $W = W_1 \oplus W_2$.

If T is a factor, one summand must be zero and if (say) $T = eW$ we may assume that e has central cover 1, in which case W must be a factor. □

The examples $x \mapsto x^t$ on $\mathcal{B}(H)$ and $x \oplus y^t \mapsto y \oplus x^t$ on $\mathcal{B}(H)e \oplus e^t\mathcal{B}(H)$, where e is a projection of rank strictly less than the dimension of H, show that neither of the two conditions imposed upon T in the next result can be removed.

Theorem 5.18. *Let $\pi\colon T \to T$ be a surjective linear isometry where T is a W^*-TRO factor not linearly isometric to a C^*-algebra. Then π is a complete isometry.*

Proof. By Lemma 5.17 we may suppose that T is eW where e is a nonzero projection in a von Neumann algebra factor W. (The argument for the left ideal case is similar.) We may further suppose that $T \subset \mathcal{B}(H)$ and that $x \mapsto x^t$ is a transposition of $\mathcal{B}(H)$ with $e^t = e$.

If there is a nonzero Hilbert space representation $\psi\colon eW \to K$, then the induced map from the factor $eWe = (eW)_2(e)$ to $K_2(\psi(e))$ is a one-dimensional Jordan *-homomorphism, implying that e is a minimal projection and W is a type I factor: in which case, eW is (linearly isometric to) a Hilbert space and the result follows from [4, Proposition 1.5].

Thus we may suppose that eW has no nonzero Hilbert space representations and so satisfy the conditions of Proposition 5.14.

Therefore, by Corollary 5.15, π is a TRO isomorphism or a TRO antiautomorphism. Assume the latter. Then there is a TRO isomorphism $\psi\colon W^te \to eW$. Putting $u = \psi(e)$ we have

$$\psi(a) = \psi(ae^*e) = \psi(a)u^*u$$

for all $a \in W^te$, giving $eW = eWu^*u$, hence $WeW = WeWu^*u$ and therefore $u^*u = 1$ because W is a factor. Since $uu^* \leq e$, this implies that $e \sim 1$ and therefore that eW is TRO isomorphic to W, a contradiction. Therefore, π is a TRO isomorphism. □

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