

UPPER BOUNDS FOR THE DIMENSION OF MODULI SPACES OF CURVES WITH SYMMETRIC WEIERSTRASS SEMIGROUPS

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ABSTRACT. We present an explicit method to produce upper bounds for the dimension of the moduli spaces of complete integral Gorenstein curves with prescribed symmetric Weierstrass semigroups.

1. INTRODUCTION

Let $\mathcal{M}_{\mathcal{H}}$ be the moduli space of the smooth complete integral pointed algebraic curves with a prescribed Weierstrass semigroup $\mathcal{H}$ of genus g . There are two important estimates on the dimension of $\mathcal{M}_{\mathcal{H}}$. On the one hand, Eisenbud and Harris [EH], arguing that locally $\mathcal{M}_{\mathcal{H}}$ is the pullback of Schubert-cycles from a suitable Grassmannian of $(g-1)$ -planes, obtained the lower bound $3g - 2 - w(\mathcal{H})$ for the dimension of any irreducible component of $\mathcal{M}_{\mathcal{H}}$, where $w(\mathcal{H})$ denotes the weight of $\mathcal{H}$. As follows from their theory of limit linear series, this bound is attained for some component of $\mathcal{M}_{\mathcal{H}}$, if $w(\mathcal{H}) \leq \frac{g}{2}$, or more generally, if $\mathcal{H}$ is a primitive semigroup of weight smaller than g (cf. [EH, theorem 3], [K]). However, if the weight is large, as in the case of symmetric semigroups, then their bound may be far from being sharp, and it may even be negative.

On the other hand, a theorem of Deligne [D, theorem 2.27], whose proof involves an interplay between three different moduli spaces, provides the upper bound $\dim \mathcal{M}_{\mathcal{H}} \leq 2g - 2 + \lambda(\mathcal{H})$, where $\lambda(\mathcal{H}) \geq 1$ stands for the number of gaps ℓ such that $\ell + n$ is a nongap for each positive nongap n (cf. [RV, section 6]). If the semigroup is symmetric, then Deligne's upper bound is equal to $2g - 1$. By using the work of Kontsevich and Zorich [KZ], it has been noticed by Bullock [Bu] that for each $g > 3$ the upper bound is attained exactly on the three symmetric semigroups with the gap sequences $\{1, \dots, g-1, 2g-1\}$, $\{1, \dots, g-2, g, 2g-1\}$ and $\{1, 3, 5, \dots, 2g-1\}$.

In this paper we assume that $\mathcal{H}$ is a numerical symmetric semigroup. Looking for an upper bound for the dimension of the moduli space $\mathcal{M}_{\mathcal{H}}$, we view $\mathcal{M}_{\mathcal{H}}$ as an open subspace of the compactified moduli space $\overline{\mathcal{M}}_{\mathcal{H}}$ which is defined by allowing arbitrary Gorenstein singularities. By varying the construction of $\overline{\mathcal{M}}_{\mathcal{H}}$ presented in [S], in Section 2 we realize $\overline{\mathcal{M}}_{\mathcal{H}}$ in a rather explicit way as the weighted projectivization $\mathbb{P}(\mathcal{X}_{\mathcal{H}})$ of an affine quasi-cone $\mathcal{X}_{\mathcal{H}}$, i.e. of a subset $\mathcal{X}_{\mathcal{H}}$ of a weighted vector space cut out by quasi-homogeneous equations. We approximate the quasi-cone $\mathcal{X}_{\mathcal{H}}$ at its vertex by a quadratic one, whose weighted projectivization provides us with an upper bound for the dimension of $\overline{\mathcal{M}}_{\mathcal{H}}$ (cf. Theorem 3.1). We explain that it is much easier to compute the quasi-homogeneous quadratic forms that

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determine the quadratic approximation than the equations of the moduli space $\overline{\mathcal{M}}_{\mathcal{H}}$.

In the last section we illustrate the method by handling explicitly with the family of symmetric semigroups

$$\mathcal{H}_\tau = \langle 6, 2 + 6\tau, 3 + 6\tau, 4 + 6\tau, 5 + 6\tau \rangle$$

of genus $g_\tau = 6\tau + 1$ ($\tau = 1, 2, \dots$). The quadratic approximation of the quasi-cone $\mathcal{X}_{\mathcal{H}_\tau}$ can be described in terms of a variety defined over an artinean algebra (cf. Theorem 4.4). Its dimension can be read off from this description, providing the upper bound $8\tau + 5$ for the dimension of $\overline{\mathcal{M}}_{\mathcal{H}_\tau}$, which for each $\tau > 1$ is better than Deligne's bound $2g_\tau - 1 = 12\tau + 1$.

2. ON THE CONSTRUCTION OF THE COMPACTIFIED MODULI SPACE $\overline{\mathcal{M}}_{\mathcal{H}}$

Let $\mathcal{C}$ be a complete integral Gorenstein curve of arithmetic genus g defined over an algebraically closed field $\mathbf{k}$, and let P be a smooth point of $\mathcal{C}$. We denote by $\mathcal{H}$ the Weierstrass semigroup of the nongaps $0 = n_0 < n_1 < n_2 < \dots$ of the pointed curve $(\mathcal{C}, P)$. Thus for each $n \in \mathcal{H}$ there is a rational function x_n on $\mathcal{C}$ with pole divisor nP . We can assume that $x_0 = 1$. For each nonnegative integer i the vector space of global sections of the divisor $n_i P$ is equal to

$$H^0(\mathcal{C}, n_i P) = \mathbf{k} x_{n_0} \oplus \mathbf{k} x_{n_1} \oplus \dots \oplus \mathbf{k} x_{n_i}$$

and in particular it has dimension $i + 1$. It follows from Riemann's theorem for complete integral curves with singularities that

$$n_i = g + i \text{ for each } i \geq g$$

and so the Weierstrass gap sequence consists of g elements, say

$$1 = \ell_1 < \ell_2 < \dots < \ell_g \leq 2g - 1.$$

We suppose that the last nongap ℓ_g is equal to $2g - 1$. This means that the semigroup $\mathcal{H}$ is symmetric, or equivalently, it has the property that a positive integer ℓ is a gap if and only if $\ell_g - \ell$ is a nongap, i.e.

$$n_i = 2g - 1 - \ell_{g-i} \quad (i = 0, \dots, g - 1).$$

Thus $n_{g-1} = 2g - 2$ and $H^0(\mathcal{C}, (2g - 2)P)$ is spanned by the g functions $x_{n_0}, x_{n_1}, \dots, x_{n_{g-1}}$. Hence $\dim H^0(\mathcal{C}, (2g - 2)P) = g$ and so $(2g - 2)P$ is a canonical divisor. We also suppose that $\ell_2 = 2$, or equivalently, the Weierstrass point P is nonhyperelliptic. Therefore by a theorem of Rosenlicht the canonical morphism of the complete integral Gorenstein curve $\mathcal{C}$ is an embedding

$$(x_{n_0} : x_{n_1} : \dots : x_{n_{g-1}}) : \mathcal{C} \hookrightarrow \mathbb{P}^{g-1}$$

(see [KM, theorem 4.3]). Thus $\mathcal{C}$ becomes a curve of degree $2g - 2$ in the projective space $\mathbb{P}^{g-1}$ and the integers $\ell_i - 1$ ($i = 1, \dots, g$) are the contact orders of the curve with the hyperplanes at $P = (0 : \dots : 0 : 1)$.

Conversely, the numerical symmetric nonhyperelliptic semigroup $\mathcal{H}$ can be realized as the Weierstrass semigroup of the canonical monomial curve

$$\mathcal{C}^{(0)} := \{(a^{n_0} b^{\ell_g - 1} : a^{n_1} b^{\ell_{g-1} - 1} : \dots : a^{n_{g-1}} b^{\ell_1 - 1}) \mid (a : b) \in \mathbb{P}^1\} \subset \mathbb{P}^{g-1}$$

at the point P . The curve $\mathcal{C}^{(0)}$ is rational and has a unique singularity, namely the unibranch point $(1 : 0 : \dots : 0)$ of multiplicity n_1 and singularity degree g (see [S, p. 190]).

To study relations between generators of the ideal of the canonical curve $\mathcal{C} \subset \mathbb{P}^{g-1}$, we consider the spaces of global sections $H^0(\mathcal{C}, r(2g-2)P)$ of the multi-canonical divisors $r(2g-2)P$, and construct P -hermitian r -monomial bases, i.e. bases consisting of r -monomial expressions in $x_{n_0}, \dots, x_{n_{g-1}}$ whose P -orders are pairwise different (cf. [O]).

Lemma 2.1. *Let τ denote the largest integer such that $n_\tau = \tau n_1$. A P -hermitian basis of $H^0(\mathcal{C}, (4g-4)P)$ is given by the products*

$$\begin{aligned} x_{n_0}x_{n_j} & (j = 0, \dots, g-1) \\ x_{n_i}x_{n_{g-j}} & (i = 1, \dots, \tau, \quad j = 1, \dots, n_1-1) \\ x_{n_i}x_{n_{g-j}} & (i = \tau+1, \dots, g-1, \quad j = 1, \dots, n_i-n_{i-1}) \\ x_{n_{\tau+1}}x_{\ell_{g-\tau-1+k}n_1} & (k = 1, \dots, \tau-1). \end{aligned}$$

Proof. Since by Riemann's theorem the dimension of $H^0(\mathcal{C}, (4g-4)P)$ is equal to $3g-3$, we have only to convince ourselves that the above products are $3g-3$ in number and have pairwise different pole orders at P (see [S, proposition 1.4]). $\square$

For each nongap $s \leq 4g-4$ we list the partitions of s as sums of two nongaps $\leq 2g-2$, say

$$s = a_{si} + b_{si} \quad \text{where} \quad a_{si} \leq b_{si} \quad (i = 0, \dots, \nu_s)$$

and abbreviate

$$a_s := a_{s0} \quad \text{and} \quad b_s := b_{s0}.$$

The $3g-3$ products $x_{a_s}x_{b_s}$ form a P -hermitian basis of the space of global sections $H^0(\mathcal{C}, (4g-4)P)$ of the bi-canonical divisor $(4g-4)P$. If $a_s < a_{s1} < \dots < a_{s\nu_s}$ then it coincides with the basis displayed in Lemma 2.1. However, we will not always work with this assumption.

Lemma 2.2. *For each integer $r \geq 3$, a P -hermitian basis of $H^0(\mathcal{C}, r(2g-2)P)$ is given by the r -monomial expressions*

$$\begin{aligned} x_{n_j} & (j = 0, \dots, g-1) \\ x_{a_s}x_{b_s}x_{n_{g-1}}^i & (i = 0, \dots, r-2, \quad s = 2g, \dots, 4g-4) \\ x_{n_1}x_{2g-n_1}x_{n_{g-2}}x_{n_{g-1}}^i & (i = 0, \dots, r-3) \end{aligned}$$

where the powers of $x_0 = 1$ have been omitted.

Proof. Let n be a nongap not larger than $r(2g-2)$. We look for a monomial expression z_n in $x_{n_1}, \dots, x_{n_{g-1}}$ of degree $\leq r$ with pole divisor nP . We proceed by induction on r . If $n \leq (r-1)(2g-2)$ then we apply the induction hypothesis and pick up the corresponding basis element of $H^0(\mathcal{C}, (r-1)(2g-2)P)$. Thus we may assume $(r-1)(2g-2) < n \leq r(2g-2)$. If $n = 4g-3$ then $r = 3$ and we take $z_n := x_{n_1}x_{2g-n_1}x_{2g-3}$. In the remaining case we apply the induction hypothesis to $n-2g+2$ and take $z_n := z_{n-2g+2}x_{2g-2}$. $\square$

As becomes clear from the preceding proof, we can normalize the functions x_n in a way that for each $r \geq 2$ the functions x_n with $n \leq r(2g-2)$ are the above basis elements of $H^0(\mathcal{C}, r(2g-2)P)$.

Let $I(\mathcal{C}) \subset \mathbf{k}[X_{n_0}, X_{n_1}, \dots, X_{n_{g-1}}]$ be the homogeneous ideal of the canonical curve $\mathcal{C} \subset \mathbb{P}^{g-1}$ i.e.

$$I = \bigoplus_{r=2}^{\infty} I_r(\mathcal{C})$$

where $I_r(\mathcal{C})$ is the vector space of the r -forms that vanish identically on $\mathcal{C}$. As an immediate consequence of the existence of a P -hermitian r -monomial basis for the space of global sections of the multi-canonical divisor $r(2g-2)P$, the homomorphism

$$\mathbf{k}[X_{n_0}, X_{n_1}, \dots, X_{n_{g-1}}]_r \rightarrow H^0(\mathcal{C}, r(2g-2)P)$$

induced by the substitutions $X_{n_i} \mapsto x_{n_i}$ is surjective for each r (as predicted by a theorem of Noether). Thus by Riemann's theorem the codimension of $I_r(\mathcal{C})$ in the $\binom{r+g-1}{r}$ -dimensional vector space $\mathbf{k}[X_{n_0}, X_{n_1}, \dots, X_{n_{g-1}}]_r$ of r -forms is equal to $(2r-1)(g-1)$. In particular, the vector space of quadratic relations has dimension

$$\dim I_2(\mathcal{C}) = \frac{(g-2)(g-3)}{2}.$$

We attach to the variable X_n the weight n . For each integer $r \geq 2$ we fix a vector space Λ_r in $\mathbf{k}[X_{n_0}, X_{n_1}, \dots, X_{n_{g-1}}]_r$ spanned by the lifting of a P -hermitian r -monomial basis of $H^0(\mathcal{C}, r(2g-2)P)$, or equivalently, a vector space spanned by r -monomials in $X_{n_0}, X_{n_1}, \dots, X_{n_{g-1}}$ whose weights are pairwise different and vary through the nongaps $n \leq r(2g-2)$. Since $\Lambda_r \cap I_r(\mathcal{C}) = 0$ and

$$\dim \Lambda_r = \dim H^0(\mathcal{C}, r(2g-2)P) = \text{codim } I_r(\mathcal{C}),$$

we obtain

$$\mathbf{k}[X_{n_0}, X_{n_1}, \dots, X_{n_{g-1}}]_r = \Lambda_r \oplus I_r(\mathcal{C}) \quad \text{for each } r \geq 2.$$

For each nongap $s \leq 4g-4$ and each integer $i = 1, \dots, \nu_s$ we have $x_{a_{si}}x_{b_{si}} \in H^0(\mathcal{C}, sP)$ and so we can write

$$x_{a_{si}}x_{b_{si}} = \sum_{n=0}^s c_{sin} x_{a_n} x_{b_n}$$

where the coefficients c_{sin} are uniquely determined constants and where the dash indicates that the summation index n only varies through nongaps. Multiplying the functions $x_{n_1}, \dots, x_{n_{g-1}}$ by constants we can normalize

$$c_{sin} = 1 \quad \text{whenever } n = s.$$

By construction the quadratic forms

$$F_{si} := X_{a_{si}}X_{b_{si}} - X_{a_s}X_{b_s} - \sum_{n=0}^{s-1} c_{sin} X_{a_n}X_{b_n}$$

vanish identically on the canonical curve $\mathcal{C}$. They are linearly independent, their number is equal to $\binom{g+1}{2} - (3g-3) = \frac{1}{2}(g-2)(g-3)$, and hence they form a basis of the vector space of quadratic relations $I_2(\mathcal{C})$.

We attach to each coefficient c_{sin} the weight $s-n$. If we consider F_{si} as a polynomial expression not only in the variables X_n but also in the coefficients c_{sin} , then it becomes quasi-homogeneous of weight s .

To assure that the canonical ideal $I(\mathcal{C})$ is generated by quadratic relations, we need assumptions on the semigroup $\mathcal{H}$. In the following we will always suppose that the symmetric semigroup $\mathcal{H}$ satisfies

$$3 < n_1 < g \quad \text{and} \quad \mathcal{H} \neq \langle 4, 5 \rangle$$

i.e. $\mathcal{H}$ is different from $\mathbb{N} \setminus \{1, \dots, g-1, 2g-1\}$, $\langle 2, 2g+1 \rangle$, $\langle 3, g+1 \rangle$ and $\langle 4, 5 \rangle$. In the excluded case where $n_1 = 3$, respectively $\mathcal{H} = \langle 4, 5 \rangle$, the canonical curve $\mathcal{C} \subset \mathbb{P}^{g-1}$ is trigonal, respectively isomorphic to a plane quintic, and one may

easily check that the intersection of the quadratic hypersurfaces containing $\mathcal{C}$ is an algebraic surface, or more precisely, a rational normal scroll, respectively the Veronese surface in $\mathbb{P}^5$.

By a theorem of Oliveira [O, theorem 1.7] it follows from the assumptions on the symmetric semigroup $\mathcal{H}$ that $\nu_s \geq 1$ whenever $s = n_i + 2g - 2$ and $i = 0, \dots, g - 3$. Thus we can assume that

$$(a_{s1}, b_{s1}) = (n_i, 2g - 2) \text{ whenever } s = n_i + 2g - 2 \text{ and } i = 0, \dots, g - 3.$$

Moreover, in this case we may assume that $b_s = b_{s0}$ is the largest nongap smaller than $n_i + \ell_{g-i-1}$ (see [S, proposition 1.7]).

The canonical curve $\mathcal{C} \subset \mathbb{P}^{g-1}$ is contained in the $g - 2$ quadratic hypersurfaces $V(F_{n_i+2g-2,1})$ ($i = 0, \dots, g - 3$), which intersect transversely at the Weierstrass point P . Thus in an open neighborhood of P , the curve $\mathcal{C}$ is the intersection of these $g - 2$ hypersurfaces. Therefore the $g - 2$ quadratic forms $F_{n_i+2g-2,1}$ determine uniquely the integral curve $\mathcal{C}$, and in particular determine the remaining $\frac{1}{2}(g-2)(g-5)$ quadratic forms of the basis of $I_2(\mathcal{C})$. We will make this explicit, by constructing syzygies of the canonical curve.

It now follows from the Enriques-Babbage theorem that the canonical integral curve $\mathcal{C}$ is non-trigonal and not isomorphic to a plane quintic (see [ACGH, p. 124]). Moreover by Petri's analysis, at least in the case where $\mathcal{C}$ is smooth, the canonical ideal $I(\mathcal{C})$ is generated by quadratic relations (see [ACGH, p. 131]).

We will give an algorithmic proof that the ideal $I(\mathcal{C})$ of the (possibly nonsmooth) canonical curve $\mathcal{C} \subset \mathbb{P}^{g-1}$ is generated by the quadratic forms F_{si} . We first treat the canonical monomial curve $\mathcal{C}^{(0)}$.

Lemma 2.3. *The ideal $I(\mathcal{C}^{(0)})$ of the canonical monomial curve $\mathcal{C}^{(0)} \subset \mathbb{P}^{g-1}$ is generated by the quadratic binomials*

$$F_{si}^{(0)} := X_{a_{si}} X_{b_{si}} - X_{a_s} X_{b_s}.$$

Proof. Let $I^{(0)}$ be the ideal generated by the binomials $F_{si}^{(0)}$. Since the ideal $I(\mathcal{C}^{(0)})$ is homogeneous and quasi-homogeneous, it is enough to show that a homogeneous and quasi-homogeneous polynomial belongs to $I^{(0)}$ if it belongs to $I(\mathcal{C}^{(0)})$, i.e. if the sum of its coefficients is equal to zero.

In this proof we do not need the assumption $a_{n_i+2g-2,1} = n_i$, and so without loss of generality we can assume that $a_s < a_{s1} < \dots < a_{s\nu_s}$ for each s . We further assume that the vector spaces Λ_r are obtained by lifting the bases displayed in Lemma 2.1 and Lemma 2.2. To apply Gröbner basis techniques, we order the monomials $\prod_{k=0}^{g-1} X_{n_k}^{i_k}$ according to the lexicographic ordering of the vectors $(\sum i_k, \sum n_k i_k, -i_0, -i_{g-1}, \dots, -i_1)$. Using our assumptions on the semigroup $\mathcal{H}$, we can enlarge the basis $\{F_{si}^{(0)}\}$ of $I^{(0)}$ to a Gröbner basis by adding certain sums of cubic binomials $\pm X_n F_{si}^{(0)}$ of the same weight (see [S, pp. 196–198]).

Let F be a homogeneous polynomial of degree r . Dividing F by the Gröbner basis, the division algorithm provides us with a decomposition

$$F = \sum_{si} G_{si} F_{si}^{(0)} + R$$

where $R \in \Lambda_r$ and G_{si} is homogeneous of degree $r - 2$ for each double index si . If F is quasi-homogeneous of weight w , then each G_{si} is quasi-homogeneous of weight $w - s$, and the remainder R is the only monomial in Λ_r of weight w whose coefficient

is equal to the sum of the coefficients of F . Therefore, if $F \in I(\mathcal{C}^{(0)})$ then $R = 0$ and $F \in I^{(0)}$. $\square$

Syzygy Lemma 2.4. *For each of the $\frac{1}{2}(g-2)(g-5)$ quadratic binomials $F_{s'i'}^{(0)}$ different from $F_{n_i+2g-2,1}^{(0)}$ ($i = 0, \dots, g-3$) there is a syzygy of the form*

$$X_{2g-2}F_{s'i'}^{(0)} + \sum_{n, si} \varepsilon_{n, si}^{(s'i')} X_n F_{si}^{(0)} = 0$$

where the coefficients $\varepsilon_{n, si}^{(s'i')}$ are integers equal to 1, -1 or 0 (which will be specified below), and where the sum is taken over the nongaps $n < 2g-2$ and the double indices si with $n+s = 2g-2+s'$.

A weak version of the Syzygy Lemma has been obtained in [S, lemma 2.3] by using Petri's analysis. We will provide an elementary purely combinatorial proof.

Proof. If we put $F := F_{s'i'}^{(0)}$ or $F := -F_{s'i'}^{(0)}$ then we can write

$$F = X_q X_r - X_m X_n$$

where q, r, m and n are nongaps satisfying $q+r = m+n$ and $m < q \leq r < n < 2g-2$. The strict inequality $n < 2g-2$ is due to the assumption $b_{s1} = 2g-2$ whenever $s = n_i + 2g-2$.

If $n+1$ is a gap then, by symmetry, the integer $k := 2g-2-n+r$ is a nongap smaller than $2g-2$, and we have the syzygy

$$X_{2g-2}F + X_n(X_m X_{2g-2} - X_q X_k) - X_q(X_r X_{2g-2} - X_n X_k) = 0$$

where the binomials in the brackets can be written as $F_{si}^{(0)} - F_{sj}^{(0)}$, $F_{si}^{(0)}$ or $-F_{sj}^{(0)}$. Analogously, if $q+1$ is a gap, then the integer $k := 2g-2-q+m$ is a nongap smaller than $2g-2$, and we get the syzygy obtained from the previous one by interchanging n with q and r with m . Now we can assume that $n+1$ and $q+1$ are nongaps. Then we have the syzygy

$$\begin{aligned} X_{2g-2}F + X_m(X_n X_{2g-2} - X_{n+1} X_{2g-3}) = \\ X_{2g-3}(X_{q+1} X_r - X_m X_{n+1}) + X_r(X_q X_{2g-2} - X_{q+1} X_{2g-3}). \end{aligned} \quad \square$$

In the remainder of this section we invert the above considerations. Given a numerical symmetric semigroup $\mathcal{H}$ of genus $g := \#(\mathbb{N} \setminus \mathcal{H})$ satisfying $3 < n_1 < g$ and $\mathcal{H} \neq \langle 4, 5 \rangle$, and given $\frac{1}{2}(g-2)(g-3)$ quadratic forms

$$F_{si} = F_{si}^{(0)} - \sum_{n=0}^{s-1} c_{sin} X_{a_n} X_{b_n},$$

we look for conditions on the constants c_{sin} in order that the intersection of the quadratic hypersurfaces $V(F_{si})$ in $\mathbb{P}^{g-1}$ is a canonical integral Gorenstein curve.

Lemma 2.5. *Let I be the ideal generated by the $\frac{1}{2}(g-2)(g-3)$ quadratic forms F_{si} . Then*

$$\mathbf{k}[X_{n_0}, X_{n_1}, \dots, X_{n_{g-1}}]_r = I_r + \Lambda_r \quad \text{for each } r \geq 2.$$

Proof. Let F be a homogeneous polynomial of degree r and weight w . Let G be its quasi-homogeneous component of weight w , and let R be the only monomial in

Λ_r of weight w whose coefficient is equal to the sum of the coefficients of G . Then $G - R \in I(C^{(0)})$, and so by Lemma 2.3 there is a partition

$$G - R = \sum_{si} G_{si} F_{si}^{(0)}.$$

Replacing each polynomial G_{si} by its homogeneous component of degree $r - 2$, we can assume that G_{si} is homogeneous of degree $r - 2$. For similar reasons, we can assume that each G_{si} is quasi-homogeneous of weight $w - s$. Hence the polynomial $F - \sum G_{si} F_{si} - R$ is homogeneous of degree r and weight $\leq w - 1$. Now the lemma follows by induction on w . $\square$

Replacing in the left hand side of the Syzygy Lemma the binomials $F_{s'i'}^{(0)}$ and $F_{si}^{(0)}$ by the quadratic forms $F_{s'i'}$ and F_{si} we obtain for each of the $\frac{1}{2}(g-2)(g-5)$ double indices $s'i'$ a linear combination of cubic monomials of weight $< s' + 2g - 2$, which by Lemma 2.5, or more precisely, by its algorithmic proof admits a decomposition

$$X_{2g-2} F_{s'i'} + \sum_{nsi} \varepsilon_{nsi}^{(s'i')} X_n F_{si} = \sum_{nsi} \eta_{nsi}^{(s'i')} X_n F_{si} + R_{s'i'}$$

where the sum on the right hand side is taken over the nongaps $n \leq 2g - 2$ and the double indices si with $n + s < s' + 2g - 2$, where the coefficients $\eta_{nsi}^{(s'i')}$ are constants, and where $R_{s'i'}$ is a linear combination of cubic monomials of pairwise different weights $< s' + 2g - 2$.

For each nongap $m < s' + 2g - 2$ we denote by $\varrho_{s'i'm}$ the only coefficient of $R_{s'i'}$ of weight m . Since we do not lose information about the coefficients of $R_{s'i'}$ if we replace the variables X_n by powers t^n of an indeterminate t , it is convenient to consider the polynomial

$$R_{s'i'}(t^{n_0}, t^{n_1}, \dots, t^{n_{g-1}}) = \sum_{m=0}^{s'+2g-3} \varrho_{s'i'm} t^m$$

where the dash indicates that the summation index only varies through nongaps.

Since $R_{s'i'}$ may be obtained as a remainder in a division procedure, we can arrange that the coefficients $\varrho_{s'i'm}$ (respectively, $\eta_{nsi}^{(s'i')}$) are quasi-homogeneous polynomial expressions of weight $s' + 2g - 2 - m$ (respectively, $s' + 2g - 2 - n - s$) in the constants c_{sin} . However, we do not specify a division procedure and we do not postulate that $R_{s'i'}$ belongs to Λ_3 , and so the $\frac{1}{2}(g-2)(g-5)$ remainders $R_{s'i'}$ are not uniquely determinate. In practice, this freedom in the construction of $R_{s'i'}$ allows us to make shortcuts, as we will illustrate in the last section.

Theorem 2.6. *Let $\mathcal{H} \subset \mathbb{N}$ be a numerical symmetric semigroup of genus g satisfying $3 < n_1 < g$ and $\mathcal{H} \neq \langle 4, 5 \rangle$. Then the $\frac{1}{2}(g-2)(g-3)$ quadratic forms $F_{si} = F_{si}^{(0)} - \sum_{n=0}^{s-1} c_{sin} X_{a_n} X_{b_n}$ cut out a canonical integral Gorenstein curve in $\mathbb{P}^{g-1}$ if and only if the coefficients c_{sin} satisfy the quasi-homogeneous equations $\varrho_{s'i'm} = 0$. In this case, the point $P = (0 : \dots : 0 : 1)$ is a smooth point of the canonical curve with Weierstrass semigroup $\mathcal{H}$.*

Proof. We first assume that the $\frac{1}{2}(g-2)(g-3)$ quadratic forms F_{si} cut out a canonical integral Gorenstein curve in $\mathbb{P}^{g-1}$. Since each $R_{s'i'}$ belongs to the ideal I generated by the quadratic forms F_{si} , we have $R_{s'i'}(x_{n_0}, x_{n_1}, \dots, x_{n_{g-1}}) = 0$ for

each double index $s'i'$. On the other hand

$$R_{s'i'}(x_{n_0}, \dots, x_{n_{g-1}}) = \sum_{m=0}^{s'+2g-3} \varrho_{s'i'm} z_{s'i'm}$$

where each $z_{s'i'm}$ is a monomial expression of weight m in the projective coordinate functions $x_{n_0}, \dots, x_{n_{g-1}}$, and hence has the pole divisor mP . Thus the coefficients $\varrho_{s'i'm}$ are zero.

Let us now assume that the coefficients c_{sin} satisfy the equations $\varrho_{s'i'm} = 0$. Since the $g-2$ quadratic hypersurfaces $V(F_{n_i+2g-2,1}) \subset \mathbb{P}^{g-1}$ ($i = 0, \dots, g-3$) intersect transversely at P , their intersection has a unique irreducible component that passes through P , and this component is a projective integral algebraic curve, say $\mathcal{C}$, which is smooth at P and whose tangent line at P is the intersection of their tangent hyperplanes $V(X_{n_i})$ ($i = 0, \dots, g-3$).

Let $y_{n_0}, \dots, y_{n_{g-1}}$ be the projective coordinate functions of $\mathcal{C}$ with $y_{n_{g-1}} = 1$. Since $n_{g-1} - n_{g-2} = \ell_2 - \ell_1 = 1$, we conclude that $t := y_{n_{g-2}}$ is a local parameter of $\mathcal{C}$ at P , and $y_{n_0}, \dots, y_{n_{g-3}}$ are power series in t of order > 1 . More precisely, comparing coefficients in the $g-2$ equations $F_{n_i+2g-2,1}(y_{n_0}, \dots, y_{n_{g-1}}) = 0$ we obtain

$$y_{n_i} = t^{n_{g-1}-n_i} + \dots = t^{\ell_{g-i}-1} + \dots \quad (i = 0, \dots, g-1)$$

where the dots stand for terms of higher orders. Thus the g integers $\ell_i - 1$ ($i = 1, \dots, g$) are the contact orders of the curve $\mathcal{C} \subset \mathbb{P}^{g-1}$ with the hyperplanes. In particular, the curve $\mathcal{C}$ is not contained in any hyperplane.

Since by assumption the $\frac{1}{2}(g-2)(g-5)$ remainders $R_{s'i'}$ are equal to zero, we have the syzygies

$$X_{2g-2}F_{s'i'} + \sum_{nsi} \varepsilon_{nsi}^{(s'i')} X_n F_{si} - \sum_{nsi} \eta_{nsi}^{(s'i')} X_n F_{si} = 0.$$

Replacing the variables $X_{n_0}, \dots, X_{n_{g-1}}$ by the projective coordinate functions $y_{n_0}, \dots, y_{n_{g-1}}$, we get a system of $\frac{1}{2}(g-2)(g-5)$ linear homogeneous equations in the $\frac{1}{2}(g-2)(g-5)$ functions $F_{s'i'}(y_{n_0}, \dots, y_{n_{g-1}})$ with coefficients in the domain $\mathbf{k}[[t]]$ of formal power series. Since the triple indices nsi of the coefficients $\varepsilon_{nsi}^{(s'i')}$, respectively $\eta_{nsi}^{(s'i')}$, satisfy $n < 2g-2$ and $n+s = s'+2g-2$, respectively $n \leq 2g-2$ and $n+s < s'+2g-2$, we conclude that the diagonal entries of the matrix of the system have constant terms equal to 1, while the remaining entries have positive orders. Thus the matrix is invertible, and so the equation $F_{si}(y_{n_0}, \dots, y_{n_{g-1}}) = 0$ holds for each double index si . This shows that $I \subseteq I(\mathcal{C})$ where I is the ideal generated by the $\frac{1}{2}(g-2)(g-3)$ quadratic forms F_{si} .

By Lemma 2.5 we have $\text{codim } I_r \leq \dim \Lambda_r$ for each $r \geq 2$. On the other hand, since $I_r(\mathcal{C}) \cap \Lambda_r = 0$, we deduce $\dim \Lambda_r \leq \text{codim } I_r(\mathcal{C})$. Since $I \subseteq I(\mathcal{C})$ we obtain

$$\text{codim } I_r(\mathcal{C}) = \text{codim } I_r = \dim \Lambda_r = (2g-2)r + 1 - g.$$

Thus $I(\mathcal{C}) = I$ and the curve $\mathcal{C} \subset \mathbb{P}^{g-1}$ has Hilbert polynomial $(2g-2)r + 1 - g$. Hence $\mathcal{C}$ has degree $2g-2$ and arithmetic genus equal to g .

Intersecting the curve $\mathcal{C} \subset \mathbb{P}^{g-1}$ with the hyperplane $V(X_{n_{g-1}})$ we get the divisor $D := (2g-2)P$ of degree $2g-2$, whose complete linear system $|D|$ has dimension $g-1$. Hence by the Riemann-Roch theorem for complete integral (not necessarily smooth) curves the Cartier divisor D is canonical, and $\mathcal{C}$ is a canonical Gorenstein curve. $\square$

To normalize the coefficients c_{sni} of the quadratic forms F_{si} , we notice that the coordinate functions x_n ($n \in \mathcal{H}$, $n \leq 2g - 2$) are not uniquely determined by their pole divisors nP . We assume that the characteristic of the constant field $\mathbf{k}$ is zero or a prime not dividing any of the differences $n - m$, where m and n are nongaps satisfying $m < n \leq 2g - 2$. Transforming

$$X_n \mapsto X_n + \sum_{m=0}^{n-1} d_{nm} X_m$$

where the coefficients d_{nm} are constants, we can normalize $\frac{1}{2}g(g-1)$ of the coefficients c_{sin} to be zero. More precisely, for each positive integer w the number of the coefficients c_{sin} of weight $s - n = w$ that can be normalized is equal to the number of nongaps m such that $m + w$ is a nongap $\leq 2g - 2$ (see [S, proposition 3.1]).

Due to these normalizations and the normalizations of the coefficients of weight zero (i.e. $c_{sis} = 1$), the only freedom left to us is to transform $x_{n_i} \mapsto c^{n_i} x_{n_i}$ ($i = 0, \dots, g - 1$) for some $c \in \mathbb{G}_m(\mathbf{k}) = \mathbf{k}^*$. We have shown:

Theorem 2.7. *Let $\mathcal{H} \subset \mathbb{N}$ be a symmetric semigroup of genus $g := \#(\mathbb{N} \setminus \mathcal{H})$ satisfying $3 < n_1 < g$ and $\mathcal{H} \neq \langle 4, 5 \rangle$. The isomorphism classes of the pointed complete integral Gorenstein curves with Weierstrass semigroup $\mathcal{H}$ correspond bijectively to the orbits of the $\mathbb{G}_m(\mathbf{k})$ -action*

$$(c, \dots, c_{sin}, \dots) \mapsto (\dots, c^{s-n} c_{sin}, \dots)$$

on the affine quasi-cone of the vectors whose coordinates are the coefficients c_{sin} of the normalized quadratic forms F_{si} that satisfy the quasi-homogeneous equations $\varrho_{s'i'm} = 0$.

3. THE METHOD

We start this section by describing a variant of the construction of the remainders $R_{s'i'}$. Instead of making induction on the weights in the variables X_n , we proceed by induction on the degrees in the constants c_{sin} . We choose for each pair $s'i'$ and each nongap $m \leq 6g - 6$ a cubic monic monomial $Z_{s'i'm}$ of weight m in $X_{n_0}, \dots, X_{n_{g-1}}$ e.g. the unique monic monomial in Λ_3 of weight m . By the Syzygy Lemma the polynomial

$$G_{s'i'}^{(1)} := X_{2g-2} F_{s'i'} + \sum_{nsi} \varepsilon_{nsi}^{(s'i')} X_n F_{si} ,$$

is a sum of cubic monomials of weight $< s' + 2g - 2$, whose coefficients are homogeneous of degree 1 in the constants c_{sin} . For each nongap $m < s' + 2g - 2$ let $\varrho_{s'i'm}^{(1)}$ be the sum of the coefficients of its terms of weight m . We notice that $\varrho_{s'i'm}^{(1)}$ is homogeneous of degree 1 and quasi-homogeneous of weight $s' + 2g - 2 - m$ in the constants c_{sin} . By Lemma 2.3, or more precisely, by its algorithmic proof, we get an equation

$$G_{s'i'}^{(1)} - \sum_m \varrho_{s'i'm}^{(1)} Z_{s'i'm} = \sum_{sim} \eta_{sim}^{(s'i')(1)} X_m F_{si}^{(0)}$$

where the coefficients $\eta_{sim}^{(s'i')(1)}$ are homogeneous of degree 1 and quasi-homogeneous of weight $s' + 2g - 2 - m - s > 0$ in the constants c_{sin} . Next we consider the polynomial

$$G_{s'i'}^{(2)} := G_{s'i'}^{(1)} - \sum_m \varrho_{s'i'm}^{(1)} Z_{s'i'm} - \sum_{sim} \eta_{sim}^{(s'i')(1)} X_m F_{si} = \sum_{sim} \eta_{sim}^{(s'i')(1)} X_m (F_{si}^{(0)} - F_{si}) ,$$

which is a sum of cubic monomials of weight $< s' + 2g - 2$, whose coefficients are homogeneous of degree 2 in the constants c_{sin} . Let $\varrho_{s'i'm}^{(2)}$ be the sum of the coefficients of its terms of weight m . Proceeding in this way, we obtain the coefficients of the remainders $R_{s'i'}$ as sums of its homogeneous components:

$$\varrho_{s'i'm} = \varrho_{s'i'm}^{(1)} + \varrho_{s'i'm}^{(2)} + \dots$$

The procedure stops after less than $s' + 2g - 2$ steps, because the degrees in the constants are not larger than $s' + 2g - 2 - m$.

To simplify the notation, we put the coefficients c_{sin} of the normalized quadratic forms F_{si} in any order, rewrite them as a finite sequence $c_0, c_1, \dots, c_\nu$ and denote their weights by $w_0, w_1, \dots, w_\nu$. We also rewrite the equations $\varrho_{s'i'm} = 0$ by using simple indices

$$\varrho_j(c_0, \dots, c_\nu) = 0 \quad (j = 1, \dots, \mu)$$

where each ϱ_j is a quasi-homogeneous polynomial in $\nu + 1$ variables, whose homogeneous components $\varrho_j^{(1)}, \varrho_j^{(2)}, \dots$ have been determined by the above procedure. There is an action of the multiplicative group $\mathbb{G}_m := \mathbf{k}^*$ on the weighted vector space $\mathbf{k}^{\nu+1}$ defined as follows

$$(c_0, \dots, c_\nu) \mapsto (c^{w_0} c_0, \dots, c^{w_\nu} c_\nu) \quad \text{for each } c \in \mathbf{k}^* .$$

By Theorem 2.7 the isomorphism classes of the pointed complete integral Gorenstein curves with Weierstrass semigroup $\mathcal{H}$ correspond bijectively to the orbits of the $\mathbb{G}_m$ -action on the affine quasi-cone

$$\{(c_0, \dots, c_\nu) \in \mathbf{k}^{\nu+1} \mid \varrho_j(c_0, \dots, c_\nu) = 0, \quad j = 1, \dots, \mu\} .$$

Linearizing the μ equations we obtain a vector space

$$\mathcal{T}_{\mathcal{H}} := \{(c_0, \dots, c_\nu) \in \mathbf{k}^{\nu+1} \mid \varrho_j^{(1)}(c_0, \dots, c_\nu) = 0, \quad j = 1, \dots, \mu\}$$

equipped with the induced $\mathbb{G}_m$ -action. To relate this with Pinkham's work [P], we notice that

$$\mathcal{T}_{\mathcal{H}} \cong T_{\mathbf{k}[\mathcal{H}]|\mathbf{k}}^{1,-} ,$$

that is, $\mathcal{T}_{\mathcal{H}}$ is isomorphic to the negatively graded part of the first cohomology group $T_{\mathbf{k}[\mathcal{H}]|\mathbf{k}}^1$ of the cotangent complex of the semigroup algebra $\mathbf{k}[\mathcal{H}]|\mathbf{k}$, where the signs of the weights have been inverted (see [S, p. 212]). To determine the vector space $\mathcal{T}_{\mathcal{H}}$, we eliminate successively coefficients c_i from the linear equation $\varrho_j^{(1)}(c_0, \dots, c_\nu) = 0$. Let r be the smallest integer such that, after an eventual permutation, the coefficients c_i with $i > r$ can be eliminated from the linear equations. Then $\mathcal{T}_{\mathcal{H}}$ becomes the $(r + 1)$ -dimensional weighted vector space

$$\mathcal{T}_{\mathcal{H}} = \{(c_0, \dots, c_r) \mid c_0, \dots, c_r \in \mathbf{k}\}$$

with the weight sequence $w_0, \dots, w_r$. If a coefficient c_i with $i > r$ has been eliminated from the linear equations $\varrho_j^{(1)}(c_0, \dots, c_\nu) = 0$, then it can also be eliminated from the corresponding polynomial equation $\varrho_j(c_0, \dots, c_\nu) = 0$, because by the quasi-homogeneity it does not occur in higher order terms of the same equation. Entering with these solutions into the remaining polynomials, after $\nu - r$ steps we obtain $\mu - (\nu - r)$ quasi-homogeneous polynomials h_j in only $r + 1$ variables, which define an affine quasi-cone

$$\mathcal{X}_{\mathcal{H}} := \{(c_0, \dots, c_r) \in \mathbf{k}^{r+1} \mid h_j(c_0, \dots, c_r) = 0 \quad \forall j\} \subseteq \mathcal{T}_{\mathcal{H}} .$$

By Theorem 2.7 the quotient $\mathcal{X}_{\mathcal{H}}/\mathbb{G}_m$ parametrizes the isomorphism classes of the pointed complete integral Gorenstein curves with Weierstrass semigroup $\mathcal{H}$. Thus the compactified moduli space $\overline{\mathcal{M}}_{\mathcal{H}}$ can be identified with the quotient of the punctured quasi-cone $\mathcal{X}_{\mathcal{H}} \setminus \{\mathfrak{o}\}$ by the $\mathbb{G}_m$ -action

$$\overline{\mathcal{M}}_{\mathcal{H}} \cong \mathbb{P}(\mathcal{X}_{\mathcal{H}}) := (\mathcal{X}_{\mathcal{H}} \setminus \{\mathfrak{o}\})/\mathbb{G}_m \subseteq \mathbb{P}(\mathcal{T}_{\mathcal{H}})$$

i.e. $\overline{\mathcal{M}}_{\mathcal{H}}$ is isomorphic to the closed subset of the r -dimensional weighted projective space $\mathbb{P}(\mathcal{T}_{\mathcal{H}}) = \mathbb{P}_{(w_0, \dots, w_r)}^r$ cut out by the quasi-homogeneous equations $h_j = 0$. Here the pointed monomial curve, which corresponds to the vertex $\mathfrak{o}$ of the affine quasi-cone $\mathcal{X}_{\mathcal{H}}$, has been excluded, and can be viewed as the "improper point" of $\overline{\mathcal{M}}_{\mathcal{H}}$.

By construction, the linear components $h_j^{(1)}$ of the quasi-homogeneous polynomials h_j are equal to zero. The quadratic components $h_j^{(2)}$ can be easily computed, by solving the linear equations $\varrho_j^{(1)} = 0$ for $c_{r+1}, \dots, c_{\nu}$, and entering into the quadratic expressions $\varrho_j^{(2)}(c_0, \dots, c_r, c_{r+1}, \dots, c_{\nu})$. We approximate $\mathcal{X}_{\mathcal{H}}$ at the vertex by the affine quadratic quasi-cone

$$\mathcal{Q}_{\mathcal{H}} := \{(c_0, \dots, c_r) \in \mathbf{k}^{r+1} \mid h_j^{(2)}(c_0, \dots, c_r) = 0 \ \forall j\} \subseteq \mathcal{T}_{\mathcal{H}}.$$

Theorem 3.1.

$$\dim \overline{\mathcal{M}}_{\mathcal{H}} \leq \dim \mathbb{P}(\mathcal{Q}_{\mathcal{H}}) \quad \text{i.e.} \quad \dim \overline{\mathcal{M}}_{\mathcal{H}} < \dim \mathcal{Q}_{\mathcal{H}}.$$

Proof. Since $\overline{\mathcal{M}}_{\mathcal{H}} \cong (\mathcal{X}_{\mathcal{H}} \setminus \{\mathfrak{o}\})/\mathbb{G}_m$ we have

$$\dim \overline{\mathcal{M}}_{\mathcal{H}} = \dim \mathcal{X}_{\mathcal{H}} - 1.$$

Due to quasi-homogeneity each irreducible component of the affine quasi-cone $\mathcal{X}_{\mathcal{H}}$ passes through the vertex $\mathfrak{o}$. Since the dimension of an integral variety coincides with its local dimension at any point (see [E, §13, Theorem A]), we conclude that the dimension of $\mathcal{X}_{\mathcal{H}}$ is equal to its local dimension at the vertex

$$\dim \mathcal{X}_{\mathcal{H}} = \dim_{\mathfrak{o}} \mathcal{X}_{\mathcal{H}}$$

which is equal to the Krull dimension of the corresponding local ring

$$\dim_{\mathfrak{o}} \mathcal{X}_{\mathcal{H}} = \dim \mathcal{O}_{\mathcal{X}_{\mathcal{H}}, \mathfrak{o}}.$$

Since by local algebra the dimension of a local ring is equal to the dimension of its associated algebra ([M, theorem 13.9]) we have

$$\dim \mathcal{O}_{\mathcal{X}_{\mathcal{H}}, \mathfrak{o}} = \dim G(\mathcal{O}_{\mathcal{X}_{\mathcal{H}}, \mathfrak{o}})$$

where

$$G(\mathcal{O}_{\mathcal{X}_{\mathcal{H}}, \mathfrak{o}}) := \bigoplus_{i=1}^{\infty} (\mathfrak{m}_{\mathcal{X}_{\mathcal{H}}, \mathfrak{o}})^i / (\mathfrak{m}_{\mathcal{X}_{\mathcal{H}}, \mathfrak{o}})^{i+1}.$$

Geometrically, this means that the local dimension is equal to the dimension of the tangent cone

$$\dim_{\mathfrak{o}} \mathcal{X}_{\mathcal{H}} = \dim C_{\mathfrak{o}}(\mathcal{X}_{\mathcal{H}})$$

and can be seen by noticing that the projectivization of the tangent cone is an effective Cartier divisor in the local blowup. Since by construction the quadratic quasi-cone $\mathcal{Q}_{\mathcal{H}}$ contains the tangent cone $C_{\mathfrak{o}}(\mathcal{X}_{\mathcal{H}})$, we conclude

$$\dim C_{\mathfrak{o}}(\mathcal{X}_{\mathcal{H}}) \leq \dim \mathcal{Q}_{\mathcal{H}}.$$

□

It is much less expensive to obtain the equations and the dimension of the quadratic quasi-cone $\mathcal{Q}_{\mathcal{H}}$ than the ones of the moduli space $\mathcal{M}_{\mathcal{H}}$. With Theorem 3.1 we get an implementable method to produce an upper bound for the dimension of the moduli space of curves with a prescribed symmetric Weierstrass semigroup. Below we summarize in a table some examples we calculated on a computer:

$\mathcal{H}$	g	E-H	$\dim \overline{\mathcal{M}}_{\mathcal{H}}$	$\dim \mathbb{P}(\mathcal{Q}_{\mathcal{H}})$	Del	$\dim T_{\mathbf{k}[\mathcal{H}] \mathbf{k}}^{1,-}$
$\langle 6, 8, 9, 10, 11 \rangle$	7	12	13	13	13	17
$\langle 6, 8, 10, 11, 13 \rangle$	8	12	14	14	15	18
$\langle 7, 9, 10, 11, 12, 13 \rangle$	8	14	15	15	15	23
$\langle 6, 8, 10, 13, 15 \rangle$	9	11	15	15	17	19
$\langle 6, 9, 10, 13, 14 \rangle$	9	12	15	15	17	19
$\langle 6, 14, 15, 16, 17 \rangle$	13	11	20	21	25	28

Here E-H stands for the lower bound $3g - 2 - w(\mathcal{H})$ of Eisenbud-Harris and Del for Deligne's upper bound $2g - 1$.

By the jacobian criterion and classical elimination theory, the moduli space $\mathcal{M}_{\mathcal{H}}$ is an open subspace of $\overline{\mathcal{M}}_{\mathcal{H}}$. If the symmetric semigroup $\mathcal{H}$ is generated by 4 elements, say $\mathcal{H} = \langle m, m_1, m_2, m_3 \rangle$, then by using Pinkham's equivariant deformation theory [P], complete intersection theory and a quasi-homogeneous version of Buchsbaum-Eisenbud's structure theorem for Gorenstein ideals of codimension 3 (see [BE, p. 466]), one can deduce that the affine monomial curve $\text{Spec } \mathbf{k}[\mathcal{H}] = \text{Spec } \mathbf{k}[t^m, t^{m_1}, t^{m_2}, t^{m_3}]$ can be negatively smoothed without any obstructions (see [B], [W1] [W2, Satz 7.1]), hence $\dim \mathcal{M}_{\mathcal{H}} = \dim \mathbb{P}(T_{\mathbf{k}[\mathcal{H}]|\mathbf{k}}^{1,-})$, and therefore

$$\overline{\mathcal{M}}_{\mathcal{H}} = \mathbb{P}(T_{\mathbf{k}[\mathcal{H}]|\mathbf{k}}^{1,-})$$

and so $\mathcal{M}_{\mathcal{H}}$ is a dense open subvariety of $\mathbb{P}(T_{\mathbf{k}[\mathcal{H}]|\mathbf{k}}^{1,-})$. However, if $\mathcal{H}$ is generated by more than four elements, then $\overline{\mathcal{M}}_{\mathcal{H}}$ tends to be a proper subspace of $\mathbb{P}(T_{\mathbf{k}[\mathcal{H}]|\mathbf{k}}^{1,-})$, as documented in the above table and discussed in the next section.

4. WORKING WITH A FAMILY OF SYMMETRIC SEMIGROUPS

In this section we apply our method to a family of symmetric semigroups of multiplicity 6 minimally generated by five elements. For each positive integer τ we consider the semigroup

$$\begin{aligned} \mathcal{H} &= \langle 6, 2 + 6\tau, 3 + 6\tau, 4 + 6\tau, 5 + 6\tau \rangle \\ &= 6\mathbb{N} \sqcup \bigsqcup_{j=2}^5 (j + 6\tau + 6\mathbb{N}) \sqcup (7 + 12\tau + 6\mathbb{N}). \end{aligned}$$

Counting the numbers of gaps and picking up the largest nongap we obtain

$$g = 6\tau + 1 \quad \text{and} \quad \ell_g = 12\tau + 1 = 2g - 1$$

and so the semigroup is symmetric.

Let $\mathcal{C}$ be a complete integral Gorenstein curve, and P a smooth point of $\mathcal{C}$ whose Weierstrass semigroup is equal to $\mathcal{H}$. As in Section 2 we choose for each nongap $n \in \mathcal{H}$ a rational function x_n on $\mathcal{C}$ with pole divisor nP . We abbreviate

$$x := x_6 \quad \text{and} \quad y_j := x_{j+6\tau} \quad (j = 2, 3, 4, 5)$$

and normalize

$$x_{6i} = x^i, \quad x_{j+6\tau+6i} = x^i y_j \quad \text{and} \quad x_{7+12\tau+6i} = x^i y_2 y_5$$

for each $i \geq 0$ and $j = 2, 3, 4, 5$. Now the P -hermitian basis $\{x_{n_0}, x_{n_1}, \dots, x_{n_{g-1}}\}$ of the vector space $H^0(\mathcal{C}, (2g-2)P)$ of global sections of the canonical divisor $(2g-2)P = 12\tau P$ consists of the products

$$x^0, \dots, x^{2\tau} \quad \text{and} \quad x^0 y_j, \dots, x^{\tau-1} y_j \quad (j = 2, 3, 4, 5).$$

Since $\ell_2 = 2$, the complete integral Gorenstein curve $\mathcal{C}$ is nonhyperelliptic, and so it can be identified with its image under the canonical embedding

$$(x_{n_0} : x_{n_1} : \dots : x_{n_{g-1}}) : \mathcal{C} \hookrightarrow \mathbb{P}^{g-1}.$$

The projection map

$$(1 : x : y_2 : y_3 : y_4 : y_5) : \mathcal{C} \rightarrow \mathbb{P}^5$$

defines an isomorphism of the canonical curve $\mathcal{C} \subset \mathbb{P}^{g-1}$ onto a curve $\mathcal{D} \subset \mathbb{P}^5$ of degree $6\tau + 5$. The image $Q := (0 : 0 : 0 : 0 : 0 : 1) \in \mathcal{D}$ of the distinguished Weierstrass point $P := (0 : \dots : 0 : 1) \in \mathcal{C}$ is the only point of $\mathcal{D}$ that does not lie on the affine space $\mathbb{A}^5 \subset \mathbb{P}^5$ of the points with nonzero first coordinate.

To study quadratic relations of the canonical curve $\mathcal{C} \subset \mathbb{P}^{g-1}$, we consider the space of global sections of the bicanonical divisor $(4g-4)P = 24\tau P$. The P -hermitian basis $\{x_n \mid n \in \mathcal{H}, n \leq 4g-4\}$ of $H^0(\mathcal{C}, (4g-4)P)$ consists of the $3g-3$ functions

$$\begin{aligned} x^i & \quad (i = 0, 1, \dots, 4\tau) \\ x^i y_j & \quad (i = 0, 1, \dots, 3\tau-1, \quad j = 2, 3, 4, 5) \\ x^i y_2 y_5 & \quad (i = 0, 1, \dots, 2\tau-2) \end{aligned}$$

which can be written as quadratic monomial expressions in the projective coordinate functions $x_{n_0}, x_{n_1}, \dots, x_{n_{g-1}}$. Let X, Y_2, Y_3, Y_4 and Y_5 be indeterminates attached with the weights $6, 2+6\tau, 3+6\tau, 4+6\tau$ and $5+6\tau$, respectively. Having in mind the normalizations of the functions x_n , we define for each nongap $n \in \mathcal{H}$ a monomial Z_n of weight n as follows

$$Z_{6i} := X^i, \quad Z_{j+6\tau+6i} := X^i Y_j \quad \text{and} \quad Z_{7+12\tau+6i} := X^i Y_2 Y_5.$$

Multiplying the functions x, y_2, y_3, y_4, y_5 by suitable constants, and writing the nine products $y_i y_j$ with $(i, j) \neq (2, 5)$ as a linear combinations of the basis elements, we obtain nine polynomials in the indeterminates X, Y_2, Y_3, Y_4, Y_5 that vanish identically on the affine curve $\mathcal{D} \cap \mathbb{A}^5 = \mathcal{D} \setminus \{Q\}$, say

$$\begin{aligned} G_i &= G_i^{(0)} - \sum_{j=1}^{12\tau+i} g_{ij} Z_{12\tau+i-j} \quad (i = 4, \dots, 8) \\ F_i &= F_i^{(0)} - \sum_{j=1}^{12\tau+i} f_{ij} Z_{12\tau+i-j} \quad (i = 6, 8, 9, 10) \end{aligned}$$

where

$$\begin{aligned} G_4^{(0)} &= Y_2^2 - X^\tau Y_4 & G_5^{(0)} &= Y_2 Y_3 - X^\tau Y_5 & G_6^{(0)} &= Y_3^2 - X^{2\tau+1} \\ F_6^{(0)} &= Y_2 Y_4 - X^{2\tau+1} & G_7^{(0)} &= Y_3 Y_4 - Y_2 Y_5 & G_8^{(0)} &= Y_4^2 - X^{\tau+1} Y_2 \\ F_8^{(0)} &= Y_3 Y_5 - X^{\tau+1} Y_2 & F_9^{(0)} &= Y_4 Y_5 - X^{\tau+1} Y_3 & F_{10}^{(0)} &= Y_5^2 - X^{\tau+1} Y_4 \end{aligned}$$

and where the summation index j only varies through integers with $12\tau+i-j \in \mathcal{H}$.

Lemma 4.1. *The ideal of the affine curve $\mathcal{D} \cap \mathbb{A}^5$ is generated by the nine polynomials G_i ($i = 4, \dots, 8$) and F_i ($i = 6, 8, 9, 10$).*

Proof. It follows by induction on descending degrees in $Y_2, \dots, Y_5$ that, modulo the ideal generated by the nine polynomials, each polynomial in $X, Y_2, \dots, Y_5$ is congruent to a polynomial whose terms are not divisible by the nine products $Y_i Y_j$ with $(i, j) \neq (2, 5)$ i.e. which is a linear combination of the monomials Z_n of pairwise different weights $n \in \mathcal{H}$. Such a linear combination $\sum c_n Z_n$ vanishes identically on the affine curve $\mathcal{D} \cap \mathbb{A}^5$ if and only if the corresponding linear combination $\sum c_n x_n$ of the rational functions $x_n \in \mathbf{k}(\mathcal{C})$ is equal to zero i.e. $c_n = 0$ for each $n \in \mathcal{H}$. $\square$

If $\mathcal{C}$ is equal to the canonical monomial curve $\mathcal{C}^{(0)} \subset \mathbb{P}^{g-1}$, then the coefficients g_{ij} and f_{ij} are equal to zero, and so the ideal of the affine monomial curve

$$\mathcal{D}^{(0)} \cap \mathbb{A}^5 = \{(c^6, c^{2+6\tau}, c^{3+6\tau}, c^{4+6\tau}, c^{5+6\tau}) \mid c \in \mathbf{k}\}$$

is generated by the nine quasi-homogeneous binomials $G_i^{(0)}$ ($i = 4, \dots, 8$) and $F_i^{(0)}$ ($i = 6, 8, 9, 10$).

In order to normalize some of the coefficients g_{ij} and f_{ij} , we notice that we have just the freedom to transform

$$\begin{aligned} x &\mapsto c^6 x + c_6 \\ y_2 &\mapsto c^{2+6\tau} y_2 + \sum_{i=0}^{\tau} c_{2+6i} x^{\tau-i} \\ y_3 &\mapsto c^{3+6\tau} y_3 + c_1 y_2 + \sum_{i=0}^{\tau} c_{3+6i} x^{\tau-i} \\ y_4 &\mapsto c^{4+6\tau} y_4 + c'_1 y_3 + c'_2 y_2 + \sum_{i=0}^{\tau} c_{4+6i} x^{\tau-i} \\ y_5 &\mapsto c^{5+6\tau} y_5 + c''_1 y_4 + c''_2 y_3 + c''_3 y_2 + \sum_{i=0}^{\tau} c_{5+6i} x^{\tau-i} \end{aligned}$$

where $c \neq 0$, c_j , c'_j and c''_j are constants. We suppose that the characteristic of the constant field is different from two and three. Then we can normalize

$$f_{81} = g_{82} = f_{92} = f_{10,3} = 0$$

(which are the only coefficients with $i - j \equiv 1 \pmod{6}$),

$$g_{41} = g_{42} = g_{46} = 0$$

and

$$f_{6,2+6i} = f_{8,3+6i} = f_{9,4+6i} = f_{9,5+6i} = 0 \quad (i = 0, \dots, \tau).$$

Now the isomorphism class of the pointed Gorenstein curve $(\mathcal{C}, P)$ determines uniquely the coefficients up to the following $\mathbb{G}_m$ -action

$$g_{ij} \mapsto c^j g_{ij} \quad \text{and} \quad f_{ij} \mapsto c^j f_{ij} \quad \text{where} \quad c \in \mathbb{G}_m = \mathbf{k}^*.$$

We attach to the coefficients g_{ij} and f_{ij} the weight j . They have to satisfy certain quasi-homogeneous polynomial equations, which we will deduce from the syzygies of the affine curve $\mathcal{D} \cap \mathbb{A}^5$.

By applying the Syzygy Lemma we conclude that the five quasi-homogeneous binomials

$$Z_{2g-2} G_4^{(0)}, Z_{2g-2} G_5^{(0)}, Z_{2g-2} (G_6^{(0)} - F_6^0), Z_{2g-2} G_7^{(0)} \quad \text{and} \quad Z_{2g-2} (G_8^{(0)} - F_8^{(0)})$$

of weight $2g - 2 + i + 12\tau = 24\tau + i$ where $i = 4, 5, 6, 7$ and 8 , respectively, are linear combinations of the binomials $Z_n G_j^{(0)}$ ($j = 4, \dots, 8$) and $Z_n F_j^{(0)}$ ($j = 6, 8, 9, 10$)

with $n = 2g - 2 + i - j < 2g - 2$ (and therefore $X^{\tau-1}$ divides Z_n). More explicitly, we write up five syzygies of the affine monomial curve $\mathcal{D}^{(0)} \cap \mathbb{A}^5$:

$$\begin{aligned} X^{\tau+1}G_4^{(0)} - Y_4F_6^{(0)} + Y_2G_8^{(0)} &= 0 \\ X^{\tau+1}G_5^{(0)} - Y_5F_6^{(0)} + Y_2F_9^{(0)} &= 0 \\ X^{\tau+1}G_6^{(0)} - X^{\tau+1}F_6^{(0)} - Y_4F_8^{(0)} + Y_3F_9^{(0)} &= 0 \\ X^{\tau+1}G_7^{(0)} - Y_5F_8^{(0)} + Y_3F_{10}^{(0)} &= 0 \\ X^{\tau+1}G_8^{(0)} - X^{\tau+1}F_8^{(0)} - Y_5F_9^{(0)} + Y_4F_{10}^{(0)} &= 0 \end{aligned}$$

Remark 4.2. *Actually, the Syzygy Lemma assures the existence of certain $\frac{1}{2}(g-2)(g-5)$ syzygies of the canonical monomial curve $\mathcal{C}^{(0)} \subset \mathbb{P}^{g-1}$. However, using the equations $Z_{n+6} = X \cdot Z_n$ and factoring out powers of X we can reduce to the five syzygies listed above.*

The five syzygies of the monomial curve $\mathcal{D}^{(0)} \cap \mathbb{A}^5$ give rise to five syzygies of the curve $\mathcal{D} \cap \mathbb{A}^5$:

$$\begin{aligned} X^{\tau+1}G_4 - Y_4F_6 + Y_2G_8 &= \\ \sum_{i=0}^{\tau} X^{\tau-i} (f_{6,1+6i}F_9 + f_{6,3+6i}G_7 + (f_{6,4+6i} - g_{8,4+6i})F_6 - g_{8,5+6i}G_5 - g_{8,6+6i}G_4) &= \\ X^{\tau+1}G_5 - Y_5F_6 + Y_2F_9 &= \\ \sum_{i=0}^{\tau} X^{\tau-i} (f_{6,1+6i}F_{10} + f_{6,3+6i}F_8 - f_{9,6+6i}G_5 - f_{9,7+6i}G_4) &= \\ X^{\tau+1}G_6 - X^{\tau+1}F_6 - Y_4F_8 + Y_3F_9 &= \\ \sum_{i=0}^{\tau} X^{\tau-i} (f_{8,4+6i}G_8 + f_{8,5+6i}G_7 - f_{9,6+6i}G_6 + f_{8,6+6i}F_6 - f_{9,7+6i}G_5) &= \\ X^{\tau+1}G_7 - Y_5F_8 + Y_3F_{10} &= \\ \sum_{i=0}^{\tau} X^{\tau-i} (f_{8,4+6i}F_9 + (f_{8,5+6i} - f_{10,5+6i})F_8 - f_{10,6+6i}G_7 - f_{10,7+6i}G_6 - f_{10,8+6i}G_5) &= \\ X^{\tau+1}G_8 - X^{\tau+1}F_8 - Y_5F_9 + Y_4F_{10} &= \\ \sum_{i=0}^{\tau} X^{\tau-i} (f_{9,6+6i}F_8 - f_{10,5+6i}F_9 - f_{10,6+6i}G_8 - f_{10,7+6i}G_7 - f_{10,8+6i}F_6) &= \end{aligned}$$

Indeed, by construction each right hand side differs from the corresponding left hand side by a linear combination of the monomials Z_n that vanishes identically on the curve $\mathcal{D} \cap \mathbb{A}^5 \cong \mathcal{C} \cap \mathbb{A}^{g-1}$ and hence is identically zero.

The vanishing of the coefficients of the five linear combinations provides us with quasi-homogeneous equations between the coefficients g_{ij} and f_{ij} . To express these equations in a concise manner, we introduce polynomials in only one variable

$$g_i := \sum_{r=1}^{12\tau+i} g_{ir} t^r = G_i(t^{-6}, t^{-2-6\tau}, t^{-3-6\tau}, t^{-4-6\tau}, t^{-5-6\tau}) t^{i+12\tau} \quad (i = 4, \dots, 8)$$

and write each one as a sum of its *partial polynomials*

$$g_i^{(j)} := \sum_{r \equiv j \pmod{6}} g_{ir} t^r \quad (j = 1, \dots, 6)$$

which are defined by collecting terms whose exponents are in the same residue class modulo 6. In a similar way we define the polynomials f_i ($i = 6, 8, 9, 10$) and its partial polynomials $f_i^{(j)}$. Due to our normalizations some of the partial polynomials

are equal to zero, remaining only 41 ones. More precisely, we can write:

$$\begin{aligned} g_4 &= g_4^{(1)} + g_4^{(2)} + g_4^{(3)} + g_4^{(4)} + g_4^{(5)} + g_4^{(6)} & f_6 &= f_6^{(1)} + f_6^{(3)} + f_6^{(4)} + f_6^{(6)} \\ g_5 &= g_5^{(1)} + g_5^{(2)} + g_5^{(3)} + g_5^{(4)} + g_5^{(5)} + g_5^{(6)} & f_8 &= f_8^{(2)} + f_8^{(4)} + f_8^{(5)} + f_8^{(6)} \\ g_6 &= g_6^{(1)} + g_6^{(2)} + g_6^{(3)} + g_6^{(4)} + g_6^{(5)} + g_6^{(6)} & f_9 &= f_9^{(1)} + f_9^{(3)} + f_9^{(6)} \\ g_7 &= g_7^{(1)} + g_7^{(2)} + g_7^{(3)} + g_7^{(4)} + g_7^{(5)} + g_7^{(6)} & f_{10} &= f_{10}^{(1)} + f_{10}^{(2)} + f_{10}^{(4)} + f_{10}^{(5)} + f_{10}^{(6)} \\ g_8 &= g_8^{(2)} + g_8^{(3)} + g_8^{(4)} + g_8^{(5)} + g_8^{(6)} \end{aligned}$$

The partial polynomials $g_i^{(j)}$ and $f_i^{(j)}$ with $i = j$ and $i = j + 6$ i.e. $g_4^{(4)}, g_5^{(5)}, g_6^{(6)}, f_6^{(6)}$ and $g_7^{(1)}, g_8^{(2)}, f_8^{(2)}, f_9^{(3)}, f_{10}^{(4)}$ have formal degree $i + 12\tau$. The formal degree of $g_4^{(5)}, g_4^{(6)}, g_5^{(6)}$, respectively $f_9^{(1)}, f_{10}^{(1)}, f_{10}^{(2)}$, is equal to $j + 6(\tau - 1)$, respectively $j + 6(\tau + 1)$. The remaining 26 partial polynomials have formal degree $j + 6\tau$. Thus the number of the coefficients that are still involved is equal to

$$4(2\tau + 1) + 5(2\tau + 2) + 3\tau + 3(\tau + 2) + 26(\tau + 1) - 3 = 50\tau + 43$$

where the discount by the number 3 is due to the 3 normalizations $g_{41} = g_{42} = g_{46} = 0$. Now applying Theorem 2.7 and Remark 4.2 we obtain an explicit description of the compactified moduli space $\overline{\mathcal{M}}_{\mathcal{H}}$.

Theorem 4.3. *Let $\mathcal{H}$ be a semigroup generated by $6, 2 + 6\tau, 3 + 6\tau, 4 + 6\tau$ and $5 + 6\tau$ where τ is a positive integer. The isomorphism classes of the pointed complete integral Gorenstein curves with Weierstrass semigroup $\mathcal{H}$ correspond bijectively to the orbits of the $\mathbb{G}_m$ -action on the quasi-cone of the vectors of length $50\tau + 43$ whose coordinates are the coefficients g_{ij} and f_{ij} of the 41 partial polynomials that satisfy the five equations:*

$$\begin{aligned} g_4 - f_6 + g_8 &= f_6^{(1)} f_9 + f_6^{(3)} g_7 + (f_6^{(4)} - g_8^{(4)}) f_6 - g_8^{(5)} g_5 - g_8^{(6)} g_4 \\ g_5 - f_6 + f_9 &= f_6^{(1)} f_{10} + f_6^{(3)} f_8 - f_9^{(6)} g_5 - f_9^{(1)} g_4 \\ g_6 - f_6 - f_8 + f_9 &= f_8^{(4)} g_8 + f_8^{(5)} g_7 - f_9^{(6)} g_6 + f_8^{(6)} f_6 - f_9^{(1)} g_5 \\ g_7 - f_8 + f_{10} &= f_8^{(4)} f_9 + (f_8^{(5)} - f_{10}^{(5)}) f_8 - f_{10}^{(6)} g_7 - f_{10}^{(1)} g_6 - f_{10}^{(2)} g_5 \\ g_8 - f_8 - f_9 + f_{10} &= f_9^{(6)} f_8 - f_{10}^{(5)} f_9 - f_{10}^{(6)} g_8 - f_{10}^{(1)} g_7 - f_{10}^{(2)} f_6 \end{aligned}$$

Thus the compactified moduli space $\overline{\mathcal{M}}_{\mathcal{H}}$ admits an embedding into a weighted projective space space of dimension $50\tau + 42$. To diminish the dimension of the ambient space, we project onto spaces of lower dimensions by eliminating some of the coordinates.

The 5 equations of Theorem 4.3 can be rewritten in terms of 30 polynomial equations between the 41 partial polynomials. Among these equations there are 5 linear ones, which we use to eliminate 5 partial polynomials as follows:

$$f_6^{(4)} = 0, \quad g_8^{(3)} = f_6^{(3)}, \quad f_8^{(5)} = 0, \quad f_{10}^{(1)} = f_9^{(1)}, \quad f_{10}^{(6)} = f_8^{(6)}$$

There remain 25 inhomogeneous equations of degree two between 36 partial polynomials, among them the following 5 equations:

$$\begin{aligned} g_4^{(2)} + g_8^{(2)} &= f_6^{(1)} f_9^{(1)} + f_6^{(3)} g_7^{(5)} - g_8^{(5)} g_5^{(3)} - g_8^{(6)} g_4^{(2)} \\ f_9^{(3)} - f_6^{(3)} &= f_6^{(1)} f_{10}^{(2)} + f_6^{(3)} f_8^{(6)} + f_9^{(1)} g_7^{(2)} \\ g_5^{(6)} - f_6^{(6)} + f_9^{(6)} &= f_6^{(1)} f_{10}^{(5)} - f_9^{(6)} g_5^{(6)} - f_9^{(1)} g_4^{(5)} \\ g_6^{(2)} - f_8^{(2)} &= f_8^{(4)} g_8^{(4)} - f_9^{(6)} g_6^{(2)} - f_9^{(1)} g_5^{(1)} \\ g_7^{(4)} - f_8^{(4)} + f_{10}^{(4)} &= f_8^{(4)} f_9^{(6)} - f_8^{(6)} g_7^{(4)} - f_9^{(1)} g_6^{(3)} - f_{10}^{(2)} g_5^{(2)} \end{aligned}$$

Since the formal degrees of $g_8^{(2)}$, $f_9^{(3)}$, $f_6^{(6)}$, $f_8^{(2)}$ and $f_{10}^{(4)}$ are equal to the formal degrees of the corresponding equations, we can eliminate these five partial polynomials, remaining only 20 equations between 31 partial polynomials. As can be read off from the formal degrees of the partial polynomials, this can be rephrased in terms of $45\tau + 40$ quasi-homogeneous equations between $35\tau + 28$ coefficients. Some of these equations may be identically zero.

Thus the compactified moduli space $\overline{\mathcal{M}}_{\mathcal{H}}$ can be realized as a closed subvariety, or more precisely, as an intersection of at most $45\tau + 40$ hypersurfaces in the weighted projective space of dimension $35\tau + 27$, whose quasi-homogeneous coordinates are the coefficients of the remaining 31 partial polynomials. We can continue in eliminating coefficients, until the remaining quasi-homogeneous equations do not admit linear terms. However, if we would do this procedure in an explicit way, our discussion would become very involved.

We first determine the weighted vector space $T_{\mathbf{k}[\mathcal{H}]|\mathbf{k}}^{1,-}$, which is (up to an isomorphism) the locus of the linearizations of the 30 equations between the partial polynomials. We can solve this system of linear equations as follows:

$$\begin{array}{llllll} f_6^{(1)} = g_4^{(1)} & f_9^{(1)} = g_4^{(1)} - g_5^{(1)} & g_6^{(1)} = g_5^{(1)} & g_7^{(1)} = g_5^{(1)} - g_4^{(1)} & f_{10}^{(1)} = g_4^{(1)} - g_5^{(1)} \\ f_8^{(2)} = g_6^{(2)} & f_{10}^{(2)} = g_4^{(2)} + g_6^{(2)} & g_5^{(2)} = 0 & g_7^{(2)} = -g_4^{(2)} & g_8^{(2)} = -g_4^{(2)} \\ f_9^{(3)} = f_6^{(3)} & g_5^{(3)} = 0 & g_6^{(3)} = 0 & g_7^{(3)} = 0 & g_8^{(3)} = f_6^{(3)} \\ f_6^{(4)} = 0 & f_8^{(4)} = g_6^{(4)} & g_4^{(4)} = -g_7^{(4)} & g_8^{(4)} = g_7^{(4)} & f_{10}^{(4)} = g_6^{(4)} - g_7^{(4)} \\ f_8^{(5)} = 0 & f_{10}^{(5)} = g_4^{(5)} & g_5^{(5)} = 0 & g_7^{(5)} = -g_4^{(5)} & g_8^{(5)} = -g_4^{(5)} \\ f_{10}^{(6)} = f_8^{(6)} & f_6^{(6)} = g_4^{(6)} + g_8^{(6)} & f_9^{(6)} = g_8^{(6)} & g_5^{(6)} = g_4^{(6)} & g_6^{(6)} = g_4^{(6)} + f_8^{(6)} \end{array}$$

Here we had to make choices, which partial polynomials should be eliminated. However, we had to take care that the formal degrees on the left are not smaller than the corresponding ones on the right. Now $T_{\mathbf{k}[\mathcal{H}]|\mathbf{k}}^{1,-}$ can be identified with the space of the vectors whose entries are the coefficients of the remaining 11 partial polynomials

$$g_4^{(1)}, g_5^{(1)}, g_4^{(2)}, g_6^{(2)}, f_6^{(3)}, g_6^{(4)}, g_7^{(4)}, g_4^{(5)}, g_4^{(6)}, g_8^{(6)} \text{ and } f_8^{(6)}.$$

The only conditions the entries have to satisfy are the three normalizations

$$g_{41} = g_{42} = g_{46} = 0.$$

Counting the coefficients that are still involved, we obtain

$$\dim T_{\mathbf{k}[\mathcal{H}]|\mathbf{k}}^{1,-} = 11\tau + 6.$$

More precisely, counting the coefficients of a given weight j , we obtain the dimension of the graded component of $T_{\mathbf{k}[\mathcal{H}]|\mathbf{k}}^1$ of negative weight $-j$:

$$\begin{aligned} \dim T_{-1-6i}^1 &= \begin{cases} 1 & \text{if } i = 0 \\ 2 & (i = 1, \dots, \tau) \end{cases} & \dim T_{-2-6i}^1 &= \begin{cases} 1 & \text{if } i = 0 \\ 2 & (i = 1, \dots, \tau) \end{cases} \\ \dim T_{-3-6i}^1 &= 1 \quad (i = 0, \dots, \tau) & \dim T_{-4-6i}^1 &= 2 \quad (i = 0, \dots, \tau) \\ \dim T_{-5-6i}^1 &= 1 \quad (i = 0, \dots, \tau - 1) & \dim T_{-6-6i}^1 &= \begin{cases} 2 & \text{if } i = 0 \\ 3 & (i = 1, \dots, \tau - 1) \\ 2 & \text{if } i = \tau \end{cases} \end{aligned}$$

In the remaining cases the dimension of T_{-j}^1 is equal to zero.

Thus the compactified moduli space $\overline{\mathcal{M}}_{\mathcal{H}}$ has been realized as a closed subvariety of the $(11\tau + 5)$ -dimensional weighted projective space $\mathbb{P}(T_{\mathbf{k}[\mathcal{H}]|\mathbf{k}}^{1,-})$. It is cut out by $21\tau + 28$ quasi-homogeneous equations, which do not admit linear terms.

As discussed below, it will be much less expensive to obtain the equations and the dimension of the quadratic quasi-cone $\mathcal{Q}_{\mathcal{H}}$ than the ones of the moduli variety $\overline{\mathcal{M}}_{\mathcal{H}}$. In particular we can make the eliminations in an explicit way.

To determine the quadratic quasi-cone $\mathcal{Q}_{\mathcal{H}}$, we just enter with our solution of the system of 30 linear equations into the quadratic terms of the original 30 equations of degree ≤ 2 , and eliminate the same partial polynomials as in the linear case. We obtain only 5 equations

$$\begin{aligned} f_9^{(1)} &= g_4^{(1)} - g_5^{(1)} + g_4^{(1)}(f_8^{(6)} - g_4^{(6)}) - g_5^{(1)}(g_8^{(6)} - g_4^{(6)}) + f_6^{(3)}g_6^{(4)} \\ f_{10}^{(2)} &= g_4^{(2)} + g_6^{(2)} + g_6^{(2)}(g_8^{(6)} - g_4^{(6)}) + g_4^{(2)}(f_8^{(6)} - g_4^{(6)}) - g_6^{(4)}g_7^{(4)} \\ g_6^{(3)} &= -g_4^{(2)}g_5^{(1)} - g_4^{(1)}g_6^{(2)} - g_6^{(4)}g_4^{(5)} \\ g_8^{(5)} &= -g_4^{(5)} - g_4^{(5)}(g_8^{(6)} - g_4^{(6)}) - g_7^{(4)}g_4^{(1)} - f_6^{(3)}g_4^{(2)} \\ f_{10}^{(5)} &= g_4^{(5)} + g_4^{(5)}(f_8^{(6)} - g_4^{(6)}) + g_7^{(4)}g_5^{(1)} - f_6^{(3)}g_6^{(2)} \end{aligned}$$

where the formal degrees on the left hand side are smaller than the corresponding formal degrees of the right hand side, while in the remaining 25 equations the formal degrees on the left are sufficiently large. Thus the quadratic quasi-cone $\mathcal{Q}_{\mathcal{H}}$ is the subvariety of $T_{\mathbf{k}[\mathcal{H}]|\mathbf{k}}^{1,-}$ given by the five conditions

$$\begin{aligned} \pi_{7+6\tau} \left(g_4^{(1)}\tilde{f}_8^{(6)} - g_5^{(1)}\tilde{g}_8^{(6)} + f_6^{(3)}g_6^{(4)} \right) &= 0 \\ \pi_{8+6\tau} \left(g_6^{(2)}\tilde{g}_8^{(6)} + g_4^{(2)}\tilde{f}_8^{(6)} - g_6^{(4)}g_7^{(4)} \right) &= 0 \\ \pi_{8+6\tau} \left(g_4^{(2)}g_5^{(1)} + g_4^{(1)}g_6^{(2)} + g_6^{(4)}g_4^{(5)} \right) &= 0 \\ \pi_{5+6\tau} \left(g_4^{(5)}\tilde{g}_8^{(6)} + g_7^{(4)}g_4^{(1)} + f_6^{(3)}g_4^{(2)} \right) &= 0 \\ \pi_{5+6\tau} \left(g_4^{(5)}\tilde{f}_8^{(6)} + g_7^{(4)}g_5^{(1)} - f_6^{(3)}g_6^{(2)} \right) &= 0 \end{aligned}$$

where

$$\tilde{f}_8^{(6)} := f_8^{(6)} - g_4^{(6)} \quad \text{and} \quad \tilde{g}_8^{(6)} := g_8^{(6)} - g_4^{(6)}$$

and where π_i denotes the projection operator on the polynomials in t that annihilates the terms of degree not larger than i .

We notice that the five conditions do not depend on the $\tau+6$ coefficients g_{51} , g_{62} , f_{63} , g_{44} , g_{74} , $\tilde{f}_{86}$, $\tilde{g}_{86}$ and $g_{4,6i}$ ($i = 2, \dots, \tau$). The five conditions on the remaining 10τ coefficients can be expressed in an elegant way in terms of five polynomial equations between ten elements of the τ -dimensional artinian algebra

$$A := \mathbf{k}[\varepsilon] = \bigoplus_{j=0}^{\tau-1} \mathbf{k} \varepsilon^j \quad \text{where} \quad \varepsilon^\tau = 0.$$

Theorem 4.4. *The quadratic quasi-cone $\mathcal{Q}_{\mathcal{H}}$ is isomorphic to the direct product*

$$\mathcal{Q}_{\mathcal{H}} = V \times W$$

where V is the $(\tau + 6)$ -dimensional weighted vector space with the weights 1, 2, 3, 4, 4, 6 and $6i$ ($i = 1, \dots, \tau$) and where W is the quasi-cone consisting of the vectors

$$(\omega_1, \dots, \omega_{10}) = \left(\sum_{j=0}^{\tau-1} w_{1j} \varepsilon^j, \dots, \sum_{j=0}^{\tau-1} w_{10,j} \varepsilon^j \right) \in \mathbb{A}^{10}$$

satisfying the 5 equations:

$$\begin{aligned} \omega_1 \omega_9 + \omega_5 \omega_6 - \omega_2 \omega_{10} &= 0 \\ \omega_4 \omega_{10} + \omega_3 \omega_9 - \omega_6 \omega_7 &= 0 \\ \omega_1 \omega_4 + \omega_2 \omega_3 + \omega_6 \omega_8 &= 0 \\ \omega_8 \omega_{10} + \omega_3 \omega_5 + \omega_1 \omega_7 &= 0 \\ \omega_2 \omega_7 + \omega_8 \omega_9 - \omega_4 \omega_5 &= 0 \end{aligned}$$

in the artinian algebra A . To the coefficients w_{ij} are attached the weights $\eta_i + 6(\tau - j)$ where $\eta_1, \dots, \eta_{10} = 1, 1, 2, 2, 3, 4, 4, -1, 6, 6$.

Proof. We define

$$\begin{aligned} w_{1j} &= g_{4,6\tau+1-6j}, & w_{2j} &= g_{5,6\tau+1-6j}, & w_{3j} &= g_{4,6\tau+2-6j}, & w_{4j} &= g_{6,6\tau+2-6j}, \\ w_{5j} &= f_{6,6\tau+3-6j}, & w_{6j} &= g_{4,6\tau+4-6j}, & w_{7j} &= g_{7,6\tau+4-6j}, & w_{8j} &= g_{4,6\tau-1-6j}, \\ w_{9j} &= f_{8,6\tau+6-6j}, & w_{10,j} &= \tilde{g}_{8,6\tau+6-6j} \end{aligned}$$

and notice that the five conditions on the 10 τ coefficients are equivalent to the five quadratic equations in the artinian algebra A . $\square$

Corollary 4.5.

$$\dim \mathcal{Q}_{\mathcal{H}} = 8\tau + 6.$$

Proof. Since $\dim V = \tau + 6$ we have to show that

$$\dim W = 7\tau.$$

For each $i = 1, \dots, 10$ let W_i be the open subset of W given by the inequality $w_{i0} \neq 0$, which means that ω_i is a unit in the local artinian algebra A . If a vector $(\omega_1, \dots, \omega_{10})$ belongs to W_1 then we can eliminate ω_9, ω_4 and ω_7 from the first, third and fourth quadratic equation, and the remaining two equations become trivial. Thus W_1 has codimension 3τ in $\mathbb{A}^{10}$ and hence dimension 7τ . In a completely analogous way we see that

$$\dim W_i = 7\tau \quad (i = 1, \dots, 10).$$

If $\tau = 1$ then $W = W_1 \cup \dots \cup W_{10}$ and therefore $\dim W = 7$.

Now we assume that $\tau > 1$. If a vector $(\omega_1, \dots, \omega_{10}) \in W$ does not belong to the union $W_1 \cup \dots \cup W_{10}$ i.e. $w_{ij} = 0$ whenever $j = 0$, then the ten coefficients w_{ij} with $j = \tau - 1$ do not enter into the five quadratic equations, and by induction we obtain

$$\dim(W \setminus (W_1 \cup \dots \cup W_{10})) = 7(\tau - 2) + 10 = 7\tau - 4 < 7\tau$$

and therefore $\dim W = 7\tau$. $\square$

Now applying Theorem 3.1 we obtain an upper bound for the dimension of the moduli variety

$$\dim \overline{\mathcal{M}}_{\mathcal{H}} \leq 8\tau + 5$$

which for each $\tau > 1$ is better than Deligne's bound $2g - 1 = 12\tau + 1$.

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