Spectral theoretic characterization of the massless Dirac operator

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Abstract

We consider an elliptic self-adjoint first order differential operator acting on pairs (2-columns) of complex-valued half-densities over a connected compact 3-dimensional manifold without boundary. The principal symbol of our operator is assumed to be trace-free. We study the spectral function which is the sum of squares of Euclidean norms of eigenfunctions evaluated at a given point of the manifold, with summation carried out over all eigenvalues between zero and a positive λ. We derive an explicit two-term asymptotic formula for the spectral function as $\lambda \to +\infty$, expressing the second asymptotic coefficient via the trace of the subprincipal symbol and the geometric objects encoded within the principal symbol — metric, torsion of the teleparallel connection and topological charge. We then address the question: is our operator a massless Dirac operator on half-densities? We prove that it is a massless Dirac operator on half-densities if and only if the following two conditions are satisfied at every point of the manifold: a) the subprincipal symbol is proportional to the identity matrix and b) the second asymptotic coefficient of the spectral function is zero.

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1 Main results

Consider a first order differential operator A acting on 2-columns $v = \begin{pmatrix} v_1 & v_2 \end{pmatrix}^T$ of complex-valued half-densities over a connected compact 3-dimensional manifold M without boundary. (See subsection 1.1.5 in [22] for definition of half-density.) We assume the coefficients of the operator A to be infinitely smooth. We also assume that the operator A is formally self-adjoint (symmetric):

$$\int_M w^* A v \, dx = \int_M (Aw)^* v \, dx \tag{1.1}$$

for all infinitely smooth $v, w : M \to \mathbb{C}^2$. Here and further on the superscript * in matrices, rows and columns indicates Hermitian conjugation in $\mathbb{C}^2$ and $dx := dx^1 dx^2 dx^3$, where $x = (x^1, x^2, x^3)$ are local coordinates on M.

Let $A_1(x, \xi)$ be the principal symbol of the operator A, i.e. matrix obtained by leaving in A only the leading (first order) derivatives and replacing each $\partial/\partial x^\alpha$ by $i\xi_\alpha$, $\alpha = 1, 2, 3$. Here $\xi = (\xi_1, \xi_2, \xi_3)$ is the variable dual to the position variable x; in physics literature the ξ would be referred to as *momentum*. Our principal symbol $A_1(x, \xi)$ is a 2×2 Hermitian matrix-function on the cotangent bundle T^*M, linear in every fibre T_x^*M (i.e. linear in ξ).

Throughout this paper we assume that the principal symbol $A_1(x, \xi)$ is trace-free for all $(x, \xi) \in T^*M$ and that

$$\det A_1(x, \xi) \neq 0, \qquad \forall (x, \xi) \in T'M, \tag{1.2}$$

where $T'M := T^*M \setminus \{\xi = 0\}$ (cotangent bundle with the zero section removed). The assumption (1.2) is a version of the ellipticity condition.

Under the above assumptions A is a self-adjoint operator in $L^2(M; \mathbb{C}^2)$ (Hilbert space of square integrable complex-valued column "functions") with domain $H^1(M; \mathbb{C}^2)$ (Sobolev space of complex-valued column "functions" which are square integrable together with their first partial derivatives) and the spectrum of A is discrete, with eigenvalues accumulating to $\pm\infty$. Let λ_k and

$v_k = \begin{pmatrix} v_{k1}(x) & v_{k2}(x) \end{pmatrix}^T$ be the eigenvalues and eigenfunctions of the operator A. The eigenvalues λ_k are enumerated in increasing order with account of multiplicity, using a positive index $k = 1, 2, \ldots$ for positive λ_k and a nonpositive index $k = 0, -1, -2, \ldots$ for nonpositive λ_k.

We will be studying the *spectral function* and the *counting function*. The spectral function is the real density defined as

$$e(\lambda, x, x) := \sum_{0<\lambda_k<\lambda} \|v_k(x)\|^2, \tag{1.3}$$

where $\|v_k(x)\|^2 := [v_k(x)]^* v_k(x)$ is the square of the Euclidean norm of the eigenfunction v_k evaluated at the point $x \in M$ and λ is a positive parameter (spectral parameter). The counting function is the function

$$N(\lambda) := \sum_{0<\lambda_k<\lambda} 1 \; = \int_M e(\lambda, x, x)\, dx\,. \tag{1.4}$$

In other words, $N(\lambda)$ is the number of eigenvalues λ_k between zero and λ.

We aim to derive, under appropriate assumptions on Hamiltonian trajectories, two-term asymptotics for the spectral function (1.3) and the counting function (1.4), i.e. formulae of the type

$$e(\lambda, x, x) = a(x)\,\lambda^3 + b(x)\,\lambda^2 + o(\lambda^2), \tag{1.5}$$

$$N(\lambda) = a\lambda^3 + b\lambda^2 + o(\lambda^2) \tag{1.6}$$

as $\lambda \to +\infty$, where the real constants a, b and real densities $a(x)$, $b(x)$ are related in accordance with

$$a = \int_M a(x)\, dx, \tag{1.7}$$

$$b = \int_M b(x)\, dx. \tag{1.8}$$

In our recent paper [10] we performed a comprehensive analysis of two-term spectral asymptotics for general first order elliptic systems. In doing this we showed that all previous publications on systems gave formulae for the second asymptotic coefficient that were either incorrect or incomplete (i.e. an algorithm for the calculation of the second asymptotic coefficient rather than an explicit formula), see Section 11 of [10] for the appropriate bibliographic review. The correct formula for the coefficient $b(x)$ was the main result of [10].

The problem examined in the current paper is a special case of that from [10]. Namely, in the current paper we make the following additional assumptions as compared to [10]:

$$\text{our manifold has dimension 3,} \tag{1.9}$$

$$\text{the number of equations in our system is 2,} \tag{1.10}$$

$$\text{our operator is differential (as opposed to pseudodifferential),} \tag{1.11}$$

$$\text{the principal symbol is trace-free.} \tag{1.12}$$

The need for a detailed analysis of the special case (1.9)–(1.12) is driven by applications to the massless Dirac operator.

The additional assumptions (1.9)–(1.12) lead to the following simplifications as compared to [10].

- The subprincipal symbol A_{sub} does not depend on the dual variable ξ (momentum) and is a function of x (position) only. Recall that the subprincipal symbol is the zeroth order term of the full symbol of the first order operator A written in a way which makes it invariant under coordinate transformations, see formula (6.2) for formal definition and subsection 2.1.3 in [22] for background material.
- The principal symbol A_1 admits a geometric description.

The first of these simplifications is trivial whereas the second is not. We list below the geometric objects encoded within the principal symbol.

Geometric object 1: the *metric.* Observe that the determinant of the principal symbol is a negative definite quadratic form

$$\det A_1(x,\xi) = -g^{\alpha\beta}\xi_\alpha\xi_\beta \tag{1.13}$$

and the coefficients $g^{\alpha\beta}(x) = g^{\beta\alpha}(x)$, $\alpha,\beta = 1,2,3$, appearing in (1.13) can be interpreted as components of a (contravariant) Riemannian metric. This implies, in particular, that our Hamiltonian (positive eigenvalue of the principal symbol) takes the form

$$h^+(x,\xi) = \sqrt{g^{\alpha\beta}(x)\,\xi_\alpha\xi_\beta} \tag{1.14}$$

and the x-components of our Hamiltonian trajectories become geodesics.

Geometric object 2: the *teleparallel connection.* This is an affine connection defined as follows. Suppose we have a covector ξ based at the point $x\in M$ and we want to construct a parallel covector $\tilde\xi$ based at the point $\tilde x\in M$. This is done by solving the linear system of equations

$$A_1(\tilde x,\tilde\xi) = A_1(x,\xi). \tag{1.15}$$

Equation (1.15) is equivalent to a system of three real linear algebraic equations for the three real unknowns, components of the covector $\tilde\xi$, and it is easy to see that this system has a unique solution. It is also easy to see that the affine connection defined by formula (1.15) preserves the Riemannian norm of covectors, i.e. $g^{\alpha\beta}(\tilde x)\,\tilde\xi_\alpha\tilde\xi_\beta = g^{\alpha\beta}(x)\,\xi_\alpha\xi_\beta$, hence, it is metric compatible. The parallel transport defined by formula (1.15) does not depend on the curve along which we transport the (co)vector, so our connection has zero curvature. The word "teleparallel" (parallel at a distance) is used in theoretical physics [18] to describe metric compatible affine connections with zero curvature. The origins of this terminology go back to the works of A. Einstein and É. Cartan [28, 23, 7], though Cartan preferred to use the term "absolute parallelism" rather than "teleparallelism".

The teleparallel connection coefficients ${\Gamma^\alpha}_{\beta\gamma}(x)$ can be written down explicitly in terms of the principal symbol, see formula (3.7), and this allows us to define yet another geometric object — the torsion tensor

$$ {T^\alpha}_{\beta\gamma} := {\Gamma^\alpha}_{\beta\gamma} - {\Gamma^\alpha}_{\gamma\beta} \,. \tag{1.16} $$

Further on we raise and lower indices of the torsion tensor using the metric.

Geometric object 3: the *topological charge.* It turns out, see Section 3, that the existence of a principal symbol implies that our manifold M is parallelizable. Parallelizability implies orientability. Having chosen a particular orientation, we allow only changes of local coordinates x^α, $\alpha = 1, 2, 3$, which preserve orientation.

We define the topological charge as

$$ \mathbf{c} := -\frac{i}{2}\sqrt{\det g_{\alpha\beta}}\ \operatorname{tr}\big((A_1)_{\xi_1}(A_1)_{\xi_2}(A_1)_{\xi_3}\big), \tag{1.17} $$

with the subscripts ξ_α indicating partial derivatives. We show in Section 3 that the number $\mathbf{c}$ defined by formula (1.17) can take only two values, $+1$ or -1, and describes the orientation of the principal symbol relative to the chosen orientation of local coordinates.

We have identified three geometric objects encoded within the principal symbol — metric, teleparallel connection and topological charge. Consequently, one would expect the coefficient $b(x)$ from formula (1.5) to be expressed via these three geometric objects and the subprincipal symbol. This assertion is confirmed by the following theorem.

Theorem 1.1 *The coefficients in the two-term asymptotics (1.5) are given by the formulae*

$$ a(x) = \frac{1}{6\pi^2}\sqrt{\det g_{\alpha\beta}(x)}\,, \tag{1.18} $$

$$ b(x) = \frac{1}{8\pi^2}\big(\big[\,3\,\mathbf{c}\,{*T^{\mathrm{ax}}} - 2\operatorname{tr}A_{\mathrm{sub}}\,\big]\sqrt{\det g_{\alpha\beta}}\,\big)(x)\,, \tag{1.19} $$

where

$$ T^{\mathrm{ax}}_{\alpha\beta\gamma} := \frac{1}{3}(T_{\alpha\beta\gamma} + T_{\gamma\alpha\beta} + T_{\beta\gamma\alpha}) \tag{1.20} $$

is axial torsion *(totally antisymmetric piece of the torsion tensor) and $*$ is the Hodge star (3.4).*

Remark 1.1 *The spectral and counting functions admit two-term asymptotic expansions (1.5) and (1.6) only under appropriate assumptions on geodesic loops* *and closed geodesics respectively, see Theorems 8.3 and 8.4 in [10]. However, one can easily reformulate asymptotic formulae (1.5) and (1.6) in such a way that they remain valid without assumptions on geodesics: this can easily be achieved, say, by taking a convolution with a function from Schwartz space $\mathcal{S}(\mathbb{R})$, see Theorems 7.1 and 7.2 in [10]. Thus, the second asymptotic coefficients of the spectral and counting functions are well-defined irrespective of how many geodesic loops or closed geodesics we have. We introduced the second asymptotic coefficients $b(x)$ and b via the unmollified asymptotic expansions (1.5) and (1.6) simply for the sake of clarity of presentation.*

The proof of Theorem 1.1 is given in Sections 2–5.

We now turn our attention to the massless Dirac operator. This operator is defined in Appendix A, see formula (A.3), and it does not fit into our scheme because it is an operator acting on a 2-component complex-valued spinor (Weyl spinor) rather than a pair of complex-valued half-densities. However, on a parallelizable manifold components of a spinor can be identified with half-densities. We call the resulting operator *the massless Dirac operator on half-densities.* The explicit formula for the massless Dirac operator on half-densities is (A.19).

The massless Dirac operator on half-densities is an operator of the type we are considering in this paper, i.e. a self-adjoint first order elliptic differential operator acting on 2-columns of complex-valued half-densities and with a trace-free principal symbol. We address the question: is a given operator A a massless Dirac operator? The answer is given by the following theorem which is our main result.

Theorem 1.2 *The operator A is a massless Dirac operator on half-densities if and only if the following two conditions are satisfied at every point of the manifold M: a) the subprincipal symbol of the operator, $A_{\mathrm{sub}}(x)$, is proportional to the identity matrix and b) the second asymptotic coefficient of the spectral function, $b(x)$, is zero.*

Note that conditions a) and b) in Theorem 1.2 are invariant under special unitary transformations, i.e. transformations of the operator

$$A \mapsto RAR^*, \tag{1.21}$$

where $R : M \to \mathrm{SU}(2)$ is an arbitrary smooth special unitary matrix-function. The invariance of condition b) is obvious. In fact, condition b) is invariant under the action of a broader group: the unitary matrix-function $R(x)$ appearing in formula (1.21) does not have to be special. As to condition a), its invariance is established by examination of formula (9.3) from [10] with the use of the special commutation properties of trace-free Hermitian 2×2 matrices (the anticommutator of a pair of trace-free Hermitian 2×2 matrices is a multiple of the identity matrix). The fact that the conditions of Theorem 1.2 are SU(2) invariant is not surprising as the massless Dirac operator is designed around the concept of SU(2) invariance, see Property 4 in Appendix A.

The proof of Theorem 1.2 is given in Sections 6 and 7.

Theorems 1.1 and 1.2 tell us that for the massless Dirac operator on half-densities formulae (1.5) and (1.6) read

$$e(\lambda,x,x)=\frac{\sqrt{\det g_{\alpha\beta}(x)}}{6\pi^2}\,\lambda^3+o(\lambda^2), \tag{1.22}$$

$$N(\lambda)=\frac{\operatorname{Vol}M}{6\pi^2}\,\lambda^3+o(\lambda^2), \tag{1.23}$$

where $\operatorname{Vol}M$ is the volume of the Riemannian 3-manifold M.

Remark 1.2 *The factor* $\sqrt{\det g_{\alpha\beta}(x)}$ *appears in the RHS of (1.22) because we are working with the massless Dirac operator on half-densities (A.19) rather than with the massless Dirac operator on spinors (A.3). For the massless Dirac operator on spinors the spectral function is a scalar field (as opposed to a density) and formula (1.22) reads* $e(\lambda,x,x)=\frac{1}{6\pi^2}\lambda^3+o(\lambda^2)$.

2 Reduction from the general setting

As explained in Section 1, the problem considered in the current paper is a special case of that from [10]. Formulae (1.23) and (1.24) from [10] in our case read

$$a(x)=\int\limits_{h^+(x,\xi)<1} đ\xi\,, \tag{2.1}$$

$$b(x)=b_1(x)+b_2(x)\,, \tag{2.2}$$

where

$$b_1(x)=-3\int\limits_{h^+(x,\xi)<1}([v^+]^*A_{\mathrm{sub}}v^+)(x,\xi)\,đ\xi\,, \tag{2.3}$$

$$b_2(x)=\frac{3i}{2}\int\limits_{h^+(x,\xi)<1}\{[v^+]^*,A_1-2h^+I,v^+\}(x,\xi)\,đ\xi\,. \tag{2.4}$$

Here $h^+(x,\xi)$ is the positive eigenvalue of the principal symbol (see also formula (1.14)), $v^+(x,\xi)$ is the corresponding normalized eigenvector (2-column), $đ\xi$ is shorthand for $đ\xi:=(2\pi)^{-3}\,d\xi=(2\pi)^{-3}\,d\xi_1d\xi_2d\xi_3$ and I is the 2×2 identity matrix. Curly brackets in formula (2.4) denote the Poisson bracket on matrix-functions

$$\{P,R\}:=P_{x^\alpha}R_{\xi_\alpha}-P_{\xi_\alpha}R_{x^\alpha} \tag{2.5}$$

and its further generalization

$$\{P,Q,R\}:=P_{x^\alpha}QR_{\xi_\alpha}-P_{\xi_\alpha}QR_{x^\alpha}\,, \tag{2.6}$$

with the subscripts x^α and ξ_α indicating partial derivatives and the repeated tensor index α indicating summation over $\alpha=1,2,3$.

Put $P^+(x,\xi) := [v^+(x,\xi)][v^+(x,\xi)]^*$, which is the orthogonal projection onto the eigenspace $\operatorname{span} v^+$ of the principal symbol. We have $A_1 - 2h^+I = 2h^+P^+ - 3h^+I$ and $\{[v^+]^*, P^+, v^+\} = 0$, so formula (2.4) can be rewritten as

$$b_2(x) = -\frac{9i}{2} \int\limits_{h^+(x,\xi)<1} (h^+\{[v^+]^*, v^+\})(x,\xi)\, đ\xi\,. \tag{2.7}$$

Our aim now is to evaluate the integrals (2.1), (2.3) and (2.7) explicitly.

Formulae (2.1) and (1.14) immediately imply (1.18).

In order to evaluate the integral (2.3) we rewrite this formula as

$$b_1(x) = -3 \int\limits_{h^+(x,\xi)<1} \operatorname{tr}(A_{\mathrm{sub}}P^+)(x,\xi)\, đ\xi$$

and use the fact that $P^+(x,\xi) = \frac{1}{2h^+(x,\xi)}(A_1(x,\xi) + h^+(x,\xi)\, I)$. We get

$$b_1(x) = -3 \int\limits_{h^+(x,\xi)<1} \frac{1}{2h^+(x,\xi)} \operatorname{tr}(A_{\mathrm{sub}}(A_1 + h^+I))(x,\xi)\, đ\xi\,.$$

But A_{sub} does not depend on ξ whereas A_1 and h^+ are, respectively, odd and even in ξ, so the term $\frac{1}{2h^+}\operatorname{tr}(A_{\mathrm{sub}}A_1)$ integrates to zero, leaving us with

$$b_1(x) = -\frac{3}{2}(\operatorname{tr} A_{\mathrm{sub}})(x) \int\limits_{h^+(x,\xi)<1} đ\xi \, = -\frac{1}{4\pi^2}\big(\operatorname{tr} A_{\mathrm{sub}} \sqrt{\det g_{\alpha\beta}}\,\big)(x)\,. \tag{2.8}$$

In order to complete the proof of Theorem 1.1 we need to evaluate explicitly the integral (2.7). The next three sections deal with this nontrivial issue.

3 Teleparallel connection

We show in this section that the principal symbol generates a teleparallel connection which allows us to reformulate the results of our spectral analysis in a much clearer geometric language.

Let us show first that the existence of a principal symbol implies that our manifold M is parallelizable. The principal symbol $A_1(x,\xi)$ is linear in ξ so it can be written as

$$A_1(x,\xi) = \sigma^\alpha(x)\,\xi_\alpha\,, \tag{3.1}$$

where $\sigma^\alpha(x)$, $\alpha = 1,2,3$, are some trace-free Hermitian 2×2 matrix-functions. Let us denote the elements of the matrices σ^α as ${\sigma^\alpha}_{\dot a b}$, where the dotted index, running through the values $\dot 1, \dot 2$, enumerates the rows and the undotted index, running through the values $1,2$, enumerates the columns; this notation is taken from [11]. Put

$${e_1}^\alpha(x) := \operatorname{Re}{\sigma^\alpha}_{\dot 1 2}(x), \quad {e_2}^\alpha(x) := -\operatorname{Im}{\sigma^\alpha}_{\dot 1 2}(x), \quad {e_3}^\alpha(x) := {\sigma^\alpha}_{\dot 1 1}(x). \tag{3.2}$$

Formula (3.2) defines a triple of smooth real vector fields $e_j(x)$, $j=1,2,3$, on the manifold M. These vector fields are linearly independent at every point x of the manifold: this follows from formula (1.2). Thus, the triple of vector fields e_j is a *frame*. The existence of a frame means that the manifold M is parallelizable.

Conversely, given a frame e_j we uniquely recover the principal symbol $A_1(x,\xi)$ via formulae (3.1), (A.1) and (A.2). Thus, a principal symbol is equivalent to a frame. Of course, this equivalence statement relies on our *a priori* assumptions (1.1), (1.2) and (1.9)–(1.12).

It is easy to see that the frame elements e_j are orthonormal with respect to the metric (1.13). Moreover, the metric can be determined directly from the frame as

$$g^{\alpha\beta}=\delta^{jk}{e_j}^{\alpha}\,{e_k}^{\beta}\,, \tag{3.3}$$

where the repeated frame indices j and k indicate summation over $j,k=1,2,3$. The two definitions of the metric, (1.13) and (3.3), are equivalent.

Parallelizability implies orientability, see Proposition 13.5 in [20]. Having chosen a particular orientation, we allow only changes of local coordinates x^α, $\alpha=1,2,3$, which preserve orientation and define the Hodge star in the standard way: the action of $*$ on a rank q antisymmetric tensor Q is

$$(*Q)_{\gamma_{q+1}\ldots\gamma_3}:=(q!)^{-1}\,\sqrt{\det g_{\alpha\beta}}\,Q^{\gamma_1\ldots\gamma_q}\varepsilon_{\gamma_1\ldots\gamma_3}\,, \tag{3.4}$$

where ε is the totally antisymmetric quantity, $\varepsilon_{123}:=+1$, and g is the Riemannian metric (1.13). Here and further on we identify differential forms with covariant antisymmetric tensors. We raise and lower tensor indices using our metric.

Substituting formulae (3.1) and (3.2) into (1.17) we get

$$\mathbf{c}=\operatorname{sgn}\det{e_j}^{\alpha}. \tag{3.5}$$

Formula (3.5) provides an equivalent (and more natural) definition of topological charge. It also explains why the topological charge, initially defined in Section 1 in accordance with formula (1.17), can only take values $+1$ or -1.

The concept of a teleparallel connection was already defined in Section 1 in accordance with formula (1.15). This connection can be equivalently defined via the frame as follows. Suppose we have a vector v based at the point $x\in M$ and we want to construct a parallel vector $\tilde v$ based at the point $\tilde x\in M$. We decompose the vector v with respect to the frame at the point x, $v=c^j e_j(x)$, and reassemble it with the same coefficients c^j at the point $\tilde x$, defining $\tilde v:=c^j e_j(\tilde x)$.

We now define the covariant derivative corresponding to the teleparallel connection. Our teleparallel connection is a special case of an affine connection, so we are looking at a covariant derivative acting on vector/covector fields in the usual manner

$$\nabla_\mu v^\alpha=\partial v^\alpha/\partial x^\mu+{\Gamma^\alpha}_{\mu\beta}\,v^\beta\,,\qquad \nabla_\mu w_\beta=\partial w_\beta/\partial x^\mu-{\Gamma^\alpha}_{\mu\beta}\,w_\alpha\,.$$

The teleparallel connection coefficients are defined from the conditions

$$\nabla_\mu {e_j}^\alpha = 0\,, \tag{3.6}$$

where the e_j are elements of our frame. Formula (3.6) gives a system of 27 linear algebraic equations for the determination of 27 unknown connection coefficients. It is known (see, for example, formula (A2) in [5]), that the unique solution of this system is

$$\Gamma^\alpha{}_{\mu\beta} = {e_k}^\alpha (\partial e^k{}_\beta / \partial x^\mu)\,, \tag{3.7}$$

where

$$e^k{}_\beta := \delta^{kj} g_{\beta\gamma} {e_j}^\gamma. \tag{3.8}$$

The triple of covector fields e^k, $k = 1, 2, 3$, is called the *coframe*. The frame and coframe uniquely determine each other via the relation

$$ {e_j}^\alpha e^k{}_\alpha = {\delta_j}^k. \tag{3.9}$$

Note that our notation for the frame and coframe is taken from [13]. We feel it necessary to mention this because there is a whole range of different notation for frames/coframes in mathematics and theoretical physics literature, which makes the subject somewhat confusing.

One can check by performing explicit calculations that the teleparallel connection has the following two important properties:

$$\nabla_\alpha g_{\beta\gamma} = 0, \tag{3.10}$$

which means that the connection is metric compatible, and

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) v^\gamma = 0 \ \text{ for any vector field } \ v\,, \tag{3.11}$$

which means that the Riemann curvature tensor is zero. Properties (3.10) and (3.11) are the defining properties of a teleparallel connection: a teleparallel connection is, by definition [18], an affine connection satisfying (3.10) and (3.11).

The tensor characterizing the "strength" of the teleparallel connection is not the Riemann curvature tensor but the torsion tensor (1.16). The teleparallel connection is, in a sense, the opposite of the more common Levi-Civita connection: the Levi-Civita connection has zero torsion but nonzero curvature, whereas the teleparallel connection has nonzero torsion but zero curvature. In our paper we distinguish these two affine connections by using different notation for connection coefficients: we write the teleparallel connection coefficients as $\Gamma^\alpha{}_{\beta\gamma}$ and the Levi-Civita connection coefficients (Christoffel symbols) as $\left\{ {\alpha \atop \beta\gamma} \right\}$, see formula (A.4). It is known, see formula (7.34) in [21], that the two sets of connection coefficients are related as $\Gamma^\alpha{}_{\beta\gamma} = \left\{ {\alpha \atop \beta\gamma} \right\} + \frac{1}{2}(T^\alpha{}_{\beta\gamma} + {T_\beta}^\alpha{}_\gamma + {T_\gamma}^\alpha{}_\beta)$.

Substituting (3.7) into (1.16) we arrive at the following explicit formula for the torsion tensor of the teleparallel connection

$$T = e_j \otimes de^j\,, \tag{3.12}$$

where the d stands for the exterior derivative. For the sake of clarity we rewrite formula (3.12) in more detailed form, retaining all tensor indices,

$$T^\alpha{}_{\beta\gamma} = e_j{}^\alpha(\partial e^j{}_\gamma/\partial x^\beta - \partial e^j{}_\beta/\partial x^\gamma)\,. \tag{3.13}$$

As always, the repeated index j appearing in formulae (3.12) and (3.13) indicates summation over $j = 1, 2, 3$.

Torsion is a rank three tensor antisymmetric in the last two indices. Because we are working in dimension three, it is convenient, as in [3], to apply the Hodge star in the last two indices and deal with the rank two tensor

$$\overset{*}{T}{}^\alpha{}_\beta := \frac{1}{2}\, T^{\alpha\gamma\delta}\, \varepsilon_{\gamma\delta\beta}\, \sqrt{\det g_{\mu\nu}} \tag{3.14}$$

instead. Substituting (3.12) into (3.14) we get

$$\overset{*}{T} = e_j \otimes \operatorname{curl} e^j\,, \tag{3.15}$$

where

$$(\operatorname{curl} e^j)_\beta := (*de^j)_\beta = \frac{1}{2}\,(de^j)^{\gamma\delta}\, \varepsilon_{\gamma\delta\beta}\, \sqrt{\det g_{\mu\nu}}\,. \tag{3.16}$$

4 Relation between curvature of the U(1) connection and torsion of the teleparallel connection

This section is devoted to the examination of the integrand in formula (2.7). Recall that the curly brackets in this integrand denote the Poisson bracket on matrix-functions (2.5).

As explained in Section 5 of [10], the expression $-i\{[v^+]^*, v^+\}$ is the scalar curvature of the U(1) connection generated by the eigenspace $\operatorname{span} v^+$ of the principal symbol. This curvature term appears in the general setting of a first order elliptic system. A feature of the particular case (1.9)–(1.12) considered in the current paper is that the scalar curvature of the U(1) connection can be expressed via torsion of the teleparallel connection. This is a substantial simplification. The teleparallel connection is a simpler geometric object than the U(1) connection because the coefficients of the teleparallel connection do not depend on the dual variable (momentum), i.e. they are "functions" on the base manifold M. The relationship between the two connections is established by the following lemma.

Lemma 4.1 *The scalar curvature of the* U(1) *connection is expressed via the torsion of the teleparallel connection, metric and topological charge as*

$$-\,i\{[v^+]^*, v^+\}(x,\xi) = \frac{\mathbf{c}}{2}\, \frac{\overset{*}{T}{}^{\alpha\beta}(x)\,\xi_\alpha\xi_\beta}{(g^{\mu\nu}(x)\,\xi_\mu\xi_\nu)^{3/2}}\,. \tag{4.1}$$

Recall that the topological charge $\mathbf{c} = \pm 1$ is defined in accordance with formula (1.17) or, equivalently, in accordance with formula (3.5).

Proof of Lemma 4.1 We give the proof for the case

$$\mathbf{c} = +1\,. \tag{4.2}$$

There is no need to give a separate proof for the case $\mathbf{c} = -1$ as the two cases reduce to one another by means of a) the observation that torsion (3.12) is invariant under inversion of the frame and b) the identity

$$\{[v^+]^*, v^+\} + \{[v^-]^*, v^-\} = 0, \tag{4.3}$$

where $v^-(x,\xi)$ is the normalized eigenvector of the principal symbol corresponding to the negative eigenvalue. Formula (4.3) is a special case of formula (1.22) from [10].

We fix an arbitrary point $Q \in T'M$ and prove formula (4.1) at this point. As the LHS and RHS of (4.1) are invariant under changes of local coordinates x, it is sufficient to prove formula (4.1) in Riemann normal coordinates, i.e. local coordinates such that $x = 0$ corresponds to the projection of the point Q onto the base manifold, $g_{\mu\nu}(0) = \delta_{\mu\nu}$ and $\frac{\partial g_{\mu\nu}}{\partial x^\lambda}(0) = 0$. Moreover, as the formula we are proving involves only first partial derivatives in x, we may assume, without loss of generality, that

$$g_{\mu\nu}(x) = \delta_{\mu\nu} \tag{4.4}$$

for all x in some neighbourhood of the origin. In other words, it is sufficient to prove formula (4.1) for the case of Euclidean metric.

As both the LHS and RHS of (4.1) have the same degree of homogeneity in ξ, namely, -1, it is sufficient to prove formula (4.1) for ξ of norm 1. Moreover, by rotating our Cartesian coordinate system we can reduce the case of general ξ of norm 1 to the case

$$\xi = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}. \tag{4.5}$$

There is one further simplification that can be made: we claim that it is sufficient to prove formula (4.1) for the case when

$$e_j{}^\alpha(0) = \delta_j{}^\alpha, \tag{4.6}$$

i.e. for the case when at the point $x = 0$ the elements of the frame are aligned with the coordinate axes. This claim follows from the observation that the LHS of formula (4.1) is invariant under rigid special unitary transformations of the column-function v^+, $v^+ \mapsto Rv^+$, where "rigid" refers to the fact that the matrix $R \in \mathrm{SU}(2)$ is constant. Of course, the column-function Rv^+ is no longer an eigenvector of the original principal symbol, but a new principal symbol obtained from the old one by the rigid special orthogonal transformation of the frame (A.14) with the 3×3 special orthogonal matrix O expressed in terms of the 2×2 special unitary matrix R in accordance with (A.15). One can always choose the special unitary matrix R so that at the point $x = 0$ the elements of the new frame are aligned with the coordinate axes (in fact, there are two possible choices of R which differ by sign). It remains only to note that direct inspection of formula (3.12) shows that torsion is also invariant under rigid special orthogonal transformations of the frame, and, hence, the tensor $\overset{*}{T}$ defined by formula (3.14) and appearing in the RHS of formula (4.1) is invariant under rigid special orthogonal transformations of the frame as well.

Having made the simplifying assumptions (4.4)–(4.6), we are now in a position to prove formula (4.1).

Let us calculate the RHS of (4.1) first. In view of (4.6) we have, in the linear approximation in x,

$$\begin{pmatrix} {e_1}^1(x) & {e_1}^2(x) & {e_1}^3(x) \\ {e_2}^1(x) & {e_2}^2(x) & {e_2}^3(x) \\ {e_3}^1(x) & {e_3}^2(x) & {e_3}^3(x) \end{pmatrix} = \begin{pmatrix} 1 & w^3(x) & -w^2(x) \\ -w^3(x) & 1 & w^1(x) \\ w^2(x) & -w^1(x) & 1 \end{pmatrix}, \tag{4.7}$$

where w is some smooth vector-function which vanishes at $x = 0$. Formula (4.7) is the standard formula for the linearization of an orthogonal matrix about the identity; see also formula (10.1) in [3]. Note that in Cosserat elasticity literature the vector-function w is called the *vector of microrotations*. Substituting (4.7) into (3.15) and (3.16) we get, at $x = 0$,

$$\overset{*}{T}_{\alpha\beta} = \partial w_\beta / \partial x^\alpha - \delta_{\alpha\beta} \operatorname{div} w, \tag{4.8}$$

which is formula (10.5) from [3]. Here we freely lower and raise tensor indices using the fact that the metric is Euclidean (in the Euclidean case (4.4) it does not matter whether a tensor index comes as a subscript or a superscript). Substituting (4.8) and (4.5) into the RHS of (4.1) we get, at our point $Q \in T'M$,

$$\frac{1}{2}\,\frac{\overset{*}{T}{}^{\alpha\beta}\xi_\alpha\xi_\beta}{(g^{\mu\nu}\xi_\mu\xi_\nu)^{3/2}} = -\frac{1}{2}(\partial w^1/\partial x^1 + \partial w^2/\partial x^2)\,. \tag{4.9}$$

Let us now calculate the LHS of (4.1). The equation for the eigenvector $v^+(x,\xi)$ of the principal symbol is

$$\begin{pmatrix} {e_3}^\alpha\xi_\alpha - \|\xi\| & (e_1 - ie_2)^\alpha\xi_\alpha \\ (e_1 + ie_2)^\alpha\xi_\alpha & -{e_3}^\alpha\xi_\alpha - \|\xi\| \end{pmatrix} \begin{pmatrix} v_1^+ \\ v_2^+ \end{pmatrix} = 0\,. \tag{4.10}$$

In view of (4.5) and (4.6) the (normalized) solution of (4.10) at our point $Q \in T'M$ is $v^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Of course, our $v^+(x,\xi)$ is defined up to the gauge transformation

$$v^+ \mapsto e^{i\phi^+} v^+, \tag{4.11}$$

where

$$\phi^+ : T'M \to \mathbb{R} \tag{4.12}$$

is an arbitrary smooth function, however the LHS of (4.1) is invariant under this gauge transformation. We now perturb equation (4.10) about the point $Q \in T'M$, that is, about $x = 0$, $\xi = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$, making use of formula (4.7), which gives us the following equation for the increment δv^+ of the eigenvector $v^+(x,\xi)$ of the principal symbol:

$$\begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} \delta v_1^+ \\ \delta v_2^+ \end{pmatrix} + \begin{pmatrix} 0 & -w^2(x) - iw^1(x) \\ -w^2(x) + iw^1(x) & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & \delta\xi_1 - i\delta\xi_2 \\ \delta\xi_1 + i\delta\xi_2 & -2\delta\xi_3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0,$$

or, equivalently,

$$\delta v_2^+ = \frac{1}{2}(-w^2(x) + iw^1(x) + \delta\xi_1 + i\delta\xi_2). \tag{4.13}$$

Formula (4.13) has to be supplemented by the normalization condition $\|v^+(x,\xi)\| = 1$, which in its linearized form reads

$$\operatorname{Re} \delta v_1^+ = 0. \tag{4.14}$$

Formulae (4.14) and (4.13) define δv^+ modulo an arbitrary $\operatorname{Im} \delta v_1^+$, with this degree of freedom being associated with the gauge transformation (4.11), (4.12). Without loss of generality we may assume that the gauge is chosen so that

$$\operatorname{Im} \delta v_1^+ = 0. \tag{4.15}$$

Combining formulae (4.14), (4.15) and (4.13) we get

$$\delta v^+ = \frac{1}{2} \begin{pmatrix} 0 \\ -w^2(x) + iw^1(x) + \delta\xi_1 + i\delta\xi_2 \end{pmatrix}. \tag{4.16}$$

Recall that the w appearing in this formula is some smooth vector-function which vanishes at $x = 0$.

Differentiation of (4.16) gives us

$$\frac{\partial v^+}{\partial x^\alpha} = \frac{1}{2} \begin{pmatrix} 0 \\ -\partial w^2/\partial x^\alpha + i\partial w^1/\partial x^\alpha \end{pmatrix}, \tag{4.17}$$

$$\frac{\partial v^+}{\partial \xi_1} = \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \frac{\partial v^+}{\partial \xi_2} = \frac{1}{2}\begin{pmatrix} 0 \\ i \end{pmatrix}, \qquad \frac{\partial v^+}{\partial \xi_3} = 0. \tag{4.18}$$

Formulae (4.17) and (4.18) imply that at our point $Q \in T'M$

$$-\, i\{[v^+]^*, v^+\} = -\frac{1}{2}(\partial w^1/\partial x^1 + \partial w^2/\partial x^2). \tag{4.19}$$

Comparing formulae (4.9) and (4.19) and recalling (4.2), we arrive at the required result (4.1). $\square$

5 Integration of the curvature term

Substituting (4.1) into (2.7) we get

$$b_2(x) = \frac{9\mathbf{c}}{4} \int\limits_{h^+(x,\xi)<1} \frac{\overset{*}{T}{}^{\alpha\beta}(x)\,\xi_\alpha\xi_\beta}{g^{\mu\nu}(x)\,\xi_\mu\xi_\nu}\,đ\xi\,. \tag{5.1}$$

Recall that $h^+(x,\xi)$ is given by formula (1.14).

The tensor $\overset{*}{T}$ can be decomposed into pure trace and trace-free pieces, i.e.

$$\overset{*}{T}{}^{\alpha\beta} = \frac{1}{3} g^{\alpha\beta} \overset{*}{T}{}^\gamma{}_\gamma + \left(\overset{*}{T}{}^{\alpha\beta} - \frac{1}{3} g^{\alpha\beta} \overset{*}{T}{}^\gamma{}_\gamma \right). \tag{5.2}$$

It is easy to see that the trace-free piece (second term in the RHS of (5.2)) does not contribute to the integral in (5.1), hence formula (5.1) becomes

$$b_2(x) = \frac{3\mathbf{c}}{4} \overset{*}{T}{}^\gamma{}_\gamma(x) \int\limits_{h^+(x,\xi)<1} đ\xi \;= \frac{\mathbf{c}}{8\pi^2} \big(\overset{*}{T}{}^\gamma{}_\gamma \sqrt{\det g_{\alpha\beta}}\,\big)(x)\,. \tag{5.3}$$

But formulae (1.20), (3.4) and (3.14) imply that

$$\overset{*}{T}{}^\gamma{}_\gamma = 3 * T^{\mathrm{ax}}. \tag{5.4}$$

Combining formulae (2.2), (2.8), (5.3) and (5.4) we arrive at formula (1.19). This completes the proof of Theorem 1.1.

6 The subprincipal symbol of the massless Dirac operator

In this section we calculate the subprincipal symbol of the massless Dirac operator, which prepares the ground for the proof of Theorem 1.2 in the next section. In view of Remark 2.1.10 from [22], defining the subprincipal symbol for the massless Dirac operator on spinors (A.3) is problematic, hence, we work with the massless Dirac operator on half-densities (A.19). For the sake of brevity we denote the massless Dirac operator on half-densities by A rather than by $W_{1/2}$.

Lemma 6.1 *The subprincipal symbol of the massless Dirac operator on half-densities (A.19) is*

$$A_{\mathrm{sub}}(x) = \frac{3\mathbf{c}}{4} \big(*T^{\mathrm{ax}}(x) \big)\, I\,, \tag{6.1}$$

where $\mathbf{c} = \pm 1$ *is the topological charge (3.5),* T^{ax} *is axial torsion (1.20),* $*$ *is the Hodge star (3.4) and* I *is the* 2×2 *identity matrix.*

Proof We give the proof of (6.1) for the case (4.2). There is no need to give a separate proof for the case $\mathbf{c} = -1$ as the two cases reduce to one another by inversion of the frame: the full symbol of the massless Dirac operator on half-densities changes sign under inversion of the frame and hence its subprincipal symbol changes sign under inversion of the frame, whereas torsion (3.12) is invariant under inversion of the frame.

According to formula (1.2) from [12] the subprincipal symbol is defined as

$$A_{\mathrm{sub}} := A_0 + \frac{i}{2}(A_1)_{x^\alpha \xi_\alpha}\,, \tag{6.2}$$

where $A_1(x,\xi)$ and $A_0(x)$ are the homogeneous (in ξ) components of the full symbol $A(x,\xi) = A_1(x,\xi) + A_0(x)$ of our first order differential operator, with the subscript indicating degree of homogeneity. For the massless Dirac operator on half-densities (A.19) these homogeneous components read (3.1) and

$$A_0(x) = -\frac{i}{4}\sigma^\alpha \sigma_\beta \left(\frac{\partial \sigma^\beta}{\partial x^\alpha} + \left\{ {\beta \atop \alpha\gamma} \right\} \sigma^\gamma \right) + \frac{i}{2}\sigma^\alpha \left\{ {\beta \atop \alpha\beta} \right\} \tag{6.3}$$

respectively. Note that in writing down (6.3) we used the standard formula

$$\frac{1}{2\det g_{\kappa\lambda}} \frac{\partial \det g_{\mu\nu}}{\partial x^\alpha} = \left\{ {\beta \atop \alpha\beta} \right\}.$$

Our task is to substitute (3.1) and (6.3) into (6.2).

We fix an arbitrary point $P \in M$ and prove formula (6.1) at this point. As the LHS and RHS of (6.1) are invariant under changes of local coordinates x, it is sufficient to check the identity (6.1) in Riemann normal coordinates, i.e. local coordinates such that $x = 0$ corresponds to the point P, $g_{\mu\nu}(0) = \delta_{\mu\nu}$ and $\frac{\partial g_{\mu\nu}}{\partial x^\lambda}(0) = 0$. Moreover, as the principal symbol is linear in ξ and the formula we are proving involves only first partial derivatives in x, we may assume, without loss of generality, that we have (4.4) for all x in some neighbourhood of the origin. In other words, it is sufficient to prove formula (6.1) for the case of Euclidean metric. Furthermore, by rotating our Cartesian coordinate system we can achieve (4.6), which opens the way to the use, in the linear approximation in x, of formula (4.7).

Substituting (4.7) into (A.1), we get, in the linear approximation in x,

$$\sigma^1 = \begin{pmatrix} w^2 & 1 + iw^3 \\ 1 - iw^3 & -w^2 \end{pmatrix} = \sigma_1\,,$$

$$\sigma^2 = \begin{pmatrix} -w^1 & -i + w^3 \\ i + w^3 & w^1 \end{pmatrix} = \sigma_2\,,$$

$$\sigma^3 = \begin{pmatrix} 1 & -iw^1 - w^2 \\ iw^1 - w^2 & -1 \end{pmatrix} = \sigma_3\,. \tag{6.4}$$

Recall that the w appearing in this formula is some smooth vector-function which vanishes at $x = 0$.

Substitution of (6.4) into (3.1) and (6.3) gives us

$$A_1(x,\xi) = \begin{pmatrix} \xi_3 & \xi_1 - i\xi_2 \\ \xi_1 + i\xi_2 & -\xi_3 \end{pmatrix}$$
$$+ \begin{pmatrix} w^2\xi_1 - w^1\xi_2 & iw^3\xi_1 + w^3\xi_2 + (-iw^1 - w^2)\xi_3 \\ -iw^3\xi_1 + w^3\xi_2 + (iw^1 - w^2)\xi_3 & -w^2\xi_1 + w^1\xi_2 \end{pmatrix}, \quad (6.5)$$

$$A_0(0) = -\frac{i}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \partial w^2/\partial x^1 & i\partial w^3/\partial x^1 \\ -i\partial w^3/\partial x^1 & -\partial w^2/\partial x^1 \end{pmatrix} + \ldots . \quad (6.6)$$

Here formula (6.5) is written in the linear approximation in x, whereas formula (6.6) displays, for the sake of brevity, only one term out of nine (the one corresponding to $\alpha = \beta = 1$ in (6.3)), with the remaining eight terms concealed within the dots $\ldots$. Note also that the Christoffel symbols disappeared because of our assumption that the metric is Euclidean.

Substituting (6.6) and (6.5) into (6.2), we get

$$A_{\mathrm{sub}}(0) = -\frac{1}{2}\,(\operatorname{div} w)\, I. \quad (6.7)$$

But, according to (4.8),

$$\overset{*}{T}{}^{\gamma}{}_{\gamma}(0) = -2\operatorname{div} w. \quad (6.8)$$

Formulae (6.7), (6.8), (5.4) and (4.2) imply formula (6.1) at $x = 0$. $\square$

7 Proof of Theorem 1.2

As Theorem 1.2 is an if and only if theorem, our proof comes in two parts.

Part 1 of the proof Let A be a massless Dirac operator on half-densities. We need to prove that a) the subprincipal symbol of this operator, $A_{\mathrm{sub}}(x)$, is proportional to the identity matrix and b) the second asymptotic coefficient of the spectral function, $b(x)$, is zero. The required result follows from Lemma 6.1 and Theorem 1.1.

Part 2 of the proof Let A be a differential operator such that a) the subprincipal symbol of this operator, $A_{\mathrm{sub}}(x)$, is proportional to the identity matrix and b) the second asymptotic coefficient of the spectral function, $b(x)$, is zero. We need to prove that A is a massless Dirac operator on half-densities.

Theorem 1.1 implies that the subprincipal symbol of our operator A is given by formula (6.1). Let e_j be the frame corresponding to the principal symbol of the operator A, see formulae (3.1) and (3.2). Now, let B be the massless Dirac operator on half-densities corresponding to the same frame. Then the principal symbols of the operators A and B coincide. But Lemma 6.1 implies that the subprincipal symbols of the operators A and B coincide as well. A first order differential operator is determined by its principal and subprincipal symbols, hence, $A = B$. $\square$

8 Explicit formula for axial torsion

Torsion is a rank three tensor antisymmetric in the last two indices. It is known [3, 18] that torsion has three irreducible pieces. Only one of the three irreducible pieces of torsion, namely, the piece which theoretical physicists label by the adjective "axial", appears in our spectral theoretic results, see Theorem 1.1 and Lemma 6.1. It is also interesting that axial torsion is the irreducible piece which is used when one models the massless neutrino [11] or the electron [6] by means of Cosserat elasticity.

Axial torsion is defined as the totally antisymmetric piece of the torsion tensor, see formula (1.20). This means that axial torsion is a 3-form. In view of the importance of axial torsion, we give an explicit formula for its Hodge dual in terms of the principal symbol $A_1(x,\xi)$. Formulae (3.15), (3.16) and (5.4) imply

$$\begin{aligned} *\,T^{\mathrm{ax}} = \frac{\delta_{kl}}{3}\sqrt{\det g^{\alpha\beta}}\,\big[& e^k{}_1\,\partial e^l{}_3/\partial x^2 + e^k{}_2\,\partial e^l{}_1/\partial x^3 + e^k{}_3\,\partial e^l{}_2/\partial x^1 \\ & - e^k{}_1\,\partial e^l{}_2/\partial x^3 - e^k{}_2\,\partial e^l{}_3/\partial x^1 - e^k{}_3\,\partial e^l{}_1/\partial x^2\big]. \end{aligned} \tag{8.1}$$

Here the coframe e^k is determined from the principal symbol in accordance with formulae (3.1), (3.2) and (3.9), whereas the contravariant metric tensor $g^{\alpha\beta}$ is determined from the principal symbol in accordance with formula (1.13).

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A The massless Dirac operator

Let M be a 3-dimensional connected compact oriented manifold equipped with a Riemannian metric $g_{\alpha\beta}$, $\alpha,\beta=1,2,3$ being the tensor indices. Note that we are more prescriptive in this appendix than in the main text of the paper: in the main text orientability emerged as a consequence of the existence of a principal symbol and the metric was defined via the principal symbol, whereas in this appendix orientability and metric are introduced *a priori*.

We work only in local coordinates with prescribed orientation.

It is known [25, 19] that a 3-dimensional oriented manifold is parallelizable, i.e. there exist smooth real vector fields e_j, $j=1,2,3$, that are linearly independent at every point x of the manifold. (This fact is often referred to as *Steenrod's theorem.*) Each vector $e_j(x)$ has coordinate components $e_j{}^\alpha(x)$, $\alpha=1,2,3$. Note that we use the Latin letter j for enumerating the vector fields (this is an *anholonomic* or *frame* index) and the Greek letter α for enumerating their components (this is a *holonomic* or *tensor* index). The triple of linearly independent vector fields e_j, $j=1,2,3$, is called a *frame*. Without loss of generality we assume further on that the vector fields e_j are orthonormal with respect to our metric: this can always be achieved by means of the Gram–Schmidt process.

Define Pauli matrices

$$\sigma^\alpha(x) := s^j\, e_j{}^\alpha(x)\,, \tag{A.1}$$

where

$$s^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = s_1\,, \quad s^2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = s_2\,, \quad s^3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = s_3\,. \tag{A.2}$$

In formula (A.1) summation is carried out over the repeated frame index $j = 1, 2, 3$, and $\alpha = 1, 2, 3$ is the free tensor index.

The massless Dirac operator is the matrix operator

$$W := -i\sigma^\alpha \left(\frac{\partial}{\partial x^\alpha} + \frac{1}{4}\sigma_\beta \left(\frac{\partial \sigma^\beta}{\partial x^\alpha} + \left\{ {\beta \atop \alpha\gamma} \right\} \sigma^\gamma \right) \right), \tag{A.3}$$

where summation is carried out over $\alpha, \beta, \gamma = 1, 2, 3$, and

$$\left\{ {\beta \atop \alpha\gamma} \right\} := \frac{1}{2} g^{\beta\delta} \left(\frac{\partial g_{\gamma\delta}}{\partial x^\alpha} + \frac{\partial g_{\alpha\delta}}{\partial x^\gamma} - \frac{\partial g_{\alpha\gamma}}{\partial x^\delta} \right) \tag{A.4}$$

are the Christoffel symbols. Here and throughout this appendix we raise and lower tensor indices using the metric. Note that we chose the letter "W" for denoting the massless Dirac operator because in theoretical physics literature it is often referred to as the *Weyl* operator.

Formula (A.3) is the formula from [11], only written in matrix notation (i.e. without spinor indices). Note that in the process of transcribing formulae from [11] into matrix notation we used the identity

$$\epsilon \sigma^\alpha \epsilon = (\sigma^\alpha)^T, \tag{A.5}$$

$\alpha = 1, 2, 3$, where

$$\epsilon := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{A.6}$$

is the "metric spinor". The identity (A.5) gives a simple way of raising/lowering spinor indices in Pauli matrices in the non-relativistic ($\alpha \neq 0$) setting.

Our definition (A.3) of the massless Dirac operator is a special case of the definition from [13]. The two definitions coincide when we work with a Spin connection as opposed to a Spin^c connection, see Propositions 2.14 and 2.15 in [13] for details.

Throughout this paper we work in dimension 3. The definition of the massless Dirac operator acting over a Riemannian manifold of arbitrary dimension can be found, for example, in [16, 14, 15].

Physically, our massless Dirac operator (A.3) describes a single massless neutrino living in a 3-dimensional compact universe M. The eigenvalues of the massless Dirac operator are the energy levels.

The massless Dirac operator (A.3) acts on 2-columns $v = \begin{pmatrix} v_1 & v_2 \end{pmatrix}^T$ of complex-valued scalar functions. In differential geometry this object is referred to as a (Weyl) spinor so as to emphasize the fact that v transforms in a particular way under transformations of the orthonormal frame e_j. However, as in our exposition the frame e_j is assumed to be chosen *a priori*, we can treat the components of the spinor as scalars. This issue will be revisited below when we state Property 4 of the massless Dirac operator.

We now list the main properties of the massless Dirac operator. We state these without proofs. The proofs can be found in Appendix 3.A of [9] or in [13].

Property 1. The massless Dirac operator is invariant under changes of local coordinates x, i.e. it maps 2-columns of smooth scalar functions $M \to \mathbb{C}^2$ to 2-columns of smooth scalar functions $M \to \mathbb{C}^2$ regardless of the choice of local coordinates.

Property 2. The massless Dirac operator is formally self-adjoint (symmetric) with respect to the inner product

$$\int_M w^* v \sqrt{\det g_{\alpha\beta}}\, dx \tag{A.7}$$

on 2-columns of smooth scalar functions $v, w : M \to \mathbb{C}^2$.

Property 3. The massless Dirac operator W commutes

$$\mathrm{C}(Wv) = W\mathrm{C}(v) \tag{A.8}$$

with the antilinear map

$$v \mapsto \mathrm{C}(v) := \epsilon\overline{v}, \tag{A.9}$$

where ϵ is the "metric spinor" (A.6). In theoretical physics the transformation (A.9) is referred to as *charge conjugation* [4, 13].

Formula (A.8) implies that v is an eigenfunction of the massless Dirac operator corresponding to an eigenvalue λ if and only if $\mathrm{C}(v)$ is an eigenfunction of the massless Dirac operator corresponding to the same eigenvalue λ. Hence, all eigenvalues of the massless Dirac operator have even multiplicity. Moreover, any eigenfunction v and its "partner" $\mathrm{C}(v)$ make the same contribution to the spectral function (1.3) at every point x of the manifold M.

If, as in [13], we introduce a magnetic field, then we lose the commutation property (A.8) and the double eigenvalues split up. This indicates that the double eigenvalues of the massless Dirac operator correspond to the two different spins.

Property 4. This property has to do with a particular behaviour under SU(2) transformations. Let $R : M \to \mathrm{SU}(2)$ be an arbitrary smooth special unitary matrix-function. Let us introduce new Pauli matrices

$$\tilde{\sigma}^\alpha := R\sigma^\alpha R^* \tag{A.10}$$

and a new operator $\tilde{W}$ obtained by replacing the σ in (A.3) by $\tilde{\sigma}$. It turns out (and this is Property 4) that the two operators, $\tilde{W}$ and W, are related in exactly the same way as the Pauli matrices, $\tilde{\sigma}$ and σ, that is,

$$\tilde{W} = RWR^*. \tag{A.11}$$

We now examine the geometric meaning of the transformation (A.10). Let us expand the new Pauli matrices $\tilde{\sigma}$ with respect to the basis (A.2):

$$\tilde{\sigma}^\alpha(x) = s^j\, \tilde{e}_j{}^\alpha(x). \tag{A.12}$$

Formulae (A.1), (A.12) and (A.10) give us the following identity relating the new vector fields $\tilde{e}_j$ and the old vector fields e_j:

$$Rs^kR^*e_k = s^j\tilde{e}_j\,. \tag{A.13}$$

Resolving (A.13) for $\tilde{e}_j$ we get

$$\tilde{e}_j = O_j{}^k e_k\,, \tag{A.14}$$

where the real scalars $O_j{}^k$ are given by the formula

$$O_j{}^k = \frac{1}{2}\operatorname{tr}(s_jRs^kR^*)\,. \tag{A.15}$$

Note that in writing formulae (A.13) and (A.14) we chose to hide the tensor index, i.e. we chose to hide the coordinate components of our vector fields. Say, formula (A.14) written in more detailed form reads $\tilde{e}_j{}^\alpha = O_j{}^k e_k{}^\alpha$.

The scalars (A.15) can be viewed as elements of a real 3×3 matrix-function O with the first index, j, enumerating rows and the second, k, enumerating columns. It is easy to check that this matrix-function O is special orthogonal. Hence, the new vector fields $\tilde{e}_j$ are orthonormal and have the same orientation as the old vector fields e_j. We have shown that the transformation (A.10) has the geometric meaning of switching from our original oriented orthonormal frame e_j to a new oriented orthonormal frame $\tilde{e}_j$.

Formula (A.15) means that the special unitary matrix R is, effectively, a square root of the special orthogonal matrix O. It is easy to see that for a given matrix $O\in\mathrm{SO}(3)$ formula (A.15) defines the matrix $R\in\mathrm{SU}(2)$ uniquely up to sign. This observation allows us to view the issue of the geometric meaning of the transformation (A.10) the other way round: given a pair of orthonormal frames, e_j and $\tilde{e}_j$, with the same orientation (i.e. with $\operatorname{sgn}\det e_j{}^\alpha = \operatorname{sgn}\det \tilde{e}_j{}^\alpha$), we can recover the special orthogonal matrix-function $O(x)$ from formula (A.14) and then attempt finding a smooth special unitary matrix-function $R(x)$ satisfying (A.15). Unfortunately, this may not always be possible due to topological obstructions. We can only guarantee the absence of topological obstructions when the two frames, e_j and $\tilde{e}_j$, are sufficiently close to each other, which is equivalent to saying that we can only guarantee the absence of topological obstructions when the special orthogonal matrix-function $O(x)$ is sufficiently close to the identity matrix for all $x \in M$.

We illustrate the possibility of a topological obstruction by means of an explicit example. Consider the unit torus $\mathbb{T}^3$ parameterized by cyclic coordinates x^α, $\alpha = 1, 2, 3$, of period 2π. The metric is assumed to be Euclidean. Define a pair of orthonormal frames

$$e_j{}^\alpha := \delta_j{}^\alpha \tag{A.16}$$

and

$$\tilde{e}_1{}^\alpha := \begin{pmatrix} \cos k_3 x^3 \\ \sin k_3 x^3 \\ 0 \end{pmatrix}, \qquad \tilde{e}_2{}^\alpha := \begin{pmatrix} -\sin k_3 x^3 \\ \cos k_3 x^3 \\ 0 \end{pmatrix}, \qquad \tilde{e}_3{}^\alpha := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{A.17}$$

where k_3 is an odd integer. Let W and $\tilde{W}$ be the massless Dirac operators corresponding to the frames (A.16) and (A.17) respectively. We claim that there does not exist a smooth matrix-function $R : \mathbb{T}^3 \to \mathrm{SU}(2)$ which would give (A.15), where $O(x)$ is the special orthogonal matrix-function defined by formula (A.14). We justify this claim in two different ways.

Justification 1. Resolving the system (A.14)–(A.17) locally for R, we get

$$R(x^3) = \pm \begin{pmatrix} e^{\frac{i}{2} k_3 x^3} & 0 \\ 0 & e^{-\frac{i}{2} k_3 x^3} \end{pmatrix}, \tag{A.18}$$

and this solution is unique modulo choice of sign; here the freedom in the choice of sign is not surprising as SU(2) is the double cover of SO(3). Formula (A.18) defines a continuous single-valued matrix-function on the unit torus $\mathbb{T}^3$ if and only if the integer k_3 is even, which it is not.

Justification 2. It is sufficient to show that the two operators, W and $\tilde{W}$, have different spectra. Straightforward separation of variables shows that zero is an eigenvalue of the operator W but not an eigenvalue of the operator $\tilde{W}$.

One can generalize the above example by introducing rotations in three different directions, which leads to eight genuinely distinct parallelizations. See also [24] page 524 or [2] page 21.

Let us emphasize that the topological obstructions we were discussing have nothing to do with Stiefel–Whitney classes. We are working on a parallelizable manifold and the Stiefel–Whitney class of such a manifold is trivial. The topological issue at hand is that our parallelizable manifold may be equipped with different spin structures.

We say that two massless Dirac operators, W and $\tilde{W}$, are equivalent if there exists a smooth matrix-function $R : M \to \mathrm{SU}(2)$ such that the corresponding Pauli matrices, σ^α and $\tilde{\sigma}^\alpha$, are related in accordance with (A.10). In view of Property 4 (see formula (A.11)) all massless Dirac operators from the same equivalence class generate the same spectral function (1.3) and the same counting function (1.4), so for the purposes of our paper viewing such operators as equivalent is most natural.

As explained above, there may be many distinct equivalence classes of massless Dirac operators, the difference between which is topological. Studying the spectral theoretic implications of these topological differences is beyond the scope of our paper. The two-term asymptotics (1.22) and (1.23) derived in the main text of our paper do not feel this topology.

In theoretical physics the SU(2) freedom involved in defining the massless Dirac operator is interpreted as a gauge degree of freedom. We do not adopt this point of view (at least explicitly) in order to fit the massless Dirac operator into the standard spectral theoretic framework.

We defined the massless Dirac operator (A.3) as an operator acting on 2-columns of scalar functions, i.e. on 2-columns of quantities which do not change under changes of local coordinates. This necessitated the introduction of the density $\sqrt{\det g_{\alpha\beta}}$ in the formula (A.7) for the inner product. In spectral theory it is more common to work with half-densities. Hence, we introduce the operator

$$W_{1/2} := (\det g_{\kappa\lambda})^{1/4}\, W\, (\det g_{\mu\nu})^{-1/4} \tag{A.19}$$

which maps half-densities to half-densities. We call the operator (A.19) *the massless Dirac operator on half-densities*.

B The spectrum for the torus and the sphere

In this appendix we examine the massless Dirac operator on the unit torus $\mathbb{T}^3$ and the unit sphere $\mathbb{S}^3$ and compare our asymptotic formulae (1.22) and (1.23) with known explicit formulae. The torus is assumed to be equipped with Euclidean metric (see also Appendix A) whereas the sphere is assumed to be equipped with metric induced by the natural embedding of $\mathbb{S}^3$ in Euclidean space $\mathbb{R}^4$. Note that in view of the obvious symmetries of the torus and the sphere the scalar function $e(\lambda, x, x)/\sqrt{\det g_{\alpha\beta}(x)}$ is constant (see also Remark 1.2), so formulae (1.22) and (1.23) are in this case equivalent, in the sense that they follow from one another. Hence, we will be dealing with formula (1.23) only.

We have $\operatorname{Vol}\mathbb{T}^3 = (2\pi)^3$, so for the torus formula (1.23) reads

$$N(\lambda) = \frac{4}{3}\pi\lambda^3 + o(\lambda^2). \tag{B.1}$$

The nonperiodicity condition (see Definitions 8.3 and 8.4 in [10]) is fulfilled for the torus, so, according to Theorem 8.4 from [10], the asymptotic formula (B.1) holds as it is, without mollification.

In order to test formula (B.1) we calculate the spectrum of the massless Dirac operator on $\mathbb{T}^3$ explicitly. We do this first for the spin structure associated with the frame (A.16). Then the spectrum is as follows.

- Zero is an eigenvalue of multiplicity two.
- For each $m \in \mathbb{Z}^3 \setminus \{0\}$ we have the eigenvalue $\|m\|$ and unique (up to rescaling) eigenfunction, with eigenfunctions corresponding to different m being linearly independent.

- For each $m \in \mathbb{Z}^3 \setminus \{0\}$ we have the eigenvalue $-\|m\|$ and unique (up to rescaling) eigenfunction, with eigenfunctions corresponding to different m being linearly independent.

Hence, $N(\lambda) + 1$ is the number of integer lattice points inside a 2-sphere of radius λ in $\mathbb{R}^3$ centred at the origin. According to [17] the latter admits the asymptotic expansion

$$\frac{4}{3}\pi\lambda^3 + O_\varepsilon(\lambda^{21/16+\varepsilon}) \tag{B.2}$$

as $\lambda \to +\infty$, with ε being an arbitrary positive number. This agrees with our asymptotic formula (B.1).

Let us now consider the spin structure associated with the frame (A.17). Then the spectrum is as follows.

- For each $m \in \mathbb{Z}^3$ we have the eigenvalue $\|m - (0, 0, 1/2)\|$ and unique (up to rescaling) eigenfunction, with eigenfunctions corresponding to different m being linearly independent.
- For each $m \in \mathbb{Z}^3$ we have the eigenvalue $-\|m-(0, 0, 1/2)\|$ and unique (up to rescaling) eigenfunction, with eigenfunctions corresponding to different m being linearly independent.

Hence, $N(\lambda)$ is the number of integer lattice points inside a 2-sphere of radius λ in $\mathbb{R}^3$ centred at $(0, 0, 1/2)$. Here the sphere is shifted from the origin so one cannot apply the result from [17]. However, as the shift is rational, one can reduce the problem to counting integer lattice points in a rational ellipsoid centred at the origin, and an application of the result from [8] gives us for the shifted sphere the same asymptotic expansion (B.2) as for the sphere centred at the origin.

As explained in Appendix A, the unit torus $\mathbb{T}^3$ admits a total of eight different spin structures. For each of these the problem of counting positive eigenvalues of the massless Dirac operator reduces to counting integer lattice points inside a 2-sphere of radius λ in $\mathbb{R}^3$ (possibly, shifted from the origin by a rational shift), so in all eight cases we do get (B.1). In fact, we can replace the remainder $o(\lambda^2)$ in (B.1) by $O_\varepsilon(\lambda^{21/16+\varepsilon})$ and this holds for all eight different spin structures.

In the remainder of this appendix we examine the massless Dirac operator on the unit sphere $\mathbb{S}^3$. We have $\operatorname{Vol}\mathbb{S}^3 = 2\pi^2$, so for the sphere formula (1.23) reads

$$N(\lambda) = \frac{\lambda^3}{3} + o(\lambda^2). \tag{B.3}$$

The nonperiodicity condition fails for the sphere because all geodesics are closed with period 2π, so formula (B.3) cannot be used in its original form and has to be mollified, see Remark 1.1. We will deal with the mollification issue later and give explicit formulae for the eigenvalues first.

It is known that $\mathbb{S}^3$ admits a unique spin structure, see Section 5 in [2]. The spectrum of the massless Dirac operator on $\mathbb{S}^3$ has been computed by different authors using different methods [26, 27, 1, 2] and reads as follows: the eigenvalues are

$$\pm\left(k+\frac{1}{2}\right), \qquad k=1,2,\ldots, \tag{B.4}$$

and their multiplicity is

$$k(k+1). \tag{B.5}$$

The mollification procedure from Section 7 of [10] goes as follows. Put $N(\lambda):=0$ for $\lambda\le 0$ and take an arbitrary real-valued even function $\rho(\lambda)$ from Schwartz space $\mathcal{S}(\mathbb{R})$ whose Fourier transform $\hat\rho(t)$ satisfies conditions $\hat\rho(0)=1$ and $\operatorname{supp}\hat\rho\subset(-2\pi,2\pi)$. Then, according to Theorem 7.2 from [10], the mollified version of formula (B.3) reads

$$\int N(\lambda-\mu)\,\rho(\mu)\,d\mu=\frac{\lambda^3}{3}+O(\lambda)$$

and this result holds notwithstanding the failure of the nonperiodicity condition. However, for the sphere there is a much simpler way of testing our asymptotic formula. Let $\lambda\ge 2$ be integer. Taking an integer λ puts us exactly in the middle of the gap between two consecutive clusters of eigenvalues, see formulae (B.4) and (B.5), and achieves the same averaging effect as convolution with a function from Schwartz space. For integer $\lambda\ge 2$ we get

$$N(\lambda)=\sum_{k=1}^{\lambda-1}k(k+1)=\frac{\lambda^3}{3}-\frac{\lambda}{3}$$

which agrees with our asymptotic formula (B.3).

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