# CHARACTER FORMULAS ON COHOMOLOGY OF DEFORMATIONS OF HILBERT SCHEMES OF $K 3$ SURFACES 

LETAO ZHANG


#### Abstract

Let $X$ be a hyperkähler manifold deformation equivalent to Hilbert scheme of $n$ points on a $K 3$ surface. We compute the graded character formula of the generic Mumford-Tate group representation on the cohomology ring of $X$, and derive a generating series for deducing the number of canonical Hodge classes on $X$. The formula indicates the number of Hodge classes on $X$ that remain Hodge under any deformation.


2010 Mathematics Subject Classification 14Q15 (primary), 14J28, 14C05, 14C25 (secondary)

## Contents

1. Introduction ..... 1
2. Cohomology of Hilbert Schemes of Points on $K 3$ surfaces ..... 33. Decomposition of Cohomology Ring
15
3. Generating Series for the Character of the Middle Cohomology ..... 13
Appendix A. Table of $G_{X}$ Representations ..... 15
Appendix B. Number of Canonical Hodge Classes ..... 17
References ..... 18

## 1. Introduction

Let $S$ be a $K 3$ surface, and $S^{[n]}$ the Hilbert scheme of $n$ points on $S$; each point in $S^{[n]}$ corresponds to a subscheme of $S$ whose Hilbert polynomial is the constant $n$. We say $X$ is of $K 3^{[n]}$-type if $X$ is hyperkähler and deformation equivalent to $S^{[n]}$.

Denote by $G_{X}$ the generic Mumford-Tate group of hyperkähler manifolds of $K 3^{[n]}$-type. The invariants of $G_{X}$ action on $H^{*}(X, \mathbb{Q})$ correspond to the canonical Hodge classes (see Section 3.2), which are Hodge classes that remain Hodge under any deformation. Chern classes of the tangent bundle $\mathcal{T}_{X}$ are examples of such canonical Hodge classes.

Now we consider the action of $G_{X}$ on the cohomology of $X$. In particular, we want to compute the characters of $G_{X}$ representation on the middle cohomology of $X$, where $X$ is of $K 3^{[n]}$-type.

The lattice of $H^{2}(X, \mathbb{Z})$ - with respect to the Beauville-Bogomolov form is $\Lambda_{S} \oplus \delta \mathbb{Z}$. Here $\Lambda_{S}:=U^{3} \oplus E_{8}^{2}(-1)$ is the lattice of $H^{2}(S, \mathbb{Z})$ where $S$ is a $K 3$ surface, and $(\delta, \delta)=-2(n-1)$. Let $G_{S}$ be the identity component of $O^{+}\left(H^{2}(S, \mathbb{Q})\right)$ with respect to the intersection form. The action of maximal torus $T_{X}$ of $G_{X}$ on $\Lambda_{S}$ is the same as the action of maximal torus $T_{S}$ of $G_{S}$ on $\Lambda_{S}$.
Theorem 1. Let $M(q):=\sum_{n=0}^{\infty} \operatorname{Char}\left(H^{2 n}(X, \mathbb{Q})\right) \cdot q^{n}$ be the generating series for the character of the $G_{X}$ representation on the middle cohomology of $X$.

$$
M(q)=\left(1+\sum_{k=1} 2(-1)^{k} q^{\frac{k(k+1)}{2}}\right)\left(\prod_{m=1} \operatorname{det}\left(I_{24}-g q^{m}\right)\right)^{-1}
$$

where $g \in T_{X}$ the maximal torus of $G_{X}, I_{N}$ is a $N \times N$ identity matrix, and $\operatorname{det}\left(I_{24}-g t^{m}\right)=\left(1-t^{m}\right)^{2} \operatorname{det}\left(I_{22}-\left.g\right|_{T_{S}} t^{m}\right)$.
Example 1.0.1. Let $X$ be of $K 3^{[7]}$-type. There are 7 Hodge classes in $H^{14}(X, \mathbb{Q})$ that remain Hodge under any deformation. Similarly, there are 5 Hodge classes in $H^{8}(X, \mathbb{Q}), 5$ Hodge classes in $H^{10}(X, \mathbb{Q})$, and 10 Hodge classes in $H^{12}(X, \mathbb{Q})$ that remain Hodge under any deformation. (cf. Appen$\operatorname{dix}$ A)

Now let $l \in H_{2}(X, \mathbb{Z})$ be a line class in $\mathbb{P}^{n} \subset X$. Hassett and Tschinkel in [13] show that $(l, l)=-\frac{5}{2}$ for the case where $n=2$ in [12]. For $n=3$, Harvey, Hassett and Tschinkel [11] show that $(l, l)=-3$ and give a concrete expression for the Lagrangian hyperplane class. For the case of $n=4$, Bakker and Jorza [2] show that $(l, l)=-\frac{7}{2}$, and also give an expression for $\left[\mathbb{P}^{4}\right]$. For $n \geq 5$, Bakker [3] shows that $(l, l)=-\frac{n+3}{2}$, which was conjectured in [13]. However, it is more difficult to compute the class $\left[\mathbb{P}^{n}\right]$ for larger $n$. One possible approach to exploring the expression for $\left[\mathbb{P}^{n}\right]$ is to find all the canonical Hodge classes in the middle cohomology of $X$ for each $n$; future work could provide possible candidates for the class of $\left[\mathbb{P}^{n}\right]$ in terms of the line class. As for the ring structure, Verbitsky [26] shows that there is an embedding $\operatorname{Sym}^{n} H^{2}(S, \mathbb{Q}) \hookrightarrow H^{2 n}\left(S^{[n]}, \mathbb{Q}\right)$, but much about the ring structure of $H^{*}(X, \mathbb{Q})^{G_{X}}$ is still unknown, e.g. relations in the subalgebra generated by $H^{*}(S, \mathbb{Q})$ for each $H^{*}(X, \mathbb{Q})^{G_{X}}$.

Acknowledgments. I am very grateful to my advisor Brendan Hassett for introducing me to this problem, and for his warm support and encouragement. Thanks are also due to Lothar Göttsche, Radu Laza, and Anthony VárillyAlvarado for interesting discussions and insightful remarks. I appreciate Eyal

Markman's illuminating questions which may lead to future research topics. The writing of this paper was supported in part by NSF grant 0901645 and 0968349.

## 2. Cohomology of Hilbert Schemes of Points on $K 3$ surfaces

In this section, we review some classical results about $S^{[n]}$, where $S$ is a $K 3$ surface. For $n>1$, the Beauville-Bogomolov form can be written as the direct sum (4]

$$
\begin{equation*}
H^{2}\left(S^{[n]}, \mathbb{Z}\right)=H^{2}(S, \mathbb{Z})_{(, ~)} \oplus_{\perp} \mathbb{Z} \delta,(\delta, \delta)=-2(n-1) \tag{2.1}
\end{equation*}
$$

where (, ) is the intersection form on $H^{2}(S, \mathbb{Z})$, and $2 \delta$ is the class of the corresponding big diagonal divisor $\Delta^{[n]} \subset S^{[n]}$ parameterizing nonreduced subschemes.

In [23], Nakajima constructs generators for the cohomology ring of Hilbert schemes of points of any projective surface. Lehn and Sorger [16] then show how $H^{*}(S, \mathbb{Q})$ generates $H^{*}\left(S^{[n]}, \mathbb{Q}\right)$ as a graded ring.

Let $A=H^{*}(S, \mathbb{Q})[2]$ denote the shifted cohomology ring weighted by $-2,0$, 2. Correspondingly, let $\mathbb{H}_{n}=H^{*}\left(S^{[n]}, \mathbb{Q}\right)[2 n]$ denote the shifted cohomology ring weighted by $-2 n, \ldots, 2 n$. Note that the weight shifting here is not the Tate twist notation for Hodge classes.

Define a linear form $T$ on $A$ by $T(a):=-\int_{[S]} a$, and let $\langle$,$\rangle be the induced$ bilinear form on the shifted cohomology $\left\langle a_{1}, a_{2}\right\rangle=T\left(a_{1} a_{2}\right)=-\int_{S} a_{1} a_{2}$. On $A^{\otimes n}$, one can define an analogous structure. Since $A$ and $\mathbb{H}_{n}$ have only graded pieces of even weights, we can simplify the algebraic model in [16].

The product is given by

$$
\left(a_{1} \otimes \cdots \otimes a_{n}\right) \cdot\left(b_{1} \otimes \cdots \otimes b_{n}\right)=\left(a_{1} b_{1}\right) \otimes \cdots \otimes\left(a_{n} b_{n}\right) .
$$

$T$ extends to $A^{\otimes n}$ via

$$
T\left(a_{1} \otimes \cdots \otimes a_{n}\right)=T\left(a_{1}\right) \cdots \cdots T\left(a_{n}\right),
$$

and the bilinear form $\langle$,$\rangle on A^{\otimes n}$ is defined accordingly:

$$
\langle a, b\rangle=T(a) T(b)
$$

We also have the symmetric group $\mathfrak{S}_{n}$ action on the $n$-fold tensor given by

$$
\pi\left(a_{1} \otimes \cdots \otimes a_{n}\right)=a_{\pi^{-1}(1)} \otimes \cdots \otimes a_{\pi^{-1}(n)} .
$$

For any partition $n=n_{1}+\cdots+n_{k}$, we have a homomorphism

$$
\begin{aligned}
A^{\otimes n} & \rightarrow A^{\otimes k} \\
a_{1} \otimes \cdots \otimes a_{n} & \mapsto\left(a_{1} \cdots a_{n_{1}}\right) \otimes \cdots \otimes\left(a_{n_{1}+\cdots+n_{k-1}+1} \cdots a_{n_{k}}\right)
\end{aligned}
$$

Given a finite set $I$ with $n$ elements, let $\left\{A_{i}\right\}_{i \in I}$ be a family of copies of $A$ indexed by $I$. Let $[n]$ denote $\{1, \ldots, n\}$; we define

$$
A^{\otimes I}:=\left(\underset{f:[n] \xrightarrow{\cong} I}{\bigoplus_{f(1) \otimes \cdots \otimes f(n)}} A_{f} / \mathfrak{S}_{n}\right.
$$

Finally, given a surjection $\phi: I \rightarrow J$ between two index sets, there is an induced multiplication

$$
\phi^{*}: A^{\otimes I} \rightarrow A^{\otimes J}
$$

and let

$$
\phi_{*}: A^{\otimes J} \rightarrow A^{\otimes I}
$$

be the adjoint of $\phi^{*}$, i.e.

$$
\left\langle\phi^{*} a, b\right\rangle=\left\langle a, \phi_{*} b\right\rangle,
$$

where $a \in A^{\otimes I}, b \in A^{\otimes^{J}}$. The projection formula

$$
\phi_{*}\left(a \cdot \phi^{*}(b)\right)=\phi_{*}(a) \cdot b
$$

holds by [16].
Denote by $\langle\pi\rangle \backslash[n]$ the set of orbits of $[n]$ under the action of $\pi$. Define

$$
A\left\{\mathfrak{S}_{n}\right\}:=\oplus_{\pi \in \mathfrak{S}_{n}} A^{\otimes\langle\pi\rangle \backslash[n]} \cdot \pi
$$

$A\left\{\mathfrak{S}_{n}\right\}$ admits an action of $\sigma \in \mathfrak{S}_{n}$, induced by the bijection

$$
\sigma:\langle\pi\rangle \backslash[n] \rightarrow\left\langle\sigma \pi \sigma^{-1}\right\rangle \backslash[n], x \mapsto \sigma x
$$

This gives an automorphism of $A\left\{\mathfrak{S}_{n}\right\}$ given by

$$
\widetilde{\sigma}: a \cdot \pi \mapsto \sigma^{*}\left(\sigma \pi \sigma^{-1}\right)
$$

Denote by $A^{[n]}$ the invariants under this action, then we have the graded isomorphism between the vector spaces [16]

$$
A^{[n]}=\sum_{\|\alpha\|=n} \bigotimes_{i} \operatorname{Sym}^{\alpha_{i}} A
$$

where $\alpha=\left(1^{\alpha_{1}}, 2^{\alpha_{2}}, \ldots, n^{\alpha_{n}}\right)$ runs all partitions of $n$ and $\|\alpha\|=\sum_{i=1}^{n} i \alpha_{i}$. For the case of $K 3$ surfaces, Lehn and Sorger prove

Theorem 2. [16] There is a canonical isomorphism of graded rings

$$
\left(H^{*}(S, \mathbb{Q})[2]\right)^{[n]} \xrightarrow{\cong} H^{*}\left(S^{[n]}, \mathbb{Q}\right)[2 n] .
$$

## 3. Decomposition of Cohomology Ring

In this section, we first review some useful results from representation theory. We then discuss generic Mumford-Tate group actions on the cohomology ring of $X$ of $K 3^{[n]}$-type. Finally, we introduce canonical Hodge classes as the invariants of the group action. Our goal is to decompose $H^{*}(X, \mathbb{Q})$ into irreducible representations and to count invariants.
3.1. Characters of Representations. We summarize general results on representations of complex (or split) orthogonal groups from [8].

Let $\mathfrak{g}$ be a semisimple Lie algebra, $\Lambda$ be its weight lattice, and $\mathbb{Z}[\Lambda]$ be the integral group ring of the abelian group $\Lambda$. For each weight $\lambda \in \Lambda$, let $e(\lambda)$ denote the basis element in $\mathbb{Z}[\Lambda]$, so that each element in $\mathbb{Z}[\Lambda]$ can be written as the finite sum $\sum_{\lambda} n_{\lambda} \cdot e(\lambda)$. Denote by $R(\mathfrak{g})$ the ring of isomorphism classes of finite-dimensional representations associated to $\mathfrak{g}$. For each class $[V]$, $[V]=\left[V^{\prime}\right]+\left[V^{\prime \prime}\right]$ whenever $V=V^{\prime} \oplus V^{\prime \prime}$, and the product of two classes is defined as $[V] \cdot[W]=[V \otimes W]$. Define the character homomorphism

$$
\text { Char : } R(g) \rightarrow \mathbb{Z}[\Lambda]
$$

by Char $[V]=\sum \#\left(V_{\lambda}\right) \cdot e(\lambda)$, where $V_{\lambda}$ is the weight space of $V$ for the weight $\lambda$ and $\#\left(V_{\lambda}\right)$ is the multiplicity of $V_{\lambda}$ in $V$. The Weyl group $\mathfrak{W}$ acts on $\mathbb{Z}[\Lambda]$ and the image of Char is contained in the ring of invariants $\mathbb{Z}[\Lambda]^{2 \mathcal{V J}}$.

Let $\omega_{1}, \ldots, \omega_{n}$ be fundamental weights of $\mathfrak{g}$. Recall that fundamental weights have the property that any highest weight may be expressed uniquely as a nonnegative integral linear combination of them; they are free generators for the lattice $\Lambda$. Let $\Gamma_{i}(i=1, \ldots, n)$ be the classes in $R(\mathfrak{g})$ of the irreducible representations of highest weight $\omega_{i}(i=1, \ldots, n)$. We have the following theorem.

Theorem 3. [8] The representation ring $R(\mathfrak{g})$ is a polynomial ring on the variables $\Gamma_{1}, \ldots, \Gamma_{n}$, and the homomorphism Char : $R(\mathfrak{g}) \rightarrow \mathbb{Z}[\Lambda]^{\mathfrak{W J}}$ is an isomorphism.

Thus decomposing $V$ into irreducible $\mathfrak{g}$ representations is equivalent to finding its character polynomial.

Example 3.1.1. [8] Let $\mathfrak{g}=\mathfrak{s o}_{2 n} \mathbb{C}$ and $V \cong \mathbb{C}^{2 n}$ be its standard representation. Its weight lattice $\Lambda$ is $\operatorname{span}\left\{L_{1}, \ldots, L_{n},\left(\sum L_{i}\right) / 2\right\}$ (see Lecture 19 in [8] for detailed explanation). For $\mathfrak{s o}_{2 n} \mathbb{C}$, fundamental weights are
$L_{1}, L_{1}+L_{2}, \ldots, L_{1}+\cdots+L_{n-2},\left(L_{1}+\cdots+L_{n}\right) / 2,\left(L_{1}+\cdots+L_{n-1}-L_{n}\right) / 2$
corresponding to irreducible representations $V, \bigwedge^{2} V, \ldots, \bigwedge^{n-2} V$ and the halfspin representations $S^{+}$and $S^{-}$. Set $t_{i}=e\left(L_{i}\right), t_{i}^{-1}=e\left(-L_{i}\right), t_{i}^{+1 / 2}=e\left(L_{i} / 2\right)$, $t_{i}^{-1 / 2}=e\left(-L_{i} / 2\right), \operatorname{Char}\left(\bigwedge^{k} V\right)$ is the $k$-th elementary symmetric polynomial

- denoted by $D_{k}$ - of the $2 n$ elements $t_{1}, t_{1}^{-1}, \ldots, t_{n}, t_{n}^{-1}$. The character $D^{+}\left(\right.$resp. $\left.D^{-}\right)$of $S^{+}\left(\right.$resp. $\left.S^{-}\right)$is the sum $\sum t_{1}^{ \pm 1 / 2} \cdots \cdots t_{n}^{ \pm 1 / 2}$, where the number of plus signs is even (resp. odd). Thus,

$$
R\left(\mathfrak{s o}_{2 n} \mathbb{C}\right)=\mathbb{Z}[\Lambda]^{\mathfrak{W J}}=\mathbb{Z}\left[D_{1}, \ldots, D_{n-2}, D^{+}, D^{-}\right]
$$

Example 3.1.2. [8] In the case of $\mathfrak{g}=\mathfrak{s o}_{2 n+1} \mathbb{C}$, its standard representation is $V \cong \mathbb{C}^{2 n+1}$ and its weight lattice is the same as $\mathfrak{s o}_{2 n} \mathbb{C}$. But the fundamental weights are

$$
L_{1}, L_{1}+L_{2}, \ldots, L_{1}+L_{2}+\cdots+L_{n-1},\left(L_{1}+\cdots+L_{n}\right) / 2
$$

corresponding to irreducible representations $V, \bigwedge^{2} V, \ldots, \bigwedge^{n-1} V$ and the spin representation $S . \operatorname{Char}\left(\bigwedge^{k} V\right)$ here is the $k$-th elementary symmetric polynomial - denoted by $B_{k}$ - of $2 n+1$ elements $t_{1}, t_{1}^{-1}, \ldots, t_{n}, t_{n}^{-1}$ and 1. Denote by $B_{n}$ the character of $S$, which is the $n$-th symmetric polynomial in variables $t_{i}^{\frac{1}{2}}+t_{i}^{-\frac{1}{2}}$. By applying Theorem 3 we obtain

$$
R\left(\mathfrak{s o}_{2 n+1} \mathbb{C}\right)=\mathbb{Z}[\Lambda]^{\mathfrak{W}}=\mathbb{Z}\left[B_{1}, \ldots, B_{n-1}, B_{n}\right]
$$

If $\Gamma_{\lambda}$ is an irreducible $\mathfrak{s o}_{2 n+1} \mathbb{C}$ representation of highest weight $\lambda=\left(\lambda_{1} \geq\right.$ $\cdots \geq \lambda_{n} \geq 0$ ), then its image in $\mathbb{Z}\left[B_{1}, \ldots, B_{n}\right]$ is $B_{1}^{\lambda_{1}-\lambda_{2}} B_{2}^{\lambda_{2}-\lambda_{3}} \cdots B_{n-1}^{\lambda_{n-1}-\lambda_{n}} B_{n}^{\lambda_{n}}$. In general, we have $\mathfrak{s o}_{2 n} \mathbb{C} \subset \mathfrak{s o}_{2 n+1} \mathbb{C}$, and the restriction representation of $\Gamma_{\lambda}$ is

$$
\operatorname{Res}_{\mathfrak{s o}_{2 n} \mathbb{C}}^{\mathfrak{s o s}_{2 n+1} \mathbb{C}} \Gamma_{\lambda}=\oplus_{\bar{\lambda}} \Gamma_{\bar{\lambda}}
$$

where $\bar{\lambda}=\left(\bar{\lambda}_{1}, \ldots, \bar{\lambda}_{n}\right)$ satisfies

$$
\lambda_{1} \geq \bar{\lambda}_{1} \geq \lambda_{2} \geq \bar{\lambda}_{2} \geq \cdots \geq \bar{\lambda}_{n-1} \geq \lambda_{n} \geq\left|\bar{\lambda}_{n}\right|
$$

and $\bar{\lambda}_{i}$ and $\lambda_{i}$ are either all integers or all half integers.
Given a finite dimensional $\mathfrak{s o}_{2 n+1} \mathbb{C}$ representation $W$, if it is induced by the inclusion $\mathfrak{s o}_{2 n} \mathbb{C} \subset \mathfrak{s o}_{2 n+1} \mathbb{C}$ and all the weights of $\mathfrak{s o}_{2 n} \mathbb{C}$ representation are integer-valued, then so are the weights of the $\mathfrak{s o}_{2 n+1} \mathbb{C}$ representation. This implies that the character of the $\mathfrak{s o}_{2 n+1} \mathbb{C}$ representation will be in $\mathbb{Z}\left[B_{1}, \ldots, B_{n-1}\right]$.
3.2. Group actions on cohomologies. Let $V$ be a $\mathbb{Q}$ vector space, and $\mathbb{S}(\mathbb{R}) \cong \mathbb{C}^{*}$ regarded as a Lie group. $S^{1}$ is a maximal compact subgroup of $\mathbb{S}(\mathbb{R})$.

Definition 3.2.1. ([10], I.A) A Hodge structure of weight $n$ is given by a representation on $V_{\mathbb{R}}:=V \otimes_{\mathbb{Q}} \mathbb{R}$

$$
\tilde{\varphi}: \mathbb{S}(\mathbb{R}) \rightarrow \mathrm{GL}\left(V_{\mathbb{R}}\right)
$$

such that for $r \in \mathbb{R}^{*} \subset \mathbb{S}(\mathbb{R}), \tilde{\varphi}(r)=r^{n} i d_{V}$.
This definition is equivalent to giving a Hodge decomposition of $V_{\mathbb{C}}:=V \otimes_{\mathbb{Q}} \mathbb{C}$, where

$$
V_{\mathbb{C}}=\oplus V^{p, q}, \text { and } V^{p, q}=\bar{V}^{q, p} .
$$

For $\varphi:=\left.\tilde{\varphi}\right|_{S^{1}}$, we obtain a representation

$$
\varphi: S^{1} \rightarrow \mathrm{SL}(V)(\mathbb{R})
$$

given by $\varphi\left(e^{i \theta}\right) v=e^{i \theta(p-q)} v$, for $v \in V^{p, q}$.
Definition 3.2.2. [10] The Mumford-Tate group $M_{\varphi}$ associated to a Hodge structure $(V, \varphi)$ of weight $n$ is the $\mathbb{Q}$-algebraic closure of

$$
\varphi: S^{1} \rightarrow \mathrm{SL}\left(V_{\mathbb{R}}\right)
$$

For a pure Hodge structure of even weight $2 p$, Hodge classes are defined as those lying in the intersection $V \cap V_{\mathbb{C}}^{p, p}$. Let $T^{k, l}=V^{\otimes k} \otimes \check{V}^{\otimes l}$ and $H g\left(V_{\varphi}\right)$ denote the direct sum of all Hodge classes in $T^{k, l}$ for all pairs of $(k, l)$. It is known that $M_{\varphi}$ fixes $\operatorname{Hg}\left(V_{\varphi}\right)$ [10].

Let $V$ denote the weight two Hodge structure on $H^{2}(X, \mathbb{Z})$, Theorem 2.2.1 in [28] indicates that

Proposition 1. For generic $X$, the Mumford-Tate group $G_{X}=S O(V,\langle\rangle$,$) ,$ where $\langle$,$\rangle is the Beauville-Bogomolov form.$

Definition 3.2.3. [19, [22] An automorphism $g$ of $H^{*}(X, \mathbb{Q})$ is called a monodromy operator (equivalently, a parallel transport operator) if there exists a smooth and proper family $\mathcal{M} \rightarrow B$ (which may depend on $g$ ) of irreducible holomorphic symplectic manifolds over a (possibly singular) complex analytic space $B$, having $X$ as a fiber over a point $b \in B$, and such that $g$ belongs to the image of $\pi_{1}(B, b)$ under the monodromy representation. The monodromy group $\operatorname{Mon}(X) \in G L\left(H^{*}(X, \mathbb{Q})\right)$ is generated by all the monodromy operators. In this context, the algebraic monodromy group $\overline{\operatorname{Mon}}(X)$ is defined as the smallest $\mathbb{Q}$-algebraic group in $G L\left(H^{*}(X, \mathbb{Q})\right)$ that contains $\operatorname{Mon}(X)$. Denote by $\overline{M o n}^{2}(X)$ the image of $\overline{M o n}(X)$ in the isometry group of $H^{2}(X, \mathbb{Q})$.

Proposition 2. Let $X$ be of $K 3^{[n]}$-type, and assume $X$ is a very general fibre in the universal family $\mathcal{X} \rightarrow B$. Let $\overline{M o n}_{0}^{2}(X)$ be the identity component of $\overline{\operatorname{Mon}}^{2}(X)$, then we have $\overline{M o n}_{0}^{2}(X)=G_{X}$.

Proof. Theorem 16 in [25] shows that any connected component of $\overline{M o n}^{2}(X)$ is a normal subgroup of the derived group of $G_{X}$. In particular, we have $\overline{M o n}_{0}^{2}(X) \subset G_{X}$.
Consider the map $\iota: O^{+}\left(H^{2}(X, \mathbb{Z})\right) \rightarrow O\left(H^{2}(X, \mathbb{Z})^{*} / H^{2}(X, \mathbb{Z})\right)$, Lemma 4.2 in [20] shows that $\operatorname{Mon}^{2}(X)$ is the inverse image of the subgroup $\{1,-1\}$ under $\iota$. Thus $S O(V,\langle\rangle,) \subset \overline{M o n}_{0}^{2}(X)$. Since $G_{X}=S O(V,\langle\rangle$,$) by Proposition [1,$ $\overline{M o n}_{0}^{2}(X)=G_{X}$.

Example 3.2.4. The simplest case is when $X$ is of $K 3$-type. The monodromy group $\overline{M o n}^{2}(X)$ is $\mathcal{O}^{+}\left(H^{2}(X, \mathbb{Q})\right)$ [5. Thus its identity component $\overline{M o n}_{0}^{2}(X)$ is $S O\left(H^{2}(X, \mathbb{Q})\right)$.

Definition 3.2.5. Let $X$ be of $K 3^{[n]}$-type, and assume $X$ is a very general fibre in the universal family $\mathcal{X} \rightarrow B$. Canonical Hodge classes of $X$ are Hodge classes that remain Hodge under any deformation.
Theorem 4. Let $X$ be of $K 3^{[n]}$-type, and assume $X$ is a very general fibre over $b$ in the universal family $\mathcal{X} \rightarrow B$. Canonical Hodge classes of $X$ are exactly the invariant classes $H^{*}(X, \mathbb{Q})^{G_{X}}$.

Proof. Since canonical Hodge classes are Hodge classes, they are contained in $H^{*}(X, \mathbb{Q})^{G_{X}}$.

Given any very general fibre $X^{\prime}$, let $\gamma \subset B$ be a path such that $\gamma(0)=b$ and $\gamma(1)=b^{\prime}$ where $\mathcal{X}_{b^{\prime}}=X^{\prime}$. Markman [22] shows that $\overline{M o n}^{2}(X)$ is a normal subgroup of $\overline{\operatorname{Mon}}(X)$, thus $\gamma \overline{M o n}_{0}^{2}(X) \bar{\gamma} \subset \overline{M o n}^{2}\left(X^{\prime}\right)$. Since $\gamma \overline{M o n}_{0}^{2}(X) \bar{\gamma}$ is connected and contains identity, $\gamma \overline{M o n}_{0}^{2}(X) \bar{\gamma}=\overline{M o n}_{0}^{2}\left(X^{\prime}\right)=G_{X^{\prime}}$.

For any $\alpha \in H^{*}(X, \mathbb{Q})^{G_{X}}, \gamma \alpha \in H^{*}\left(X^{\prime}, \mathbb{Q}\right)$. Given $h^{\prime} \in G_{X^{\prime}}$, there exists $h \in G_{X}$ such that $h^{\prime}=\gamma h \bar{\gamma}$. This implies

$$
\begin{aligned}
h^{\prime}(\gamma \alpha) & =\gamma h \bar{\gamma}(\gamma \alpha) \\
& =\gamma h \alpha \\
& =\gamma \alpha
\end{aligned}
$$

Thus $\gamma \alpha \in H^{*}\left(X^{\prime}, \mathbb{Q}\right)^{G_{X^{\prime}}}$ is a Hodge class.
Now let $\mathcal{A} \in H^{*}(\mathcal{X}, \mathbb{Q})$ be the class such that $\mathcal{A}_{b}=\alpha \in H^{*}(\mathcal{X}, \mathbb{Q})$. For any very general fibre $\mathcal{X}_{b^{\prime}}, \mathcal{A}_{b^{\prime}} \in H^{*}\left(\mathcal{X}_{b^{\prime}}, \mathbb{Q}\right)$ is obtained by a path from $b$ to $b^{\prime}$. Passing through finite étale cover of $B, \mathcal{A}_{b^{\prime}}$ is unique. By the above analysis, $\mathcal{A}_{b}$ are Hodge for all very general fibres. By Deligne-Cattani-Kaplan theorem in [7], Hodge loci of $\mathcal{A}$ is closed. Thus $\mathcal{A}_{s}$ are Hodge for all special fibres $\mathcal{X}_{s}$. Thus $\alpha$ remains Hodge under any deformation.

Thus $H^{*}(X, \mathbb{Q})^{G_{X}}$ is the collection of all canonical Hodge classes.
Note that the same statement was proved in Lemma 3.2 of [21].
By the above analysis, the invariants of the $G_{X}$ representation on $H^{*}(X, \mathbb{Q})$ are the canonical Hodge classes. By computing number of trivial $G_{X}$ representations on $H^{2 p}(X, \mathbb{Q})$, we can obtain number of canonical Hodge classes of type $(p, p)$ of $X$.

Example 3.2.6. In Table 1 , the first row of data shows the number of canonical Hodge classes. For instance, if $X$ is of $K 3^{[5]}$-type, then there are 2 canonical Hodge classes in $H^{2,2}(X), 1$ in $H^{3,3}(X), 4$ in $H^{4,4}(X)$, and 2 in $H^{5,5}(X)$.
3.3. Decomposition of the Cohomology Representation. Denote by $G_{S}$ the identity component of the special orthogonal group associated with the intersection form on $H^{2}(S, \mathbb{Z})$. (cf. Example 3.2.4)

We can decompose $H^{*}(X, \mathbb{Q})$ into irreducible representations for the action of $G_{X}$. 11] provides an explicit method for writing the decomposition:
(1) Use the isomorphism $H^{2}(X, \mathbb{Z}) \cong \Lambda_{S} \oplus_{\perp} \mathbb{Z} \delta$ and compatible maximal torus of $G_{S}$ and $G_{X}$ to fix the embedding $G_{S} \subset G_{X}$. (cf. Section (1)
(2) Decompose $H^{*}\left(S^{[n]}, \mathbb{Q}\right)$ into the highest-weight $G_{S}$ representations using $H^{*}\left(S^{[n]}, \mathbb{Q}\right)[2 n] \cong\left(H^{*}(S, \mathbb{Q})[2]\right)^{[n]}$ in Theorem 2, The highestweight irreducible representation $V_{S}(\lambda)$ will lie in the summand of an irreducible $G_{X}$ representation $V_{X}(\lambda)$ in $H^{*}(X, \mathbb{Q})$.
(3) Repeat step 1 and 2 on $H^{*}(X, \mathbb{Q}) / V_{X}(\lambda)$.

Let $\mathbb{V}_{2 k, n}:=H^{2 k}(X, \mathbb{Q})$ where $X$ of $K 3^{[n]}$-type, $\mathbb{V}_{2 k, n}$ is of weight $2 k-2 n$ in $H^{*}(X, \mathbb{Q})[2 n]$. Let $V_{\lambda}$ be the irreducible $G_{X}$ representation of the highest weight $\lambda$; we get the following computational results

| $\lambda$ | $\operatorname{dim} V_{\lambda}$ | $V_{4,5}$ | $\mathbb{V}_{6,5}$ | $\mathbb{V}_{8,5}$ | $\mathbb{V}_{10,5}$ | $\mathbb{V}_{6,6}$ | $\mathbb{V}_{8,6}$ | $\mathbb{V}_{10,6}$ | $\mathbb{V}_{12,6}$ | $\ldots$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $(0,0,0, \ldots)$ | 1 | 2 | 1 | 4 | 2 | 2 | 5 | 4 | 7 | $\ldots$ |
| $(1,0,0, \ldots)$ | 23 | 1 | 3 | 3 | 5 | 3 | 4 | 7 | 7 | $\ldots$ |
| $(2,0,0, \ldots)$ | 275 | 1 | 1 | 3 | 2 | 1 | 4 | 4 | 7 | $\ldots$ |
| $(1,1,0, \ldots)$ | 253 |  | 1 | 1 | 2 | 1 | 1 | 3 | 2 | $\ldots$ |
| $(3,0,0, \ldots)$ | 2277 |  | 1 | 1 | 2 | 1 | 1 | 3 | 3 | $\ldots$ |
| $(2,1,0, \ldots)$ | 4025 |  |  | 1 | 1 |  | 1 | 2 | 2 | $\ldots$ |
| $(1,1,1, \ldots)$ | 1771 |  |  |  |  |  |  |  | 1 | $\ldots$ |
| $(4,0,0, \ldots)$ | 14674 |  |  | 1 |  |  | 1 | 1 | 2 | $\ldots$ |
| $(3,1,0, \ldots)$ | 256795 |  |  |  | 1 |  |  | 1 | 1 | $\ldots$ |
| $(2,2,0, \ldots)$ | 2193763 |  |  |  |  |  |  |  | 1 | $\ldots$ |
| $(5,0,0, \ldots)$ | 7804350225 |  |  |  | 1 |  |  | 1 |  | $\ldots$ |
| $(4,1,0, \ldots)$ | $\ldots$ |  |  |  |  |  |  | $\ldots$ | $\ldots$ | $\ldots$ |
| $(6,0,0, \ldots)$ | $\ldots$ |  |  |  |  |  |  |  | $\ldots$ | $\ldots$ |

Table 1: $G_{X}$ Representations

Here "..." denotes truncated data, and each integer denotes the number of times $V_{\lambda}$ appears in $\mathbb{V}_{2 k, n}=H^{2 k}(X, \mathbb{Q})$. In particular, the first row in the table indicates the number of copies of trivial $G_{X}$ representation in each $H^{2 k}(X, \mathbb{Q})$, and corresponds to the number of canonical Hodge classes.
$H^{2}(S, \mathbb{Q})$ corresponds to the standard $G_{S}$ representation $V_{S}(1)$. Let $\mathbb{H}_{n}=$ $H^{*}\left(S^{[n]}, \mathbb{Q}\right)[2 n]$, which is a bigraded algebra associated to $H^{*}\left(S^{[n]}, \mathbb{Q}\right)$. By Theorem 2, we can decompose $\mathbb{H}_{n}$ into $G_{S}$ representations. By using the same
convention as Example 3.1.1, we know that

$$
\operatorname{Char}_{G_{S}}\left(H^{2 k}\left(S^{[n]}, \mathbb{R}\right)\right) \in \mathbb{Z}\left[D_{1}, \ldots, D_{9}, D^{+}, D^{-}\right]
$$

Remark 1. When $k \leq 10$, spin representations do not appear in our representation as $\wedge^{k} V_{S}(1,0, \ldots)$ is irreducible (Theorem 19.2 [8]).

By compatibility of maximal tori of $G_{X}$ and $G_{S}$ (cf. Section (1), we have

$$
\begin{equation*}
\operatorname{Char}_{G_{S}}\left(H^{2 k}\left(S^{[n]}, \mathbb{R}\right)\right)=\operatorname{Char}_{G_{X}}\left(\mathbb{V}_{2 k, n}\right) \tag{3.1}
\end{equation*}
$$

By 3.1, we will not distinguish notations between $\operatorname{Char}_{G_{S}}$ and Char $_{G_{X}}$, and will denote both by Char in the following discussion. Let

$$
p(z)_{n}:=\sum_{k=-n}^{n} \operatorname{Char}\left(\mathbb{V}_{2 k+2 n, n}\right) \cdot z^{2 k} \in \mathbb{Z}\left[D_{1}, \ldots, D_{9}, D^{+}, D^{-}\right]\left[z, \frac{1}{z}\right]
$$

be the graded character of $H^{*}(X, \mathbb{R})$. Now we take the sum

$$
\begin{equation*}
p(z, t):=\sum_{n=0}^{\infty} p(z)_{n} t^{n} \tag{3.2}
\end{equation*}
$$

following the grading of each $\mathbb{V}_{\bullet, n}$.
Example 3.3.1. Consider $X$ of $K 3^{[3]}$-type, and $\mathbb{H}_{3}=H^{*}\left(S^{[3]}, \mathbb{Q}\right)[6]$. By the above analysis, we obtain

$$
\begin{aligned}
H^{0}\left(S^{[3]}, \mathbb{Q}\right)[6] & =1_{S} \\
H^{2}\left(S^{[3]}, \mathbb{Q}\right)[6] & =1_{S} \oplus V_{S}(1) \\
H^{4}\left(S^{[3]}, \mathbb{Q}\right)[6] & =1_{S}^{3} \oplus V_{S}(1)^{2} \oplus V_{S}(2) \\
H^{6}\left(S^{[3]}, \mathbb{Q}\right)[6] & =1_{S}^{3} \oplus V_{S}(1)^{3} \oplus V_{S}(2) \oplus V_{S}(1,1) \oplus V_{S}(3)
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
p(z)_{3}= & \sum_{k=-3}^{3} \operatorname{Char}_{G_{S}}\left(H^{6+2 k}\left(S^{[3]}\right)\right) \cdot z^{2 k} \\
= & z^{-6}+\left(1+D_{1}\right) z^{-4}+\left(3+2 D_{1}+D_{1}^{2}-D_{2}\right) z^{-2} \\
& +\left(3+3 D_{1}+D_{1}^{2}+D_{1}^{3}-2 D_{1} D_{2}+D_{3}\right) \\
& +z^{6}+\left(1+D_{1}\right) z^{4}+\left(3+2 D_{1}+D_{1}^{2}-D_{2}\right) z^{2}
\end{aligned}
$$

The induced $G_{X}$ representations are

$$
\begin{aligned}
& \mathbb{V}_{0,3}=H^{0}(X, \mathbb{Q})[6]=1_{X} \\
& \mathbb{V}_{2,3}=H^{2}(X, \mathbb{Q})[6]=V_{X}(1) \\
& \mathbb{V}_{4,3}=H^{4}(X, \mathbb{Q})[6]=1_{X} \oplus V_{X}(1) \oplus V_{X}(2) \\
& \mathbb{V}_{6,3}=H^{6}(X, \mathbb{Q})[6]=1_{X} \oplus V_{X}(1) \oplus V_{X}(1,1) \oplus V_{X}(3)
\end{aligned}
$$

The character formula is

$$
\begin{aligned}
p(z)_{3}= & \sum_{k=-3}^{3} \operatorname{Char}_{G_{X}}\left(\mathbb{V}_{2 k+6,3}\right) \cdot z^{2 k} \\
= & z^{-6}+B_{1} z^{-4}+\left(1+B_{1}+B_{1}^{2}-B_{2}\right) z^{-2}+\left(1+B_{1}+B_{2}+B_{1}^{3}-2 B_{1} B_{2}+B_{3}\right) \\
& +z^{6}+B_{1} z^{4}+\left(1+B_{1}+B_{1}^{2}-B_{2}\right) z^{2}
\end{aligned}
$$

and the number of canonical Hodge classes of $H^{k}(X, \mathbb{Q})$ corresponds to the constant term in the coefficient of $z^{k}$.

Remark 2. The computation in example 3.3 .1 is reversible. Explicitly, the number of copies of the highest-weight representation $V_{X}(\lambda)$ appearing in $\mathbb{V}_{n, k}$ corresponds to the coefficient of $\operatorname{Char}\left(V_{X}(\lambda)\right)$

Proposition 3. The character of the $G_{X}$ representation is

$$
\begin{equation*}
p_{g}(z, t):=\sum_{n=0}^{\infty} p(z)_{n} t^{n}=\prod_{m=1}^{\infty} \frac{1}{\operatorname{det}\left(I_{24}-g t^{m}\right)} \tag{3.3}
\end{equation*}
$$

where $g \in T_{X}$ the maximal torus of $G_{X}, I_{N}$ is a $N \times N$ identity matrix, and $\operatorname{det}\left(I_{24}-g t^{m}\right)=\left(1-z^{-2} t^{m}\right) \cdot\left(1-z^{2} t^{m}\right) \operatorname{det}\left(I_{22}-\left.g\right|_{T_{S}} t^{m}\right)$.

Proof. Recall that $\mathbb{V}=H^{*}(S, \mathbb{Q})[2]=H^{0}(S, \mathbb{Q})+H^{2}(S, \mathbb{Q})+H^{4}(S, \mathbb{Q})$ is bigraded, where $H^{0}(S, \mathbb{Q})$ and $H^{4}(S, \mathbb{Q})$ are of weight -2 and +2 respectively, and $H^{2}(S, \mathbb{Q})$ is of weight 0 . For every symmetric power $\operatorname{Sym}^{k}(\mathbb{V})$, it is sufficient to show the formula holds when $g$ is diagonal. Since $G_{S}$ acts trivially on $H^{0}$ and $H^{4}$, let $u_{-2} \in H^{0}(S, \mathbb{Q})$ be the eigenvector with eigenvalue 1 and weight -2 , and $u_{2} \in H^{4}(S, \mathbb{Q})$ be eigenvector with eigenvalue 1 and weight 2 . $H^{2}(S, \mathbb{Q})$ corresponds to the standard $G_{S}$ representation $V_{S}(1,0, \ldots, 0)$, and let $v_{i}(i=1, \ldots, 22)$ be its eigenvectors; when $i$ is even, $v_{i}$ has eigenvalue $t_{\frac{i}{2}}$, and when $i$ is odd, $v_{i}$ has eigenvalue $t_{\left\lfloor\frac{i}{2}\right\rfloor}^{-1}$.
Molien's Formula in [15] indicates that for a representation $W$ of a group $G$, and given a linear operator $g \in G$, its action on the symmetric algebra
$\operatorname{Sym}^{\bullet}(W)$ has the graded character

$$
\sum_{i=0}^{\infty} \operatorname{Char}\left(\operatorname{Sym}^{i}(g)\right) t^{i}=\frac{1}{\operatorname{det}(I-t g)}
$$

The symmetric algebra on $\mathbb{V}$, denoted by $\operatorname{Sym}^{\bullet}(\mathbb{V})$, has the form

$$
\operatorname{Sym}^{\bullet}\left(u_{2}\right) \otimes \operatorname{Sym}^{\bullet}\left(v_{1}\right) \otimes \operatorname{Sym}^{\bullet}\left(v_{2}\right) \otimes \cdots \otimes \operatorname{Sym}^{\bullet}\left(v_{22}\right) \otimes \operatorname{Sym}^{\bullet}\left(u_{-2}\right) .
$$

Since $v_{i} \in \mathbb{V}$ is of weight 0 , the bigraded character of $\operatorname{Sym}^{\bullet}\left(v_{i}\right)$ is

$$
\sum_{k=0}^{\infty}\left(\mu_{i} z^{0}\right)^{k} \cdot t^{k}=\frac{1}{1-\mu_{i} t}
$$

where $\mu_{i}$ is the eigenvalue of $v_{i}$. Since $u_{-2}$ (resp. $u_{2}$ ) has weight -2 (resp. weight +2 ) in $\mathbb{V}$, we have

$$
\begin{aligned}
\operatorname{Char}\left(\operatorname{Sym}^{\bullet}\left(u_{-2}\right)\right)(t) & =\sum_{k=0}^{\infty}\left(1 \cdot z^{-2}\right)^{k} t^{k}=\frac{1}{1-z^{-2} t}, \text { and } \\
\operatorname{Char}\left(\operatorname{Sym}^{\bullet}\left(u_{2}\right)\right)(t) & =\sum_{k=0}^{\infty}\left(1 \cdot z^{2}\right)^{k} t^{k}=\frac{1}{1-z^{2} t}
\end{aligned}
$$

Thus, we obtain

$$
\begin{equation*}
\operatorname{Char}\left(\operatorname{Sym}^{\bullet}(\mathbb{V})\right)(t)=\frac{1}{\left(1-z^{2} t\right)\left(1-z^{-2} t\right) \operatorname{det}\left(I_{22}-\left.g\right|_{T_{S}} t\right)} \tag{3.4}
\end{equation*}
$$

By Theorem 2, the graded character $p(z)_{n}$ of $H^{*}(X, \mathbb{Q})$ is given by

$$
\begin{aligned}
p(z)_{n} t^{n} & =\operatorname{Char}\left(\sum_{\|\alpha\|=n} \bigotimes_{i} \operatorname{Sym}^{\alpha_{i}} \mathbb{V}\right) t^{n} \\
& =\sum_{\|\alpha\|=n} \prod_{i=1}^{n}\left(\operatorname{Char}\left(\operatorname{Sym}^{\alpha_{i}} \mathbb{V}\right) t^{i \alpha_{i}}\right) .
\end{aligned}
$$

Note $\operatorname{Char}\left(\operatorname{Sym}^{\alpha_{i}} \mathbb{V}\right) t^{i \alpha_{i}}$ is the $\alpha_{i}$-th term in $\operatorname{Char}\left(\operatorname{Sym}^{\bullet}(\mathbb{V})\right)\left(t^{i}\right)$. Then for each $\alpha=\left(1^{\alpha_{1}}, 2^{\alpha_{2}}, \ldots, n^{\alpha_{n}}\right)$ where $\|\alpha\|=1 \alpha_{1}+\cdots+n \alpha_{n}=n$,

$$
\prod_{i=1}^{n}\left(\operatorname{Char}\left(\operatorname{Sym}^{\alpha_{i}} \mathbb{V}\right) t^{i \alpha_{i}}\right)
$$

corresponds to the $t^{n}$-th term in

$$
\prod_{j=1}^{n}\left(\operatorname{Char}\left(\operatorname{Sym}^{\bullet}(\mathbb{V})\right)\left(t^{j}\right)\right)
$$

Together with equation 3.4, one can obtain

$$
\sum_{n=0}^{\infty} p(z)_{n} t^{n}=\prod_{m=1}^{\infty} \frac{1}{\left(1-z^{2} t^{m}\right)\left(1-z^{-2} t^{m}\right) \operatorname{det}\left(I_{22}-\left.g\right|_{T_{S}} t^{m}\right)}
$$

Remark 3. Given a smooth projective complex surface $S^{\prime}$, let $p\left(S^{[n]}, z\right)$ be the Poincaré polynomial $\sum_{i=0}^{4 n} \beta_{i}\left(S^{\prime[n]}\right) z^{i}$ of $S^{\prime[n]}$. Göttsche [9] shows that $\sum_{n=0} p\left(S^{\prime[n]}, z\right) q^{n}$ has the expression

$$
\begin{equation*}
\prod_{m=1}^{\infty} \frac{\left(1+z^{2 m-1} q^{m}\right)^{b_{1}\left(S^{\prime}\right)}\left(1+z^{2 m+1} q^{m}\right)^{b_{1}\left(S^{\prime}\right)}}{\left(1-z^{2 m-2} q^{m}\right)^{b_{0}\left(S^{\prime}\right)}\left(1-z^{2 m} q^{m}\right)^{b_{2}\left(S^{\prime}\right)}\left(1-z^{2 m+2} q^{m}\right)^{b_{4}\left(S^{\prime}\right)}} \tag{3.5}
\end{equation*}
$$

For the case when $S^{\prime}$ is a $K 3$ surface, $b_{0}\left(S^{\prime}\right)=b_{4}\left(S^{\prime}\right)=1, b_{1}\left(S^{\prime}\right)=b_{3}\left(S^{\prime}\right)=0$, and $b_{2}\left(S^{\prime}\right)=22$. Letting $z^{2} q=t$, Equation 3.5 becomes

$$
\prod_{m=1}^{\infty} \frac{1}{\left(1-z^{-2} t^{m}\right)\left(1-t^{m}\right)^{22}\left(1-z^{2} t^{m}\right)}
$$

This is the same as taking the character of the identity in Proposition 3.

## 4. Generating Series for the Character of the Middle Cohomology

Proof of Theorem 1: For $X$ is of $K 3^{[n]}$-type, we consider the middle cohomology $H^{2 n}(X, \mathbb{Q})$ of $X$. The character of $H^{2 n}(X, \mathbb{Q})$ is of weight 0 in Equation 3.3 in Proposition 3. The character formula can be written as

$$
\begin{equation*}
\left(\prod_{m=1} \frac{\left(1-q^{m}\right)^{2}}{\left(1-z^{-2} q^{m}\right)\left(1-z^{2} q^{m}\right)}\right)\left(\prod_{k=1}\left(1-q^{k}\right)^{2} \cdot \operatorname{det}\left(I-\left.g\right|_{T_{S}} q^{k}\right)\right)^{-1} \tag{4.1}
\end{equation*}
$$

Lemma 1 in [1] indicates

$$
\begin{equation*}
\prod_{m=1} \frac{\left(1-q^{m}\right)^{2}}{\left(1-z^{-2} q^{m-1}\right)\left(1-z^{2} q^{m}\right)}=\sum_{\substack{N, r=-\infty \\ r \geq|N|}}^{\infty}(-1)^{r+N} z^{2 N} q^{\frac{r^{2}-N^{2}+r+N}{2}} \tag{4.2}
\end{equation*}
$$

Multiplying both sides of Equation 4.2 by $\left(1-z^{-2}\right)$, we obtain

$$
\begin{aligned}
& \prod_{m=1} \frac{\left(1-q^{m}\right)^{2}}{\left(1-z^{-2} q^{m}\right)\left(1-z^{2} q^{m}\right)} \\
= & \sum_{\substack{N, r=-\infty \\
r \geq|N|}}^{\infty}(-1)^{r+N} q^{\frac{r^{2}-N^{2}+r+N}{2}}\left(z^{2 N}-z^{2(N-1)}\right) \\
= & \sum_{N=-\infty}^{\infty}\left(\sum_{\substack{l=0 \\
r=|N|+l}}^{\infty}(-1)^{r+N} q^{\frac{r^{2}-N^{2}+r+N}{2}}-\sum_{\substack{l=0 \\
r=|N+1|+l}}^{\infty}(-1)^{r+N+1} q^{\frac{r^{2}-(N+1)^{2}+r+N+1}{2}}\right) z^{2 N} .
\end{aligned}
$$

Fix $N \geq 0$, the coefficient of $z^{2 N}$ is

$$
\begin{equation*}
\sum_{l=1}^{\infty}(-1)^{l+1} q^{\frac{l(l-1+2 N)}{2}}-\sum_{l=1}^{\infty}(-1)^{l+1} q^{\frac{l(l-1+2(N+1))}{2}} \tag{4.3}
\end{equation*}
$$

Let $a_{k}(q)=\sum_{l=1}(-1)^{l+1} q^{\frac{l(l-1+2 k)}{2}}$ for $k \geq 0$, the coefficient of $z^{2 k}$ can be written as $a_{k}(q)-a_{k+1}(q)$. In particular, the coefficient of $z^{0}$ is

$$
a_{0}(q)-a_{1}(q)=1+\sum_{l=1} 2(-1)^{l} q^{\frac{l(l+1)}{2}}
$$

Thus the coefficient of $z^{0}$ in Equation 4.1 is

$$
\frac{1+\sum_{l=1} 2(-1)^{l} q^{\frac{l(l+1)}{2}}}{\operatorname{det}\left(I-g q^{k}\right)}
$$

where $\operatorname{det}\left(I-g q^{k}\right)=\left(1-q^{k}\right)^{2} \cdot \operatorname{det}\left(I-\left.g\right|_{T_{S}} q^{k}\right)$.
Corollary 5. Let $\beta_{i}\left(S^{[n]}\right)$ denote the $i$-th Betti number of $S^{[n]}$. We have

$$
\sum_{n=0}^{\infty} \beta_{2 n+2 k}\left(S^{[n]}\right) q^{n}=\frac{q}{\Delta(q)}\left(a_{k}(q)-a_{k+1}(q)\right), k \geq 0
$$

where $\Delta(q)=q \prod_{m}\left(1-q^{m}\right)^{24}$ is a cusp form of weight 12 for $S L_{2}(\mathbb{Z})$, and $a_{k}(q)=\sum_{l=1}(-1)^{l+1} q^{\frac{l(l-1+2 k)}{2}}$.
Proof. The corollary follows from the proof of Theorem 1 (see Equation 4.3) by taking the trivial representation.
Remark 4. Göttsche [9] shows that the generating series for the Euler numbers of $S^{[n]}$ is

$$
\sum_{n=0}^{\infty} e\left(S^{[n]}\right) q^{n}=\frac{q}{\Delta(q)}
$$

According to [27] and remark 3.7 of Appendix in [14] we have

$$
\Delta(q)=4096 \epsilon\left(\delta^{2}-\epsilon\right)^{2}
$$

where $\epsilon$ and $\delta$ are modular forms for $\Gamma_{0}(2)$ of weights 4 and 2 with the following forms:

$$
\begin{gathered}
\epsilon=\sum_{n=1}^{\infty}\left(\sum_{d \mid n, \frac{n}{d} \text { odd }} d^{3}\right) q^{n} \\
\delta=-\frac{1}{8}-3 \sum_{n=1}^{\infty}\left(\sum_{d \mid n, d \text { odd }} d\right) q^{n}
\end{gathered}
$$

It is interesting that the modular form $\Delta$ appears many times in computations related to the cohomology rings of $S^{[n]}$.

## Appendix A. Table of $G_{X}$ Representations

The following computations carried out using MAGMA 6]. Table 2 records $H^{*}(X, \mathbb{Q})$ as a decomposition of $G_{X}$ representations, where $X$ is of $K 3^{[n]}$-type. $\lambda$ in each row denotes the highest weight of the $G_{X}$ representation, and $\mathbb{V}_{k, n}$ in each column denotes $H^{k}(X, \mathbb{Q})$. Each integer datum indicates the number of copies of the highest-weight representation $V_{X}(\lambda)$ in $H^{k}(X, \mathbb{Q})$.

| $\lambda$ | $\mathbb{V}_{8,7}$ | $\mathbb{V}_{10,7}$ | $\mathbb{V}_{12,7}$ | $\mathbb{V}_{14,7}$ | $\mathbb{V}_{8,8}$ | $\mathbb{V}_{10,8}$ | $\mathbb{V}_{12,8}$ | $\mathbb{V}_{14,8}$ | $\mathbb{V}_{16,8}$ | $\mathbb{V}_{10,9}$ | $\mathbb{V}_{12,9}$ | $\mathbb{V}_{14,9}$ | $\mathbb{V}_{16,9}$ | $\mathbb{V}_{18,9}$ | $\ldots$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $(0,0,0, .)$. | 5 | 5 | 10 | 7 | 6 | 6 | 13 | 12 | 18 | 6 | 15 | 15 | 25 | 21 | $\ldots$ |
| $(1,0,0, .)$. | 5 | 9 | 11 | 14 | 5 | 10 | 14 | 21 | 21 | 11 | 16 | 27 | 33 | 39 | $\ldots$ |
| $(2,0,0, .)$. | 4 | 5 | 10 | 9 | 4 | 6 | 13 | 15 | 21 | 6 | 14 | 19 | 31 | 30 | $\ldots$ |
| $(1,1,0, .)$. | 1 | 4 | 4 | 7 | 1 | 4 | 5 | 10 | 9 | 4 | 6 | 13 | 15 | 21 | $\ldots$ |
| $(3,0,0, .)$. | 1 | 4 | 5 | 7 | 1 | 4 | 6 | 11 | 11 | 4 | 7 | 14 | 18 | 24 | $\ldots$ |
| $(2,1,0, .)$. | 1 | 2 | 4 | 5 | 1 | 2 | 5 | 8 | 10 | 2 | 5 | 10 | 16 | 18 | $\ldots$ |
| $(1,1,1, .)$. |  |  | 1 |  |  |  | 1 | 1 | 2 |  | 1 | 1 | 3 | 3 | $\ldots$ |
| $(4,0,0, .)$. | 1 | 1 | 3 | 3 | 1 | 1 | 4 | 5 | 8 | 1 | 4 | 6 | 12 | 11 | $\ldots$ |
| $(3,1,0, .)$. |  | 1 | 2 | 3 |  | 1 | 2 | 5 | 5 | 1 | 2 | 6 | 9 | 13 | $\ldots$ |
| $(2,2,0, .)$. |  |  | 1 |  |  |  | 1 | 1 | 3 |  | 1 | 1 | 4 | 3 | $\ldots$ |
| $(2,1,1, .)$. |  |  |  | 1 |  |  |  | 1 | 1 |  |  | 1 | 2 | 3 | $\ldots$ |
| $(5,0,0, .)$. | 1 | 1 | 2 |  | 1 | 1 | 3 | 3 | 1 | 1 | 4 | 5 | 8 | $\ldots$ |  |
| $(4,1,0, .)$. |  | 1 | 1 |  |  | 1 | 2 | 3 |  | 1 | 2 | 5 | 6 | $\ldots$ |  |
| $(3,2,0, .)$. |  |  |  | 1 |  |  |  | 1 | 1 |  |  | 1 | 2 | 3 | $\ldots$ |
| $(3,1,1, .)$. |  |  |  |  |  |  |  | 1 |  |  |  | 1 | 1 | $\ldots$ |  |
| $(2,2,1, .)$. |  |  |  |  |  |  |  |  |  |  |  |  |  | 1 | $\ldots$ |
| $(6,0,0, .)$. |  |  | 1 |  |  |  | 1 | 1 | 2 |  | 1 | 1 | 3 | 3 | $\ldots$ |
| $(5,1,0, .)$. |  |  |  | 1 |  |  |  | 1 | 1 |  |  | 1 | 2 | 3 | $\ldots$ |
| $(4,2,0, .)$. |  |  |  |  |  |  |  |  | 1 |  |  |  | 1 | 1 | $\ldots$ |
| $(4,1,1, .)$. |  |  |  |  |  |  |  |  |  |  |  |  |  | 1 | $\ldots$ |
| $(3,3,0, .)$. |  |  |  |  |  |  |  |  |  |  |  |  |  | 1 | $\ldots$ |
| $(7,0,0, .)$. |  |  |  | 1 |  |  |  | 1 |  |  |  | 1 | 1 | 2 | $\ldots$ |
| $(6,1,0, .)$. |  |  |  |  |  |  |  |  | 1 |  |  |  | 1 | 1 | $\ldots$ |
| $(5,2,0, .)$. |  |  |  |  |  |  |  |  |  |  |  |  | 1 | $\ldots$ |  |
| $(8,0,0, .)$. |  |  |  |  |  |  |  |  | 1 |  |  |  | 1 |  | $\ldots$ |
| $(7,1,0, .)$. |  |  |  |  |  |  |  |  |  |  |  |  |  | 1 | $\ldots$ |
| $(9,0,0, .)$. |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |

Table 2. $G_{X}$ Representations

Remark 5. 24] and [17] discuss the stable cohomology of $S^{[n]}$. Here we only listed data for unstable parts. In general, the ring structure of $H^{*}\left(S^{[n]}, \mathbb{Q}\right)$ is still unknown using irreducible representations. Section 5 in [18] gives partial results on relations.

## Appendix B. Number of Canonical Hodge Classes

The following computations were done using MAGMA [6]. In Table 3, each row is indexed by $n$ - corresponding to $X$ which is of $K 3^{[n]}$-type; each column is indexed by $k$ - corresponding to the $k$-th cohomology group $H^{k}(X, \mathbb{Q})$. Since all the odd cohomologies vanish, we only listed the even values of $k$. Each integer datum in Table 3 refers to the number of canonical Hodge classes in $H^{k}(X, \mathbb{Q})$, i.e. the number of copies trivial $G_{X}$ representation.

| nk | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 24 | 26 | 28 | 30 | 32 | $\cdots$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 2 | 0 | 1 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| 3 | 0 | 1 | 1 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| 4 | 0 | 2 | 1 | 3 |  |  |  |  |  |  |  |  |  |  |  |  |  |
| 5 | 0 | 2 | 1 | 4 | 2 |  |  |  |  |  |  |  |  |  |  |  |  |
| 6 | 0 | 2 | 2 | 5 | 4 | 7 |  |  |  |  |  |  |  |  |  |  |  |
| 7 | 0 | 2 | 2 | 5 | 5 | 10 | 7 |  |  |  |  |  |  |  |  |  |  |
| 8 | 0 | 2 | 2 | 6 | 6 | 13 | 12 | 18 |  |  |  |  |  |  |  |  |  |
| 9 | 0 | 2 | 2 | 6 | 6 | 15 | 15 | 25 | 21 |  |  |  |  |  |  |  |  |
| 10 | 0 | 2 | 2 | 6 | 7 | 16 | 18 | 33 | 33 | 43 |  |  |  |  |  |  |  |
| 11 | 0 | 2 | 2 | 6 | 7 | 16 | 20 | 37 | 42 | 61 | 56 |  |  |  |  |  |  |
| 12 | 0 | 2 | 2 | 6 | 7 | 17 | 21 | 41 | 51 | 79 | 84 | 104 |  |  |  |  |  |
| 13 | 0 | 2 | 2 | 6 | 7 | 17 | 21 | 43 | 55 | 91 | 108 | 146 | 138 |  |  |  |  |
| 14 | 0 | 2 | 2 | 6 | 7 | 17 | 22 | 44 | 59 | 101 | 129 | 188 | 205 | 238 |  |  |  |
| 15 | 0 | 2 | 2 | 6 | 7 | 17 | 22 | 44 | 61 | 106 | 142 | 219 | 262 | 335 | 333 |  |  |
| 16 | 0 | 2 | 2 | 6 | 7 | 17 | 22 | 45 | 62 | 110 | 152 | 244 | 312 | 432 | 480 | 538 |  |
| 17 | $\ldots$ |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |

Table 3. Number of Canonical Hodge Classes

Remark 6. For a fixed $k$ and for $n \geq k$, the number of copies of trivial $G_{X}$ representations stabilizes. This also can be seen by results in stable cohomology of Hilbert schemes of points on $K 3$ surfaces ([24], [17]).
Remark 7. The Beauville-Bogomolov class $\alpha_{X}$ and the Chern classes of the tangent bundle are canonical Hodge classes, but there are more canonical Hodge classes in addition to these.

Example B.0.2. Consider $X$ of $K 3^{[8]}$-type. There are 6 canonical Hodge classes in $H^{8}(X, \mathbb{Q})$ according to the table. On the other hand, there are only 4 canonical Hodge classes which can be expressed in terms of Chern classes and $\alpha_{X}$, namely

$$
c_{4}\left(T_{X}\right), c_{2}^{2}\left(T_{X}\right), c_{2} \alpha_{X}, \alpha_{X}^{2}
$$

Future work will be devoted to finding algebraic expression of all canonical Hodge classes i.e. to express these classes as polynomials in Chern classes of certain coherent sheaves.

## References

[1] George E Andrews. Hecke modular forms and the kac-peterson identities. Transactions of the American Mathematical Society, pages 451-458, 1984.
[2] B. Bakker and A. Jorza. Lagrangian hyperplanes in holomorphic symplectic varieties. arXiv preprint arXiv:1111.0047, 2011.
[3] Benjamin Bakker. A classification of lagrangian planes in holomorphic symplectic varieties. arXiv preprint arXiv:1310.6341, 2013.
[4] Arnaud Beauville. Variétés Kähleriennes dont la première classe de Chern est nulle. J. Differential Geom., 18(4):755-782 (1984), 1983.
[5] Ciprian Borcea. Diffeomorphisms of a K3 surface. Math. Ann., 275(1):1-4, 1986.
[6] Wieb Bosma, John Cannon, and Catherine Playoust. The Magma algebra system. I. The user language. J. Symbolic Comput., 24(3-4):235-265, 1997. Computational algebra and number theory (London, 1993).
[7] Eduardo Cattani, Pierre Deligne, and Aroldo Kaplan. On the locus of Hodge classes. J. Amer. Math. Soc., 8(2):483-506, 1995.
[8] William Fulton and Joe Harris. Representation theory, a first course, volume 129 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1991.
[9] Lothar Göttsche. The Betti numbers of the Hilbert scheme of points on a smooth projective surface. Math. Ann., 286(1-3):193-207, 1990.
[10] Mark Green, Phillip Griffiths, and Matt Kerr. Mumford-Tate groups and domains, volume 183 of Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2012.
[11] David Harvey, Brendan Hassett, and Yuri Tschinkel. Characterizing projective spaces on deformations of Hilbert schemes of K3 surfaces. Comm. Pure Appl. Math., 65(2):264286, 2012.
[12] Brendan Hassett and Yuri Tschinkel. Moving and ample cones of holomorphic symplectic fourfolds. Geom. Funct. Anal., 19(4):1065-1080, 2009.
[13] Brendan Hassett and Yuri Tschinkel. Intersection numbers of extremal rays on holomorphic symplectic varieties. Asian J. Math., 14(3):303-322, 2010.
[14] Friedrich Hirzebruch, Thomas Berger, and Rainer Jung. Manifolds and modular forms. Aspects of Mathematics, E20. Friedr. Vieweg \& Sohn, Braunschweig, 1992. With appendices by Nils-Peter Skoruppa and by Paul Baum.
[15] Stavros Kousidis. A closed character formula for symmetric powers of irreducible representations. In 22nd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2010), Discrete Math. Theor. Comput. Sci. Proc., AN, pages 833-844. Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2010.
[16] Manfred Lehn and Christoph Sorger. The cup product of Hilbert schemes for $K 3$ surfaces. Invent. Math., 152(2):305-329, 2003.
[17] Wei-Ping Li, Zhenbo Qin, and Weiqiang Wang. Stability of the cohomology rings of Hilbert schemes of points on surfaces. J. Reine Angew. Math., 554:217-234, 2003.
[18] Eyal Markman. Generators of the cohomology ring of moduli spaces of sheaves on symplectic surfaces. J. Reine Angew. Math., 544:61-82, 2002.
[19] Eyal Markman. On the monodromy of moduli spaces of sheaves on $K 3$ surfaces. $J$. Algebraic Geom., 17(1):29-99, 2008.
[20] Eyal Markman. Integral constraints on the monodromy group of the hyperKähler resolution of a symmetric product of a K3 surface. Internat. J. Math., 21(2):169-223, 2010.
[21] Eyal Markman. The beauville-bogomolov class as a characteristic class. arXiv preprint arXiv:1105.3223, 2011.
[22] Eyal Markman. A survey of Torelli and monodromy results for holomorphic-symplectic varieties. In Complex and differential geometry, volume 8 of Springer Proc. Math., pages 257-322. Springer, Heidelberg, 2011.
[23] Hiraku Nakajima. Heisenberg algebra and Hilbert schemes of points on projective surfaces. Ann. of Math. (2), 145(2):379-388, 1997.
[24] Hiraku Nakajima. Lectures on Hilbert schemes of points on surfaces. Number 18. American Mathematical Soc., 1999.
[25] C. A. M. Peters and J. H. M. Steenbrink. Monodromy of variations of Hodge structure. Acta Appl. Math., 75(1-3):183-194, 2003. Monodromy and differential equations (Moscow, 2001).
[26] Mikhail Sergeevic Verbitsky. Cohomology of compact hyperkaehler manifolds. ProQuest LLC, Ann Arbor, MI, 1995. Thesis (Ph.D.)-Harvard University.
[27] Don Zagier. Note on the Landweber-Stong elliptic genus. In Elliptic curves and modular forms in algebraic topology (Princeton, NJ, 1986), volume 1326 of Lecture Notes in Math., pages 216-224. Springer, Berlin, 1988.
[28] Yu. G. Zarhin. Hodge groups of K3 surfaces. J. Reine Angew. Math., 341:193-220, 1983.

Department of Mathematics, Stony Brook University, Stony Brook, NY 11794, U.S.A.

E-mail address: letao.zhang@stonybrook.edu

