CHARACTER FORMULAS ON COHOMOLOGY OF DEFORMATIONS OF HILBERT SCHEMES OF K3 SURFACES

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ABSTRACT. Let X be a hyperkähler manifold deformation equivalent to Hilbert scheme of n points on a K3 surface. We compute the graded character formula of the generic Mumford-Tate group representation on the cohomology ring of X, and derive a generating series for deducing the number of canonical Hodge classes on X. The formula indicates the number of Hodge classes on X that remain Hodge under any deformation.

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1. INTRODUCTION

Let S be a K3 surface, and $S^{[n]}$ the Hilbert scheme of n points on S; each point in $S^{[n]}$ corresponds to a subscheme of S whose Hilbert polynomial is the constant n. We say X is of $K3^{[n]}$ -type if X is hyperkähler and deformation equivalent to $S^{[n]}$.

Denote by G_X the generic Mumford-Tate group of hyperkähler manifolds of $K3^{[n]}$ -type. The invariants of G_X action on $H^*(X, \mathbb{Q})$ correspond to the *canonical Hodge classes* (see Section 3.2), which are Hodge classes that remain Hodge under any deformation. Chern classes of the tangent bundle \mathcal{T}_X are examples of such canonical Hodge classes.

Now we consider the action of G_X on the cohomology of X. In particular, we want to compute the characters of G_X representation on the middle cohomology of X, where X is of $K3^{[n]}$ -type.

The lattice of $H^2(X,\mathbb{Z})$ – with respect to the Beauville-Bogomolov form – is $\Lambda_S \oplus \delta\mathbb{Z}$. Here $\Lambda_S := U^3 \oplus E_8^2(-1)$ is the lattice of $H^2(S,\mathbb{Z})$ where S is a K3 surface, and $(\delta, \delta) = -2(n-1)$. Let G_S be the identity component of $O^+(H^2(S,\mathbb{Q}))$ with respect to the intersection form. The action of maximal torus T_X of G_X on Λ_S is the same as the action of maximal torus T_S of G_S on Λ_S .

Theorem 1. Let $M(q) := \sum_{n=0}^{\infty} Char(H^{2n}(X, \mathbb{Q})) \cdot q^n$ be the generating series for the character of the G_X representation on the middle cohomology of X.

$$M(q) = \left(1 + \sum_{k=1}^{k-1} 2(-1)^k q^{\frac{k(k+1)}{2}}\right) \left(\prod_{m=1}^{m-1} \det(I_{24} - gq^m)\right)^{-1},$$

where $g \in T_X$ the maximal torus of G_X , I_N is a $N \times N$ identity matrix, and $\det(I_{24} - gt^m) = (1 - t^m)^2 \det(I_{22} - g|_{T_S}t^m).$

Example 1.0.1. Let X be of $K3^{[7]}$ -type. There are 7 Hodge classes in $H^{14}(X, \mathbb{Q})$ that remain Hodge under any deformation. Similarly, there are 5 Hodge classes in $H^8(X, \mathbb{Q})$, 5 Hodge classes in $H^{10}(X, \mathbb{Q})$, and 10 Hodge classes in $H^{12}(X, \mathbb{Q})$ that remain Hodge under any deformation. (cf. Appendix A)

Now let $l \in H_2(X,\mathbb{Z})$ be a line class in $\mathbb{P}^n \subset X$. Hassett and Tschinkel in [13] show that $(l,l) = -\frac{5}{2}$ for the case where n = 2 in [12]. For n = 3, Harvey, Hassett and Tschinkel [11] show that (l,l) = -3 and give a concrete expression for the Lagrangian hyperplane class. For the case of n = 4, Bakker and Jorza [2] show that $(l,l) = -\frac{7}{2}$, and also give an expression for $[\mathbb{P}^4]$. For $n \geq 5$, Bakker [3] shows that $(l,l) = -\frac{n+3}{2}$, which was conjectured in [13]. However, it is more difficult to compute the class $[\mathbb{P}^n]$ for larger n. One possible approach to exploring the expression for $[\mathbb{P}^n]$ is to find all the canonical Hodge classes in the middle cohomology of X for each n; future work could provide possible candidates for the class of $[\mathbb{P}^n]$ in terms of the line class. As for the ring structure, Verbitsky [26] shows that there is an embedding Symⁿ $H^2(S, \mathbb{Q}) \hookrightarrow H^{2n}(S^{[n]}, \mathbb{Q})$, but much about the ring structure of $H^*(X, \mathbb{Q})^{G_X}$ is still unknown, e.g. relations in the subalgebra generated by $H^*(S, \mathbb{Q})$ for each $H^*(X, \mathbb{Q})^{G_X}$.

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2. Cohomology of Hilbert Schemes of Points on K3 surfaces

In this section, we review some classical results about $S^{[n]}$, where S is a K3 surface. For n > 1, the Beauville-Bogomolov form can be written as the direct sum [4]

(2.1)
$$H^2(S^{[n]}, \mathbb{Z}) = H^2(S, \mathbb{Z})_{(,,)} \oplus_{\perp} \mathbb{Z}\delta, \ (\delta, \delta) = -2(n-1),$$

where (,) is the intersection form on $H^2(S,\mathbb{Z})$, and 2δ is the class of the corresponding big diagonal divisor $\Delta^{[n]} \subset S^{[n]}$ parameterizing nonreduced subschemes.

In [23], Nakajima constructs generators for the cohomology ring of Hilbert schemes of points of any projective surface. Lehn and Sorger [16] then show how $H^*(S, \mathbb{Q})$ generates $H^*(S^{[n]}, \mathbb{Q})$ as a graded ring.

Let $A = H^*(S, \mathbb{Q})[2]$ denote the shifted cohomology ring weighted by -2, 0, 2. Correspondingly, let $\mathbb{H}_n = H^*(S^{[n]}, \mathbb{Q})[2n]$ denote the shifted cohomology ring weighted by -2n, ..., 2n. Note that the weight shifting here is not the Tate twist notation for Hodge classes.

Define a linear form T on A by $T(a) := -\int_{[S]} a$, and let \langle, \rangle be the induced bilinear form on the shifted cohomology $\langle a_1, a_2 \rangle = T(a_1a_2) = -\int_S a_1a_2$. On $A^{\otimes n}$, one can define an analogous structure. Since A and \mathbb{H}_n have only graded pieces of *even* weights, we can simplify the algebraic model in [16].

The product is given by

$$(a_1 \otimes \cdots \otimes a_n) \cdot (b_1 \otimes \cdots \otimes b_n) = (a_1 b_1) \otimes \cdots \otimes (a_n b_n)$$
.

T extends to $A^{\otimes n}$ via

$$T(a_1 \otimes \cdots \otimes a_n) = T(a_1) \cdots T(a_n)$$
,

and the bilinear form \langle , \rangle on $A^{\otimes n}$ is defined accordingly:

$$\langle a, b \rangle = T(a)T(b).$$

We also have the symmetric group \mathfrak{S}_n action on the *n*-fold tensor given by

$$\pi(a_1\otimes\cdots\otimes a_n)=a_{\pi^{-1}(1)}\otimes\cdots\otimes a_{\pi^{-1}(n)}$$

For any partition $n = n_1 + \cdots + n_k$, we have a homomorphism

$$A^{\otimes n} \to A^{\otimes k}$$
$$a_1 \otimes \cdots \otimes a_n \mapsto (a_1 \cdots a_{n_1}) \otimes \cdots \otimes (a_{n_1 + \cdots + n_{k-1} + 1} \cdots a_{n_k})$$

Given a finite set I with n elements, let $\{A_i\}_{i \in I}$ be a family of copies of A indexed by I. Let [n] denote $\{1, \ldots, n\}$; we define

$$A^{\otimes I} := \left(\bigoplus_{f:[n] \xrightarrow{\cong} I} A_{f(1) \otimes \dots \otimes f(n)} \right) / \mathfrak{S}_n$$

Finally, given a surjection $\phi: I \to J$ between two index sets, there is an induced multiplication

$$\phi^*: A^{\otimes I} \to A^{\otimes J} ,$$

and let

$$\phi_*: A^{\otimes J} \to A^{\otimes I}$$

be the *adjoint* of ϕ^* , i.e.

 $\langle \phi^* a, b \rangle = \langle a, \phi_* b \rangle,$

where $a \in A^{\otimes I}, b \in A^{\otimes J}$. The projection formula

$$\phi_*(a \cdot \phi^*(b)) = \phi_*(a) \cdot b$$

holds by [16].

Denote by $\langle \pi \rangle \setminus [n]$ the set of orbits of [n] under the action of π . Define

$$A\{\mathfrak{S}_n\} := \bigoplus_{\pi \in \mathfrak{S}_n} A^{\otimes \langle \pi \rangle \setminus [n]} \cdot \pi$$

 $A{\mathfrak{S}_n}$ admits an action of $\sigma \in \mathfrak{S}_n$, induced by the bijection

$$\sigma: \langle \pi \rangle \setminus [n] \to \left\langle \sigma \pi \sigma^{-1} \right\rangle \setminus [n], \ x \mapsto \sigma x.$$

This gives an automorphism of $A\{\mathfrak{S}_n\}$ given by

$$\widetilde{\sigma}: a \cdot \pi \mapsto \sigma^*(\sigma \pi \sigma^{-1}).$$

Denote by $A^{[n]}$ the invariants under this action, then we have the graded isomorphism between the vector spaces [16]

$$A^{[n]} = \sum_{\|\alpha\|=n} \bigotimes_{i} \operatorname{Sym}^{\alpha_{i}} A$$

where $\alpha = (1^{\alpha_1}, 2^{\alpha_2}, \dots, n^{\alpha_n})$ runs all partitions of n and $\|\alpha\| = \sum_{i=1}^n i\alpha_i$. For the case of K3 surfaces, Lehn and Sorger prove

Theorem 2. [16] There is a canonical isomorphism of graded rings

$$(H^*(S,\mathbb{Q})[2])^{[n]} \xrightarrow{\cong} H^*(S^{[n]},\mathbb{Q})[2n]$$
.

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3. Decomposition of Cohomology Ring

In this section, we first review some useful results from representation theory. We then discuss generic Mumford-Tate group actions on the cohomology ring of X of $K3^{[n]}$ -type. Finally, we introduce canonical Hodge classes as the invariants of the group action. Our goal is to decompose $H^*(X, \mathbb{Q})$ into irreducible representations and to count invariants.

3.1. Characters of Representations. We summarize general results on representations of complex (or split) orthogonal groups from [8].

Let \mathfrak{g} be a semisimple Lie algebra, Λ be its weight lattice, and $\mathbb{Z}[\Lambda]$ be the integral group ring of the abelian group Λ . For each weight $\lambda \in \Lambda$, let $e(\lambda)$ denote the basis element in $\mathbb{Z}[\Lambda]$, so that each element in $\mathbb{Z}[\Lambda]$ can be written as the finite sum $\sum_{\lambda} n_{\lambda} \cdot e(\lambda)$. Denote by $R(\mathfrak{g})$ the ring of isomorphism classes of finite-dimensional representations associated to \mathfrak{g} . For each class [V], [V] = [V'] + [V''] whenever $V = V' \oplus V''$, and the product of two classes is defined as $[V] \cdot [W] = [V \otimes W]$. Define the character homomorphism

Char :
$$R(g) \to \mathbb{Z}[\Lambda]$$

by $\operatorname{Char}[V] = \sum \#(V_{\lambda}) \cdot e(\lambda)$, where V_{λ} is the weight space of V for the weight λ and $\#(V_{\lambda})$ is the multiplicity of V_{λ} in V. The Weyl group \mathfrak{W} acts on $\mathbb{Z}[\Lambda]$ and the image of Char is contained in the ring of invariants $\mathbb{Z}[\Lambda]^{\mathfrak{W}}$.

Let $\omega_1, \ldots, \omega_n$ be fundamental weights of \mathfrak{g} . Recall that fundamental weights have the property that any highest weight may be expressed uniquely as a nonnegative integral linear combination of them; they are free generators for the lattice Λ . Let Γ_i $(i = 1, \ldots, n)$ be the classes in $R(\mathfrak{g})$ of the irreducible representations of highest weight ω_i $(i = 1, \ldots, n)$. We have the following theorem.

Theorem 3. [8] The representation ring $R(\mathfrak{g})$ is a polynomial ring on the variables $\Gamma_1, \ldots, \Gamma_n$, and the homomorphism Char : $R(\mathfrak{g}) \to \mathbb{Z}[\Lambda]^{\mathfrak{W}}$ is an isomorphism.

Thus decomposing V into irreducible \mathfrak{g} representations is equivalent to finding its character polynomial.

Example 3.1.1. [8] Let $\mathfrak{g} = \mathfrak{so}_{2n}\mathbb{C}$ and $V \cong \mathbb{C}^{2n}$ be its standard representation. Its weight lattice Λ is span $\{L_1, \ldots, L_n, (\sum L_i)/2\}$ (see Lecture 19 in [8] for detailed explanation). For $\mathfrak{so}_{2n}\mathbb{C}$, fundamental weights are

 $L_1, L_1 + L_2, \ldots, L_1 + \cdots + L_{n-2}, (L_1 + \cdots + L_n)/2, (L_1 + \cdots + L_{n-1} - L_n)/2$ corresponding to irreducible representations $V, \bigwedge^2 V, \ldots, \bigwedge^{n-2} V$ and the halfspin representations S^+ and S^- . Set $t_i = e(L_i), t_i^{-1} = e(-L_i), t_i^{+1/2} = e(L_i/2),$ $t_i^{-1/2} = e(-L_i/2), \text{ Char}(\bigwedge^k V)$ is the k-th elementary symmetric polynomial - denoted by D_k - of the 2*n* elements $t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}$. The character D^+ (resp. D^-) of S^+ (resp. S^-) is the sum $\sum t_1^{\pm 1/2} \cdots t_n^{\pm 1/2}$, where the number of plus signs is even (resp. odd). Thus,

$$R(\mathfrak{so}_{2n}\mathbb{C}) = \mathbb{Z}[\Lambda]^{\mathfrak{W}} = \mathbb{Z}[D_1, \dots, D_{n-2}, D^+, D^-]$$

Example 3.1.2. [8] In the case of $\mathfrak{g} = \mathfrak{so}_{2n+1}\mathbb{C}$, its standard representation is $V \cong \mathbb{C}^{2n+1}$ and its weight lattice is the same as $\mathfrak{so}_{2n}\mathbb{C}$. But the fundamental weights are

$$L_1, L_1 + L_2, \dots, L_1 + L_2 + \dots + L_{n-1}, (L_1 + \dots + L_n)/2$$

corresponding to irreducible representations $V, \bigwedge^2 V, \ldots, \bigwedge^{n-1} V$ and the spin representation S. Char $(\bigwedge^k V)$ here is the k-th elementary symmetric polynomial – denoted by B_k – of 2n + 1 elements $t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}$ and 1. Denote by B_n the character of S, which is the n-th symmetric polynomial in variables $t_i^{\frac{1}{2}} + t_i^{-\frac{1}{2}}$. By applying Theorem 3 we obtain

$$R(\mathfrak{so}_{2n+1}\mathbb{C}) = \mathbb{Z}[\Lambda]^{\mathfrak{W}} = \mathbb{Z}[B_1, \dots, B_{n-1}, B_n]$$

If Γ_{λ} is an irreducible $\mathfrak{so}_{2n+1}\mathbb{C}$ representation of highest weight $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$, then its image in $\mathbb{Z}[B_1, \ldots, B_n]$ is $B_1^{\lambda_1 - \lambda_2} B_2^{\lambda_2 - \lambda_3} \cdots B_{n-1}^{\lambda_{n-1} - \lambda_n} B_n^{\lambda_n}$. In general, we have $\mathfrak{so}_{2n}\mathbb{C} \subset \mathfrak{so}_{2n+1}\mathbb{C}$, and the restriction representation of Γ_{λ} is

$$\operatorname{Res}_{\mathfrak{so}_{2n}\mathbb{C}}^{\mathfrak{so}_{2n+1}\mathbb{C}}\Gamma_{\lambda}=\oplus_{\bar{\lambda}}\Gamma_{\bar{\lambda}}$$

where $\bar{\lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_n)$ satisfies

$$\lambda_1 \geq \bar{\lambda}_1 \geq \lambda_2 \geq \bar{\lambda}_2 \geq \cdots \geq \bar{\lambda}_{n-1} \geq \lambda_n \geq |\bar{\lambda}_n|$$

and λ_i and λ_i are either all integers or all half integers.

Given a finite dimensional $\mathfrak{so}_{2n+1}\mathbb{C}$ representation W, if it is induced by the inclusion $\mathfrak{so}_{2n}\mathbb{C} \subset \mathfrak{so}_{2n+1}\mathbb{C}$ and all the weights of $\mathfrak{so}_{2n}\mathbb{C}$ representation are integer-valued, then so are the weights of the $\mathfrak{so}_{2n+1}\mathbb{C}$ representation. This implies that the character of the $\mathfrak{so}_{2n+1}\mathbb{C}$ representation will be in $\mathbb{Z}[B_1, \ldots, B_{n-1}]$.

3.2. Group actions on cohomologies. Let V be a \mathbb{Q} vector space, and $\mathbb{S}(\mathbb{R}) \cong \mathbb{C}^*$ regarded as a Lie group. S^1 is a maximal compact subgroup of $\mathbb{S}(\mathbb{R})$.

Definition 3.2.1. ([10], I.A) A *Hodge structure* of weight n is given by a representation on $V_{\mathbb{R}} := V \otimes_{\mathbb{Q}} \mathbb{R}$

$$\tilde{\varphi}: \mathbb{S}(\mathbb{R}) \to \mathrm{GL}(V_{\mathbb{R}})$$

such that for $r \in \mathbb{R}^* \subset \mathbb{S}(\mathbb{R})$, $\tilde{\varphi}(r) = r^n i d_V$. This definition is equivalent to giving a Hodge decomposition of $V_{\mathbb{C}} := V \otimes_{\mathbb{Q}} \mathbb{C}$, where

$$V_{\mathbb{C}} = \oplus V^{p,q}$$
, and $V^{p,q} = \overline{V}^{q,p}$.

For $\varphi := \tilde{\varphi}|_{S^1}$, we obtain a representation

$$\varphi: S^1 \to \mathrm{SL}(V)(\mathbb{R})$$

given by $\varphi(e^{i\theta})v = e^{i\theta(p-q)}v$, for $v \in V^{p,q}$.

Definition 3.2.2. [10] The Mumford-Tate group M_{φ} associated to a Hodge structure (V, φ) of weight n is the Q-algebraic closure of

$$\varphi: S^1 \to \mathrm{SL}(V_{\mathbb{R}}).$$

For a pure Hodge structure of even weight 2p, Hodge classes are defined as those lying in the intersection $V \cap V_{\mathbb{C}}^{p,p}$. Let $T^{k,l} = V^{\otimes k} \otimes \check{V}^{\otimes l}$ and $Hg(V_{\varphi})$ denote the direct sum of all Hodge classes in $T^{k,l}$ for all pairs of (k,l). It is known that M_{φ} fixes $Hg(V_{\varphi})$ [10].

Let V denote the weight two Hodge structure on $H^2(X, \mathbb{Z})$, Theorem 2.2.1 in [28] indicates that

Proposition 1. For generic X, the Mumford-Tate group $G_X = SO(V, \langle, \rangle)$, where \langle, \rangle is the Beauville-Bogomolov form.

Definition 3.2.3. [19, 22] An automorphism g of $H^*(X, \mathbb{Q})$ is called a monodromy operator (equivalently, a parallel transport operator) if there exists a smooth and proper family $\mathcal{M} \to B$ (which may depend on g) of irreducible holomorphic symplectic manifolds over a (possibly singular) complex analytic space B, having X as a fiber over a point $b \in B$, and such that g belongs to the image of $\pi_1(B, b)$ under the monodromy representation. The monodromy group $Mon(X) \in GL(H^*(X, \mathbb{Q}))$ is generated by all the monodromy operators. In this context, the algebraic monodromy group $\overline{Mon}(X)$ is defined as the smallest \mathbb{Q} -algebraic group in $GL(H^*(X, \mathbb{Q}))$ that contains Mon(X). Denote by $\overline{Mon}^2(X)$ the image of $\overline{Mon}(X)$ in the isometry group of $H^2(X, \mathbb{Q})$.

Proposition 2. Let X be of $K3^{[n]}$ -type, and assume X is a very general fibre in the universal family $\mathcal{X} \to B$. Let $\overline{Mon}_0^2(X)$ be the identity component of $\overline{Mon}^2(X)$, then we have $\overline{Mon}_0^2(X) = G_X$.

Proof. Theorem 16 in [25] shows that any connected component of $\overline{Mon}^2(X)$ is a normal subgroup of the derived group of G_X . In particular, we have $\overline{Mon}_0^2(X) \subset G_X$.

Consider the map $\iota : O^+(H^2(X,\mathbb{Z})) \to O(H^2(X,\mathbb{Z})^*/H^2(X,\mathbb{Z}))$, Lemma 4.2 in [20] shows that $Mon^2(X)$ is the inverse image of the subgroup $\{1, -1\}$ under ι . Thus $SO(V, \langle, \rangle) \subset \overline{Mon}_0^2(X)$. Since $G_X = SO(V, \langle, \rangle)$ by Proposition 1, $\overline{Mon}_0^2(X) = G_X$.

Example 3.2.4. The simplest case is when X is of K3-type. The monodromy group $\overline{Mon}^2(X)$ is $\mathcal{O}^+(H^2(X,\mathbb{Q}))$ [5]. Thus its identity component $\overline{Mon}^2_0(X)$ is $SO(H^2(X,\mathbb{Q}))$.

Definition 3.2.5. Let X be of $K3^{[n]}$ -type, and assume X is a very general fibre in the universal family $\mathcal{X} \to B$. Canonical Hodge classes of X are Hodge classes that remain Hodge under any deformation.

Theorem 4. Let X be of $K3^{[n]}$ -type, and assume X is a very general fibre over b in the universal family $\mathcal{X} \to B$. Canonical Hodge classes of X are exactly the invariant classes $H^*(X, \mathbb{Q})^{G_X}$.

Proof. Since canonical Hodge classes are Hodge classes, they are contained in $H^*(X, \mathbb{Q})^{G_X}$.

Given any very general fibre X', let $\gamma \subset B$ be a path such that $\gamma(0) = b$ and $\gamma(1) = b'$ where $\mathcal{X}_{b'} = X'$. Markman [22] shows that $\overline{Mon}^2(X)$ is a normal subgroup of $\overline{Mon}(X)$, thus $\gamma \overline{Mon}_0^2(X) \overline{\gamma} \subset \overline{Mon}^2(X')$. Since $\gamma \overline{Mon}_0^2(X) \overline{\gamma}$ is connected and contains identity, $\gamma \overline{Mon}_0^2(X) \overline{\gamma} = \overline{Mon}_0^2(X') = G_{X'}$.

For any $\alpha \in H^*(X, \mathbb{Q})^{G_X}$, $\gamma \alpha \in H^*(X', \mathbb{Q})$. Given $h' \in G_{X'}$, there exists $h \in G_X$ such that $h' = \gamma h \bar{\gamma}$. This implies

$$h'(\gamma \alpha) = \gamma h \bar{\gamma}(\gamma \alpha)$$
$$= \gamma h \alpha$$
$$= \gamma \alpha$$

Thus $\gamma \alpha \in H^*(X', \mathbb{Q})^{G_{X'}}$ is a Hodge class.

Now let $\mathcal{A} \in H^*(\mathcal{X}, \mathbb{Q})$ be the class such that $\mathcal{A}_b = \alpha \in H^*(\mathcal{X}_b, \mathbb{Q})$. For any very general fibre $\mathcal{X}_{b'}, \mathcal{A}_{b'} \in H^*(\mathcal{X}_{b'}, \mathbb{Q})$ is obtained by a path from b to b'. Passing through finite étale cover of $B, \mathcal{A}_{b'}$ is unique. By the above analysis, \mathcal{A}_b are Hodge for all very general fibres. By Deligne-Cattani-Kaplan theorem in [7], Hodge loci of \mathcal{A} is closed. Thus \mathcal{A}_s are Hodge for all special fibres \mathcal{X}_s . Thus α remains Hodge under any deformation.

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Thus $H^*(X, \mathbb{Q})^{G_X}$ is the collection of all canonical Hodge classes.

Note that the same statement was proved in Lemma 3.2 of [21].

By the above analysis, the invariants of the G_X representation on $H^*(X, \mathbb{Q})$ are the canonical Hodge classes. By computing number of trivial G_X representations on $H^{2p}(X, \mathbb{Q})$, we can obtain number of canonical Hodge classes of type (p, p) of X.

Example 3.2.6. In Table 1, the first row of data shows the number of canonical Hodge classes. For instance, if X is of $K3^{[5]}$ -type, then there are 2 canonical Hodge classes in $H^{2,2}(X)$, 1 in $H^{3,3}(X)$, 4 in $H^{4,4}(X)$, and 2 in $H^{5,5}(X)$.

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3.3. Decomposition of the Cohomology Representation. Denote by G_S the identity component of the special orthogonal group associated with the intersection form on $H^2(S, \mathbb{Z})$. (cf. Example 3.2.4)

We can decompose $H^*(X, \mathbb{Q})$ into irreducible representations for the action of G_X . [11] provides an explicit method for writing the decomposition:

- (1) Use the isomorphism $H^2(X, \mathbb{Z}) \cong \Lambda_S \oplus_{\perp} \mathbb{Z}\delta$ and compatible maximal torus of G_S and G_X to fix the embedding $G_S \subset G_X$. (cf. Section 1)
- (2) Decompose $H^*(S^{[n]}, \mathbb{Q})$ into the highest-weight G_S representations using $H^*(S^{[n]}, \mathbb{Q})[2n] \cong (H^*(S, \mathbb{Q})[2])^{[n]}$ in Theorem 2. The highest-weight irreducible representation $V_S(\lambda)$ will lie in the summand of an irreducible G_X representation $V_X(\lambda)$ in $H^*(X, \mathbb{Q})$.
- (3) Repeat step 1 and 2 on $H^*(X, \mathbb{Q})/V_X(\lambda)$.

Let $\mathbb{V}_{2k,n} := H^{2k}(X, \mathbb{Q})$ where X of $K3^{[n]}$ -type, $\mathbb{V}_{2k,n}$ is of weight 2k - 2nin $H^*(X, \mathbb{Q})[2n]$. Let V_{λ} be the irreducible G_X representation of the highest weight λ ; we get the following computational results

λ	$\mathrm{dim}V_\lambda$	$V_{4,5}$	$\mathbb{V}_{6,5}$	$\mathbb{V}_{8,5}$	$\mathbb{V}_{10,5}$	$\mathbb{V}_{6,6}$	$\mathbb{V}_{8,6}$	$\mathbb{V}_{10,6}$	$\mathbb{V}_{12,6}$	
$(0,0,0,\dots)$	1	2	1	4	2	2	5	4	7	
$(1,0,0,\dots)$	23	1	3	3	5	3	4	7	7	
$(2,0,0,\dots)$	275	1	1	3	2	1	4	4	7	
$(1,1,0,\dots)$	253		1	1	2	1	1	3	2	
$(3,0,0,\dots)$	2277		1	1	2	1	1	3	3	
$(2,1,0,\dots)$	4025			1	1		1	2	2	
$(1, 1, 1, \ldots)$	1771								1	
$(4,0,0,\dots)$	14674			1			1	1	2	
$(3,1,0,\dots)$	256795				1			1	1	
$(2,2,0,\dots)$	2193763								1	
$(5, 0, 0, \dots)$	7804350225				1			1		
$(4, 1, 0, \dots)$										
$(6, 0, 0, \dots)$										

Table 1: G_X Representations

Here "..." denotes truncated data, and each integer denotes the number of times V_{λ} appears in $\mathbb{V}_{2k,n} = H^{2k}(X, \mathbb{Q})$. In particular, the first row in the table indicates the number of copies of trivial G_X representation in each $H^{2k}(X, \mathbb{Q})$, and corresponds to the number of canonical Hodge classes.

 $H^2(S, \mathbb{Q})$ corresponds to the standard G_S representation $V_S(1)$. Let $\mathbb{H}_n = H^*(S^{[n]}, \mathbb{Q})[2n]$, which is a bigraded algebra associated to $H^*(S^{[n]}, \mathbb{Q})$. By Theorem 2, we can decompose \mathbb{H}_n into G_S representations. By using the same

convention as Example 3.1.1, we know that

$$\operatorname{Char}_{G_S}(H^{2k}(S^{[n]},\mathbb{R})) \in \mathbb{Z}[D_1,\ldots,D_9,D^+,D^-]$$

Remark 1. When $k \leq 10$, spin representations do not appear in our representation as $\wedge^k V_S(1,0,\ldots)$ is irreducible (Theorem 19.2 [8]).

By compatibility of maximal tori of G_X and G_S (cf. Section 1), we have

(3.1)
$$\operatorname{Char}_{G_S}(H^{2k}(S^{[n]},\mathbb{R})) = \operatorname{Char}_{G_X}(\mathbb{V}_{2k,n})$$

By 3.1, we will not distinguish notations between $\operatorname{Char}_{G_S}$ and $\operatorname{Char}_{G_X}$, and will denote both by Char in the following discussion. Let

$$p(z)_n := \sum_{k=-n}^n \operatorname{Char}(\mathbb{V}_{2k+2n,n}) \cdot z^{2k} \in \mathbb{Z}[D_1, \dots, D_9, D^+, D^-][z, \frac{1}{z}]$$

be the graded character of $H^*(X, \mathbb{R})$. Now we take the sum

(3.2)
$$p(z,t) := \sum_{n=0}^{\infty} p(z)_n t^n$$

following the grading of each $\mathbb{V}_{\bullet,n}$.

Example 3.3.1. Consider X of $K3^{[3]}$ -type, and $\mathbb{H}_3 = H^*(S^{[3]}, \mathbb{Q})[6]$. By the above analysis, we obtain

$$H^{0}(S^{[3]}, \mathbb{Q})[6] = 1_{S}$$

$$H^{2}(S^{[3]}, \mathbb{Q})[6] = 1_{S} \oplus V_{S}(1)$$

$$H^{4}(S^{[3]}, \mathbb{Q})[6] = 1_{S}^{3} \oplus V_{S}(1)^{2} \oplus V_{S}(2)$$

$$H^{6}(S^{[3]}, \mathbb{Q})[6] = 1_{S}^{3} \oplus V_{S}(1)^{3} \oplus V_{S}(2) \oplus V_{S}(1, 1) \oplus V_{S}(3)$$

Thus we have

$$p(z)_{3} = \sum_{k=-3}^{3} \operatorname{Char}_{G_{S}}(H^{6+2k}(S^{[3]})) \cdot z^{2k}$$

= $z^{-6} + (1+D_{1})z^{-4} + (3+2D_{1}+D_{1}^{2}-D_{2})z^{-2}$
+ $(3+3D_{1}+D_{1}^{2}+D_{1}^{3}-2D_{1}D_{2}+D_{3})$
+ $z^{6} + (1+D_{1})z^{4} + (3+2D_{1}+D_{1}^{2}-D_{2})z^{2}$

The induced G_X representations are

$$\mathbb{V}_{0,3} = H^0(X, \mathbb{Q})[6] = 1_X
\mathbb{V}_{2,3} = H^2(X, \mathbb{Q})[6] = V_X(1)
\mathbb{V}_{4,3} = H^4(X, \mathbb{Q})[6] = 1_X \oplus V_X(1) \oplus V_X(2)
\mathbb{V}_{6,3} = H^6(X, \mathbb{Q})[6] = 1_X \oplus V_X(1) \oplus V_X(1, 1) \oplus V_X(3)$$

The character formula is

$$p(z)_{3} = \sum_{k=-3}^{3} \operatorname{Char}_{G_{X}}(\mathbb{V}_{2k+6,3}) \cdot z^{2k}$$

= $z^{-6} + B_{1}z^{-4} + (1 + B_{1} + B_{1}^{2} - B_{2})z^{-2} + (1 + B_{1} + B_{2} + B_{1}^{3} - 2B_{1}B_{2} + B_{3})$
+ $z^{6} + B_{1}z^{4} + (1 + B_{1} + B_{1}^{2} - B_{2})z^{2}$

and the number of canonical Hodge classes of $H^k(X, \mathbb{Q})$ corresponds to the constant term in the coefficient of z^k .

Remark 2. The computation in example 3.3.1 is reversible. Explicitly, the number of copies of the highest-weight representation $V_X(\lambda)$ appearing in $\mathbb{V}_{n,k}$ corresponds to the coefficient of $\operatorname{Char}(V_X(\lambda))$

Proposition 3. The character of the G_X representation is

(3.3)
$$p_g(z,t) := \sum_{n=0}^{\infty} p(z)_n t^n = \prod_{m=1}^{\infty} \frac{1}{\det(I_{24} - gt^m)}$$

where $g \in T_X$ the maximal torus of G_X , I_N is a $N \times N$ identity matrix, and $\det(I_{24} - gt^m) = (1 - z^{-2}t^m) \cdot (1 - z^2t^m) \det(I_{22} - g|_{T_S}t^m).$

Proof. Recall that $\mathbb{V} = H^*(S, \mathbb{Q})[2] = H^0(S, \mathbb{Q}) + H^2(S, \mathbb{Q}) + H^4(S, \mathbb{Q})$ is bigraded, where $H^0(S, \mathbb{Q})$ and $H^4(S, \mathbb{Q})$ are of weight -2 and +2 respectively, and $H^2(S, \mathbb{Q})$ is of weight 0. For every symmetric power $\operatorname{Sym}^k(\mathbb{V})$, it is sufficient to show the formula holds when g is diagonal. Since G_S acts trivially on H^0 and H^4 , let $u_{-2} \in H^0(S, \mathbb{Q})$ be the eigenvector with eigenvalue 1 and weight -2, and $u_2 \in H^4(S, \mathbb{Q})$ be eigenvector with eigenvalue 1 and weight 2. $H^2(S, \mathbb{Q})$ corresponds to the standard G_S representation $V_S(1, 0, \ldots, 0)$, and let v_i $(i = 1, \ldots, 22)$ be its eigenvector; when i is even, v_i has eigenvalue $t_{\frac{i}{2}}^{-1}$.

Molien's Formula in [15] indicates that for a representation W of a group G, and given a linear operator $g \in G$, its action on the symmetric algebra

 $\operatorname{Sym}^{\bullet}(W)$ has the graded character

$$\sum_{i=0}^{\infty} \operatorname{Char}(\operatorname{Sym}^{i}(g))t^{i} = \frac{1}{\det(I - tg)}$$

The symmetric algebra on \mathbb{V} , denoted by $\operatorname{Sym}^{\bullet}(\mathbb{V})$, has the form

$$\operatorname{Sym}^{\bullet}(u_2) \otimes \operatorname{Sym}^{\bullet}(v_1) \otimes \operatorname{Sym}^{\bullet}(v_2) \otimes \cdots \otimes \operatorname{Sym}^{\bullet}(v_{22}) \otimes \operatorname{Sym}^{\bullet}(u_{-2})$$
.

Since $v_i \in \mathbb{V}$ is of weight 0, the bigraded character of $Sym^{\bullet}(v_i)$ is

$$\sum_{k=0}^{\infty} (\mu_i z^0)^k \cdot t^k = \frac{1}{1 - \mu_i t} ,$$

where μ_i is the eigenvalue of v_i . Since u_{-2} (resp. u_2) has weight -2 (resp. weight +2) in \mathbb{V} , we have

$$\operatorname{Char}(\operatorname{Sym}^{\bullet}(u_{-2}))(t) = \sum_{k=0}^{\infty} (1 \cdot z^{-2})^k t^k = \frac{1}{1 - z^{-2}t} , \text{ and}$$
$$\operatorname{Char}(\operatorname{Sym}^{\bullet}(u_2))(t) = \sum_{k=0}^{\infty} (1 \cdot z^2)^k t^k = \frac{1}{1 - z^2t} .$$

Thus, we obtain

(3.4)
$$\operatorname{Char}(\operatorname{Sym}^{\bullet}(\mathbb{V}))(t) = \frac{1}{(1-z^{2}t)(1-z^{-2}t)\det(I_{22}-g|_{T_{S}}t)}$$

By Theorem 2, the graded character $p(z)_n$ of $H^*(X, \mathbb{Q})$ is given by

$$p(z)_{n}t^{n} = \operatorname{Char}\left(\sum_{\|\alpha\|=n} \bigotimes_{i} \operatorname{Sym}^{\alpha_{i}} \mathbb{V}\right) t^{n}$$
$$= \sum_{\|\alpha\|=n} \prod_{i=1}^{n} \left(\operatorname{Char}\left(\operatorname{Sym}^{\alpha_{i}} \mathbb{V}\right) t^{i\alpha_{i}}\right)$$

Note Char (Sym^{α_i} \mathbb{V}) $t^{i\alpha_i}$ is the α_i -th term in Char(Sym[•](\mathbb{V})) (t^i) . Then for each $\alpha = (1^{\alpha_1}, 2^{\alpha_2}, \dots, n^{\alpha_n})$ where $\| \alpha \| = 1\alpha_1 + \dots + n\alpha_n = n$,

$$\prod_{i=1}^{n} \left(\operatorname{Char} \left(\operatorname{Sym}^{\alpha_{i}} \mathbb{V} \right) t^{i\alpha_{i}} \right)$$

corresponds to the t^n -th term in

$$\prod_{j=1}^{n} \left(\operatorname{Char}(\operatorname{Sym}^{\bullet}(\mathbb{V}))(t^{j}) \right) .$$

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Together with equation 3.4, one can obtain

$$\sum_{n=0}^{\infty} p(z)_n t^n = \prod_{m=1}^{\infty} \frac{1}{(1-z^2 t^m)(1-z^{-2} t^m) \det(I_{22}-g|_{T_S} t^m)} \,.$$

Remark 3. Given a smooth projective complex surface S', let $p(S'^{[n]}, z)$ be the Poincaré polynomial $\sum_{i=0}^{4n} \beta_i(S'^{[n]}) z^i$ of $S'^{[n]}$. Göttsche [9] shows that $\sum_{n=0} p(S'^{[n]}, z)q^n$ has the expression

(3.5)
$$\prod_{m=1}^{\infty} \frac{(1+z^{2m-1}q^m)^{b_1(S')}(1+z^{2m+1}q^m)^{b_1(S')}}{(1-z^{2m-2}q^m)^{b_0(S')}(1-z^{2m}q^m)^{b_2(S')}(1-z^{2m+2}q^m)^{b_4(S')}} .$$

For the case when S' is a K3 surface, $b_0(S') = b_4(S') = 1$, $b_1(S') = b_3(S') = 0$, and $b_2(S') = 22$. Letting $z^2q = t$, Equation 3.5 becomes

$$\prod_{m=1}^{\infty} \frac{1}{(1-z^{-2}t^m)(1-t^m)^{22}(1-z^2t^m)}$$

This is the same as taking the character of the identity in Proposition 3.

4. Generating Series for the Character of the Middle Cohomology

Proof of Theorem 1: For X is of $K3^{[n]}$ -type, we consider the middle cohomology $H^{2n}(X, \mathbb{Q})$ of X. The character of $H^{2n}(X, \mathbb{Q})$ is of weight 0 in Equation 3.3 in Proposition 3. The character formula can be written as

(4.1)
$$\left(\prod_{m=1} \frac{(1-q^m)^2}{(1-z^{-2}q^m)(1-z^2q^m)}\right) \left(\prod_{k=1} (1-q^k)^2 \cdot \det(I-g|_{T_S}q^k)\right)^{-1}$$

Lemma 1 in [1] indicates

(4.2)
$$\prod_{m=1} \frac{(1-q^m)^2}{(1-z^{-2}q^{m-1})(1-z^2q^m)} = \sum_{N,r=-\infty\atop r \ge |N|}^{\infty} (-1)^{r+N} z^{2N} q^{\frac{r^2-N^2+r+N}{2}} .$$

*

.

Multiplying both sides of Equation 4.2 by $(1 - z^{-2})$, we obtain

$$\begin{split} &\prod_{m=1} \frac{(1-q^m)^2}{(1-z^{-2}q^m)(1-z^2q^m)} \\ &= \sum_{\substack{N,r=-\infty\\r\geq |N|}}^{\infty} (-1)^{r+N} q^{\frac{r^2-N^2+r+N}{2}} \left(z^{2N}-z^{2(N-1)}\right) \\ &= \sum_{\substack{N=-\infty\\r\geq |N|+l}}^{\infty} \left(\sum_{\substack{l=0\\r=|N|+l}}^{\infty} (-1)^{r+N} q^{\frac{r^2-N^2+r+N}{2}} - \sum_{\substack{l=0\\r=|N+1|+l}}^{\infty} (-1)^{r+N+1} q^{\frac{r^2-(N+1)^2+r+N+1}{2}} \right) z^{2N} \end{split}$$

Fix $N \ge 0$, the coefficient of z^{2N} is

m) 9

(4.3)
$$\sum_{l=1}^{\infty} (-1)^{l+1} q^{\frac{l(l-1+2N)}{2}} - \sum_{l=1}^{\infty} (-1)^{l+1} q^{\frac{l(l-1+2(N+1))}{2}}$$

Let $a_k(q) = \sum_{l=1}^{k} (-1)^{l+1} q^{\frac{l(l-1+2k)}{2}}$ for $k \ge 0$, the coefficient of z^{2k} can be written as $a_k(q) - a_{k+1}(q)$. In particular, the coefficient of z^0 is

$$a_0(q) - a_1(q) = 1 + \sum_{l=1}^{l} 2(-1)^l q^{\frac{l(l+1)}{2}}$$

Thus the coefficient of z^0 in Equation 4.1 is

$$\frac{1 + \sum_{l=1}^{l} 2(-1)^l q^{\frac{l(l+1)}{2}}}{\det(I - gq^k)}$$

where $\det(I - gq^k) = (1 - q^k)^2 \cdot \det(I - g|_{T_S}q^k)$.

Corollary 5. Let $\beta_i(S^{[n]})$ denote the *i*-th Betti number of $S^{[n]}$. We have

$$\sum_{n=0}^{\infty} \beta_{2n+2k}(S^{[n]}) \ q^n = \frac{q}{\Delta(q)}(a_k(q) - a_{k+1}(q)), \ k \ge 0$$

where $\Delta(q) = q \prod_m (1-q^m)^{24}$ is a cusp form of weight 12 for $SL_2(\mathbb{Z})$, and $a_k(q) = \sum_{l=1} (-1)^{l+1} q^{\frac{l(l-1+2k)}{2}}$.

Proof. The corollary follows from the proof of Theorem 1 (see Equation 4.3) by taking the trivial representation.

Remark 4. Göttsche [9] shows that the generating series for the Euler numbers of $S^{[n]}$ is

$$\sum_{n=0}^{\infty} e(S^{[n]})q^n = \frac{q}{\Delta(q)}$$

According to [27] and remark 3.7 of Appendix in [14] we have

$$\Delta(q) = 4096\epsilon(\delta^2 - \epsilon)^2$$

where ϵ and δ are modular forms for $\Gamma_0(2)$ of weights 4 and 2 with the following forms:

$$\epsilon = \sum_{n=1}^{\infty} \left(\sum_{d|n, \frac{n}{d} \text{ odd}} d^3 \right) q^n$$
$$\delta = -\frac{1}{8} - 3\sum_{n=1}^{\infty} \left(\sum_{d|n, d \text{ odd}} d \right) q^n$$

It is interesting that the modular form Δ appears many times in computations related to the cohomology rings of $S^{[n]}$.

Appendix A. Table of G_X Representations

The following computations carried out using MAGMA [6]. Table 2 records $H^*(X, \mathbb{Q})$ as a decomposition of G_X representations, where X is of $K3^{[n]}$ -type. λ in each row denotes the highest weight of the G_X representation, and $\mathbb{V}_{k,n}$ in each column denotes $H^k(X, \mathbb{Q})$. Each integer datum indicates the number of copies of the highest-weight representation $V_X(\lambda)$ in $H^k(X, \mathbb{Q})$.

λ	$\mathbb{V}_{8,7}$	$\mathbb{V}_{10,7}$	$\mathbb{V}_{12,7}$	$\mathbb{V}_{14,7}$	$\mathbb{V}_{8,8}$	$\mathbb{V}_{10,8}$	$\mathbb{V}_{12,8}$	$\mathbb{V}_{14,8}$	$\mathbb{V}_{16,8}$	$\mathbb{V}_{10,9}$	$\mathbb{V}_{12,9}$	$\mathbb{V}_{14,9}$	$\mathbb{V}_{16,9}$	$\mathbb{V}_{18,9}$	
0, 0, 0,)	5	5	10	7	6	6	13	12	18	6	15	15	25	21	
1, 0, 0,)	5	9	11	14	5	10	14	21	21	11	16	27	33	39	
2, 0, 0,)	4	5	10	9	4	6	13	15	21	6	14	19	31	30	
[1, 1, 0,)	1	4	4	7	1	4	5	10	9	4	6	13	15	21	
3, 0, 0,)	1	4	5	7	1	4	6	11	11	4	7	14	18	24	
2, 1, 0,)	1	2	4	5	1	2	5	8	10	2	5	10	16	18	
1, 1, 1,)			1				1	1	2		1	1	3	3	
4, 0, 0,)	1	1	3	3	1	1	4	5	8	1	4	6	12	11	
3, 1, 0,)		1	2	3		1	2	5	5	1	2	6	9	13	
2, 2, 0,)			1				1	1	3		1	1	4	3	
2, 1, 1,)				1				1	1			1	2	3	
5, 0, 0,)		1	1	2		1	1	3	3	1	1	4	5	8	
4, 1, 0,)			1	1			1	2	3		1	2	5	6	
3, 2, 0,)				1				1	1			1	2	3	
3, 1, 1,)									1				1	1	
2, 2, 1,)														1	
6, 0, 0,)			1				1	1	2		1	1	3	3	
5, 1, 0,)				1				1	1			1	2	3	
4, 2, 0,)									1				1	1	
4, 1, 1,)														1	
3, 3, 0,)														1	
7, 0, 0,)				1				1				1	1	2	
6, 1, 0,)									1				1	1	
5, 2, 0,)														1	
8, 0, 0,)									1				1		
7, 1, 0,)														1	.
(9, 0, 0,)														1	

TABLE 2. G_X Representations

Remark 5. [24] and [17] discuss the stable cohomology of $S^{[n]}$. Here we only listed data for unstable parts. In general, the ring structure of $H^*(S^{[n]}, \mathbb{Q})$ is still unknown using irreducible representations. Section 5 in [18] gives partial results on relations.

APPENDIX B. NUMBER OF CANONICAL HODGE CLASSES

The following computations were done using MAGMA [6]. In Table 3, each row is indexed by n – corresponding to X which is of $K3^{[n]}$ -type; each column is indexed by k – corresponding to the k-th cohomology group $H^k(X, \mathbb{Q})$. Since all the odd cohomologies vanish, we only listed the even values of k. Each integer datum in Table 3 refers to the number of canonical Hodge classes in $H^k(X, \mathbb{Q})$, i.e. the number of copies trivial G_X representation.

nk	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	• • •
2	0	1															
3	0	1	1														
4	0	2	1	3													
5	0	2	1	4	2												
6	0	2	2	5	4	7											
7	0	2	2	5	5	10	7										
8	0	2	2	6	6	13	12	18									
9	0	2	2	6	6	15	15	25	21								
10	0	2	2	6	7	16	18	33	33	43							
11	0	2	2	6	7	16	20	37	42	61	56						
12	0	2	2	6	7	17	21	41	51	79	84	104					
13	0	2	2	6	7	17	21	43	55	91	108	146	138				
14	0	2	2	6	7	17	22	44	59	101	129	188	205	238			
15	0	2	2	6	7	17	22	44	61	106	142	219	262	335	333		
16	0	2	2	6	7	17	22	45	62	110	152	244	312	432	480	538	
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TABLE 3. Number of Canonical Hodge Classes

Remark 6. For a fixed k and for $n \ge k$, the number of copies of trivial G_X representations stabilizes. This also can be seen by results in stable cohomology of Hilbert schemes of points on K3 surfaces ([24], [17]).

Remark 7. The Beauville-Bogomolov class α_X and the Chern classes of the tangent bundle are canonical Hodge classes, but there are more canonical Hodge classes in addition to these.

Example B.0.2. Consider X of $K3^{[8]}$ -type. There are 6 canonical Hodge classes in $H^8(X, \mathbb{Q})$ according to the table. On the other hand, there are only 4 canonical Hodge classes which can be expressed in terms of Chern classes and α_X , namely

$$c_4(T_X), c_2^2(T_X), c_2\alpha_X, \alpha_X^2$$
.

Future work will be devoted to finding algebraic expression of all canonical Hodge classes i.e. to express these classes as polynomials in Chern classes of certain coherent sheaves.

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