THE DISTRIBUTION OF S-INTEGRAL POINTS ON SL_2-ORBIT CLOSURES OF BINARY FORMS

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Abstract. We study the distribution of S-integral points on SL_2-orbit closures of binary forms and prove an asymptotic formula for the number of S-integral points. This extends a result of Duke, Rudnick, and Sarnak. The main ingredients of the proof are the method of mixing developed by Eskin-McMullen and Benoist-Oh, Chambert-Loir-Tschinkel's study of asymptotic volume of height balls, and Hassett-Tschinkel's description of log resolutions of SL_2-orbit closures of binary forms.

Introduction

The distribution of integral points on homogeneous spaces has been studied by several researchers, and important developments are [DRS93] and [EM93], which used different techniques to settle the problem of asymptotic formulae for the number of integral points on affine symmetric spaces. [EM93] uses an ergodic theoretic approach based on *mixing* and this method is extended to the S-integral points setting in [BO12]. On the other hand, another approach based on the height zeta functions method, has been developed in [CLT12], [CLT10b], [TBT13]. The advantage of this method is that one can analyze more general (D, S)-integral points while ergodic methods are only available when D is the full boundary divisor. However, the height zeta functions method is also limited in that it is only applicable to bi-equivariant compactifications of connected linear algebraic groups.

In this paper, we study (D, S)-integral points on one sided equivariant compactifications of connected semisimple groups assuming that D is the full boundary divisor. Our method is a variant of the method of mixing in [EM93] and [BO12]. To demonstrate our method, we solve the problem of counting S-integral points of bounded height on SL_2-orbit closures of binary forms, which is considered in the integral case by Duke, Rudnick and Sarnak in [DRS93].

Let us explain the problem in detail. Let V be a two-dimensional vector space over $\mathbb{Q}$ with coordinates x and y. We consider the standard SL_2 action on V. Let $W_n = \mathrm{Sym}^n(V^*)$ be the space of binary forms of degree n,

$$f = a_0x^n + a_1x^{n-1}y + \cdots + a_ny^n,$$

where $n \geq 3$. Here the left action of SL_2 on W_n is given by

$$(g \cdot f)\begin{pmatrix} x \\ y \end{pmatrix} = f\left(g^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right).$$

Consider the projective space $\mathbb{P}(W_n) \cong \mathbb{P}^n$, and let $[f] \in \mathbb{P}(W_n)(\mathbb{Q})$ be a binary form of degree n with coefficients in $\mathbb{Q}$ and with distinct roots. We define X_f to be the Zariski

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closure of the SL_2-orbit of $[f]$, i.e,

$$X_f = \overline{\mathrm{SL}_2 \cdot [f]} \subset \mathbb{P}(W_n).$$

This is a projective threefold defined over $\mathbb{Q}$. The complement of the open orbit $U_f := \mathrm{SL}_2 \cdot [f]$ in X_f forms a geometrically irreducible divisor, and set theoretically it is an intersection of X_f with the discriminant divisor in $\mathbb{P}(W_n)$.(See [MU83, Lemma 1.5].) We denote this Weil divisor by D_f. We fix the integral model $\mathbb{P}_{\mathbb{Z}}(W_n)$ of $\mathbb{P}(W_n)$ as

$$\mathbb{P}_{\mathbb{Z}}(W_n) := \mathrm{Proj}(\mathbb{Z}[a_0, \cdots, a_n]),$$

and let $\mathcal{X}_f$ and $\mathcal{D}_f$ be closures of X_f and D_f in $\mathbb{P}_{\mathbb{Z}}(W_n)$ respectively. They form flat integral models of X_f and D_f respectively, and we define

$$\mathcal{U}_f = \mathcal{X}_f \setminus \mathcal{D}_f.$$

Let S be a finite set of places including the archimedean place, and we denote the ring of S-integers by $\mathbb{Z}_S$. One can consider the counting function of the number of S-integral points with respect to a height function $\mathsf{H} : \mathbb{P}(W_n)(\mathbb{Q}) \to \mathbb{R}_{>0}$:

$$N(\mathcal{U}_f, T) = \#\{P \in \mathcal{U}_f(\mathbb{Z}_S) \mid \mathsf{H}(P) \leq T\},$$

where the height function H is introduced in Section 1. In [DRS93], Duke, Rudnick, and Sarnak studied the asymptotic formula of this counting function when S consists of the archimedean place:

Theorem 0.1. *[DRS93, Theorem 1.9] When $S = \{\infty\}$, there exists a constant $c \geq 0$ such that*

$$N(\mathcal{U}_f, T) \sim cT^{\frac{2}{n}}.$$

Duke, Rudnick, and Sarnak studied the counting problem of integral points on affine symmetric spaces([DRS93, Theorem 1.2]) using techniques from automorphic forms. The above remarkable theorem is an example of an asymptotic formula for a non-symmetric space. Their method is based on equidistribution of lattice points in angular sectors on the hyperbolic plane and elementary approximation arguments using the polar decomposition of $\mathrm{SL}_2(\mathbb{R})$. In this paper, we give a new proof of Theorem 0.1 and extend their result to any S:

Theorem 0.2. *Suppose that $n \geq 5$ and f is general so that the stabilizer of $[f]$ is trivial. Then there exists a constant $c \geq 0$ such that*

$$N(\mathcal{U}_f, T) \sim cT^{\frac{2}{n}} (\log T)^{\#S-1}.$$

Our proof is based on the method of mixing developed in [EM93], [BO12]. In [EM93], Eskin and McMullen introduced axiomatic treatments of the counting problem and the method of mixing. Using mixing, they independently solved the question of distribution of integral points on affine symmetric spaces([EM93, Theorem 1.4]). Benoist and Oh generalized this method to S-adic Lie group settings in [BO12], and solved the counting problem of S-integral points on affine symmetric spaces $H\backslash G$([BO12, Theorem 1.1, Corollary 1.2, Theorem 1.4]). An important property used in their proofs is the wavefront property for symmetric spaces([BO12, Definition 3.1, Proposition 3.2]). It is the key to establishing equidistribution of translations of H-orbits. We consider a special height function which is invariant under the action of a compact subgroup H satisfying the wavefront property, and reduce the counting problem on G to the counting problem on $H\backslash G$, where G is a S-adic Lie group associated to SL_2. This proves that the function $N(\mathcal{U}_f, T)$ is approximated by the asymptotic volume of height balls.

The computation of asymptotic volume of height balls is a subject of [CLT10a]. Chambert-Loir and Tschinkel showed that the global geometric data, which is so-called the Clemens complex, controls the volume of height balls, and one can compute the asymptotic formula based on that. However, to use their machinery, we need to describe a log resolution of singularities for a pair (X_f, D_f). This has been discussed in [HT03]. The variety X_f admits a moduli interpretation as a subvariety of a moduli space of stable maps, and Hassett and Tschinkel used this moduli interpretation to construct a log resolution of a pair (X_f, D_f). We will recall their result and provide its refinement in Section 3. It is straightforward to generalize our method to arbitrary number fields, but we restrict ourselves over the field of rational numbers for notational reasons.

Let us outline the contents of the paper. In Section 1, we define the height function H and discuss its basic properties. Then in Section 2 we explain the method of mixing and its application. In Section 3, we recall the construction of a log resolution of a pair (X_f, D_f) in [HT03] and explain how moduli spaces of stable curves and stable maps can be used to obtain a log resolution of (X_f, D_f). In Section 4, we recall results of [CLT10a] and apply them to obtain asymptotic formulae. In Section 5, we discuss some generalizations of results in Section 2.

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1. Height functions

In this section, we introduce a height function of $\mathcal{O}(1)$ on $\mathbb{P}(W_n)$ to count S-integral points on $\mathcal{U}_f$. First let us recall some definitions regarding height functions in general.

Definition 1.1. [CLT10a, Section 2.1.3] Let F be a locally compact field and X a smooth projective variety defined over F. One can consider $X(F)$ as a compact analytic manifold over F. Let L be a line bundle on X. The $L(F)$ is endowed with the structure of the analytic line bundle on $X(F)$. A metric on $L(F)$ to be a collection of functions $L_P(F) \to \mathbb{R}_+$ for all $P \in X(F)$, denoted by $l \mapsto \|l\|$, such that

- $\|\cdot\|$ is a norm on the F-vector space $L_P(F)$;
- for any open subset $U \subset X(F)$ and any non vanishing analytic section $\mathsf{f} \in \Gamma(U, L(F))$, the function $U \ni P \mapsto \|\mathsf{f}(P)\|$ is smooth, i.e., it is locally constant if F is non-archimedean, otherwise it is C^∞.

With metrizations, one can define local height functions:

Definition 1.2. [CLT10a, Section 2.2.6] Let F be a locally compact field and X a smooth projective variety defined over F, $\mathcal{L} = (L, \|\cdot\|)$ a metrized line bundle on X, and a nonzero section $\mathsf{f} \in \Gamma(X, \mathcal{L})$. Let U be the complement of the support of f. The local height function of $\mathcal{L}$ associated to f is given by

$$\mathsf{H} : U(F) \to \mathbb{R}_+, \quad P \mapsto \|\mathsf{f}(P)\|^{-1}.$$

We define height functions of $\mathcal{O}(1)$ on $\mathbb{P}(W_n)$. For a nonarchimedean place v, we define a metrization on $\mathcal{O}(1)$ by requiring the following property: for any linear form $\mathsf{f} \in \Gamma(\mathcal{O}(1), \mathbb{P}(W_n))$, we have

$$\|\mathsf{f}\|(a_0, \cdots, a_n) = \frac{|\mathsf{f}(a_0, \cdots, a_n)|_v}{\max\{|a_0|_v, \cdots, |a_n|_v\}}$$

At the archimedean place, we define our metrization by

$$\|\mathsf{f}\|(a_0, \cdots, a_n) = \frac{|\mathsf{f}(a_0, \cdots, a_n)|_v}{\sqrt{\sum_{i=0}^{n} \binom{n}{i}^{-1} a_i^2}}$$

For $v = p$ a prime, $\mathcal{O}(1)$ is endowed with the standard metric induced from the integral model $\mathbb{P}_{\mathbb{Z}}(W_n)$.([CLT10a, Section 2.3])

Let $D \subset \mathbb{P}(W_n)$ be the discriminant divisor and s_D the corresponding section of $\mathcal{O}(D)$. The section s_D is a homogeneous polynomial of degree $2n-2$ with $\mathbb{Z}$-coefficients. Let S be a finite set of places including the archimedean place and $U = \mathbb{P}(W_n) \setminus D$. For each $v \in S$, we define the local height $\mathsf{H}_v : U(\mathbb{Q}_v) \to \mathbb{R}_{>0}$ associated to $\frac{1}{2n-2}D$ by

$$\mathsf{H}_v(a_0, \cdots, a_n) = \frac{\max\{|a_0|_v, \cdots, |a_n|_v\}}{|s_D(a_0, \cdots, a_n)|_v^{\frac{1}{2n-2}}}$$

when v is a non-archimedean place, and

$$\mathsf{H}_v(a_0, \cdots, a_n) = \frac{\sqrt{\sum_{i=0}^{n} \binom{n}{i}^{-1} a_i^2}}{|s_D(a_0, \cdots, a_n)|_\infty^{\frac{1}{2n-2}}}$$

when v is the archimedean place, where

$$[a_0x^n + a_1x^{n-1}y + \cdots + a_ny^n] \in U(\mathbb{Q}_v).$$

The function H_v is the local height function of $\mathcal{O}(1)$ associated to $\frac{1}{2n-2}D$.

One important property of these local heights is that they are invariant under the action of a maximal compact subgroup:

Lemma 1.3. *For $v = p$ a prime, H_p is invariant under the action of $\mathrm{SL}_2(\mathbb{Z}_p)$. For $v = \infty$, H_∞ is invariant under the action of $\mathrm{SO}_2(\mathbb{R})$.*

Proof. Let $\mathcal{L}$ be the metrized line bundle associated to the invertible sheaf $\mathcal{O}(1)$ on $\mathbb{P}(W_n)$. First note that s_D is SL_2-invariant, i.e., for any $P \in \mathbb{P}(W_n)(\mathbb{Q}_v)$ and $g \in \mathrm{SL}_2(\mathbb{Q}_v)$, we have

$$g^{-1} \cdot (s_D(g \cdot P)) = s_D(P),$$

where the group SL_2 acts on the line bundle $\mathcal{L}$ in a way that the action is compatible with the one on $\mathbb{P}(W_n)$. Suppose that the place $v = p$ is a non-archimedean place. The group scheme $\mathrm{SL}_2/\mathrm{Spec}(\mathbb{Z}_p)$ acts on $\mathbb{P}_{\mathbb{Z}_p}(W_n)$ as well as the line bundle $\mathcal{L}/\mathrm{Spec}(\mathbb{Z}_p)$, hence $\mathrm{SL}_2(\mathbb{Z}_p)$ acts on the $\mathbb{Z}_p$-bundle $\mathcal{L}(\mathbb{Z}_p)$. This means that the action of $\mathrm{SL}_2(\mathbb{Z}_p)$ on $\mathcal{L}(\mathbb{Q}_p)$ is isometric. Thus using the definition of the height function above, we obtain that for $P \in U(\mathbb{Q}_p)$ and $g \in \mathrm{SL}_2(\mathbb{Z}_p)$,

$$\mathsf{H}_p(gP) = \|s_D(gP)\|_p^{-1} = \|g^{-1} \cdot s_D(gP)\|_p^{-1} = \|s_D(P)\|_p^{-1} = \mathsf{H}_p(P).$$

For the archimedean place, see [DRS93, Section 4] where this special height function was used to reduce the counting problem on SL_2 to the counting problem on the hyperbolic plane $\mathfrak{H}$. $\square$

Define $\mathbb{A}_S := \prod_{v\in S}\mathbb{Q}_v$, and we call it the S-adic ring. Let $U(\mathbb{A}_S) = \prod_{v\in S} U(\mathbb{Q}_v)$ be a S-adic manifold, and we define the global height $\mathsf{H} : U(\mathbb{A}_S) \to \mathbb{R}_{>0}$ as the product of local heights :

$$\mathsf{H}(P) := \prod_{v\in S} \mathsf{H}_v(P_v),$$

where $P = (P_v)_{v\in S} \in U(\mathbb{A}_S)$. This is a continuous function on $U(\mathbb{A}_S)$. A key property of height functions is that the set of S-integral points of bounded height is finite, hence we can count the number of S-integral points of bounded height:

$$N(\mathcal{U}_f, T) = \#\{P \in \mathcal{U}_f(\mathbb{Z}_S) \mid \mathsf{H}(P) \leq T\},$$

for any binary form f of degree $n \geq 3$ with distinct roots. We are interested in studying the asymptotic behavior of $N(\mathcal{U}_f, T)$ as $T \to \infty$.

2. The method of mixing

Let $\mathcal{D}$ be the integral model of the discriminant divisor in $\mathbb{P}_{\mathbb{Z}}(W_n)$ and $\mathcal{U}$ the complement of $\mathcal{D}$. By a theorem of Borel, Harish-Chandra, there are only finitely many $\mathrm{SL}_2(\mathbb{Z}_S)$-orbits on $\mathcal{U}_f(\mathbb{Z}_S)$.(See [BHC62, Theorem 6.9] and [PR94, Theorem 5.8].) Also, the stabilizer of f is a finite group. Therefore, our problem concerning the asymptotic of $N(\mathcal{U}_f, T)$ can be reduced to evaluating the asymptotic of

$$N(f, T) = \#\{g \in \mathrm{SL}_2(\mathbb{Z}_S) \mid \mathsf{H}(g \cdot [f]) \leq T\},$$

for any $f \in \mathcal{U}(\mathbb{Z}_S)$.

To study this counting function, we will use the method of mixing developed in [EM93] and [BO12] for symmetric varieties. Specifically, [EM93] used mixing and the so-called wavefront property to study the distribution of integral points for sufficiently "nice" sets, and [BO12] developed this theory for S-integral points.

Let $G = \mathrm{SL}_2(\mathbb{A}_S) = \prod_{v\in S}\mathrm{SL}_2(\mathbb{Q}_v)$ be a S-adic Lie group and $\Gamma = \mathrm{SL}_2(\mathbb{Z}_S)$ be diagonally embedded in G. Then Γ is a lattice in G, i.e., Γ is discrete in G and $X := \Gamma\backslash G$ has finite volume with respect to the invariant measure μ_X on X.(See [Mar91, Theorem I.3.2.4].) For the group action of G, the mixing property is as follows:

Theorem 2.1. *The action of G on X is mixing, i.e., for any $\alpha, \beta \in \mathsf{L}^2(X)$, we have*

$$\lim_{g\to\infty} \int_X \alpha(xg)\beta(x)\,\mathrm{d}\mu_X(x) = \frac{\int_X \alpha\,\mathrm{d}\mu_X \int_X \beta\,\mathrm{d}\mu_X}{\mu_X(X)}.$$

Proof. This is a consequence of [BO12, Proposition 2.4]. To apply [BO12, Proposition 2.4], one needs to check two conditions: (i) our S-adic Lie group G satisfies the Howe-Moore property; (ii) our lattice Γ is irreducible.

The condition (i) is stated in [BO12, Theorem 2.5] which claims the Howe-Moore property for any S-adic Lie group associated to a semisimple group.

To check the condition (ii), see [BO12, Lemma 9.4] whose assumptions are all satisfied by SL_2. $\square$

To count S-integral points, we consider the height balls:

$$\mathsf{B}(T) = \{g \in G \mid \mathsf{H}(g \cdot [f]) \leq T\}.$$

We denote the volume of these height balls with respect to the Haar measure μ_G on G by $V(T)$. Here we assume that $\mu_G = \mu_X$ holds locally. The asymptotic of this volume function will be studied in Section 4. To apply results of [BO12], these height balls need to satisfy the following condition:

Proposition 2.2. *For any $\epsilon > 0$, there exists a neighborhood U of the identity e in G such that*

$$(1-\epsilon)\mu_G(\cup_{g\in U}\mathsf{B}(T)g) \leq V(T) \leq (1+\epsilon)\mu_G(\cap_{g\in U}\mathsf{B}(T)g),$$

for all $T \gg 1$.

Proof. The above condition is refereed as *well-roundedness* in [EM93] and [BO12]. This follows from the precise asymptotic formula in Corollary 4.5. Indeed for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\frac{V(T+\delta)}{V(T)} < \frac{1}{1-\epsilon}, \quad \frac{V(T)}{V(T-\delta)} < 1+\epsilon$$

for sufficiently large $T \gg 1$. Now choose a neighborhood U of e so that

$$\cup_{g\in U}\mathsf{B}(T)g \subset \mathsf{B}(T+\delta), \quad \mathsf{B}(T-\delta) \subset \cap_{g\in U}\mathsf{B}(T)g.$$

Our assertion follows from this. $\square$

Let $\chi_{\mathsf{B}(T)}$ be the characteristic function of the height ball $\mathsf{B}(T)$. Consider the counting function:

$$F_T(g) = \sum_{\gamma\in\Gamma} \chi_{\mathsf{B}(T)}(\gamma g) = \#\Gamma \cap (\mathsf{B}(T)g^{-1}).$$

This defines a function on $X = \Gamma\backslash G$. Our goal in this section is to prove the following theorem:

Theorem 2.3. *Let $V^*(T) = V(T)/\mu_X(X)$. Then we have point-wise convergence*

$$\frac{F_T(g_0)}{V^*(T)} \to 1 \quad \text{as } T \to \infty,$$

for any $g_0 \in G$.

To prove this theorem, we consider a compact subgroup of G satisfying the wavefront property as in [BO12, Definition 3.1]:

Definition 2.4. Let $\mathcal{G}$ be a locally compact group, and $\mathcal{H}$ a closed subgroup of $\mathcal{G}$. The group $\mathcal{G}$ has the wavefront property in $\mathcal{H}\backslash\mathcal{G}$ if there exists a Borel subset F such that $\mathcal{G} = \mathcal{H}F$ and, for every neighborhood U of the identity in $\mathcal{G}$, there exists a neighborhood V of the identity such that

$$\mathcal{H}Vg \subset \mathcal{H}gU,$$

for all $g \in F$.

This property first appeared in [EM93] to establish equidistribution of $\mathcal{H}$-orbits:

Theorem 2.5. *[BO12, Theorem 4.1] Let $\mathcal{G}$ be a locally compact group, $\mathcal{H} \subset \mathcal{G}$ a closed subgroup, $\Gamma \subset \mathcal{G}$ a lattice such that $\Gamma_{\mathcal{H}} = \Gamma \cap \mathcal{H}$ is a lattice in $\mathcal{H}$. Set $\mathcal{X} := \Gamma\backslash\mathcal{G}$ and $\mathcal{Y} := \Gamma_{\mathcal{H}}\backslash\mathcal{H}$, and we denote the invariant measures on $\mathcal{X}$ and $\mathcal{Y}$ by $\mu_{\mathcal{X}}$ and $\mu_{\mathcal{Y}}$ respectively.*

Suppose that the action of $\mathcal{G}$ on $\mathcal{X}$ is mixing and that $\mathcal{G}$ has the wavefront property on $\mathcal{H}\backslash\mathcal{G}$. Then the translates $\mathcal{Y}g$ become equidistributed in $\mathcal{X}$ as $g \to \infty$ in $\mathcal{H}\backslash\mathcal{G}$, i.e., for any $\psi \in C_c(\mathcal{X})$, we have

$$\frac{1}{\mu_{\mathcal{Y}}(\mathcal{Y})} \int_{\mathcal{Y}} \psi(yg)\, \mathrm{d}\mu_{\mathcal{Y}}(y) \to \frac{1}{\mu_{\mathcal{X}}(\mathcal{X})} \int_{\mathcal{X}} \psi\, \mathrm{d}\mu_{\mathcal{X}},$$

as the image of g in $\mathcal{H}\backslash\mathcal{G}$ leaves every compact subset.

Define a subgroup H of G by

$$H = \prod_{p \in S_{\mathrm{fin}}} \mathrm{SL}_2(\mathbb{Z}_p) \times \mathrm{SO}_2(\mathbb{R}),$$

where $S_{\mathrm{fin}} = S \setminus \{\infty\}$. Note that Lemma 1.3 implies that our height function H is invariant under the action of the subgroup H, i.e.,

$$\mathsf{H}(hg \cdot [f]) = \mathsf{H}(g \cdot [f])$$

for $g \in G$ and $h \in H$. Moreover, we have

Lemma 2.6. *The group G has the wavefront property in $H\backslash G$.*

Proof. For SO(2), this property is established in [EM93, Theorem 3.1]. It follows from the definition of the wavefront property that $\mathrm{SL}_2(\mathbb{Q}_p)$ has the wavefront property in $\mathrm{SL}_2(\mathbb{Z}_p)\backslash\mathrm{SL}_2(\mathbb{Q}_p)$ because $\mathrm{SL}_2(\mathbb{Z}_p)$ is open. Now our assertion follows from [BO12, Proposition 3.5] which claims that if each factor satisfies the wavefront property, then their product also satisfies the wavefront property. □

We consider any sequence of non-negative integrable functions φ_n on G satisfying

- each φ_n has a compact support,
- $\max_n \|\varphi_n\|_\infty < \infty$,
- $\lim_{n\to\infty} \int_G \varphi_n \, \mathrm{d}\mu_G = \infty$,
- $\varphi_n(hg) = \varphi_n(g)$ for any $g \in G$ and $h \in H$.

The sequence $\{\chi_{\mathsf{B}(T_n)}\}$ is an example. We define a function F_{φ_n} on $X = \Gamma\backslash G$ by

$$F_{\varphi_n}(x) := \sum_{\gamma \in \Gamma} \varphi_n(\gamma g),$$

where $x = \Gamma g$. Let $I_n := \int_G \varphi_n \, \mathrm{d}\mu_G / \mu_X(X)$. The following proposition is essentially proved in [BO12, Proposition 5.3]. The only difference is that we are not counting S-integral points on $H\backslash G$, but on G.

Proposition 2.7. *F_{φ_n}/I_n converges weakly to 1 as $n \to \infty$.*

Proof. Let $\alpha \in C_c(X)$. We want to prove that

$$\lim_{n\to\infty} \frac{1}{I_n} \int_X F_{\varphi_n}(x)\alpha(x)\, \mathrm{d}\mu_X(x) = \int_X \alpha(x)\, \mathrm{d}\mu_X(x).$$

Let μ_H and $\mu_{H\backslash G}$ be the Haar measure and the invariant measure on H and $H\backslash G$ respectively such that $\mu_G = \mu_H \mu_{H\backslash G}$ and $\mu_H(H) = 1$.

$$\begin{aligned}\int_X F_{\varphi_n} \alpha \, \mathrm{d}\mu_X &= \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} \varphi_n(\gamma g)\alpha(\Gamma g) \, \mathrm{d}\mu_X(\Gamma g) \\ &= \int_G \varphi_n(g)\alpha(g) \, \mathrm{d}\mu_G(g) \\ &= \int_{H\backslash G} \varphi_n(Hg) \int_H \alpha(hg) \, \mathrm{d}\mu_H(h) \, \mathrm{d}\mu_{H\backslash G}(Hg) \\ &= \#\Gamma_H \int_{H\backslash G} \varphi_n(Hg) \int_{\Gamma_H\backslash H} \alpha(\Gamma_H hg) \, \mathrm{d}\mu_H(h) \, \mathrm{d}\mu_{H\backslash G}(Hg),\end{aligned}$$

where $\Gamma_H = \Gamma \cap H$. Note that since H is compact, Γ_H is a finite group and a lattice in H. Since the action of G on X is mixing and G has the wavefront property in $H\backslash G$, the translates $\Gamma_H\backslash Hg$ become equidistributed in X as $g \to \infty$ in $H\backslash G$(Theorem 2.5). This means that

$$\int_{\Gamma_H\backslash H} \alpha(\Gamma_H hg) \, \mathrm{d}\mu_H(h) \to \frac{\mu_H(H)}{\#\Gamma_H \mu_X(X)} \int_X \alpha \, \mathrm{d}\mu_X,$$

as $g \to \infty$. Since φ_n is uniformly bounded, our assertion follows from the dominated convergence theorem. □

Finally we prove Theorem 2.3:

Proof. The following proof mirrors the proof of [BO12, Proposition 6.2]. Fix $\epsilon > 0$. Let U be a symmetric neighborhood of the identity e such that

$$(1-\epsilon)\mu_G(\cup_{g\in U} \mathsf{B}(T)g) \leq V(T) \leq (1+\epsilon)\mu_G(\cap_{g\in U} \mathsf{B}(T)g).$$

The existence of such U is guaranteed by Proposition 2.2. We consider the functions $\varphi_T^{\pm}$ on G given by

$$\varphi_T^+(g) = \sup_{u\in U} \chi_{\mathsf{B}(T)}(gu^{-1}), \quad \varphi_T^-(g) = \inf_{u\in U} \chi_{\mathsf{B}(T)}(gu^{-1})$$

We let $I_T^{\pm} = \int_G \varphi_T^{\pm} \, \mathrm{d}\mu_G / \mu_X(X)$. Then we have

$$(1-\epsilon)I_T^+ \leq V^*(T) \leq (1+\epsilon)I_T^-.$$

On the other hand, we define

$$F_T^{\pm}(g) = \sum_{\gamma\in\Gamma} \varphi_T^{\pm}(\gamma g).$$

It is easy to verify that

$$F_T^-(gu) \leq F_T(g) \leq F_T^+(gu),$$

for all $g \in G$ and $u \in U$. Pick a non-negative continuous function α on X such that $\int_G \alpha \, \mathrm{d}\mu_G = 1$ and the support of α is included in $g_0 U$. Then we have

$$\int_X \alpha F_T^- \, \mathrm{d}\mu_X \leq F_T(g_0) \leq \int_X \alpha F_T^+ \, \mathrm{d}\mu_X.$$

Applying Proposition 2.7 to $\varphi_T^{\pm}$ we obtain that for $T \gg 1$

$$(1-\epsilon)I_T^- \leq F_T(g_0) \leq (1+\epsilon)I_T^+.$$

We can conclude that for $T \gg 1$

$$\frac{1-\epsilon}{1+\epsilon} \leq \frac{F_T(g_0)}{V^*(T)} \leq \frac{1+\epsilon}{1-\epsilon}.$$

Thus $F_T(g_0)/V^*(T)$ converges to 1. $\square$

3. Moduli interpretations

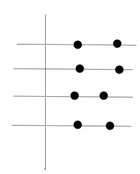
In [HT03], using moduli interpretations, Hassett and Tschinkel constructed a partial desingularization of a pair (X_f, D_f) to give a geometric explanation for the result of Duke, Rudnick and Sarnak (Theorem 0.1). This is not exactly what we need, however their geometry is quite important for establishing the asymptotic formula for $V(T)$, the volume of height balls. In this section, we recall the geometry of (X_f, D_f) described in [HT03], and then provide its refinement. We assume that the ground field is an algebraically closed field of characteristic zero throughout this section.

First let us recall some definitions of stable curves and stable maps. In this paper, a curve of genus zero is a connected projective curve C such that (i) C is the union of a finite number of $\mathbb{P}^1$'s; (ii) each component of C meets with other components transversally; (iii) C is a tree of smooth rational curves, i.e., there are no loops of rational curves.

A curve of genus zero with n marked points is a curve C of genus zero together with mutually distinct n smooth points on C. A special point on a curve C of genus zero with n marked points $(p_1, \cdots, p_n)$ is either a marked point p_i or an intersection of two irreducible components of C. A stable curve of genus zero with n marked points is a curve of genus zero together with n marked points

$$(C, p_1, \cdots, p_n)$$

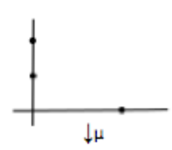
such that each component has at least three special points. This condition is equivalent to say that the automorphism group of $(C, p_1, \cdots, p_n)$ is finite.



A stable map of degree one from genus zero curves with n marked points to $\mathbb{P}^1$ is a tuple

$$(C, p_1, \cdots, p_n, \mu : C \to \mathbb{P}^1).$$

such that (1) $(C, p_1, \cdots, p_n)$ is a curve of genus zero with n marked points; (ii) the morphism μ has degree one, i.e., all components except one component L are collapsed by μ and the restriction of μ to L is an isomorphism to $\mathbb{P}^1$; (iii) all components except L has at least three special points. Again this condition (iii) is equivalent to say that the automorphism group of $(C, p_1, \cdots, p_n, \mu)$ is finite.



Fix $n \geq 3$. Let $\overline{\mathcal{M}}_{0,n}$ be the Knudsen-Mumford moduli space of stable curves of genus zero with n marked points. Let $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1)$ denote the Kontsevich moduli space of stable maps of degree one from genus zero curves with n marked points to $\mathbb{P}^1$. The action of SL_2 on $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1)$ is defined by

$$g \cdot (C, p_1, \cdots, p_n, \mu) = (C, p_1, \cdots, p_n, g \circ \mu).$$

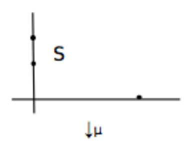
We consider the forgetting map

$$\begin{aligned} \psi : \overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1) &\to \overline{\mathcal{M}}_{0,n} \\ (C, p_1, \cdots, p_n, \mu) &\mapsto (C', p_1, \cdots, p_n) \end{aligned}$$

where C' is formed from C by collapsing the irreducible components that are destabilized, i.e., irreducible components with at most two special points.

For each subset $S \subset N = \{1, \cdots, n\}$ such that $|S| \geq 2$, consider stable maps of degree one $\mu : C \to \mathbb{P}^1$ such that

- C is the union of two $\mathbb{P}^1$s,
- marked points in S are on the one component, and remaining marked points are on another component,
- The component containing marked points in S is collapsed by μ.



The Zariski closure of the locus of such stable maps becomes an irreducible divisor $B_S \subset \overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1)$. For $s = 2, \cdots n$, we define

$$B[s] := \sum_{|S|=s} B_S, \quad B := \sum_{s=2}^{n} B[s].$$

Theorem 3.1. *The moduli spaces $\overline{\mathcal{M}}_{0,n}$ and $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1)$ are smooth projective algebraic varieties, and the boundary divisor B is a divisor with strict normal crossings.*

The evaluation map ev

$$\begin{aligned} \mathrm{ev} : \overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1) &\to (\mathbb{P}^1)^n \\ (C, p_1, \cdots, p_n, \mu) &\mapsto (\mu(p_1), \cdots, \mu(p_n)) \end{aligned}$$

is an SL_2-equivariant birational morphism. This is because for any $(p_1, \cdots, p_n) \in (\mathbb{P}^1)^n$ such that p_i's are mutually distinct, its preimage by ev is simply

$$(\mathbb{P}^1, p_1, \cdots, p_n, \mathrm{id})$$

so ev is an isomorphism on this locus. The divisor $B[2]$ is the proper transform of the big diagonal Δ and other boundary divisors $B[s] (s \geq 3)$ are contracted by the evaluation map ev. The symmetric group $\mathfrak{S}_n$ acts on $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1)$ by permuting marked points. Note that the actions of SL_2 and $\mathfrak{S}_n$ commute. We consider $\mathfrak{S}_n$-quotients of ev to obtain an SL_2-equivariant birational map:

$$\varrho : \tilde{X} = \overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1)/\mathfrak{S}_n \to (\mathbb{P}^1)^n/\mathfrak{S}_n \cong \mathbb{P}(W_n)$$

Let $\tilde{D}[s]$ be the image of $B[s]$. Then $\tilde{D}[2]$ is the proper transform of the discriminant divisor $D \subset \mathbb{P}(W_n)$ and other boundary divisors $\tilde{D}[s](s \geq 3)$ are contracted by ϱ. We write $\tilde{D}$ for $\sum_{s=2}^{n} \tilde{D}[s]$.

Let f be a binary form of degree n with distinct roots. As we defined in introduction, X_f is the Zariski closure of the SL_2-orbit of f in $\mathbb{P}(W_n)$. Let $\tilde{X}_f \subset \tilde{X}$ denote the strict transform of X_f, and define $\tilde{D}_f = \tilde{X}_f \cap \tilde{D}$. Here the intersection is the set theoretic intersection. A pair $(\tilde{X}_f, \tilde{D}_f)$ is a partial desingularization of (X_f, D_f) constructed by Hassett and Tschinkel, and in particular $(\tilde{X}_f, \tilde{D}_f)$ is log canonical for a generic f([HT03, Proposition 2.8]). Write $\alpha = (\alpha_1, \cdots, \alpha_n)$ for distinct roots of f, and let $C_\alpha \in \overline{\mathcal{M}}_{0,n}$ be the corresponding pointed rational curve

$$C_\alpha = (\mathbb{P}^1, \alpha_1, \cdots, \alpha_n).$$

We define Y_α to be $\psi^{-1}(C_\alpha)$. Then $\tilde{X}_f$ is a finite quotient of Y_α, and Y_α is an equivariant compactification of PGL_2. More precisely, consider the stabilizer of C_α in the symmetric group $\mathfrak{S}_n$:

$$\mathfrak{H} = \{\sigma \in \mathfrak{S}_n \mid \sigma \cdot C_\alpha \cong C_\alpha\}.$$

Then $\tilde{X}_f$ is isomorphic to the quotient $Y_\alpha/\mathfrak{H}$. When $n \geq 5$ and f is general, the stabilizer $\mathfrak{H}$ is trivial. Hence for such f, we have $\tilde{X}_f \cong Y_\alpha$. Each Y_α only meets with $B[n]$ and $B[n-1]$. Indeed, for any stable map $C = (C, p_1, \cdots, p_n, \mu) \in B[s]$ such that $s \leq n-2$, the image of the forgetting map $\psi(C)$ has at least two irreducible components, and hence it cannot be equal to C_α. The following theorem will be used later:

Theorem 3.2. *[HT03, Proposition 2.6] Y_α is smooth and its boundary has strict normal crossings contained in $B[n-1] \cup B[n]$.*

The proof of the above theorem gives us an explicit description of the Y_α. When $n = 3$, then $Y_\alpha = \overline{\mathcal{M}}_{0,3}(\mathbb{P}^1, 1)$. The evaluation map $\mathrm{ev} : \overline{\mathcal{M}}_{0,3}(\mathbb{P}^1, 1) \to (\mathbb{P}^1)^3$ is the blow-up of $(\mathbb{P}^1)^3$ along the small diagonal Δ_{small}, and $E := B[3]$ is the exceptional divisor. Since the normal bundle $\mathcal{N}_{\Delta_{\mathrm{small}}}$ is isomorphic to $\mathcal{O}(2) \oplus \mathcal{O}(2)$, E is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Let

$$\pi_1 : E \to \mathbb{P}^1,$$

be the projection to the cross-ratio of marked points and the node, and let

$$\pi_2 : E \to \mathbb{P}^1,$$

be the projection to the image of marked points.

For arbitrary n, write E for the intersection of Y_α and $B[n]$. This is the Zariski closure of the locus of stable maps $(C, p_1, \cdots, p_n, \mu)$ such that

- C is the union of two $\mathbb{P}^1$s,
- every marked point is on the collapsed component,
- $\psi(C, p_1, \cdots, p_n, \mu) \cong C_\alpha$.

We define F_i for $i = 1, \cdots, n$ as the intersection of Y_α and $B_{N\setminus\{i\}}$. Then F_i is the Zariski closure of the locus of stable maps $(C, p_1, \cdots, p_n, \mu)$ such that

- C is the union of two $\mathbb{P}^1$s,

- every marked point except i-th marked point is on the collapsed component, and i-th marked point is on the component which is isomorphically mapped to $\mathbb{P}^1$ by μ,
- $\psi(C, p_1, \cdots, p_n, \mu) \cong C_\alpha$.

Consider the sequence of the forgetting maps:

$$Y_{\alpha_1,\cdots,\alpha_n} \to Y_{\alpha_1,\cdots,\alpha_{n-1}} \to \cdots \to Y_{\alpha_1,\alpha_2,\alpha_3,\alpha_4} \to Y_{\alpha_1,\alpha_2,\alpha_3} \cong \overline{\mathcal{M}}_{0,3}(\mathbb{P}^1, 1).$$

Each morphism is a SL_2-equivariant birational morphism, and $\phi_i : Y_{\alpha_1,\cdots,\alpha_i} \to Y_{\alpha_1,\cdots,\alpha_{i-1}}$ contracts a divisor F_i onto $\mathbb{P}^1$. Indeed, for PGL_2-equivariant compactifications, the number of boundary components is equal to Picard rank of the underlying variety([HTT14, Proposition 5.1]), so each birational morphism ϕ_i must be an extremal contraction. Moreover F_i is contracted by ϕ_i because if we forget α_i, then we cannot recover the value $\mu(\alpha_i)$. Hence ϕ_i is a divisorial contraction contracting F_i. On $Y_{\alpha_1,\alpha_2,\alpha_3}$, E is identified with $B[3]$ and $F_1 + F_2 + F_3$ is identified with $B[2]$. Then each ϕ_i can be considered as the blow-up of the proper transform of $\pi_1^{-1}(\alpha_i) \subset B[3]$.

This blow-up description shows that E, $F_1, \cdots, F_n$ form a basis for $\mathrm{Pic}(Y_\alpha)_{\mathbb{Q}}$, and the canonical bundle is equal to

$$K_{Y_\alpha} = -\sum_{i=1}^{n} F_i - 2E.$$

Consider the morphism

$$\beta : Y_\alpha \subset \overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1) \to \tilde{X} \to \mathbb{P}(W_n),$$

and let $L := \beta^*\mathcal{O}_{\mathbb{P}(W_n)}(1)$. We have

Proposition 3.3. *[HT03, Lemma 3.3] The pullback of the hyperplane class is of the form*

$$L = \frac{n-2}{2}\sum_{i=1}^{n} F_i + \frac{n}{2}E.$$

Proof. We include a proof for completeness. Since L is the pullback from a $\mathfrak{S}_n$-quotient of $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1)$, L must take the form

$$L = a\sum_{i=1}^{n} F_i + bE.$$

Note that the scheme-theoretic intersection $Y_\alpha \cap B_{N\setminus\{i\}}$ is reduced. Indeed, the divisor $B_{N\setminus\{i\}}$ is isomorphic to $\overline{\mathcal{M}}_{0,n} \times \overline{\mathcal{M}}_{0,2}(\mathbb{P}^1, 1)$, so the intersection $Y_\alpha \cap B_{N\setminus\{i\}}$ is isomorphic to $\overline{\mathcal{M}}_{0,2}(\mathbb{P}^1, 1) \cong \mathbb{P}^1 \times \mathbb{P}^1$. Let $R \subset E$ be the proper transform of the general fiber of $\pi_2 : E \to \mathbb{P}^1$. Then we have

$$F_i.R = 1, \quad \text{and} \quad (E + F_4 + \cdots + F_n).R = -1.$$

The second identity follows from [Har77, Theorem 8.24(c)]. Hence we find $E.R = 2 - n$. On the other hand, R is contracted by β, so we have $L.R = 0$. Thus we conclude that L is of the form

$$L = c((n-2)\sum_{i=1}^{n} F_i + nE)$$

Next, let $C \subset E$ be the proper transform of the general fiber of $\pi_1 : E \to \mathbb{P}^1$. The β maps C isomorphically onto its image and the image has degree n. Thus we obtain

$L.C = n$. It is easy to see that $F_i.C = 0$. Moreover, since $\mathcal{N}_{\Delta_{\mathrm{small}}} \cong \mathcal{O}(2) \oplus \mathcal{O}(2)$, it follows from [Har77, Proposition 7.12 and Theorem 8.24(c)] that

$$\mathcal{O}(E)|_C \cong \mathcal{O}(2).$$

Therefore, we find $E.C = 2$ and we conclude that $c = \frac{1}{2}$. □

Let Z be any smooth equivariant compactification of SL_2. The degree 2 map $\mathrm{SL}_2 \to \mathrm{PGL}_2$ extends to a SL_2-equivariant rational map $\varphi : Z \dashrightarrow Y_\alpha$. After applying SL_2-equivariant resolution, if necessary, we may assume that φ is an honest SL_2-equivariant morphism and the boundary divisor of Z is a divisor with strict normal crossings.

Lemma 3.4. *$\varphi : Z \to Y_\alpha$ is ramified along E and F_i for each i.*

Proof. We prove that E is ramified. Recall that the forgetting map $Y_\alpha \to Y_{\alpha_1,\alpha_2,\alpha_3} = \overline{\mathcal{M}}_{0,3}(\mathbb{P}^1, 1)$ is a SL_2-equivariant birational morphism and E is identified with $B[3]$. Moreover, the evaluation map $\mathrm{ev} : \overline{\mathcal{M}}_{0.3}(\mathbb{P}^1, 1) \to (\mathbb{P}^1)^3$ is the blow-up along the small diagonal Δ_{small} and E is the exceptional divisor of this blow up. Thus, we only need to consider $Z \to \overline{\mathcal{M}}_{0,3}(\mathbb{P}^1, 1)$ and prove that the exceptional divisor $B[3]$ is ramified. After changing a base point, if necessary, we may assume that the map $\mathrm{SL}_2 \to (\mathbb{P}^1)^3$ is given by

$$\mathrm{SL}_2 \ni g \mapsto \left(g \begin{pmatrix} 1 \\ 0 \end{pmatrix}, g \begin{pmatrix} 0 \\ 1 \end{pmatrix}, g \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \in (\mathbb{P}^1)^3.$$

The function field of Z is $k(a, b, c, d)$ where k is the ground field and a, b, c, d satisfy $ad - bc = 1$. Then the function field of PGL_2 is a subfield of $k(a, b, c, d)$:

$$k\left(\frac{a}{c}, \frac{b}{d}, \frac{a+b}{c+d}\right).$$

Let $x = a/c$, $y = b/d$, and $z = (a + b)/(c + d)$. The divisor E is the exceptional locus over the small diagonal Δ_{small} defined by $x - y = 0$ and $y - z = 0$. Write t for $(y - z)/(x - y)$. Then the local ring $\mathcal{O}_E$ at E is

$$k[t, x, y]_{(x-y)}.$$

Note that $k(a, b, c, d) = k(x, y, z)(c^{-1})$ and c^{-1} satisfies

$$c^{-2} = -(x - y)(1 + t^{-1}).$$

Therefore, we conclude that E is ramified. The divisors F_i can be studied similarly. □

Let $Z \to \tilde{Z} \to Y_\alpha$ be the stein factorization of $\varphi : Z \to Y_\alpha$. We denote a SL_2-equivariant birational map $Z \to \tilde{Z}$ by h and a degree 2 finite morphism $\tilde{Z} \to Y_\alpha$ by $\tilde{\varphi}$. Let $\tilde{E}$ and $\tilde{F}_i$ be inverse images of E and F_i on $\tilde{Z}$ respectively, and we identify them with proper transforms of $\tilde{E}$ and $\tilde{F}_i$ on Z respectively. Write $\{G_j\}$ for exceptional divisors of h. Define M to be a sum of $\tilde{E}$, $\tilde{F}_i$'s and G_j's so that M is the boundary divisor of Z. Consider a divisor $\frac{2}{n}\varphi^* L + K_Z + M$, and write it as a linear combination of components of M:

$$\frac{2}{n}\varphi^* L + K_Z + M = a\tilde{E} + \sum_{i=1}^{n} b_i \tilde{F}_i + \sum_j c_j G_j.$$

Lemma 3.5. *We have*

$$a = 0, \quad b_i > 0, \quad \text{and} \quad c_j > 0.$$

Proof. It follows from [KM98, Corollary 2.31] that a pair $(Y_\alpha, E + \sum_i F_i)$ is log canonical. Then [Kol92, Proposition 20.2 and Proposition 20.3] implies that a pair $(\tilde{Z}, \tilde{E} + \sum_i \tilde{F}_i)$ is also log canonical. We obtain that

$$\frac{2}{n}\varphi^* L + K_Z + M = \frac{2}{n}\varphi^* L + h^*(K_{\tilde{Z}} + \tilde{E} + \sum_i \tilde{F}_i) + \sum_j d_j G_j,$$

where $d_j \geq 0$. Then again it follows from [Kol92, Proposition 20.2] that

$$\frac{2}{n}\varphi^* L + K_Z + M = \varphi^* \left(\frac{2}{n} L + K_{Y_\alpha} + E + \sum_i F_i\right) + \sum_j d_j G_j.$$

Now Proposition 3.3 implies that $a = 0$ and $b_i > 0$. Next we prove that $c_j > 0$. If $h(G_j) \subset \cup_i \tilde{F}_i$, then the support of $\varphi^*\left(\frac{2}{n}L + K_{Y_\alpha} + E + \sum_i F_i\right)$ contains G_j so $c_j > 0$ follows. Suppose that $h(G_j) \not\subset \cup_i \tilde{F}_i$. Let $V = Y_\alpha \setminus \cup_i \tilde{F}_i$. It follows from [KM98, Corollary 2.31] that (V, E) is purely log terminal. Hence [Kol92, Proposition 20.2 and Proposition 20.3] show that a pair $(\tilde{\varphi}^{-1}(V), \tilde{E})$ is also purely log terminal. Thus we obtain that $d_j > 0$. □

4. Asymptotic volume of height balls

In this section, we recall the results of [CLT10a] and apply them to study the volume function $V(T)$ defined in Section 2.

4.1. **Clemens complexes and height zeta functions.** Let F be a local field of characteristic zero and we fix an algebraic closure $F \subset \bar{F}$. Consider a smooth projective variety X defined over F with a reduced effective divisor D over F. Let

$$D_{\bar{F}} = \cup_{\alpha \in \bar{\mathcal{A}}} D_\alpha,$$

be the irreducible decomposition of $D_{\bar{F}}$. We assume that $D_{\bar{F}} = \sum_{\alpha\in\bar{\mathcal{A}}} D_\alpha$ is a divisor with strict normal crossings. For any $A \subset \bar{\mathcal{A}}$, define $D_A = \cap_{\alpha\in A} D_\alpha$. Note that $D_\emptyset = X_{\bar{F}}$. Because of strict normal crossings, D_A is a disjoint union of smooth projective varieties of codimension $|A|$. We define the geometric Clemens complex $\mathcal{C}_{\bar{F}}(D)$ as the set of all pairs (A, Z) where $A \subset \bar{\mathcal{A}}$ and Z is an irreducible component of D_A([CLT10a, Section 3.1.3]). The geometric Clemens complex is endowed with the following partial order relation: $(A, Z) \prec (A', Z')$ if $A \subset A'$ and $Z \supset Z'$. Thus, $\mathcal{C}_{\bar{F}}(D)$ is a poset and its elements are called faces. When $a \prec b$, we say that a is a face of b. The Galois group $\mathrm{Gal}(F)$ acts on $\mathcal{C}_{\bar{F}}(D)$ naturally. Since F is a pe rfect field, an integral scheme Z of $X_{\bar{F}}$ is defined over F if and only if Z is fixed by the Galois action Gal(F). We define the rational Clemens complex $\mathcal{C}_F(D)$ as the sub-poset of $\mathcal{C}_{\bar{F}}(D)$ consisting of $\mathrm{Gal}(F)$-fixed faces([CLT10a, Section 3.1.4]). For any $(A, Z) \in \mathcal{C}_F(D)$, D_A and Z are defined over F, so $Z(F)$ makes sense. We define the analytic Clemens complex $\mathcal{C}^{\mathrm{an}}_F(D)$ to be a sub-poset of $\mathcal{C}_F(D)$ consisting of pairs $(A, Z) \in \mathcal{C}_F(D)$ such that $Z(F) \neq \emptyset$.([CLT10a, Section 3.1.5]). In general, let $\mathcal{P}$ be a poset. The dimension of a face $p \in \mathcal{P}$ is defined as the supremum of the lengths n of chains $p_0 \prec \cdots \prec p_n$, where p_i are distinct and $p = p_n$. The dimension of $\mathcal{P}$ is the supremum of all dimensions of all faces.

Let $\mathcal{A} = \bar{\mathcal{A}}/\mathrm{Gal(F)}$ be the quotient of $\bar{\mathcal{A}}$ by the Galois group Gal(F). This set can be identified with the set of irreducible components of D. For each $\alpha \in \mathcal{A}$, we denote the corresponding divisor by Δ_α. Let $\mathcal{L} = (L, \|\cdot\|)$ be a metrized line bundle with a global section f_L whose support coincide with the support of D (see Section 1 for a definition of metrized line bundles). Let ω be a non-vanishing top degree differential form on $U = X \setminus D$. We are interested in the following height zeta function:

$$\mathsf{Z}(s) = \int_{U(F)} \|\mathsf{f}_L\|^s \,\mathrm{d}|\omega|,$$

where s is a complex number and $|\omega|$ is a measure associated to ω(see [CLT10a, Section 2.1.7] for a definition). The connection between height zeta functions and asymptotic volume of height balls is given by Tauberian theorems [CLT10a, Appendix A].

Theorem 4.1. *[CLT10a, Theorem A.1] Suppose that $\mathsf{Z}(s)$ admits a meromorphic continuation to the half plane $\{\Re(s) > a - \delta\}$, where $a > 0$ and $\delta > 0$, with the unique pole at $s = a$ of order b. Then the volume function*

$$V(T) = \int_{\{\mathsf{H}_{\mathsf{f}_L}(P) = \|\mathsf{f}_L(P)\|^{-1} \leq T\}} \mathrm{d}|\omega|$$

behaves like $\Theta T^a \log(T)^{b-1}$ as $T \to \infty$.

Hence the meromorphic continuation of $\mathsf{Z}(s)$ is the key to understand the asymptotic behavior of volume of height balls, and its properties are governed by Clemens complexes of D. More precisely write

$$\mathrm{div}\,(\mathsf{f}_\mathrm{L}) = \sum_{\alpha \in \mathcal{A}} \lambda_\alpha \Delta_\alpha, \quad -\mathrm{div}\,(\omega) = \sum_{\alpha \in \mathcal{A}} \kappa_\alpha \Delta_\alpha.$$

Note that we are assuming that $\lambda_\alpha > 0$ for any $\alpha \in \mathcal{A}$. Define

$$a(L, \omega) = \max_{\alpha \in \mathcal{A}} \frac{\kappa_\alpha - 1}{\lambda_\alpha}.$$

Then $\mathsf{Z}(s)$ is holomorphic when $\Re(s) > a(L, \omega)$([CLT10a, Lemma 4.1]). Let $\mathcal{A}(L, \omega)$ denote the set of all $\alpha \in \mathcal{A}$ where the maximum is obtained, i.e., $a(L, \omega) = (\kappa_\alpha - 1)/\lambda_\alpha$. Let $\mathcal{C}^{\mathrm{an}}_{F,(L,\omega)}(D)$ be a subposet of $\mathcal{C}^{\mathrm{an}}_F(D)$ consisting of (A, Z) such that $A \subset \mathcal{A}(L, \omega)$. [CLT10a, Proposition 4.3 and Corollary 4.4] claims that the height zeta function $\mathsf{Z}(s)$ admits a meromorphic continuation extended to a half plane $\Re(s) > a(L, \omega) - \delta$ for some $\delta > 0$ and its order of the pole at $s = a(L, \omega)$ is given by 1+ the dimension of the poset $\mathcal{C}^{\mathrm{an}}_{F,(L,\omega)}(D)$. We summarize the above discussion in the following theorem:

Theorem 4.2. *[CLT10a, Lemma 4.1, Proposition 4.3, and Corollary 4.4] The height zeta function $\mathsf{Z}(s)$ is holomorphic on a half plane $\Re(s) > a(L, \omega)$. Moreover, it admits a meromorphic continuation extended to a half plane $\Re(s) > a(L, \omega) - \delta$ for some $\delta > 0$ and the order of the pole at $s = a(L, \omega)$ is*

$$1 + \dim \mathcal{C}^{\mathrm{an}}_{F,(L,\omega)}(D).$$

4.2. **Asymptotic volume.** We retain the notations in Section 1 and Section 3. For our arithmetic applications, we need to construct moduli spaces $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1)$ and $\overline{\mathcal{M}}_{0,n}$ over $\mathrm{Spec}(\mathbb{Q})$. This is done in [Bal08]. In fact, the moduli spaces of stable maps are constructed over $\mathrm{Spec}(\mathbb{Z})$ via geometric invariant theory.

Let $[f] \in \mathbb{P}(W_n)(\mathbb{Q})$ be a binary form of degree n with $\mathbb{Q}$-coefficients and distinct roots. Then X_f is the Zariski closure of the SL_2-orbit of $[f]$ and it is defined over $\mathbb{Q}$. We consider the SL_2-equivariant birational morphism

$$\varrho : \tilde{X} = \overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1)/\mathfrak{S}_n \to \mathbb{P}(W_n),$$

which is a $\mathfrak{S}_n$-quotient of the evaluation map $\mathrm{ev} : \overline{\mathcal{M}}_{0,\mathrm{n}}(\mathbb{P}^1, 1) \to (\mathbb{P}^1)^{\mathrm{n}}$. We denote the quotient map $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1) \to \tilde{X}$ by q. Let $\tilde{X}_f \subset \tilde{X}$ be the strict transform of X_f which is again defined over $\mathbb{Q}$. Write F_f for the splitting field of f and $\alpha = (\alpha_1, \cdots, \alpha_n)$ for roots of f. Then the pointed rational curve $C_\alpha = (\mathbb{P}^1, \alpha_1, \cdots, \alpha_n)$ is defined over F_f, hence $Y_\alpha = \psi^{-1}(C_\alpha)$ is defined over F_f where $\psi : \overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1) \to \overline{\mathcal{M}}_{0,n}$ is the forgetting map. Define

$$A[n] = \tilde{X}_f \cap \tilde{D}[n], \quad A[n-1] = \tilde{X}_f \cap \tilde{D}[n-1].$$

Note that since $\tilde{D}[n]$ and $\tilde{D}[n-1]$ are defined over $\mathbb{Q}$, $A[n]$ and $A[n-1]$ are also defined over $\mathbb{Q}$. The divisor $A[n]$ is geometrically irreducible, but $A[n-1]$ may not be. We have

Lemma 4.3. *The set $A[n](\mathbb{Q})$ is Zariski dense in $A[n]$.*

Proof. Let $(C, \alpha_1, \cdots, \alpha_n, \mu)$ be a stable map such that

- C is the union of two $\mathbb{P}^1$s and both $\mathbb{P}^1$s are defined over $\mathbb{Q}$,
- marked points $\alpha_1, \cdots, \alpha_n$ are on the collapsed component,
- a map μ is also defined over $\mathbb{Q}$.

Then (C, α, μ) is a stable map defined over the splitting field F_f and it corresponds to a F_f-rational point P on $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^1, 1)$. Consider a Galois action $\sigma \in \mathrm{Gal}(F_f/\mathbb{Q})$. A point σP corresponds to $(C, \sigma\alpha, \mu)$, hence we have $q(P) = q(\sigma P)$. This means that $q(P)$ is $\mathrm{Gal}(F_f/\mathbb{Q})$-fixed, thus $q(P)$ is a $\mathbb{Q}$-rational point. Now we can vary the intersection of C and the value of μ so that $A[n](\mathbb{Q})$ is Zariski dense. $\square$

Let Z be a smooth SL_2-equivariant compactification of SL_2 defined over $\mathbb{Q}$. We have a SL_2-equivariant rational map $\varphi : Z \dashrightarrow \tilde{X}_f$ mapping $\mathrm{SL}_2 \ni g \mapsto g[f] \in X_f$ and after applying SL_2-equivariant resolution, if necessary, we may assume that φ is an honest morphism. We denote the morphism from $Z \to X_f$ by φ too. Write ω for the top invariant differential form on SL_2. Let S be a finite set of places including the archimedean place. For each $v \in S$, we are interested in the following height zeta function:

$$\mathsf{Z}_v(s) = \int_{Z(\mathbb{Q}_v)} \mathsf{H}_v(\varphi(z))^{-s} \, \mathrm{d}|\omega|_v(z)$$

where H_v is the local height function defined in Section 1. We have

Theorem 4.4. *Assume that either*

- *$n \geq 5$ and f is general enough so that $Y_\alpha \cong \tilde{X}_f \otimes F_f$, or*
- *all roots of f are $\mathbb{Q}$-rational.*

Then the height zeta function $\mathsf{Z}_v(s)$ is holomorphic on a half plane $\Re(s) > \frac{2}{n}$ and it admits a meromorphic continuation extended to a half plane $\Re(s) > \frac{2}{n} - \delta$ for some $\delta > 0$. Moreover the order of the pole at $s = \frac{2}{n}$ is 1.

Proof. Suppose that $n \geq 5$ and f is general. Let $\{\Delta_\alpha\}_{\alpha \in \mathcal{A}}$ be the irreducible decomposition of the boundary divisor D of Z. Let f be the pullback of the discriminant divisor on Z. Let

$$\mathrm{div}(\mathsf{f}) = \sum_{\alpha \in \mathcal{A}} \lambda_\alpha \Delta_\alpha, \quad -\mathrm{div}(\omega) = \sum_{\alpha \in \mathcal{A}} \kappa_\alpha \Delta_\alpha$$

Then Lemma 3.5 implies that

$$\max_{\alpha\in\mathcal{A}}\left\{\frac{\kappa_\alpha-1}{\lambda_\alpha}\right\}=\frac{2}{n}.$$

Moreover, the Clemens complex $C^{\mathrm{an}}_{\mathbb{Q}_v,(L,\omega)}(D)$ consists of one element corresponding to $A[n]$; indeed, Lemma 4.3 implies that $A[n]$ has a dense set of rational points, and Lemma 3.4 concludes that the ramification divisor $\tilde{E}$ above $A[n]$ also has a dense set of rational points. This means that this ramification divisor $\tilde{E}$ is an element of the analytic Clemens complex. Lemma 3.5 guarantees that $\tilde{E}$ is the only divisor which achieves $\frac{\kappa_\alpha-1}{\lambda_\alpha}=\frac{2}{n}$. Now our assertion follows from Theorem 4.2.

Assume that all roots of f are $\mathbb{Q}$-rational. Then the Y_α is defined over $\mathbb{Q}$ and the divisor $E=Y_\alpha\cap B_N$ contains Zariski dense $\mathbb{Q}$-rational points. Thus our assertion follows from Lemma 3.4, Lemma 3.5 and Theorem 4.2. □

We define

$$G=\prod_{v\in S}\mathrm{SL}_2(\mathbb{Q}_v)$$

and consider the height ball

$$\mathsf{B}(T)=\{g\in G\mid \mathsf{H}(g\cdot[f])\leq T\}.$$

where H is the global height function defined by $\mathsf{H}=\prod_{v\in S}\mathsf{H}_v$. Let $\mu_G=\prod_{v\in S}|\omega|_v$ be a Haar measure. We denote the volume function of the height ball $\mathsf{B}(T)$ with respect to a Haar measure μ_G by $V(T)$. Then we have

Corollary 4.5. *Suppose that $n\geq 5$ and f is general, or all roots of f are $\mathbb{Q}$-rational. Then we have*

$$V(T)\sim cT^{\frac{2}{n}}(\log T)^{\#S-1},$$

for some $c>0$.

Proof. To prove this corollary, we need to consider the following height zeta function:

$$\int_G \mathsf{H}(g\cdot[f])^{-s}\,\mathrm{d}\mu_G=\prod_{v\in S}\int_{\mathrm{SL}_2(\mathbb{Q}_v)}\mathsf{H}_v(g_v\cdot[f])^{-s}\,\mathrm{d}|\omega|_v=:\prod_{v\in S}\mathsf{Z}_v(s).$$

Then it follows from Theorem 4.4 that this zeta function has a pole at $s=2/n$ of order $\#S$. If S consists of the real place, then we can apply Theorem 4.1 to conclude our assertion. However when S contains a non-archimedean place p, then the local zeta function $\mathsf{Z}_p(s)$ is $\frac{2\pi i}{\log p}$-periodic and we cannot apply Theorem 4.1 since $\mathsf{Z}_p(s)$ has infinitely many poles on the vertical line $\Re(s)=2/n$. More precisely the following function

$$(1-p^{-(s-\frac{2}{n})})\mathsf{Z}_p(s)$$

admits an analytic continuation to the half plane $\{\Re(s)>\frac{2}{n}-\delta\}$ for some $\delta>0$([CLT10a, Proposition 4.2]). Instead, we apply [CLT10a, Theorem A.7] to $\mathsf{Z}(s)$. To apply this theorem, one needs to verify that $\log p/\log q$ is not Liouville number for distinct primes p,q. This fact is proved in [Bak90]. [Bak90, Theorem 3.1] claims that the irrationality measure of $\frac{\log p}{\log q}$ is bounded by a constant depending on p,q. Thus our assertion follows. □

5. Generalizations

The method to prove the main result in Section 2 generalizes to semisimple groups. Let F be a number field and G a simply connected, almost F-simple group. Let S be a finite set of places containing all archimedean places v such that $G(F_v)$ is non-compact. We denote the ring of integers of F by $\mathfrak{o}_F$ and the ring of S-integers of F by $\mathfrak{o}_{F,S}$. We fix an integral model of G so that $G(\mathfrak{o}_F)$ makes sense. Denote the S-adic Lie group $\prod_{v\in S} G(F_v)$ by G_S. We embed $G(\mathfrak{o}_{F,S})$ into G_S diagonally. Then $G(\mathfrak{o}_{F,S})$ is a lattice in G_S. Let X be a smooth projective equivariant compactification of G defined over F and $\mathcal{L} = (L, \|\cdot\|)$ an adelically metrized big line bundle on X with a global section s whose support coincides with $X \setminus G$. We define local height functions $\mathsf{H}_v : G(F_v) \to \mathbb{R}_{>0}$ and the global height $\mathsf{H} : G_S \to \mathbb{R}_{>0}$ by

$$\mathsf{H}_{\mathcal{L},s,v}(P_v) = \|s(P_v)\|^{-1}, \quad \mathsf{H}_{\mathcal{L},s}((P_v)_{v\in S}) = \prod_{v\in S} \mathsf{H}_{\mathcal{L},s,v}(P_v).$$

We suppose that for any archimedean place $v \in S$, the local height function $\mathsf{H}_{\mathcal{L},s,v}$ is invariant under the action of a maximal compact subgroup K_v. It is always possible to choose a metrization to satisfy this property. It is also a property of height functions that for any non-archimedean place v, the local height $\mathsf{H}_{\mathcal{L},s,v}$ is invariant under the action of a compact open subgroup K_v. We are interested in a counting function $N(T)$ of $G(\mathfrak{o}_{F,S})$ with respect to $\mathsf{H}_{\mathcal{L},s}$,

$$N(T) = \#\{\gamma \in G(\mathfrak{o}_{F,S}) \mid \mathsf{H}_{\mathcal{L},s}(\gamma) \leq T\}.$$

When X is a biequivariant compactification of G, this counting function has been studied in [TBT13] and [BO12]. However, the case of one-sided equivariant compactifications remained open. Our technique in Section 2 can solve this case.

The action of G_S on $Y := G(\mathfrak{o}_{F,S})\backslash G_S$ is mixing([BO12, Proposition 2.4]). We define

$$H_S = \prod_{v\in S} K_v.$$

Then G_S has the wavefront property in $H_S\backslash G_S$. Thus translates of H_S-orbits are equidistributed in Y(Theorem 2.5). Let μ_S be a haar measure on G_S and μ_Y an invariant measure on Y such that $\mu_S = \mu_Y$ holds locally. We consider height balls

$$\mathsf{B}(T) = \{g \in G_S) \mid \mathsf{H}_{\mathcal{L},s}(g) \leq T\}.$$

We denote the volume function of these height balls by $V(T)$. Now the discussion in Section 2 leads to the following theorem:

Theorem 5.1. *Let $V^*(T) = V(T)/\mu_Y(Y)$. Then we have*

$$\frac{N(T)}{V^*(T)} \to 1 \quad \text{as } T \to +\infty.$$

References

[Bak90] Alan Baker. *Transcendental number theory.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1990.

[Bal08] Elizabeth Baldwin. A GIT construction of moduli spaces of stable maps in positive characteristic. *J. Lond. Math. Soc. (2)*, 78(1):107–124, 2008.

[BHC62] Armand Borel and Harish-Chandra. Arithmetic subgroups of algebraic groups. *Ann. of Math. (2)*, 75:485–535, 1962.

[BO12] Yves Benoist and Hee Oh. Effective equidistribution of S-integral points on symmetric varieties. *Ann. Inst. Fourier (Grenoble)*, 62(5):1889–1942, 2012.

[CLT10a] Antoine Chambert-Loir and Yuri Tschinkel. Igusa integrals and volume asymptotics in analytic and adelic geometry. *Confluentes Mathematici*, 2, no. 3:351–429, 2010.

[CLT10b] Antoine Chambert-Loir and Yuri Tschinkel. Integral points of bounded height on toric varieties, 2010. `arXiv 1006.3345`.

[CLT12] Antoine Chambert-Loir and Yuri Tschinkel. Integral points of bounded height on partial equivariant compactifications of vector groups. *Duke Math. J.*, 161(15):2799–2836, 2012.

[DRS93] William Duke, Zeév Rudnick, and Peter Sarnak. Density of integer points on affine homogeneous varieties. *Duke Math. J.*, 71(1):143–179, 1993.

[EM93] Alex Eskin and Curt McMullen. Mixing, counting, and equidistribution in Lie groups. *Duke Math. J.*, 71(1):181–209, 1993.

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[HT03] Brendan Hassett and Yuri Tschinkel. Integral points and effective cones of moduli spaces of stable maps. *Duke Math. J.*, 120(3):577–599, 2003.

[HTT14] Brendan Hassett, Sho Tanimoto, and Yuri Tschinkel. Balanced line bundles and equivariant compactifications of homogeneous spaces. *Internat. Math. Res. Notices*, 2014.

[KM98] János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

[Kol92] János Kollár. *Flips and abundance for algebraic threefolds*. Société Mathématique de France, Paris, 1992. Papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991, Astérisque No. 211 (1992).

[Mar91] Grigori A. Margulis. *Discrete subgroups of semisimple Lie groups*, volume 17 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1991.

[MU83] Shigeru Mukai and Hiroshi Umemura. Minimal rational threefolds. In *Algebraic geometry (Tokyo/Kyoto, 1982)*, volume 1016 of *Lecture Notes in Math.*, pages 490–518. Springer, Berlin, 1983.

[PR94] Vladimir Platonov and Andrei Rapinchuk. *Algebraic groups and number theory*, volume 139 of *Pure and Applied Mathematics*. Academic Press Inc., Boston, MA, 1994. Translated from the 1991 Russian original by Rachel Rowen.

[TBT13] Ramin Takloo-Bighash and Yuri Tschinkel. Integral points of bounded height on compactifications of semi-simple groups. *Amer. J. Math.*, 135(5):1433–1448, 2013.

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