# ESSENTIAL EXTENSIONS, THE NILPOTENT FILTRATION AND THE ARONE-GOODWILLIE TOWER

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ABSTRACT. The spectral sequence associated to the Arone-Goodwillie tower for the *n*-fold loop space functor is used to show that the first two non-trivial layers of the nilpotent filtration of the reduced mod 2 cohomology of a (sufficiently connected) space with nilpotent cohomology are comparable. This relies upon the theory of unstable modules over the mod 2 Steenrod algebra, together with properties of a generalized class of almost unstable modules which is introduced here.

An essential ingredient of the proof is a non-vanishing result for certain extension groups in the category of unstable modules localized away from nilpotents.

### 1. Introduction

A fundamental question in algebraic topology is to ask what modules over the mod p Steenrod algebra  $\mathscr{A}$  can be realized as the reduced mod p cohomology  $H^*(X)$  of a space. First obstructions are provided by the fact that the module must be unstable and that this structure should be compatible with the cup product. For example, Steenrod asked what polynomial algebras can be realized as the cohomology of a space. At the opposite extreme one can ask what unstable algebras with trivial cup square (at the prime two) can be realized; a partial response is provided below in Theorem 2.

The structure theory of the category  $\mathscr{U}$  of unstable modules allows the formulation of precise questions of particular interest in the case where  $H^*(X)$  is nilpotent. Kuhn [Kuh95b] proposed a series of highly-influential non-realization conjectures, postulating significant restrictions on the structure of  $H^*(X)$  as an unstable module. Many of these are now theorems [Sch98, CGPS14]. The conjectures are phrased in terms of the nilpotent filtration of the category  $\mathscr{U}$ ; this is a decreasing filtration, where  $\mathscr{N}il_n$  is the smallest localizing subcategory of  $\mathscr{U}$  containing all n-fold suspensions. A general question is the following: what can be said about the structure of  $H^*(X)$  as an unstable module if it belongs to  $\mathscr{N}il_n$ ?

The condition  $H^*(X) \in \mathcal{N}il_n$  has topological significance when Lannes' mapping space technology can be applied: it is equivalent to  $\max(BV, X)$  being (n-1)-connected for all elementary abelian p-groups V. It is clear that n-fold suspensions  $\Sigma^n Y$  satisfy the hypothesis and, since the algebraic suspension restricts to a functor  $\Sigma: \mathcal{N}il_n \to \mathcal{N}il_{n+1}$ , it is most interesting to consider the case where X is not a suspension.

The largest submodule of an unstable module M which lies in  $\mathcal{N}il_n$  is written  $\operatorname{nil}_n M$ ; if M lies in  $\mathcal{N}il_n \setminus \mathcal{N}il_{n+1}$ , then there is a short exact sequence of unstable modules:

$$0 \to \mathrm{nil}_{n+1} M \to M \to \Sigma^n \rho_n M \to 0$$

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where  $\rho_n M$  is a reduced unstable module (ie contains no non-trivial nilpotent sub-module) which is non-zero. If in addition M is n-connected, then  $\rho_n M$  is connected (trivial in degree zero).

It is now known that (at least up to nilpotent unstable modules) there are few restrictions which can be placed on  $\rho_n M$  (for  $n \geq 1$ ), following the affirmation of the Lannes and Schwartz Artinian conjecture [PS14, SS14]. Namely, generalizing Kuhn's observation [Kuh95b], examples can be manufactured by considering the topological realization of the beginning of an injective resolution (modulo nilpotents) of a given reduced module and forming the homotopy cofibre:

$$\operatorname{hocofib}\left\{\bigvee_{i} \Sigma^{n} BV(i)_{+} \to \bigvee_{j} \Sigma^{n} BV(j)_{+}\right\}$$

where  $\{V(i)\}$ ,  $\{V(j)\}$  are finite sets of finite rank elementary abelian p-groups. The Artinian conjecture ensures that all finitely cogenerated modules admit such finite type presentations (modulo nilpotents).

Kuhn's non-realization conjectures highlight the interest of the case where  $\rho_n M$  is finitely generated over the Steenrod algebra. If M is n-connected, the cases which have already been proved show that  $\operatorname{nil}_{n+1} M$  must actually be large; for example, M cannot itself be finitely generated under these hypotheses. These results rely upon Lannes' T-functor technology to reduce to the smallest non-trivial case [CGPS14].

The purpose of this paper is to show how the analysis of the first two non-trivial columns of the spectral sequence associated to the Arone-Goodwillie tower for the functor  $X \mapsto \Sigma^\infty \Omega^n X$  impose conditions on the first two non-trivial layers of the nilpotent filtration of  $H^*(X)$ . In the case n=1, this is an application of the Eilenberg-Moore spectral sequence and the result recovered is a generalization of the main result of [CGPS14, Section 6]. The case n>1 is new and arose as an offshoot of the author's programme to express the non-realization results obtained by Kuhn [Kuh08] (for p=2) and by Büscher, Hebestreit, Röndigs and Stelzer [BHRS13] (for odd primes), in terms of obstruction classes in suitable Ext groups. The key result is provided, on passage to the quotient category  $\mathscr{U}/\mathscr{N}il$ , by a generalization of a theorem of Kuhn (see Theorem 8.3) showing that certain Ext groups in  $\mathscr{U}/\mathscr{N}il$  are non-trivial.

An application of the case n=1 is the following:

**Theorem 1.** Let M be a connected unstable module over  $\mathbb{F}_2$  of finite type such that  $\rho_0 M$  is non-zero and finitely generated. Then  $U(\Sigma M)$ , the Massey-Peterson enveloping algebra of the suspension of M, is not realizable as the  $\mathbb{F}_2$ -cohomology of a space.

The enveloping algebra  $U(\Sigma M)$  is isomorphic to the exterior algebra  $\Lambda^*(\Sigma M)$ , so this is a case of the following Theorem (for a more refined statement, see Corollary 9.18), in which QK denotes the module of indecomposables:

**Theorem 2.** Let K be a connected unstable algebra of finite type over  $\mathbb{F}_2$  such that the cup square  $Sq_0$  acts trivially on the augmentation ideal K. If QK is a 1-connected unstable module such that  $\rho_1QK$  is non-zero and finitely generated, then K is not realizable as the  $\mathbb{F}_2$ -cohomology of a space.

Note that the non-realization result of Gaudens and Schwartz [GS12, CGPS14] does not apply directly here, since no restriction is placed upon the higher nilpotent filtration of QK.

Remark 1.1. The hypothesis that  $\rho_1 QK$  is a finitely generated unstable module is essential. For example, consider  $H^*(X)$  for X = SU, so that  $\Omega X \simeq BU$ .

The main results of the paper are Theorem 9.9 (for n > 1) and Theorem 9.16 (for n = 1). For these, the prime is taken to be two; the modifications necessary in the odd primary case are indicated in Section 10.

To give an idea of the flavour of the results, consider the following:

**Theorem 3.** For  $1 \leq n \in \mathbb{N}$  and X an n-connected space such that  $H^*(X)$  is of finite type and  $\rho_n H^*(X)$  is finitely generated and non-trivial,

$$H^*(X)/\operatorname{nil}_{n+2}H^*(X)$$

is not the n-fold suspension of an unstable module.

At first sight, this result may not appear surprising: the structure theory of unstable algebras (modulo nilpotents) implies that, under the given hypotheses,  $\rho_n H^*(X)$  cannot be finitely generated if X is the n-fold suspension of a connected space. The theorem shows that this is already exhibited algebraically by the structure of  $H^*(X)/\text{nil}_{n+2}H^*(X)$ .

This is a fundamental point: in the spectral sequence the columns will not in general be unstable modules. For n=1, this is not a serious difficulty, since it is known that the columns of the  $E_2$ -term of the Eilenberg-Moore spectral sequence are unstable. For the general case, this is no longer true, yet the spectral sequence converges to an unstable module. To allow the nilpotent filtration of unstable modules to be brought to bear, the notion of an almost unstable module is introduced here, which is shown to be sufficient to cover the cases of interest.

Theorem 9.9 and Theorem 9.16 are much more precise, relating  $\rho_n H^*(X)$  and the next layer of the nilpotent filtration,  $\rho_{n+1}H^*(X)$ . Roughly speaking, for n>1 the result states that  $\rho_{n+1}H^*(X)$  is at least as large as  $\rho_n H^*(X)$ ; this is exhibited by the injectivity (modulo smaller objects relative to the Krull filtration of  $\mathscr{U}$ ) of the natural transformation

$$\Phi \rho_n H^*(X) \to \rho_{n+1} H^*(X)$$

that arises from the non-exactness of the iterated loop functor  $\Omega^n: \mathcal{U} \to \mathcal{U}$ .

For n = 1, the result is stronger, here the relevant transformation is

$$S^{2}(\rho_{1}H^{*}(X)) \to \rho_{2}H^{*}(X),$$

induced by the cup product of  $H^*(X)$ . The approach is unified here, showing how the two cases are related.

This gives information on the beginning of the nilpotent filtration, whereas the proofs of the known cases of Kuhn's non-realization conjectures give global information. Where applicable, Lannes' mapping space technology can be used to study the higher parts of the nilpotent filtration; the ideas involved will be transparent to the experts and to the readers of [CGPS14] and are not developed here.

Organization of the paper: Background is surveyed in Section 2; readers should consult this as and when is necessary. The technical notion of an almost unstable module is introduced in Section 3; this is necessary to be able to control the image of differentials in the spectral sequence, as is explained in Section 4. The spectral sequence derived from the Arone-Goodwillie tower is reviewed in Section 5 and the calculational input is provided in Section 6, namely the calculation of the  $E_1$ -term of the spectral sequence via the cohomology of extended powers. In particular, it is shown that, for the case at hand, the columns of the  $E_1$ -term are almost unstable. The case of the second extended power admits an explicit algebraic model, as explained in Section 7, which not only makes the results of the previous section more explicit in this case, but also provides a model for the differential  $d_1$ . The input from homological algebra is explained in Section 8, refining a theorem of Kuhn; this is the key ingredient in the proofs of the two main theorems, which

are given in Section 9. Section 10 sketches the modifications required in the odd primary case.

#### 2. Algebraic preliminaries

This section reviews background, referring to the literature (in particular [Sch94] and [Kuh14]) for details. As usual,  $\mathscr U$  denotes the full subcategory of unstable modules in  $\mathscr M$ , the category of graded modules over the mod p Steenrod algebra  $\mathscr A$ . The inclusion  $\mathscr U \to \mathscr M$  has left adjoint  $\Omega^\infty: \mathscr M \to \mathscr U$ , the destabilization functor.

The suspension functor  $\Sigma: \mathcal{M} \to \mathcal{M}$  restricts to  $\Sigma: \mathcal{U} \to \mathcal{U}$  and the iterated suspension functor  $\Sigma^t: \mathcal{U} \to \mathcal{U}$  ( $t \in \mathbb{N}$ ) has left adjoint the iterated loop functor  $\Omega^t: \mathcal{U} \to \mathcal{U}$ , which identifies with the composite functor  $\Omega^\infty \Sigma^{-t}$  restricted to  $\mathcal{U}$ .

The category of unstable algebras is denoted  $\mathscr{K}$  and the Massey-Peterson enveloping algebra  $U: \mathscr{U} \to \mathscr{K}$  is the left adjoint to the forgetful functor  $\mathscr{K} \to \mathscr{U}$ ; U takes values in the category  $\mathscr{K}_a$  of augmented unstable algebras. The indecomposables functor  $Q: \mathscr{K}_a \to \mathscr{U}$  is given explicitly by  $QK := \overline{K}/\overline{K}^2$ , where  $\overline{K}$  is the augmentation ideal.

2.1. The nilpotent filtration. The category of unstable modules  $\mathscr U$  has nilpotent filtration:

$$\ldots \subset \mathcal{N}il_{i+1} \subset \mathcal{N}il_i \subset \ldots \subset \mathcal{N}il_1 \subset \mathcal{N}il_0 = \mathcal{U},$$

where  $\mathcal{N}il_s$  is the smallest localizing subcategory containing all s-fold suspensions [Sch94, Kuh14]. In particular  $\mathcal{N}il_1$  is the subcategory of nilpotent unstable modules  $\mathcal{N}il$ .

The inclusion  $\mathscr{N}il_s \hookrightarrow \mathscr{U}$  admits a right adjoint  $\operatorname{nil}_s : \mathscr{U} \to \mathscr{N}il_s \subset \mathscr{U}$  so that an unstable module M has a natural, convergent decreasing filtration:

$$\ldots \subset \operatorname{nil}_{s+1} M \subset \operatorname{nil}_s M \subset \ldots \subset \operatorname{nil}_0 M = M$$

and, for  $s \in \mathbb{N}$ ,  $\operatorname{nil}_s M/\operatorname{nil}_{s+1} M \cong \Sigma^s \rho_s M$ , where  $\rho_s M$  is a reduced unstable module<sup>1</sup>. (An unstable module is reduced if it contains no non-trivial suspension.)

2.2. Functors between  $\mathbb{F}$ -vector spaces. Functors on vector spaces over a finite field  $\mathbb{F}$  arise naturally in the study of unstable modules via Lannes' T-functor [Sch94].

Notation 2.1. For  $\mathbb{F}$  a finite field, let  $\mathscr{F}$  denote the category of functors from finite-dimensional  $\mathbb{F}$ -vector spaces to  $\mathbb{F}$ -vector spaces and  $\mathscr{F}_{\omega} \subset \mathscr{F}$  the full subcategory of locally finite (or analytic) functors.

The category  $\mathscr{F}$  is tensor abelian with enough projectives and injectives. A functor is finite if it has a finite composition series and is locally finite if it is the colimit of its finite subobjects.

A functor F is polynomial of degree d if  $\Delta^{d+1}F = 0$ , where  $\Delta : \mathscr{F} \to \mathscr{F}$  is the difference functor defined by  $\Delta F(V) := F(V \oplus \mathbb{F})/F(V)$ .

Over the prime field  $\mathbb{F}_p$ , the quotient category  $\mathscr{U}/\mathscr{N}il$  is equivalent to  $\mathscr{F}_{\omega}$  and the localization functor  $\mathscr{U} \to \mathscr{U}/\mathscr{N}il$  gives the exact functor  $l: \mathscr{U} \to \mathscr{F}$  which can be identified in terms of Lannes' T-functor as  $M \mapsto \{V \mapsto (T_V M)^0\}$  [Sch94].

Notation 2.2. For  $d \in \mathbb{N}$ , denote by  $\mathscr{F}_d$  the full subcategory of  $\mathscr{F}$  of functors of polynomial degree d.

<sup>&</sup>lt;sup>1</sup>The notation  $\rho_s$  is used to avoid possible confusion with the Singer functors.

- 2.3. Examples of polynomial functors. A number of polynomial functors arise here, which are closely related to the nth tensor power functor  $T^n: V \mapsto V^{\otimes n}$ . The symmetric group  $\mathfrak{S}_n$  acts naturally by place permutations on  $T^n$ , giving:
  - (1) the *n*th divided power  $\Gamma^n := (T^n)^{\mathfrak{S}_n}$ ;
  - (2) the *n*th symmetric power  $S^n := (T^n)/\mathfrak{S}_n$ .

These functors are dual under the Kuhn duality functor  $D: \mathscr{F}^{\text{op}} \to \mathscr{F}$ , given by  $DF(V) := F(V^*)^*$  (see [Kuh94]).

Similarly, there is the nth exterior power functors  $\Lambda^n$ , which is self-dual. The functors  $S^1, \Lambda^1, \Gamma^1$  coincide with  $\mathrm{Id}: V \mapsto V$ .

The Frobenius pth power map induces a natural transformation  $S^n \to S^{np}$ . Henceforth taking p = 2, there is a non-split short exact sequence

$$(1) 0 \to \mathrm{Id} \to S^2 \to \Lambda^2 \to 0,$$

representing a non-zero class  $\varphi \in \operatorname{Ext}^1_{\mathscr{F}}(\Lambda^2, \operatorname{Id})$ . The composite of  $S^2 \to \Lambda^2$  with its dual is the norm map  $S^2 \to \Gamma^2$ ; this occurs in the top row of the following pullback diagram of exact sequences:

$$(2) \qquad 0 \longrightarrow \operatorname{Id} \longrightarrow S^{2} \longrightarrow \Gamma^{2} \longrightarrow \operatorname{Id} \longrightarrow 0$$

$$\parallel \qquad \qquad \downarrow \qquad \qquad$$

The rows represent non-zero classes in  $\operatorname{Ext}_{\mathscr{Z}}^2$ :

$$\tilde{e}_1 \in \operatorname{Ext}_{\mathscr{F}}^2(S^2, \operatorname{Id}) \mapsto e_1 \in \operatorname{Ext}_{\mathscr{F}}^2(\operatorname{Id}, \operatorname{Id}).$$

These classes appear for example in [FLS94].

2.4. The Krull filtration. The category  $\mathcal U$  of unstable modules has Krull filtration:

$$\mathcal{U}_0 \subset \mathcal{U}_1 \subset \ldots \subset \mathcal{U}_n \subset \ldots \subset \mathcal{U}$$

(see [Sch94, Kuh14]);  $\mathcal{U}_0$  identifies as the full subcategory of locally finite modules. For current purposes, the following is the key result:

**Proposition 2.3.** [Sch94] For  $d \in \mathbb{N}$ , the functor  $l : \mathcal{U} \to \mathscr{F}$  restricts to

$$l: \mathcal{U}_d \to \mathcal{F}_d$$
.

Moreover, if M is a reduced unstable module, then  $M \in \text{Ob } \mathcal{U}_d$  if and only if f(M)has polynomial degree d.

2.5. Functors on  $\mathscr{A}$ -modules. The categories  $\mathscr{M}$  and  $\mathscr{U}$  are tensor abelian; in particular, for  $n \in \mathbb{N}$ , the nth tensor functor  $T^n : \mathcal{M} \to \mathcal{M}, M \mapsto M^{\otimes n}$  is defined, which restricts to  $T^n: \mathcal{U} \to \mathcal{U}$ . Again,  $\mathfrak{S}_n$  acts naturally by place permutations on  $T^n$ , giving the nth symmetric invariants  $\Gamma^n := (T^n)^{\mathfrak{S}_n}$  and the nth symmetric coinvariants  $S^n := (T^n)/\mathfrak{S}_n$ . The functors  $\Gamma^n, S^n : \mathcal{M} \to \mathcal{M}$ restrict to  $\Gamma^n, S^n: \mathcal{U} \to \mathcal{U}$ . Similarly, the exterior power functor  $\Lambda^n: \mathcal{M} \to \mathcal{M}$ restricts to  $\Lambda^n: \mathcal{U} \to \mathcal{U}$ .

For the remainder of the section, the prime p is taken to be 2. Thus, the Frobenius functor  $\Phi: \mathcal{M} \to \mathcal{M}$  [Sch94, Section 1.7] is the usual degree-doubling functor.

**Lemma 2.4.** For p=2, the Frobenius functor  $\Phi: \mathcal{M} \to \mathcal{M}$  is exact, commutes with tensor products and there is a natural isomorphism

$$\Phi \Sigma \cong \Sigma^2 \Phi$$
.

Moreover,  $\Phi$  restricts to  $\Phi: \mathcal{U} \to \mathcal{U}$  and, if M is a reduced unstable module,  $\Phi M$ 

For  $i \in \mathbb{N}$ ,  $\Phi : \mathcal{U} \to \mathcal{U}$  restricts to:  $\Phi : \mathcal{N}il_i \to \mathcal{N}il_{2i}$ .

The Frobenius short exact sequence (1) and its dual have analogues in  $\mathcal{M}$ :

**Lemma 2.5.** For p=2 and  $M \in Ob \mathcal{M}$ , there are natural short exact sequences

$$0 \to \Lambda^2 M \to \Gamma^2 M \to \Phi M \to 0$$
$$0 \to \Phi M \to S^2 M \to \Lambda^2 M \to 0.$$

2.6. The Singer functors. (In this section, to simplify presentation, p is taken to be 2.) The Singer functor  $R_1$  was introduced for unstable modules in the form used here by Lannes and Zarati [LZ87]. For M an unstable module,  $R_1M$  is the sub  $\mathbb{F}[u]$ -module of  $\mathbb{F}[u] \otimes M$  generated by the image of the total Steenrod square  $\operatorname{St}_1(x) := \sum_i u^{|x|-i} \otimes Sq^ix \in \mathbb{F}[u] \otimes M$ , for  $x \in M$ . A key fact is that  $R_1M$  is stable under the action of  $\mathscr{A}$  on  $\mathbb{F}[u] \otimes M$ .

The extension to all  $\mathscr{A}$ -modules requires  $\mathbb{F}[u] \otimes M$  to be replaced by a half-completed tensor product, since the sum in  $\operatorname{St}_1(x)$  is no longer finite in general. This is reviewed in [Pow15] and details are given (for odd primes) in [Pow14].

Remark 2.6. This Singer functor must not be confused with the stabilized version which occurs in [BMMS86, Chapter II, Section 5], following ideas of Miller. For the current situation, compare the treatment (in homology) in [KM13].

Notation 2.7. Write  $\mathbb{F}[u]$  for the unstable algebra generated by u of degree 1 and  $\mathbb{F}[u]$ - $\mathscr{M}$  for the category of  $\mathbb{F}[u]$ -modules in  $\mathscr{M}$  (respectively  $\mathbb{F}[u]$ - $\mathscr{U}$  for  $\mathbb{F}[u]$ -modules in  $\mathscr{U}$ ).

**Lemma 2.8.** The categories  $\mathbb{F}[u]$ - $\mathcal{M}$  and  $\mathbb{F}[u]$ - $\mathcal{U}$  are abelian and there are exact forgetful functors  $\mathbb{F}[u]$ - $\mathcal{M} \to \mathbb{F}[u]$ - $\mathcal{M} \to \mathcal{M}$  and  $\mathbb{F}[u]$ - $\mathcal{U} \to \mathcal{U}$ .

Proposition 2.9. [LZ87, Pow14, Pow15]

- (1) The functor  $R_1 : \mathcal{M} \to \mathbb{F}[u]$ - $\mathcal{M}$  is exact and restricts to an exact functor  $R_1 : \mathcal{U} \to \mathbb{F}[u]$ - $\mathcal{U}$ .
- (2) There is a natural surjection  $R_1 \to \Phi$  which, for  $M \in Ob \mathcal{M}$ , fits into a short exact sequence:

$$0 \to uR_1M \to R_1M \to \Phi M \to 0$$
.

In particular, the u-adic filtration of  $R_1M$  has filtration quotients of the form  $\Sigma^j \Phi M$ .

Remark 2.10. Forgetting the  $\mathbb{F}[u]$ -module structure, the Singer functor can be considered as an exact functor  $R_1: \mathcal{M} \to \mathcal{M}$  which restricts to  $R_1: \mathcal{U} \to \mathcal{U}$ .

Notation 2.11. [Pow15] For  $n \in \mathbb{N}$  and  $M \in \text{Ob } \mathcal{M}$ , write:

$$R_{1/n}M := (R_1M) \otimes_{\mathbb{F}[u]} \mathbb{F}[u]/(u^n).$$

**Lemma 2.12.** For  $1 \leq n \in \mathbb{N}$ ,  $R_{1/n}$  is an exact functor  $R_{1/n} : \mathcal{M} \to \mathcal{M}$  which restricts to  $R_{1/n} : \mathcal{U} \to \mathcal{U}$ .

For  $M \in Ob \mathcal{M}$ 

(1) for  $n \geq 2$ , there is a natural short exact sequence

$$0 \to \Sigma^{n-1} \Phi M \to R_{1/n} M \to R_{1/n-1} M \to 0;$$

- (2) the u-adic filtration of  $R_{1/n}M$  has filtration quotients  $\Sigma^i \Phi M$ ,  $0 \le i < n$ ;
- (3) the natural surjection  $R_1M \rightarrow \Phi M$  factors across  $R_1M \rightarrow R_{1/n}M$  as

$$R_{1/n}M \rightarrow \Phi M$$
,

which is an isomorphism mod  $\mathcal{N}il$ .

#### 3. Almost unstable modules

Suppose that  $\mathbb{F} = \mathbb{F}_2$ . (The results of this section have analogues for odd primes.)

**Definition 3.1.** An unstable module  $M \in \text{Ob } \mathcal{U}$  is almost unstable if it admits a finite filtration with subquotients of the form  $\Sigma^{-i}N$  for some  $i \in \mathbb{N}$  and  $N \in \mathcal{N}il_i$ . The full subcategory of almost unstable modules in  $\mathcal{M}$  is denoted  $\widetilde{\mathcal{U}}$ .

**Proposition 3.2.** There are inclusions of subcategories  $\mathscr{U} \subset \widetilde{\mathscr{U}} \subset \mathscr{M}$  and  $\widetilde{\mathscr{U}}$  is an abelian Serre subcategory.

Moreover,

- (1)  $\widetilde{\mathscr{U}}$  is closed under  $\otimes$  and hence under  $\Sigma$ ;
- (2) any almost unstable module is concentrated in non-negative degrees;
- (3) a module  $M \in \text{Ob } \mathcal{M}$  concentrated in non-negative degrees and bounded above is almost unstable.

*Proof.* It is clear that  $\widetilde{\mathscr{U}}$  contains  $\mathscr{U}$ . To show that  $\widetilde{\mathscr{U}}$  is a Serre subcategory, it suffices to show that it is closed under formation of subobjects and quotients, since closure under extension is clear.

For  $M \in \text{Ob } \mathscr{U}$ , write  $f_iM$  for an increasing finite filtration that satisfies the defining property of Definition 3.1. If  $K \subset M$  is a submodule, consider the induced filtration  $f_iK := K \cap f_iM$ . Then, by construction,  $f_iK/f_{i-1}K \hookrightarrow f_iM/f_{i-1}M$ . By hypothesis the right hand module is of the form  $\Sigma^{-t}N$  for some  $t \in \mathbb{N}$  and  $N \in \mathscr{N}il_t$ ; since  $\mathscr{N}il_t$  is a Serre subcategory [Sch94],  $f_iK$  is a filtration of the required form.

Similarly, for  $M \to Q$ , consider the quotient filtration  $f_iQ := \text{image}\{f_iM \to Q\}$ . Then  $f_iQ/f_{i-1}Q$  is a quotient of  $f_iM/f_{i-1}M$ , whence the result as before, mutatis mutandis.

Closure under tensor product is a consequence of the fact that  $\otimes$  restricts to  $\otimes: \mathcal{N}il_i \times \mathcal{N}il_j \to \mathcal{N}il_{i+j}$  [Sch94, Kuh14]. Closure under  $\Sigma$  follows since the suspension functor identifies with  $\Sigma \mathbb{F} \otimes -$ .

The remaining statements are straightforward.

**Example 3.3.** As usual, extend the unstable algebra structure of  $\mathbb{F}[u]$  to an algebra structure in  $\mathscr{M}$  on  $\mathbb{F}[u^{\pm 1}]$ . Consider the submodule  $M := \mathbb{F}[u^{\pm 1}]_{\geq -1} \subset \mathbb{F}[u^{\pm 1}] \in \mathrm{Ob}\,\mathscr{M}$ , so that  $\Sigma M$  occurs in the short exact sequence:

$$0 \to \Sigma \mathbb{F}[u] \to \Sigma M \to \mathbb{F} \to 0.$$

This exhibits  $\Sigma M$  as an almost unstable module, whereas M is not, since it is non-zero in degree -1. Note that  $\Sigma^t M$   $(t \in \mathbb{Z})$  is never unstable.

Remark 3.4. Proposition 3.2 implies that  $\widetilde{\mathscr{U}}$  is closed under the formation of finite limits and finite colimits.

- (1) That closure under inverse limits fails in general is clear from Proposition 3.2 (3), since any  $\mathscr{A}$ -module M is the inverse limit of its system of truncations  $M^{\leq k}$  (the quotient of M by elements of degrees > k) as  $k \to \infty$ .
- (2) Closure under colimits also fails in general, as exhibited by the following example. For  $0 < t \in \mathbb{N}$ , let N(t) denote the subquotient  $\Phi^t F(1)/\Phi^{2t} F(1)$  of the free unstable module F(1); this has total dimension t, with classes in degrees  $2^i$ ,  $t \leq i < 2t$ , linked by the operation  $Sq_0$ .

Consider the  $\mathscr{A}$ -module:

$$M := \bigoplus_{t>1} \Sigma^{t-2^t} N(t).$$

Proposition 3.2 (3) implies that each  $\Sigma^{t-2^t}N(t)$  is almost unstable (and the choice of the desuspension ensures that M is of finite type).

However, M is not almost unstable; if it were, there would exist  $T \in \mathbb{N}$  such that  $\Sigma^T M$  admits a finite filtration (say of length  $l \in \mathbb{N}$ ) such that each subquotient is unstable. Choosing  $t \in \mathbb{N}$  such that t > l and  $t - 2^t + T < 0$ , consideration of the factor  $\Sigma^{t-2^t} N(t)$  leads to a contradiction.

3.1. The nilpotent filtration of  $\widetilde{\mathscr{U}}$ . The above notions can be refined by introducing an analogue of the nilpotent filtration of  $\mathscr{U}$ .

**Definition 3.5.** For  $i \in \mathbb{N}$ , let  $\widetilde{\mathscr{N}il_i} \subset \widetilde{\mathscr{U}}$  be the full subcategory of objects which admit a finite filtration with subquotients of the form  $\Sigma^{-t}N$  for some  $t \in \mathbb{N}$  and  $N \in \mathscr{N}il_{i+t}$ .

By definition, there is a decreasing filtration:

$$\ldots \subset \widetilde{\mathscr{N}il_{t+1}} \subset \widetilde{\mathscr{N}il_t} \subset \ldots \subset \widetilde{\mathscr{N}il_0} = \widetilde{\mathscr{U}}.$$

Proposition 3.2 generalizes to:

**Proposition 3.6.** For  $s, t \in \mathbb{N}$ :

- (1)  $\widetilde{\mathcal{N}il_t}$  is a Serre subcategory of  $\widetilde{\mathcal{U}}$ ;
- (2) tensor product restricts to  $\otimes : \widetilde{\mathcal{N}il_s} \times \widetilde{\mathcal{N}il_t} \to \widetilde{\mathcal{N}il_{s+t}}:$
- (3) suspension induces  $\Sigma : \widetilde{\mathcal{N}il}_s \to \widetilde{\mathcal{N}il}_{s+1}$  which is an equivalence of categories, with inverse  $\Sigma^{-1}$ .

*Proof.* Once established that  $\Sigma$  induces an equivalence of categories, the properties follow from the case of  $\widetilde{\mathscr{U}} = \widetilde{\mathscr{N}il_0}$ .

To show that  $\Sigma^{-1}: \mathcal{M} \to \mathcal{M}$  induces a functor  $\widetilde{\mathcal{N}il_{s+1}} \to \widetilde{\mathcal{N}il_s}$ , since  $\Sigma^{-1}$  is exact, it suffices to check on an almost unstable module of the form  $\Sigma^{-i}N$  with  $i \in \mathbb{N}$  and  $N \in \mathcal{N}il_{s+1+i}$ . Then  $\Sigma^{-1}(\Sigma^{-i}N)$  can be written  $\Sigma^{-(i+1)}N$  with N considered as lying in  $\mathcal{N}il_{s+(i+1)}$ .

The category  $\mathscr{U}$  is not stable under  $\Sigma^{-1}$ . In combination with Proposition 3.8 below, the above result should be compared with the fact [Sch94] that the loop functor  $\Omega: \mathscr{U} \to \mathscr{U}$  restricts to  $\Omega: \mathscr{N}il_{i+1} \to \mathscr{N}il_i$ .

**Proposition 3.7.** For  $i \in \mathbb{N}$ , the Frobenius functor  $\Phi : \mathcal{M} \to \mathcal{M}$  restricts to

$$\Phi: \widetilde{\mathscr{N}il_i} \to \widetilde{\mathscr{N}il_{2i}}.$$

Hence, for  $1 \leq n \in \mathbb{N}$ , the truncated Singer functor  $R_{1/n} : \mathcal{M} \to \mathcal{M}$  restricts to:

$$R_{1/n}: \widetilde{\mathscr{N}il_i} \to \widetilde{\mathscr{N}il_{2i}}.$$

*Proof.* The functors considered are exact, hence it suffices to consider behaviour on a module of the form  $\Sigma^{-t}N$  with  $N \in \text{Ob } \mathcal{N}il_{i+t}$ . Now  $\Phi\Sigma^{-t}N \cong \Sigma^{-2t}\Phi N$  and  $\Phi N \in \mathcal{N}il_{2(i+t)}$  by Lemma 2.4, which implies the first statement. The corresponding statement for  $R_{1/n}$  then follows using the u-adic filtration (cf. Lemma 2.12).

**Proposition 3.8.** For  $i \in \mathbb{N}$ , the destabilization functor restricts to

$$\Omega^{\infty}: \widetilde{\mathscr{N}il_i} \to \mathscr{N}il_i.$$

Proof. The category  $\mathscr{N}il_i$  is localizing and  $\Omega^{\infty}$  is right exact, hence it suffices to consider  $\Omega^{\infty}$  applied to a module of the form  $\Sigma^{-t}N$  with  $N \in \text{Ob } \mathscr{N}il_{i+t}$ . By construction, the composite functor  $\Omega^{\infty}\Sigma^{-t}$  restricted to  $\mathscr{U}$  is the iterated loop functor  $\Omega^t$ . Since  $\Omega^t$  restricts to  $\Omega^t$ :  $\mathscr{N}il_{i+t} \to \mathscr{N}il_i$  [Sch94], this establishes the result.

Recall that  $\operatorname{nil}_i: \mathscr{U} \to \mathscr{N}il_i \subset \mathscr{U}$  denotes the right adjoint to  $\mathscr{N}il_i \hookrightarrow \mathscr{U}$ .

Corollary 3.9. For  $M \in \text{Ob } \widetilde{\mathscr{N}il_i}$  and  $N \in \text{Ob } \mathscr{U}$ , the inclusion  $\text{nil}_i N \hookrightarrow N$  induces an isomorphism

$$\operatorname{Hom}_{\mathscr{M}}(M,\operatorname{nil}_{i}N) \stackrel{\cong}{\to} \operatorname{Hom}_{\mathscr{M}}(M,N).$$

*Proof.* Since N and  $nil_iN$  are unstable, the morphism identifies with

$$\operatorname{Hom}_{\mathscr{M}}(\Omega^{\infty}M, \operatorname{nil}_{i}N) \to \operatorname{Hom}_{\mathscr{M}}(\Omega^{\infty}M, N).$$

Proposition 3.8 shows that  $\Omega^{\infty}M \in \text{Ob } \mathcal{N}il_i$ , whence the result.

As a particular case of Corollary 3.9, one obtains:

Corollary 3.10. For  $M \in Ob \widetilde{\mathscr{N}il_1}$  and  $N \in Ob \mathscr{U}$  a reduced unstable module,

$$\operatorname{Hom}_{\mathscr{M}}(M,N) = 0.$$

3.2. Good almost unstable modules. For  $M \in \text{Ob } \widetilde{\mathscr{U}}$ , there is a canonical surjection to a reduced unstable module, namely:

$$M \to (\Omega^{\infty} M)/\mathrm{nil}_1(\Omega^{\infty} M).$$

In many cases of interest, the kernel of this map lies in  $\widetilde{\mathcal{N}il_1}$ . This motivates the following:

**Definition 3.11.** A module  $M \in \text{Ob } \mathcal{M}$  is a good almost unstable module if it is almost unstable and the kernel of  $M \to (\Omega^{\infty} M)/\text{nil}_1(\Omega^{\infty} M)$  lies in  $\widetilde{\mathcal{N}il}_1$ .

**Lemma 3.12.** A module  $M \in \text{Ob } \mathcal{M}$  is a good almost unstable module if and only if the kernel of  $M \to \Omega^{\infty} M$  lies in  $\widetilde{\mathcal{N}il}_1$ .

*Proof.* By definition  $\operatorname{nil}_1(\Omega^{\infty}M)$  lies in  $\mathcal{N}il_1$ , whence the result.

Notation 3.13. If  $M \in \text{Ob } \widetilde{\mathscr{U}}$  is good, write the associated short exact sequence:

$$0 \to M' \to M \to \rho_0 M \to 0$$
,

where  $M' \in \text{Ob } \widetilde{\mathscr{N}il_1}$  and  $\rho_0 M \in \text{Ob } \mathscr{U}$  is reduced.

**Example 3.14.** Every unstable module is good when considered as an almost unstable module. More generally, if M is of the form  $\Sigma^{-t}N$  with  $N \in \mathcal{N}il_t$ , then M is good almost unstable, with associated exact sequence:

$$0 \to \Sigma^{-t} \operatorname{nil}_{t+1} N \to M \to \rho_t N \to 0.$$

In particular, if t = 0 (so that M is unstable), there is no conflict with the notation  $\rho_0 M$ .

**Proposition 3.15.** A subquotient of a good almost unstable module is good almost unstable. Moreover, if  $f: M \to Q$  is a surjection from a good almost unstable module, then f induces a surjection  $\rho_0 f: \rho_0 M \to \rho_0 Q$  which is an isomorphism if and only if ker f lies in  $\widetilde{\mathcal{N}il}_1$ .

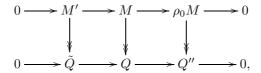
*Proof.* Let M be a good almost unstable module with associated short exact sequence as in Notation 3.13. Consider a submodule K; setting  $K' := K \cap M'$ , one has the morphism of short exact sequences:

$$0 \longrightarrow K' \longrightarrow K \longrightarrow K'' \longrightarrow 0$$

$$0 \longrightarrow M' \longrightarrow M \longrightarrow \rho_0 M \longrightarrow 0,$$

which shows that  $K'' \subset \rho_0 M$  is a reduced unstable module and  $K' \subset M'$  belongs to  $\widetilde{\mathcal{N}il_1}$ , thus K is good.

Similarly, for a surjection  $M \to Q$ , there is a morphism of short exact sequences:



where  $\tilde{Q}$  is defined by the commutative square on the left, hence belongs to  $\widetilde{\mathcal{N}il_1}$  since M' does, and Q'' is unstable, as a quotient of  $\rho_0 M$ . Now Q'' is a quotient of  $\Omega^{\infty}Q$ , by Lemma 3.12, thus Q is good.

By construction there is a surjection  $\rho_0 M \rightarrow \rho_0 Q$ . The final statement is clear.

Proposition 3.16. The class of good almost unstable modules is stable under finite direct sums and under  $\otimes$ . Moreover, it is preserved by the functors:

- (1)  $\Phi: \widetilde{\mathscr{U}} \to \widetilde{\mathscr{U}};$
- (2)  $R_{1/n}: \widetilde{\mathscr{U}} \to \widetilde{\mathscr{U}}$ , for  $1 \leq n \in \mathbb{N}$ .

If M is good almost unstable, then

$$\begin{array}{rcl} \rho_0(\Phi M) & \cong & \Phi(\rho_0 M) \\ \rho_0(R_{1/n} M) & \cong & R_{1/n}(\rho_0 M) & \cong & \Phi(\rho_0 M). \end{array}$$

*Proof.* Straightforward. The statement for  $\Phi$  and  $R_{1/n}$  is a generalization of Proposition 3.7, using the fact that  $\Phi$  preserves reduced unstable modules.

# 4. Almost unstable spectral sequences

The main interest in this section is in spectral sequences which converge to an unstable module and the following natural question: to what extent can the theory of unstable modules be used to understand the structure of the spectral sequence?

Hypothesis 4.1. Suppose that the spectral sequence  $(E_r^{*,*}, d_r)$  satisfies the following conditions:

- (1) it is second quadrant  $(E_r^{s,t}=0 \text{ if } s>0 \text{ or } t<0)$  and cohomological  $d_r:E_r^{*,*}\to E_r^{*+r,*+1-r};$ (2)  $E_1^{0,*}=0;$ (3) each  $E_r^{-k,*}$   $(k\in\mathbb{N})$  is an  $\mathscr{A}$ -module and  $d_r$  is  $\mathscr{A}$ -linear, namely:

$$d_r: \Sigma(\Sigma^{-k}E_r^{-k,*}) \to (\Sigma^{-k+r}E_r^{-k+r,*})$$

is a (degree zero) morphism of  $\mathcal{M}$ ;

(4) the spectral sequence converges strongly and  $\bigoplus_k \Sigma^{-k} E_{\infty}^{-k,*}$  is the associated graded of an unstable module, in particular each  $\Sigma^{-k} E_{\infty}^{-k,*}$  is unstable.

**Definition 4.2.** A spectral sequence  $(E_r^{*,*}, d_r)$  satisfying Hypothesis 4.1 is almost unstable (respectively good almost unstable) if  $\Sigma^{-k}E_1^{-k,*}$  is almost unstable (resp. good almost unstable) for all  $k \in \mathbb{N}$ .

**Proposition 4.3.** For a spectral sequence  $(E_r^{*,*}, d_r)$  satisfying Hypothesis 4.1, which is good almost unstable, and  $k \in \mathbb{N}$ ,

- (1)  $\Sigma^{-k}E_r^{-k,*}$  is a good unstable module for all  $1 \leq r \in \mathbb{N}$ ; (2) for  $1 \leq r \in \mathbb{N}$ ,  $\rho_0(\Sigma^{-k}E_{r+1}^{-k,*}) \subset \rho_0(\Sigma^{-k}E_r^{-k,*})$  with equality if r > k; in particular,  $\rho_0(\Sigma^{-k}E_\infty^{-k,*}) = \rho_0(\Sigma^{-k}E_r^{-k,*})$  for  $r \geq k$ .

*Proof.* The first statement follows from Proposition 3.15.

For the second, the differential is of the form

$$d_r: \Sigma(\Sigma^{-k} E_r^{-k,*}) \to (\Sigma^{-k+r} E_r^{-k+r,*}),$$

where  $\Sigma(\Sigma^{-k}E_r^{-k,*}) \in \text{Ob } \widetilde{\mathscr{N}il_1}$  and  $\Sigma^{-k+r}E_r^{-k+r,*} \in \widetilde{\mathscr{U}}$  is a good unstable module. In particular, by Corollary 3.10, the composite map

$$d_r: \Sigma(\Sigma^{-k}E_r^{-k,*}) \to (\Sigma^{-k+r}E_r^{-k+r,*}) \twoheadrightarrow \rho_0(\Sigma^{-k+r}E_r^{-k+r,*})$$

is trivial, thus the image of  $d_r$  lies in  $(\Sigma^{-k+r}E_r^{-k+r,*})'$ .

It follows from the final statement of Proposition 3.15 that  $\rho_0(\Sigma^{-k}E_{r+1}^{-k,*})$  identifies with  $\rho_0(\ker d_r)$ , which is a submodule of  $\rho_0(\Sigma^{-k}E_r^{-k,*})$ , by the argument employed in the proof of *loc. cit.*.

The second point follows since the spectral sequence is concentrated in the second quadrant with trivial column  $E_1^{0,*}$ , by hypothesis.

**Corollary 4.4.** For a spectral sequence  $(E_r^{*,*}, d_r)$  satisfying Hypothesis 4.1, which is good almost unstable, such that  $E_1^{-1,*} = \Sigma(\Sigma^{-t}M)$  for some  $t \in \mathbb{N}$  and  $M \in \mathcal{N}il_t$ ,

$$\rho_0(\Sigma^{-1} E_{\infty}^{-1,*}) = \rho_0(\Omega^t M) \cong \rho_t(M) 
\rho_0(\Sigma^{-2} E_{\infty}^{-2,*}) = \rho_0(\Sigma^{-2} E_2^{-2,*})$$

Moreover, the differential  $d_1: \Sigma(\Sigma^{-2}E_1^{-2,*}) \to \Sigma^{-1}E_1^{-1,*} = \Sigma^{-t}M$  factors across the inclusion

$$\Sigma^{-t} \operatorname{nil}_{t+1} M \hookrightarrow \Sigma^{-t} M$$

and induces a morphism  $d_1: \rho_0(\Sigma^{-2}E_1^{-2,*}) \to \rho_{t+1}M$  and

$$\rho_0(\Sigma^{-2}E_{\infty}^{-2,*}) \cong \ker\{\rho_0(\Sigma^{-2}E_1^{-2,*}) \to \rho_{t+1}M\}.$$

*Proof.* The first part follows from Proposition 4.3. By hypothesis,  $(\Sigma^{-2}E_1^{-2,*})$  is a good almost unstable module, hence there is a short exact sequence

$$0 \to (\Sigma^{-2} E_1^{-2,*})' \to (\Sigma^{-2} E_1^{-2,*}) \to \rho_0(\Sigma^{-2} E_1^{-2,*}) \to 0$$

with  $(\Sigma^{-2}E_1^{-2,*})' \in \text{Ob } \widetilde{\mathscr{N}il_1}$ . The differential  $d_1$  is of the form:

$$d_1: \Sigma(\Sigma^{-2}E_1^{-2,*}) \to \Sigma^{-t}M$$

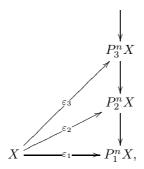
hence  $\Sigma^t d_1: \Sigma^{t+1}(\Sigma^{-2}E_1^{-2,*}) \to M$ . As  $(\Sigma^{-2}E_1^{-2,*})$  is good almost unstable,  $\Sigma^{t+1}(\Sigma^{-2}E_1^{-2,*})$  lies in  $\widetilde{\mathscr{N}il_{t+1}}$  and  $\Sigma^{t+1}((\Sigma^{-2}E_1^{-2,*})')$  in  $\widetilde{\mathscr{N}il_{t+2}}$ . Since M is unstable, Proposition 3.8 implies that  $\Sigma^t d_1$  maps to  $\operatorname{nil}_{t+1} M$  and its restriction to  $\Sigma^{t+1}(\Sigma^{-2}E_1^{-2,*})'$  maps to  $\operatorname{nil}_{t+2} M$ .

The result follows as in the proof of Proposition 4.3.

# 5. The Arone-Goodwillie spectral sequence

In this section, the presentation of [Kuh08] is followed, since the results of Section 6 use *loc. cit.*.

For X a pointed space (respectively a spectrum), the Arone-Goodwillie tower associated to the functor  $X \mapsto \Sigma^{\infty} \Omega^n X$  for  $n \in \mathbb{N}$  has the following form:



where  $P_1^n X = \Sigma^{-n} X$  for  $n < \infty$  (ie the spectrum  $\Sigma^{-n} \Sigma^{\infty} X$ ) and  $P_0^n = 0$ .

Ahearn and Kuhn [AK02] identify the fibres of the tower in terms of the extended power construction via the cofibre sequence:

$$D_{n,j}\Sigma^{-n}X \to P_j^nX \to P_{j-1}^nX$$

for  $1 \leq j \in \mathbb{N}$ , where, for a spectrum Y,

$$D_{n,j}Y := \left(\mathfrak{C}(n,j)_+ \wedge Y^{\wedge j}\right)_{h\mathfrak{S}_j}$$

 $\mathfrak{C}(n,j)$  the Boardman-Vogt space of j little n-cubes in an n-cube.

For a pointed space X, the adjunction unit  $X \to \Omega \Sigma X$  induces an n-fold loop map  $\Omega^n X \to \Omega^{n+1} \Sigma X$ , for  $n \in \mathbb{N}$ ; by [AK02, Corollary 1.2], this induces a natural map of towers  $P^n X \to P^{n+1} \Sigma X$  which identifies on the level of the fibres as the natural transformation  $D_{n,j} \Sigma^{-n} X \to D_{n+1,j} \Sigma^{-n} X$  induced by the inclusion  $\mathfrak{C}(n,j) \hookrightarrow \mathfrak{C}(n+1,j)$ . Similarly, the natural evaluation map  $\varepsilon : \Sigma \Sigma^{\infty} \Omega^{n+1} X \to \Sigma^{\infty} \Omega^n X$  induces a map of towers  $\Sigma P^{n+1} X \to P^n X$  and, on fibres,  $\varepsilon : \Sigma D_{n+1,j} X \to D_{n,j} \Sigma X$  (see [AK02]).

If X is n-connected for  $n \in \mathbb{N}$ , the connectivity of the maps  $\varepsilon_j$  increases linearly with j, hence:

**Proposition 5.1.** [Kuh08] For X an n-connected space with  $H^*(X)$  of finite type, the spectral sequence associated to the Arone-Goodwillie tower satisfies Hypothesis 4.1 with

$$E_1^{-j,*} = H^*(\Sigma^j D_{n,j} \Sigma^{-n} X)$$

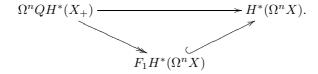
and converges strongly to  $H^*(\Omega^n X)$ .

The associated filtration of  $H^*(\Omega^n X)$  is

$$0 = F_0 H^*(\Omega^n X) \subset F_1 H^*(\Omega^n X) \subset F_2 H^*(\Omega^n X) \subset \ldots \subset H^*(\Omega^n X),$$

where 
$$F_jH^*(\Omega^nX) = \operatorname{image}\{H^*(P_i^nX) \to H^*(\Omega^nX)\}.$$

There is a commutative diagram in  $\mathcal{U}$ , in which  $QH^*(X_+)$  denotes the module of indecomposables of the unstable algebra  $H^*(X_+)$ :



The functor  $U: \mathcal{U} \to \mathcal{K}$  induces a morphism of unstable algebras:

(3) 
$$U(\Omega^n Q H^*(X_+)) \to H^*((\Omega^n X)_+).$$

If M is connected, there is a natural inclusion of unstable modules

$$M \to \overline{UM} \subset UM$$

which induces a surjection onto the indecomposables Q(UM); the product of UM induces an increasing filtration of the augmentation ideal  $\overline{UM}$ :

$$M = F_1 \overline{UM} \subseteq F_2 \overline{UM} \subseteq F_3 \overline{UM} \subseteq \ldots \subseteq F_i \overline{UM} \subseteq \ldots \subseteq \overline{UM}$$
.

The results of Ahearn and Kuhn [AK02] imply that this filtration is compatible with the filtration  $F_jH^*(\Omega^nX)$ ; namely, for  $1 \leq j \in \mathbb{N}$ , the morphism (3) restricts to a morphism of unstable modules:

$$F_j \overline{U\Omega^n QH^*(X_+)} \to F_j H^*(\Omega^n X).$$

At the prime p=2, it is the submodule  $F_2\overline{UM}$  which is of interest. The construction of UM implies the following:

**Lemma 5.2.** For  $M \in \text{Ob } \mathscr{U}$  and  $\lambda : \Phi M \to M$  the morphism of unstable modules induced by  $Sq_0$ ,  $F_2\overline{UM}$  occurs in the pushout of short exact sequences:

$$\Phi M \longrightarrow S^2 M \longrightarrow \Lambda^2 M$$

$$\downarrow \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \downarrow$$

$$M \longrightarrow F_2 \overline{UM} \longrightarrow \Lambda^2 M.$$

Remark 5.3. Taking  $M = F_1 H^*(\Omega^n X)$ , one obtains the fundamental morphism of short exact sequences:

The identification of  $\overline{\cup}$  in terms of the structure of the spectral sequence will be important in Section 9.

# 6. Cohomology of extended powers at p=2

Fix an integer  $n \geq 1$  and a spectrum Y. In [Kuh08, Section 3], Kuhn describes the mod 2 cohomology of the extended powers  $H^*(D_{n,j}Y)$ ; we follow *loc. cit.* in considering only spectra with  $H^*(Y)$  bounded below and of finite type.

The structure of  $H^*(D_{n,\bullet}Y)$  is determined (see [Kuh08, Theorem 3.14]) in terms of the following morphisms:

(1) The product [Kuh08, Definition 3.3]

$$\star: H^*(D_{n,i}Y) \otimes H^*(D_{n,j}Y) \to H^*(D_{n,i+j}Y),$$

which is a morphism of  $\mathscr{A}$ -modules and induces a commutative (bi)graded algebra structure on  $H^*(D_{n,\bullet}Y)$ .

(2) For  $1 \leq j \in \mathbb{N}$ , the dual Browder operation [Kuh08, Definition 3.5]

$$L_{n-1}: H^*(Y)^{\otimes j} \to H^*(\Sigma^{(1-j)(n-1)}D_{n,j}Y),$$

which is  $\mathscr{A}$ -linear.

(3) The dual Dyer-Lashof operations (for  $r \ge 0$ ) [Kuh08, Definition 3.2]

$$\mathfrak{Q}_r: H^d(D_{n,j}Y) \to H^{2d+r}(D_{n,2j}Y).$$

The operation  $\mathfrak{Q}_r$  is trivial for  $r \geq n$  [Kuh08, Proposition 3.8].

Remark 6.1. The dual Dyer-Lashof operations  $\mathfrak{Q}_r$  are not  $\mathscr{A}$ -linear, but satisfy Nishida relations. Moreover, the operation  $\mathfrak{Q}_0$  is not  $\mathbb{F}$ -linear; the default of linearity is given by the interaction with the ★-product:

$$\mathfrak{Q}_0(x+y) = \mathfrak{Q}_0(x) + \mathfrak{Q}_0(y) + x \star y.$$

Thus  $\mathfrak{Q}_0$  behaves like a divided square operation.

**Lemma 6.2.** For  $1 \leq j \in \mathbb{N}$ , the  $\star$ -product induces a morphism of  $\mathscr{A}$ -modules:

$$\star: \Lambda^2 H^*(D_{n,i}Y) \to H^*(D_{n,2i}Y).$$

Proof. Follows from [Kuh08, Proposition 3.11 (iii)].

**Lemma 6.3.** For  $1 \leq j \in \mathbb{N}$  and an integer l > 0, the sub vector space of  $H^*(D_{n,2j}Y)$  generated by the images of

$$\mathfrak{Q}_r: H^*(D_{n,j}Y) \to H^{2*+r}(D_{n,2j}Y)$$

$$(r \ge l)$$
 is a sub  $\mathscr{A}$ -module  $\left(\sum_{r \ge l} \operatorname{image}(\mathfrak{Q}_r)\right)$  of  $H^*(D_{n,2j}Y)$ .

*Proof.* Follows from the Nishida relation given in [Kuh08, Proposition 3.1(ii)].

Remark 6.4. The higher dual Dyer-Lashof operations  $\mathfrak{Q}_r$ ,  $r \geq n$  act trivially by [Kuh08, Proposition 3.8], hence the sum  $\sum_{r>l} image(\mathfrak{Q}_r)$  is finite (and zero for

**Proposition 6.5.** For  $1 \leq j \in \mathbb{N}$ , the dual Dyer-Lashof operations induce  $\mathscr{A}$ -linear maps:

$$\overline{\mathfrak{Q}_0} : \Phi H^*(D_{n,j}Y) \to H^*(D_{n,2j}Y) / \left(\Lambda^2 H^*(D_{n,j}Y) + \sum_{r \ge 1} \operatorname{image}(\mathfrak{Q}_r)\right)$$

$$\overline{\mathfrak{Q}_l} : \Sigma^l \Phi H^*(D_{n,j}Y) \hookrightarrow H^*(D_{n,2j}Y) / \left(\sum_{r \ge l+1} \operatorname{image}(\mathfrak{Q}_r)\right),$$

$$\overline{\mathfrak{Q}_l} : \Sigma^l \Phi H^*(D_{n,j}Y) \hookrightarrow H^*(D_{n,2j}Y) / \Big( \sum_{r > l+1} \operatorname{image}(\mathfrak{Q}_r) \Big)$$

where l > 0.

*Proof.* By [Kuh08, Proposition 3.11(i)], the operation  $\mathfrak{Q}_0$  becomes  $\mathbb{F}$ -linear after the passage to the quotient by the submodule  $\Lambda^2 H^*(D_{n,j}Y)$ . Moreover, the Nishida relation for  $\mathfrak{Q}_0$  [Kuh08, Proposition 3.15(i)] establishes the  $\mathscr{A}$ -linearity, after passing to the additional quotient by the image of the higher dual Dyer-Lashof operations, which is a sub  $\mathcal{A}$ -module by Lemma 6.3.

The argument for  $\mathfrak{Q}_l$  (l > 0) is similar, using the Nishida relation [Kuh08, Proposition 3.1(ii)].

**Theorem 6.6.** Suppose that Y is a spectrum such that  $H^*(Y) \in Ob \mathscr{U}$  is almost unstable and is of finite type. Then for  $1 \leq j \in \mathbb{N}$ :

- (1)  $H^*(D_{n,i}Y) \in \text{Ob } \widetilde{\mathscr{U}} \text{ is almost unstable};$
- (2) if  $H^*(Y)$  is a good almost unstable module, then  $H^*(D_{n,j}Y)$  is good and
  - (a)  $\rho_0(H^*(D_{n,j}Y)) \cong \Gamma^j \rho_0(H^*(Y)), n \geq 2;$
  - (b)  $\rho_0(H^*(D_{1,j}Y)) \cong T^j \rho_0(H^*(Y)).$

*Proof.* The case n=1 is straightforward (compare [Kuh08, Remark 2.1]), hence suppose that n > 2.

The proof that the modules are almost unstable is based on [Kuh08, Theorem 3.14], which states that,  $H^*(D_{n,\bullet}X)$  is generated as a (bi)graded commutative algebra (under the  $\star$ -product) by elements of the form

$$\mathfrak{Q}_{r_1}\ldots\mathfrak{Q}_{r_t}L_{n-1}(x_1\otimes\ldots\otimes x_k)\in H^*(D_{n,2^tk}Y),$$

subject to the relations given in [Kuh08, Section 3.3].

The category  $\widetilde{\mathscr{U}}$  is a Serre subcategory of  $\mathscr{M}$  and is stable under  $\otimes$ , thus an increasing induction upon j implies that it is sufficient to work modulo  $\star$ -decomposables. Hence one is reduced to considering  $\star$ -indecomposables of the above form (note that Adem-type relations intervene in considering the words in the dual Dyer-Lashof operations, by [Kuh08, Proposition 3.13]). Moreover, since the dual Dyer-Lashof operations double the j degree and  $\Omega_r$  is trivial for  $r \geq n$ , in a given j-degree, there are only finitely many words  $\Omega_{r_1} \dots \Omega_{r_t}$  which arise.

For k tensor factors, the morphism  $L_{n-1}$  is  $\mathscr{A}$ -linear:

$$L_{n-1}: \Sigma^{(k-1)(n-1)} H^*(Y)^{\otimes k} \to H^*(D_{n,k}Y).$$

The hypothesis implies that  $\Sigma^{(k-1)(n-1)}H^*(Y)^{\otimes k}$  lies in  $\widetilde{\mathcal{N}il}_{(k-1)(n-1)}$  (in particular is almost unstable), hence so does the image of  $L_{n-1}$ , by Proposition 3.6. For k > 1, (k-1)(n-1) > 0, whereas for k = 1,  $L_{n-1}$  is an isomorphism.

A straightforward filtration argument based on Proposition 6.5 allows words in dual Dyer-Lashof operations to be treated. Namely, up to higher terms, the image of an operation  $\mathfrak{Q}_r$  is a quotient of the functor  $\Sigma^r \Phi$  and, by Proposition 3.7, the functor  $\Sigma^r \Phi$  induces:

$$\Sigma^r \Phi : \widetilde{\mathscr{N}il_i} \to \widetilde{\mathscr{N}il_{2i+r}}.$$

Since  $\widetilde{\mathcal{N}il_{2i+r}}$  is a Serre subcategory, up to filtration, this exhibits the image under  $\mathfrak{Q}_r$  of an element of  $\widetilde{\mathcal{N}il_i}$  as lying in  $\widetilde{\mathcal{N}il_{2i+r}}$ .

Putting these facts together, one concludes that  $H^*(D_{n,j}Y)$  is almost unstable. Moreover, the argument shows that the only possible contributions not in  $\widetilde{\mathcal{N}il_1}$  arise from  $\star$ -products of terms from the image of iterates of  $\mathfrak{Q}_0$ . If  $H^*(Y)$  is good almost unstable, then there is an associated short exact sequence in  $\mathscr{M}$ :

$$0 \to (H^*Y)' \to H^*Y \to \rho_0(H^*Y) \to 0.$$

Any terms arising from  $(H^*Y)'$  also lie in  $\widetilde{\mathcal{N}il}_1$ . Hence, to prove the result, it suffices to show that the projection  $H^*Y \twoheadrightarrow \rho_0(H^*Y)$  induces a surjection (recall  $n \geq 2$ , by hypothesis)

$$H^*(D_{n,j}Y) \twoheadrightarrow \Gamma^j(\rho_0(H^*Y)),$$

since  $\Gamma^j(\rho_0(H^*Y))$  is a reduced unstable module. This is clear as graded vector spaces; to check that the morphism is  $\mathscr{A}$ -linear, use [Kuh08, Proposition 3.15(i)], which shows that the action of the Steenrod squares on  $\mathfrak{Q}_0$  is correct modulo the higher dual Dyer-Lashof operations.

**Corollary 6.7.** For  $1 \leq n \in \mathbb{N}$  and X an n-connected space such that  $H^*(X) \in \mathcal{N}il_n$  and is of finite type, the spectral sequence calculating  $H^*(\Omega^n X)$  associated to the Arone-Goodwillie tower is good almost unstable.

In particular, the morphism  $d_1$  from the (-2)-column to the (-1)-column induces

$$\Gamma^{2}(\rho_{n}H^{*}(X)) \to \rho_{n+1}H^{*}(X) \qquad n \ge 2$$
$$T^{2}(\rho_{1}H^{*}(X)) \to \rho_{2}H^{*}(X) \qquad n = 1.$$

*Proof.* The final statement follows from Corollary 4.4.

## 7. Algebraic Models

In the case of the second extended power, it is possible to give explicit algebraic models for their cohomology. For current purposes, this is not strictly necessary; it is included since it makes the results of Section 6 much more explicit.

7.1. The case of the second extended power. The calculation of  $H^*(D_{\infty,2}Y)$  (see [KM13], where homology is used) is a stable version of the calculation of the quadratic construction [Mil74, GLZ89, HLS95]. Here  $H^*(D_{n,2}Y)$  is considered for finite n; this brings the dual Browder operations into the picture.

**Lemma 7.1.** For Y a spectrum with  $H^*(Y)$  bounded below and of finite type,

(1) the  $\star$ -product induces a monomorphism of  $\mathscr{A}$ -modules:

$$\star: \Lambda^2 H^*(Y) \hookrightarrow H^*(D_{n,2}Y);$$

(2) the dual Browder operation induces a monomorphism of  $\mathscr{A}$ -modules:

$$L_{n-1}: \Sigma^{n-1}S^2H^*(Y) \hookrightarrow H^*(D_{n,2}Y);$$

(3) the sum of these induces a monomorphism of  $\mathscr{A}$ -modules:

$$\star \coprod L_{n-1} : \Lambda^2 H^*(Y) \oplus \Sigma^{n-1} S^2 H^*(Y) \hookrightarrow H^*(D_{n,2}Y).$$

*Proof.* The morphisms are provided respectively by [Kuh08, Proposition 3.11(iii)] and [Kuh08, Proposition 3.12]. The injectivity is a consequence of [Kuh08, Theorem 3.14].

Recall from Section 2 that, for  $M \in \text{Ob } \mathscr{M}$  and  $1 \leq n \in \mathbb{N}$ , there are natural morphisms:

Here the middle row is not in general a sequence (for n = 1 the morphisms are isomorphisms) and not in general exact (for  $n \ge 3$ ).

**Definition 7.2.** For  $1 \leq n \in \mathbb{N}$  and  $M \in \text{Ob } \mathcal{M}$ , let  $\mathscr{E}_n M$  denote the  $\mathscr{A}$ -module given by forming the pushout and pullback of diagram (5).

**Example 7.3.** For n = 1 and  $M \in \text{Ob } \mathcal{M}$ ,  $\mathcal{E}_1 M$  is naturally isomorphic to  $M^{\otimes 2}$ .

**Proposition 7.4.** For  $1 \leq n \in \mathbb{N}$ , the above construction defines a functor  $\mathscr{E}_n : \mathscr{M} \to \mathscr{M}$  which restricts to  $\mathscr{E}_n : \mathscr{U} \to \mathscr{U}$ .

For  $M \in \text{Ob } \mathcal{M}$ , there is a natural short exact sequence:

$$0 \to \Lambda^2 M \oplus \Sigma^{n-1} S^2 M \to \mathscr{E}_n M \to R_{1/n-1} M \to 0.$$

*Proof.* Straightforward.

The functor  $\mathscr{E}_n$  provides an algebraic model for  $H^*(D_{n,2}Y)$ :

**Proposition 7.5.** For  $n \in \mathbb{N} \cup \{\infty\}$  and Y a spectrum with  $H^*(Y)$  bounded below and of finite type, there is a natural isomorphism

$$\mathscr{E}_n H^*(Y) \cong H^*(D_{n,2}Y)$$

which extends the inclusion  $\star \coprod L_{n-1} : \Lambda^2 H^*(Y) \oplus \Sigma^{n-1} S^2 H^*(Y) \hookrightarrow H^*(D_{n,2}Y)$  of Lemma 7.1.

*Proof.* The result is essentially a restatement of the results of [Kuh08, Section 3], using the algebraic functors introduced in Section 2.  $\Box$ 

**Example 7.6.** For n = 1, one recovers from Example 7.3 and Proposition 7.5 the standard identification  $H^*(D_{1,2}X) \cong H^*(X)^{\otimes 2}$ .

Remark 7.7. For  $n \geq 1$ , there are natural transformations  $\mathscr{E}_{n+1} \to \mathscr{E}_n$ ,  $\mathscr{E}_n \Sigma \to \Sigma \mathscr{E}_{n+1}$  that provide algebraic models (via Proposition 7.5) for the morphisms in cohomology induced respectively by  $D_{n,2}Y \to D_{n+1,2}Y$  and  $\Sigma D_{n+1,2}Y \to D_{n,2}\Sigma Y$ .

7.2. The algebraic differential. There is an algebraic differential which is related to the differential used by Singer (see [Pow15] for references).

Recall that  $\mathbb{F}[u^{\pm 1}]$  has an  $\mathscr{A}$ -module structure extending that of  $\mathbb{F}[u]$ ; it is a fundamental fact that the residue map  $\mathbb{F}[u^{\pm 1}] \to \Sigma^{-1}\mathbb{F}$  is  $\mathscr{A}$ -linear. This gives rise to a natural transformation  $d_M : R_1M \to \Sigma^{-1}M$  in  $\mathscr{M}$ , since  $R_1M$  embeds in the half-completed tensor product  $\mathbb{F}[u^{\pm 1}] \underline{\otimes} M$ . If M is unstable then  $d_M$  is trivial.

**Lemma 7.8.** [Pow15] For  $1 \leq n \in \mathbb{N}$  and  $N \in \text{Ob } \mathscr{U}$ , the differential  $d_{\Sigma^{-n}N}$  induces a natural transformation  $d_{1/n}: R_{1/n}(\Sigma^{-n}N) \to \Sigma^{-n-1}N$  which fits into a commutative diagram

$$R_1(\Sigma^{-n}N) \xrightarrow{d_{\Sigma^{-n}N}} \Sigma^{-n-1}N$$

$$\downarrow \qquad \qquad \downarrow$$

$$R_{1/n}(\Sigma^{-n}N) \xrightarrow{d_{1/n}} \Sigma^{-n-1}N.$$

The cokernel of  $\Sigma d_{1/n}: \Sigma R_{1/n}(\Sigma^{-n}N) \to \Sigma^{-n}N$  is  $\Omega^n N$ .

Proof. Straightforward.

By the definition of  $\mathscr{E}_n M$  (for general  $M \in \mathrm{Ob}\,\mathscr{M}$ ), the quotient  $\mathscr{E}_n M/\Lambda^2 M$  occurs as the pushout of the diagram:

$$\sum^{n-1} \Phi M \xrightarrow{\longrightarrow} R_{1/n} M$$

$$\sum^{n-1} S^2 M.$$

**Proposition 7.9.** For  $1 \leq n \in \mathbb{N}$  and K a connected unstable algebra with augmentation ideal  $\overline{K}$ , the natural transformation

$$d_{1/n}: R_{1/n}(\Sigma^{-n}\overline{K}) \to \Sigma^{-n-1}\overline{K}$$

together with the product  $S^2(\overline{K}) \to \overline{K}$  induce a natural transformation in  $\mathcal{M}$ :

$$d_1: \mathscr{E}_n(\Sigma^{-n}\overline{K}) \to \Sigma^{-n-1}\overline{K}.$$

*Proof.* The subobject  $\Sigma^{n-1}S^2(\Sigma^{-n}\overline{K})$  of  $\mathscr{E}_n(\Sigma^{-n}\overline{K})$  is naturally isomorphic to  $\Sigma^{-n-1}S^2(\overline{K})$ , hence the product induces a natural morphism of  $\mathscr{A}$ -modules

$$\Sigma^{n-1}S^2(\Sigma^{-n}\overline{K}) \to \Sigma^{-n-1}\overline{K}.$$

The verification that this is compatible with  $d_{1/n}$  given by Lemma 7.8 is straightforward.

7.3. The spectral sequence differential  $d_1$ . Consider the first stages of the Arone-Goodwillie tower for  $\Omega^n X$ , with X an n-connected space. There is a cofibre sequence of spectra

$$D_{n,2}\Sigma^{-n}X \to P_2^nX \to \Sigma^{-n}X$$

and the differential  $d_1$  from the -2-column to the -1-column of the spectral sequence is the connecting morphism

$$d_1: \Sigma H^*(D_{n,2}\Sigma^{-n}X) \to H^*(\Sigma^{-n}X).$$

This can be identified algebraically in terms of the isomorphism of Proposition 7.5.

**Proposition 7.10.** For  $1 \le n \in \mathbb{N}$  and X a connected space with  $H^*(X)$  of finite type, the following diagram commutes:

$$\Sigma \mathcal{E}_n(\Sigma^{-n}H^*(X)) \xrightarrow{\Sigma d_1} \Sigma^{-n}H^*(X)$$

$$\cong \bigvee_{d_1} \bigoplus_{d_1} H^*(\Sigma^{-n}X)$$

in which the  $d_1$  of the top row indicates the algebraic differential of Proposition 7.9.

*Proof.* This result corresponds to [Kuh08, Proposition 4.3].

# 7.4. Exploiting the nilpotent filtration.

**Proposition 7.11.** For  $1 \leq n \in \mathbb{N}$  and an unstable module  $N \in \text{Ob } \mathcal{N}il_n$ , the algebraic differential  $d_{1/n} : R_{1/n}(\Sigma^{-n}N) \to \Sigma^{-n-1}N$  factors across  $\Sigma^{-n-1}\text{nil}_{n+1}N$  and the resulting map fits into a natural commutative diagram:

$$R_{1/n}(\Sigma^{-n}N) \xrightarrow{} \Sigma^{-n-1} \operatorname{nil}_{n+1}N$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$\Phi \rho_n N \xrightarrow{\delta_n} \rho_{n+1}N,$$

where the vertical morphisms are induced by the natural projections  $R_{1/n} \to \Phi$  of Lemma 2.12 together with  $\Sigma^{-n}N \to \rho_n N$ .

The natural transformation  $\delta_n: \Phi \rho_n N \to \rho_{n+1} N$  is induced by the linear transformation  $x \in \Sigma^{-n} N \mapsto Sq^{|x|+1}(x)$ . In particular, if  $\delta_n$  is non-trivial, then N is not an n-fold suspension.

*Proof.* Straightforward, unravelling definitions to identify the morphism  $\delta_n$ .

Remark 7.12. Using the notation of Proposition 7.11, if  $x = \Sigma^{-n}y$  for  $y \in N$ , then  $Sq^{|x|+1}(\Sigma^{-n}y) = \Sigma^{-n}Sq^{|y|+1-n}y$ . In particular, for n = 1, the map  $\Phi \rho_1 N \to \rho_2 N$  is induced by the operation  $Sq_0$  on N.

**Corollary 7.13.** In the situation of Corollary 4.4 for  $n \ge 2$ , the induced morphism factors as

$$\Gamma^2(\rho_n H^*(X)) \twoheadrightarrow \Phi(\rho_n H^*(X)) \xrightarrow{\delta_n} \rho_{n+1} H^*(X).$$

There is an alternative viewpoint on the natural transformation  $\delta_n: \Phi \rho_n N \to \rho_{n+1} N$  for  $N \in \mathcal{N}il_n$ , based on the following result, in which  $\Omega_1^n: \mathcal{U} \to \mathcal{U}$  denotes the first left derived functor of the iterated loop functor  $\Omega^n$ .

**Proposition 7.14.** For  $1 \leq n \in \mathbb{N}$  and  $s \geq n$ , the functor  $\Omega_1^n : \mathcal{U} \to \mathcal{U}$  restricts to

$$\Omega_1^n: \mathscr{N}il_s \to \mathscr{N}il_{2(s-n)+1}.$$

Moreover, for  $N \in \text{Ob } \mathcal{N}il_n$  (so that  $\Omega_1^n N \in \mathcal{N}il_1$ )

$$\rho_1(\Omega_1^n N) \cong \Phi(\rho_n N).$$

Proof. The proof is by induction on n; for n=1 this is [Sch94, Lemma 6.1.3] (which also states that  $\Omega_1$  takes values in  $\mathcal{N}il_1$ ). The inductive step uses the short exact sequence  $\Omega\Omega_1^{n-1} \to \Omega_1^n \to \Omega_1\Omega^{n-1}$  associated to the identification  $\Omega^n = \Omega \circ \Omega^{n-1}$  (see [Pow15] and the references therein). Namely, if  $M \in \mathcal{N}il_s$  with  $s \geq n$ , then  $\Omega_1^{n-1}M \in \mathcal{N}il_{2(s-(n-1))+1} = \mathcal{N}il_{2(s-n)+3}$ , hence  $\Omega\Omega_1^{n-1}M \in \mathcal{N}il_{2(s-n)+2}$ . Similarly,  $\Omega^{n-1}M \in \mathcal{N}il_{s-(n-1)} = \mathcal{N}il_{s-(n-1)}$ , hence  $\Omega_1\Omega^{n-1}M \in \mathcal{N}il_{2(s-n)+1}$ . The result follows, since  $\mathcal{N}il_{2(s-n)+1}$  is closed under extensions.

Corollary 7.15. For  $1 \leq n \in \mathbb{N}$  and  $N \in Ob \mathcal{N}il_n$ , there is a natural exact sequence

$$\Sigma \Phi \rho_n M \stackrel{\Sigma \delta_n}{\to} \Sigma \rho_{n+1} M \to \Omega^n(M/\mathrm{nil}_{n+2} M) \to \rho_n M \to 0.$$

*Proof.* This follows by considering the long exact sequence for  $\Omega^n_{ullet}$  associated to the short exact sequence

$$0 \to \Sigma^{n+1} \rho_{n+1} M \to M/\mathrm{nil}_{n+2} M \to \Sigma^n \rho_n M \to 0$$

together with the factorization provided by Proposition 7.14.

### 8. Essential extensions

Let  $\mathbb{F}$  be a finite field and recall that  $\mathscr{F}$  is the category of functors from finite-dimensional  $\mathbb{F}$ -vector spaces to  $\mathbb{F}$ -vector spaces.

The aim of this section is to give a generalization of the following result:

**Theorem 8.1.** [Kuh95a, Theorem 4.8] For F a non-constant finite functor, precomposition with F induces a (naturally split) monomorphism:

$$\operatorname{Ext}_{\mathscr{F}}^*(G,H) \hookrightarrow \operatorname{Ext}_{\mathscr{F}}^*(G \circ F, H \circ F).$$

This is refined by using the observation that the result only depends on the top polynomial degree behaviour of F.

Remark 8.2. For  $f: F_1 \to F_2$  a morphism between functors taking finite-dimensional values, for any morphism  $\alpha: G \to H$ , there is a commutative diagram:

$$\begin{array}{ccc} G \circ F_1 & \xrightarrow{\alpha_{F_1}} & H \circ F_1 \\ Gf & & \downarrow Hf \\ G \circ F_2 & \xrightarrow{\alpha_{F_2}} & H \circ F_2 \end{array}$$

which corresponds to the two (equivalent) ways to define a natural transformation:  $\operatorname{Hom}_{\mathscr{F}}(G,H) \to \operatorname{Hom}_{\mathscr{F}}(G\circ F_1,H\circ F_2)$ . By naturality this extends to  $\operatorname{Ext}_{\mathscr{F}}^*$ .

Recall that a functor F has polynomial degree exactly d if it is polynomial of degree d but not of degree d-1 (ie  $F \in \mathscr{F}_d \backslash \mathscr{F}_{d-1}$ ).

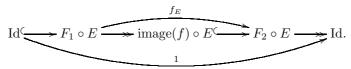
**Theorem 8.3.** For  $f: F_1 \to F_2$  a morphism between finite functors of polynomial degree exactly d > 0, if image(f) has polynomial degree exactly d, then precomposition with  $F_1$  together with f induce a (naturally split) monomorphism:

$$\operatorname{Ext}_{\mathscr{F}}^*(G,H) \hookrightarrow \operatorname{Ext}_{\mathscr{F}}^*(G \circ F_1, H \circ F_2).$$

*Proof.* The proof follows that of [Kuh95a, Theorem 4.8], which relies upon the fact that the full subcategory of functors of polynomial degree at most 1 (which contains the constant and additive functors) is semisimple. The hypotheses ensure that  $\Delta^{d-1}F_1$  and  $\Delta^{d-1}F_2$  are non-constant functors of polynomial degree  $\leq 1$  and that  $\Delta^{d-1}f$  maps to a non-constant functor.

Applying [Kuh95a, Lemma 4.12], there exists a finite functor E such that the identity functor Id is a direct summand of  $\operatorname{image}(f) \circ E$ . (The proof of the lemma is based on the fact that  $\Delta^{d-1}F$  is a natural direct summand of the functor  $V \mapsto F(V \oplus \mathbb{F}^{d-1})$ , for  $F \in \operatorname{Ob} \mathscr{F}$ .)

By semi-simplicity the splitting factors:



The proof is completed, mutatis mutandis, as in [Kuh95a, Section 4.3].

**Example 8.4.** Consider a finite functor F of polynomial degree exactly d > 0 and let  $q_{d-1}F$  be the largest quotient of F of polynomial degree  $\leq d-1$ , with associated short exact sequence:

$$0 \to \overline{F} \to F \to q_{d-1}F \to 0$$

where  $\overline{F} \hookrightarrow F$  satisfies the hypotheses of Theorem 8.3. The theorem shows that:

$$\operatorname{Ext}_{\mathscr{F}}^*(G,H) \hookrightarrow \operatorname{Ext}_{\mathscr{F}}^*(G \circ \overline{F}, H \circ F).$$

is a split monomorphism.

Taking  $\mathbb{F} = \mathbb{F}_2$ , there is a non-zero class  $\varphi \in \operatorname{Ext}^1_{\mathscr{F}}(\Lambda^2, \operatorname{Id})$  given by the short exact sequence (1). Applying the above result gives a pull-back diagram of short exact sequences

$$0 \longrightarrow F \longrightarrow \mathcal{E} \longrightarrow \Lambda^2 \circ \overline{F} \longrightarrow 0$$

$$0 \longrightarrow F \longrightarrow S^2 \circ F \longrightarrow \Lambda^2 \circ F \longrightarrow 0$$

in which both short exact sequences are essential, in particular the top row corresponds to a non-zero class of  $\operatorname{Ext}^1_{\mathscr{F}}(\Lambda^2 \circ \overline{F}, F)$ .

For  $\mathbb{F} = \mathbb{F}_2$ , recall the extension classes from Section 2.3, which are related by the Frobenius morphism  $\mathrm{Id} \to S^2$ :

$$\tilde{e}_1 \in \operatorname{Ext}_{\mathscr{F}}^2(S^2, \operatorname{Id}) \mapsto e_1 \in \operatorname{Ext}_{\mathscr{F}}^2(\operatorname{Id}, \operatorname{Id}).$$

Corollary 8.5. For  $\mathbb{F} = \mathbb{F}_2$  and  $\tilde{F} \subset F$  finite functors of polynomial degree exactly d > 0, the following classes are non-trivial

(1) 
$$i^*(e_1 \circ F) \in \operatorname{Ext}_{\mathscr{F}}^2(\tilde{F}, F)$$

(1) 
$$i^*(e_1 \circ F) \in \operatorname{Ext}_{\mathscr{F}}^2(\tilde{F}, F);$$
  
(2)  $i^*(\tilde{e}_1 \circ F) \in \operatorname{Ext}_{\mathscr{F}}^2(S^2 \circ \tilde{F}, F),$ 

where  $i^*$  denotes the pullback induced by the inclusion  $i: \tilde{F} \hookrightarrow F$ .

Moreover, under the Frobenius morphism  $Id \to S^2$ , these classes are related by

$$i^*(\tilde{e}_1 \circ F) \in \operatorname{Ext}_{\mathscr{F}}^2(S^2 \circ \tilde{F}, F) \mapsto i^*(e_1 \circ F) \in \operatorname{Ext}_{\mathscr{F}}^2(\tilde{F}, F).$$

Proof. Follows directly from Theorem 8.3, as in Example 8.4.

# 9. The Main results

This section gives the proofs of the main results of the paper, Theorems 9.9 and 9.16, based upon the non-triviality result of Section 8. Namely an obstruction class  $\omega_X$  living in a suitable group  $\operatorname{Ext}_{\mathscr{F}}^2(-,-)$  is introduced, which must vanish. Combined with the non-vanishing result Theorem 8.3, this provides restrictions on the structure of  $H^*(X)$ , in particular on the relationship between the first two non-trivial layers of its nilpotent filtration.

9.1. Compatibility with the \*-product. In the following, note that [Kuh08, Proposition 4.1 shows that the spectral sequence associated to the Arone-Goodwillie tower is a spectral sequence of bigraded algebras with respect to the ⋆-product.

**Proposition 9.1.** For X an n-connected space with  $H^*(X)$  of finite type, there are identifications via the Arone-Goodwillie spectral sequence calculating  $H^*(\Omega^n X)$ :

$$\begin{array}{ccc} F_1H^*(\Omega^nX) &\cong & \Sigma E_\infty^{-1,*} \\ F_2H^*(\Omega^nX)/F_1H^*(\Omega^nX) &\cong & \Sigma E_\infty^{-2,*} \end{array}$$

and the morphism

$$\overline{\cup}:\Lambda^2(F_1H^*(\Omega^nX))\to F_2H^*(\Omega^nX)/F_1H^*(\Omega^nX)$$

coincides with the morphism induced by the  $\star$ -product in the spectral sequence.

*Proof.* The result follows from the compatibility of the spectral sequence with the multiplicative structure, established by Ahearn and Kuhn [AK02], as used in [Kuh08].

9.2. Working modulo nilpotents. Throughout this section, the following is supposed:

Hypothesis 9.2. For fixed  $1 \leq n \in \mathbb{N}$ , X is an n-connected space such that  $H^*(X) \in$  $\mathcal{N}il_n$  and is of finite type.

The first condition ensures strong convergence of the spectral sequence calculating  $H^*(\Omega^n X)$  and the second specifies the class of spaces of interest here. In particular, there is a surjection  $H^*(X) \to \Sigma^n \rho_n H^*(X)$ .

Notation 9.3. For  $j \in \{1, 2\}$ , set  $F_i := l(F_i H^*(\Omega^n X))$ .

Corollary 4.4 implies:

**Lemma 9.4.** There is a natural isomorphism  $F_1 \cong l(\rho_n H^*(X))$ .

For clarity of presentation, the case n=1 is postponed to Section 9.3.

Notation 9.5. Set  $K := l(\ker\{\Phi\rho_n H^*(X) \xrightarrow{\delta_n} \rho_{n+1} H^*(X)\})$ , so that  $K \subset F_1$  and write  $\Gamma^2 \circ F_1$  for the functor defined by the pullback diagram:

$$0 \longrightarrow \Lambda^{2} \circ F_{1} \longrightarrow \widetilde{\Gamma^{2} \circ F_{1}} \longrightarrow K \longrightarrow 0$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$0 \longrightarrow \Lambda^{2} \circ F_{1} \longrightarrow \Gamma^{2} \circ F_{1} \longrightarrow F_{1} \longrightarrow 0$$

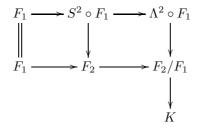
**Lemma 9.6.** For  $n \geq 2$  there is an isomorphism  $F_2/F_1 \cong \widetilde{\Gamma^2 \circ F_1}$  and the morphism induced by  $\overline{\cup}: \Lambda^2(F_1H^*(\Omega^nX)) \to F_2H^*(\Omega^nX)/F_1H^*(\Omega^nX)$  identifies with the inclusion  $\Lambda^2 \circ F_1 \hookrightarrow \widetilde{\Gamma^2} \circ F_1$ .

Proof. The result follows from Corollary 4.4, using the identification of the differential  $d_1$  modulo nilpotents, which follows from Corollary 7.13.

Notation 9.7. For  $n \geq 2$ , let  $\omega_X \in \operatorname{Ext}_{\mathscr{F}}^2(K, F_1)$  be the Yoneda product of the class  $\varphi \circ F_1 \in \operatorname{Ext}^1_{\mathscr{F}}(\Lambda^2 \circ F_1, F_1)$  with the class in  $\operatorname{Ext}^1_{\mathscr{F}}(K, \Lambda^2 \circ F_1)$  representing  $F_2/F_1$ .

**Lemma 9.8.** *For* n > 2,

- (1)  $\omega_X = i^*(e_1 \circ F_1)$ , where  $i^* : \operatorname{Ext}_{\mathscr{F}}^2(F_1, F_1) \to \operatorname{Ext}_{\mathscr{F}}^2(K, F_1)$  is induced by the inclusion  $i: K \hookrightarrow F_1$ ;
- (2) there is a commutative diagram in which the three-term rows and columns are short exact:



and  $S^2 \circ F_1 \to F_2$  is induced by  $\cup : S^2(F_1H^*(\Omega^nX)) \to F_2H^*(\Omega^nX)$ .

(3) In particular  $\omega_X = 0 \in \operatorname{Ext}_{\mathscr{F}}^2(K, F_1)$ .

*Proof.* The first point follows from the identification of  $F_2/F_1$  given in Lemma 9.6 and the definition of  $\widetilde{\Gamma^2 \circ F_1}$ .

The second point is a consequence of the compatibility between the cup product and the  $\star$ -product and follows by combining Proposition 9.1 with the identifications of  $F_1$  (Lemma 9.4) and  $F_2/F_1$ .

The final point follows from homological algebra.

**Theorem 9.9.** Suppose that  $n \geq 2$  and X is a topological space satisfying Hypothesis 9.2. If  $F_1 = l(\rho_n H^*(X))$  is a finite functor of polynomial degree exactly d > 0, then  $K := l\left(\ker\{F_1 \xrightarrow{\delta_n} \rho_{n+1} H^*(X)\}\right)$  has polynomial degree < d.

Equivalently, if  $\rho_n H^*(X)$  is finitely generated over  $\mathscr{A}$  and lies in  $\mathscr{U}_d \setminus \mathscr{U}_{d-1}$  for d > 0, then

$$\delta_n: \Phi \rho_n H^*(X) \to \rho_{n+1} H^*(X)$$

has kernel in  $\mathcal{U}_{d-1}$ , in particular  $\delta_n$  is non-trivial.

Hence

- (1)  $\rho_{n+1}H^*(X) \notin \mathscr{U}_{d-1};$
- (2)  $H^*(X)/\text{nil}_{n+2}H^*(X)$  is not an n-fold suspension.

*Proof.* Lemma 9.8 shows that the obstruction class  $\omega_X$  is trivial and also that it identifies with  $i^*(e_1 \circ F_1)$ . If  $F_1$  is a finite functor then, by Theorem 8.3, this class is non-trivial if K has exact polynomial degree d.

The fact that  $H^*(X)/\rho_{n+2}H^*(X)$  is not an n-fold suspension follows from the identification of  $\delta_n$  in Corollary 7.15.

Corollary 9.10. Let M be an unstable module of finite type such that

- (1)  $M \in \text{Ob } \mathcal{N}il_n$ , for  $2 \leq n \in \mathbb{N}$ ;
- (2) M is n-connected;
- (3)  $\rho_n M$  is finitely generated over  $\mathscr{A}$  and lies in  $\mathscr{U}_d \setminus \mathscr{U}_{d-1}$  for some d > 0;
- (4) the morphism induced by  $Sq_{n-1}$ ,  $\Phi M \to \Sigma^{1-n}(M/\text{nil}_{n+2}M)$  has image in  $\mathcal{U}_{d-1}$ .

Then M cannot be the reduced  $\mathbb{F}_2$ -cohomology of an n-connected space.

*Proof.* A consequence of Theorem 9.9, using the identification of  $\delta_n$  from Section 7.4 (cf. Remark 7.12) to express the condition in terms of  $Sq_{n-1}$ .

Remark 9.11. The statement has an unavoidably technical nature, due to the identification of the morphism  $\delta_n$  in terms of an operation  $Sq_i$  with i > 0. In the case n = 1, the analogous result is conceptually simpler, since the corresponding operation is the cup square  $Sq_0$  (see Corollary 9.18).

9.3. The case n=1. The exceptional case (n=1) corresponds to the Eilenberg-Moore spectral sequence. Theorem 6.6 implies that the cohomology of  $D_{1,2}(\Sigma^{-1}X)$  is the  $\mathscr{A}$ -module  $H^*(\Sigma^{-1}X)^{\otimes 2}$ . The differential  $d_1$  is induced by the product  $\mu: H^*(X)^{\otimes 2} \to H^*(X)$ , which factors by commutativity as  $S^2(H^*(X)) \to H^*(X)$ . Since  $H^*(X)$  is nilpotent, by hypothesis, this induces

$$S^2(\rho_1 H^*(X)) \to \rho_2 H^*(X).$$

Notation 9.12. Write K for the kernel of the corresponding morphism in  $\mathscr{F}$ :

$$S^2 \circ F_1 \to l(\rho_2 H^*(X));$$

**Lemma 9.13.** The quotient  $F_2/F_1$  fits into the pullback diagram of short exact sequences:

$$0 \longrightarrow \Lambda^{2} \circ F_{1} \longrightarrow F_{2}/F_{1} \longrightarrow K \longrightarrow 0$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow j$$

$$0 \longrightarrow \Lambda^{2} \circ F_{1} \longrightarrow T^{2} \circ F_{1} \longrightarrow S^{2} \circ F_{1} \longrightarrow 0$$

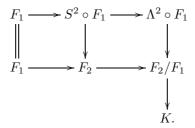
Proof. Straightforward.

Notation 9.14. For n=1, let  $\omega_X \in \operatorname{Ext}^2_{\mathscr{F}}(K,F_1)$  be the Yoneda product of the class  $\varphi \circ F_1 \in \operatorname{Ext}^1_{\mathscr{F}}(\Lambda^2 \circ F_1,F_1)$  with the class in  $\operatorname{Ext}^1_{\mathscr{F}}(K,\Lambda^2 \circ F_1)$  representing  $F_2/F_1$ .

The following result is the analogue for n = 1 of Lemma 9.8:

# **Lemma 9.15.** *For* n = 1,

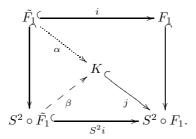
- (1)  $\omega_X = j^*(\tilde{e}_1 \circ F_1)$ , where  $j^* : \operatorname{Ext}_{\mathscr{F}}^2(S^2 \circ F_1, F_1) \to \operatorname{Ext}_{\mathscr{F}}^2(K, F_1)$  is induced by the inclusion  $j : K \hookrightarrow S^2 \circ F_1$ ;
- (2) there is a commutative diagram in which the three-term rows and columns are short exact:



(3) In particular  $\omega_X = 0 \in \operatorname{Ext}_{\mathscr{F}}^2(K, F_1)$ .

Proof. Analogous to the proof of Lemma 9.8, mutatis mutandis, using Lemma 9.13.

Before stating the theorem, it is worth resuming the situation;  $F_1$  is of polynomial degree exactly d>0 and  $\tilde{F_1} \stackrel{i}{\hookrightarrow} F_1$  an arbitrary subfunctor. There is a commutative square of inclusions:



Applying  $\operatorname{Ext}_{\mathscr{F}}^2(-,F_1)$  gives a diagram of Ext-groups. Of particular interest are the following observations:

- (1) If the dotted factorization  $\alpha$  exists, then the class  $\alpha^*(\omega_X) \in \operatorname{Ext}^2_{\mathscr{F}}(\tilde{F}_1, F_1)$  coincides with  $i^*(e_1 \circ F_1)$ , in the notation of Corollary 8.5.
- (2) If the dashed factorization  $\beta$  exists (and hence  $\alpha$ ), then the class  $\beta^*(\omega_X) \in \operatorname{Ext}_{\mathscr{F}}^2(S^2 \circ \tilde{F}_1, F_1)$  coincides with  $i^*(\tilde{e}_1 \circ F_1)$ , in the notation of Corollary

**Theorem 9.16.** Let X be a simply-connected space such that  $H^*(X)$  is nilpotent and of finite type and  $F_1 = l(\rho_1 H^*(X))$  is finite of polynomial degree exactly d > 0. For a subfunctor  $\tilde{F}_1$  of  $F_1$  of polynomial degree exactly d, the following properties hold:

(1) the subfunctor  $S^2 \circ \tilde{F}_1 \subset S^2 \circ F_1$  is not contained within K, equivalently the morphism

$$S^2 \circ \tilde{F_1} \to l(\rho_2 H^*(X))$$

is non trivial;

(2) the subfunctor  $\tilde{F}_1 \subset S^2 \circ F_1$  is not contained within K, equivalently the composite:

$$\tilde{F}_1 \hookrightarrow S^2 \circ \tilde{F}_1 \to l(\rho_2 H^*(X))$$

is non-trivial.

Hence

- (1) the image of  $S^2 \circ \tilde{F}_1 \to l(\rho_2 H^*(X))$  has polynomial degree exactly 2d, in particular  $\rho_2 H^*(X) \notin \mathcal{U}_{2d-1}$ ;
- (2) the morphism induced by  $Sq_0$ :

$$\Phi \rho_1 H^*(X) \to \rho_2 H^*(X)$$

has kernel in  $\mathcal{U}_{d-1}$ , in particular is non-trivial, so that  $H^*(X)/\text{nil}_3H^*(X)$  is not a suspension.

*Proof.* The argument follows the proof of Theorem 9.9, mutatis mutandis.

Suppose that  $\tilde{F}_1 \subset F_1$  is a subfunctor such that  $S^2 \circ \tilde{F}_1 \subset K$ . Then the class  $\omega_X \in \operatorname{Ext}_{\mathscr{F}}^2(K, F_1)$  pulls back to the class of  $\operatorname{Ext}_{\mathscr{F}}^2(S^2 \circ \tilde{F}_1, F_1)$  which is the pullback of  $\tilde{e}_1 \circ F_1$  under the morphism induced by  $\tilde{F}_1 \subset F_1$ . Suppose that  $\tilde{F}_1$  has polynomial degree exactly d, then this class is non-trivial, by Corollary 8.5; this contradicts Lemma 9.15.

The argument using the hypothesis that  $\tilde{F}_1 \subset K$  is similar mutatis mutandis, again using Corollary 8.5.

To show that the polynomial degree of the image of  $S^2 \circ \tilde{F_1} \to l(\rho_2 H^*(X))$  is exactly 2d, without loss of generality we may assume that  $\tilde{F_1}$  has no non-trivial quotient of polynomial degree < d. In this case, the functor  $S^2 \circ \tilde{F_1}$  has no non-trivial quotient of degree < 2d (see [CGPS14], for example).

Finally, the composite  $\tilde{F}_1 \hookrightarrow S^2 \circ \tilde{F}_1 \to l(\rho_2 H^*(X))$  is given, upon passage to  $\mathscr{U}/\mathscr{N}il$ , by the morphism  $\Phi\Sigma\rho_1 H^*(X) \to \Sigma^2\rho_2 H^*(X)$  induced by  $Sq_0$ , noting that the natural isomorphism  $\Phi\Sigma \cong \Sigma^2\Phi$  allows the suspensions to be removed. Taking  $\tilde{F}_1 := K \cap F_1$ , the above argument shows that this has polynomial degree  $\leq d-1$ , whence the result.

Remark 9.17. Theorem 9.16 is an improvement upon the main result of [CGPS14, Section 6], where only the case  $\tilde{F}_1 = \overline{F_1}$ , the kernel of  $F_1 \twoheadrightarrow q_{d-1}F_1$  (see Example 9.15 for this notation) is considered, for the morphism  $S^2 \circ \overline{F_1} \to l(\rho_2 H^*(X))$ . The above theorem unifies this with the case  $n \geq 2$ .

The improvement is obtained by using Theorem 8.3 rather than [Kuh95a, Theorem 4.8].

Corollary 9.18. Let K be a connected unstable algebra of finite type over  $\mathbb{F}_2$  such that  $\overline{K}$  is 1-connected and  $\rho_1\overline{K}$  is finitely generated over  $\mathscr{A}$  and lies in  $\mathscr{U}_d\backslash\mathscr{U}_{d-1}$  for d>0 (in particular is non-zero).

If the morphism induced by the cup square,  $Sq_0: \Phi \overline{K} \to K/\text{nil}_3K$ , has image in  $\mathcal{U}_{d-1}$ , then  $\overline{K}$  cannot be the reduced  $\mathbb{F}_2$ -cohomology of a simply-connected space.

*Proof.* Follows from Theorem 9.16 (cf. Corollary 9.10).  $\Box$ 

Remark 9.19. Corollary 9.18 applies if the image of  $Sq_0: \overline{\Phi K} \to K$  lies in  $\mathcal{U}_{d-1}$ . This occurs, for example, if  $Sq_0$  is zero, the case considered in Theorem 2. Under this hypothesis, QK is a quotient of  $\Sigma\Omega\overline{K}$ , and it follows that  $\rho_1\overline{K}$  is isomorphic to  $\rho_1QK$ .

10. The case 
$$p$$
 odd

This section sketches the modifications necessary for p an odd prime, writing  $H^*(-)$  for singular cohomology with  $\mathbb{F}_p$ -coefficients. Here the difference stems from the fact that the enveloping algebra UM of an unstable module M is the quotient of the free symmetric algebra  $S^*(M)$  by the relation  $x^p = P_0 x$  for x of even degree; in particular, the relation depends only upon the even degree elements of M.

The inclusion of the full subcategory  $\mathscr{U}' \subset \mathscr{U}$  of modules concentrated in even degree admits a right adjoint and induces an equivalence  $\mathscr{U}'/\mathscr{N}il' \cong \mathscr{U}/\mathscr{N}il$ , where  $\mathscr{N}il' = \mathscr{U}' \cap \mathscr{N}il$  (see [Sch94, Section 5.1]). Since the final part of the proof works modulo nilpotents, this allows the avoidance of problems at p odd associated with the action of the Bockstein (cf. [Sch98] and [BHRS13]).

The fundamental exact sequence of functors (cf. Section 2.3) is now

$$0 \to \operatorname{Id} \xrightarrow{x \mapsto x^p} S^p \longrightarrow \overline{S^p} \to 0,$$

where the truncated symmetric power  $\overline{S^p}$  is simple and self dual, together with the norm sequence:

$$0 \longrightarrow \operatorname{Id} \longrightarrow S^p \xrightarrow{N} \Gamma^p \longrightarrow \operatorname{Id} \longrightarrow 0$$

which represents a non-zero class in  $\operatorname{Ext}^2_{\mathscr F}(\operatorname{Id},\operatorname{Id})$  [FLS94].

The  $E^1$ -page of the Arone-Goodwillie spectral sequence for  $X \mapsto \Sigma^\infty \Omega^n X$  at p odd is calculated as in [BHRS13]. The arguments again use almost unstable modules (for the odd primary case); although only a very limited part of the structure of the  $E^1$ -page intervenes, it is necessary to establish that the spectral sequence is good almost unstable (as in Section 6).

One considers the pth filtration  $F_pH^*(\Omega^nX) \subset H^*(\Omega^nX)$ ; the multiplicative structure induces a morphism of unstable modules

$$\bigoplus_{j=2}^{p-1} S^j(F_1 H^*(\Omega^n X)) \to F_p(H^*(\Omega^n X))$$

which is a monomorphism in  $\mathscr{U}/\mathscr{N}il$ . The cokernel  $\overline{F_p}(H^*(\Omega^nX))$  lies in a short exact sequence in  $\mathscr{U}/\mathscr{N}il$ :

$$0 \to \Sigma^{-1} E_{\infty}^{-1,*} \to \overline{F_p}(H^*(\Omega^n X)) \to \Sigma^{-p} E_{\infty}^{-p,*} \to 0.$$

Moreover the cup product induces

$$S^p(F_1H^*(\Omega^nX)) \to \overline{F_p}(H^*(\Omega^nX))$$

which is a monomorphism in  $\mathcal{U}/\mathcal{N}il$ .

The identification of  $\Sigma^{-p}E_{\infty}^{-p,*}$  modulo nilpotents relies on the calculation of  $d_{p-1}: \Sigma(\Sigma^{-p}E_{p-1}^{-p,*}) \to \Sigma^{-1}E_{p-1}^{-1,*}$ , since [BHRS13, Proposition 4.1] shows that lower differentials act trivially on the dual Dyer-Lashof operations.

Hereafter, the argument proceeds as for the case p=2, mutatis mutandis, by analysing  $\overline{F_p}(H^*(\Omega^n X))$  (for p=2, this is simply  $F_2(H^*(\Omega^n X))$ ).

Since the proof reduces to an argument in  $\mathscr{U}/\mathscr{N}il$ , the ability to work in  $\mathscr{U}'$  provides a useful simplification, making the parallel with the case p=2 transparent. For example, for n>1, the analogue of Corollary 6.7 for  $d_{p-1}$  reduces to considering a morphism

$$\Gamma^p(\rho'_n H^*(X)) \to \rho'_{n+1} H^*(X).$$

where, for M an unstable module,  $\rho'_k M$  denotes the largest submodule of  $\rho_k M$  concentrated in even degrees. Moreover, Lemma 2.5 has the following analogue: for  $M' \in \text{Ob } \mathscr{U}'$  there is a short exact sequence:

$$0 \to \overline{S^p}M' \to \Gamma^pM' \to \Phi M' \to 0$$

where the Frobenius functor  $\Phi$  restricted to  $\mathscr{U}'$  is directly analogous to that for p=2. As in the case p=2, the contribution  $\Phi(\rho'_n H^*(X))$  is provided by  $\mathfrak{Q}_0$ .

The proofs of the analogues of the results of Section 9 follow an identical strategy, depending upon Theorem 8.3, which is prime independent.

Remark 10.1. A refinement of the results can be obtained by exploiting the weight splitting of  $\mathscr{U}/\mathscr{N}il$  associated to the action of the multiplicative group  $\mathbb{F}_n^{\times}$ .

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