A LOCAL PROOF OF THE BREUIL-MÉZARD CONJECTURE IN THE SCALAR SEMI-SIMPLIFICATION CASE

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Abstract. We give a new local proof of the Breuil-Mézard conjecture in the case of a reducible representation of the absolute Galois group of $\mathbb{Q}_p$, $p > 2$, that has scalar semi-simplification, via a formalism of Paškūnas.

1. Introduction

Let $p > 2$ be a prime number, k be a finite field of characteristic p and L a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and uniformizer ϖ. Let $\rho\colon G_{\mathbb{Q}_p} \to \mathrm{GL}_2(k)$ be a continuous representation of the form

$$\rho(g) = \begin{pmatrix} \chi(g) & \phi(g) \\ 0 & \chi(g) \end{pmatrix}, \forall g \in G_{\mathbb{Q}_p}, \tag{1}$$

so that the semi-simplification of ρ is isomorphic to $\chi \oplus \chi$. Let $R_\rho^\square$ denote the associated universal framed deformation ring of ρ and let $\rho^\square$ be the universal framed deformation. For any $\mathfrak{p} \in \text{m-Spec}(R_\rho^\square[1/p])$, the set of maximal ideals, the residue field $\kappa(\mathfrak{p})$ is a finite extension of $\mathbb{Q}_p$. We denote its ring of integers by $\mathcal{O}_\mathfrak{p}$ and get an associated representation $\rho_\mathfrak{p}^\square\colon G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathcal{O}_\mathfrak{p})$ that lifts ρ. Let $\tau\colon I_{\mathbb{Q}_p} \to \mathrm{GL}_2(L)$ be a representation of the inertia group of $\mathbb{Q}_p$ with an open kernel, $\psi\colon G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ a continuous character and let $\mathbf{w} = (a, b)$ be a pair of integers with $b > a$. We say that $\rho_\mathfrak{p}^\square$ is of p-adic Hodge type $(\mathbf{w}, \tau, \psi)$ if it is potentially semi-stable with Hodge-Tate weights $\mathbf{w}$, $\det \rho_\mathfrak{p} \cong \psi\epsilon$, $\psi|_{I_{\mathbb{Q}_p}} = \epsilon^{a+b} \det \tau$ and $\mathrm{WD}(\rho_\mathfrak{p}^\square)\big|_{I_{\mathbb{Q}_p}} \cong \tau$, where ϵ is the cyclotomic character and $\mathrm{WD}(\rho_\mathfrak{p}^\square)$ is the Weil-Deligne representation associated to $\rho_\mathfrak{p}^\square$ by Fontaine [8].

By a result of Henniart [11] there exists a unique smooth irreducible $K := \mathrm{GL}_2(\mathbb{Z}_p)$-representation $\sigma(\tau)$ and a modification $\sigma^{\mathrm{cr}}(\tau)$ defined by Kisin [13, 1.1.4] such that for any smooth absolutely irreducible $\mathrm{GL}_2(\mathbb{Q}_p)$-representation π with associated Weil-Deligne representation $\mathrm{LL}(\pi)$ via the classical local Langlands correspondence, we have $\mathrm{Hom}_K(\sigma(\tau), \pi) \neq 0$ (resp. $\mathrm{Hom}_K(\sigma^{\mathrm{cr}}(\tau), \pi) \neq 0$) if and only if $\mathrm{LL}(\pi)|_{I_{\mathbb{Q}_p}} \cong \tau$ (resp. $\mathrm{LL}(\pi)|_{I_{\mathbb{Q}_p}} \cong \tau$ and the monodromy operator N on $\mathrm{LL}(\pi)$ is trivial). We have $\sigma(\tau) \ncong \sigma^{\mathrm{cr}}(\tau)$ only if $\tau = \chi \oplus \chi$, in which case $\sigma(\tau) = \tilde{\mathrm{st}} \otimes \chi \circ \det$ and $\sigma^{\mathrm{cr}}(\tau) = \chi \circ \det$, where $\tilde{\mathrm{st}}$ is the Steinberg representation of $\mathrm{GL}_2(\mathbb{F}_p)$, inflated to $\mathrm{GL}_2(\mathbb{Z}_p)$, and χ is considered as a character of $\mathbb{Z}_p^\times$ via local class field theory. By enlarging L if necessary, we can assume that $\sigma(\tau)$ (resp. $\sigma^{\mathrm{cr}}(\tau)$) is defined over L. We define $\sigma(\mathbf{w}, \tau) := \sigma(\tau) \otimes \mathrm{Sym}^{b-a-1} L^2 \otimes \det^a$ and let $\overline{\sigma(\mathbf{w}, \tau)}$ be the semi-simplification of the reduction of a K-invariant $\mathcal{O}$-lattice modulo ϖ. One can show that $\overline{\sigma(\mathbf{w}, \tau)}$ is independent of the choice of the lattice. For every irreducible smooth finite-dimensional K-representation σ over k we let $m_\sigma(\mathbf{w}, \tau)$

denote the multiplicity with which σ occurs in $\overline{\sigma(\mathbf{w},\tau)}$. Analogously we define $\sigma^{\mathrm{cr}}(\mathbf{w},\tau) := \sigma^{\mathrm{cr}}(\tau) \otimes \mathrm{Sym}^{b-a-1} L^2 \otimes \det^a$ and let $m^{\mathrm{cr}}_\sigma(\mathbf{w},\tau)$ denote the multiplicity with which σ occurs in $\overline{\sigma^{\mathrm{cr}}(\mathbf{w},\tau)}$.

We prove the following theorem.

Theorem 1.1. *Let $p > 2$ and let $(\mathbf{w},\tau,\psi)$ be a Hodge type. There exists a reduced $\mathcal{O}$-torsion free quotient $R^\square_\rho(\mathbf{w},\tau,\psi)$ (resp. $R^{\square,\mathrm{cr}}_\rho(\mathbf{w},\tau,\psi)$) of $R^\square_\rho$ such that for all $\mathfrak{p} \in \text{m-Spec}(R^\square_\rho[1/p])$, $\mathfrak{p}$ is an element of $\text{m-Spec}\left(R^\square_\rho(\mathbf{w},\tau,\psi)[1/p]\right)$ (resp. $\text{m-Spec}\left(R^{\square,\mathrm{cr}}_\rho(\mathbf{w},\tau,\psi)[1/p]\right)$) if and only if $\rho^\square_\mathfrak{p}$ is potentially semi-stable (resp. potentially crystalline) of p-adic Hodge type $(\mathbf{w},\tau,\psi)$. If $R^\square_\rho(\mathbf{w},\tau,\psi)$ (resp. $R^{\square,\mathrm{cr}}_\rho(\mathbf{w},\tau,\psi)$) is non-zero, then it has Krull dimension 5.*

Furthermore, there exists a four-dimensional cycle $z(\rho)$ of $R^\square_\rho$ such that there are equalities of four-dimensional cycles

$$z_4\left(R^\square_\rho(\mathbf{w},\tau,\psi)/(\varpi)\right) = m_\lambda(\mathbf{w},\tau)z(\rho), \tag{2}$$

$$z_4\left(R^{\square,\mathrm{cr}}_\rho(\mathbf{w},\tau,\psi)/(\varpi)\right) = m^{\mathrm{cr}}_\lambda(\mathbf{w},\tau)z(\rho), \tag{3}$$

where $\lambda := \mathrm{Sym}^{p-2} k^2 \otimes \chi \circ \det$.

The equality of cycles also implies the analogous equality of Hilbert-Samuel multiplicities. Hence the above theorem proves the Breuil-Mézard conjecture [2], as stated in [13], in our case. This case has also been handled by Kisin in [13] using global methods, see also the errata in [10]. However, our proof is purely local and the results of this paper, together with works of Paškūnas [18], Yongquan Hu and Fucheng Tan [12], the whole conjecture is now proved in the 2-dimensional case only by local methods, when $p \geq 5$.

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2. Formalism

We quickly recall a formalism due to Paškūnas used by him to prove the Breuil-Mézard conjecture for residual representations with scalar endomorphisms in [18]. Let R be a complete local notherian commutative $\mathcal{O}$-algebra with residue field k. Let G be a p-adic analytic group, K be a compact open subgroup and P its pro-p Sylow subgroup. Let N be a finitely generated $R[\![K]\!]$-module, V be a continuous finite dimensional L-representation of K, and Θ be an $\mathcal{O}$-lattice in V which is invariant under the action of K. Let

$$M(\Theta) := \mathrm{Hom}_\mathcal{O}(\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(N, \mathrm{Hom}_\mathcal{O}(\Theta,\mathcal{O})),\mathcal{O}). \tag{4}$$

This is a finitely generated R-module [18, Lemma 2.15]. Let d denote the Krull dimension of $M(\Theta)$. Recall that Pontryagin duality $\lambda \mapsto \lambda^\vee$ induces an anti-equivalence of categories between discrete $\mathcal{O}$-modules and compact $\mathcal{O}$-modules [15, (5.2.2)-(5.2.3)]. For any λ in $\mathrm{Mod}^{\mathrm{sm}}_K(\mathcal{O})$, the category of smooth K-representations on $\mathcal{O}$-torsion modules, we define

$$M(\lambda) := \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(N, \lambda^\vee)^\vee. \tag{5}$$

Then $M(\lambda)$ is also a finitely generated R-module [18, Cor. 2.5]. We define $\mathrm{Mod}^{\mathrm{pro}}_G(\mathcal{O})$ to be the category of compact $\mathcal{O}[\![K]\!]$-modules with an action of $\mathcal{O}[G]$, such that the restriction to $\mathcal{O}[K]$ of both actions coincide. Pontryagin duality induces an anti-equivalence of categories between $\mathrm{Mod}^{\mathrm{sm}}_G(\mathcal{O})$ and $\mathrm{Mod}^{\mathrm{pro}}_G(\mathcal{O})$. For any $R[1/p]$-module m of finite length, we choose a finitely generated R-submodule m^0 with $\mathrm{m} \cong \mathrm{m}^0 \otimes_{\mathcal{O}} L$ and define

$$\Pi(\mathrm{m}) := \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}}(\mathrm{m}^0 \otimes_R N, L). \tag{6}$$

By [18, Lemma 2.21], $\Pi(\mathrm{m})$ is an admissible unitary L-Banach space representation of G.

Theorem 2.1 (Paškūnas,[18])**.** *Let $\mathfrak{a}$ be the R-annihilator of $M(\Theta)$. If the following hold*

(a) *N is projective in $\mathrm{Mod}^{\mathrm{pro}}_K(\mathcal{O})$,*
(b) *$R/\mathfrak{a}$ is equidimensional and all the associated primes are minimal,*
(c) *there exists a dense subset Σ of $\mathrm{Supp}\, M(\Theta)$, contained in $\text{m-Spec}\, R[1/p]$, such that for all $\mathfrak{n} \in \Sigma$ the following hold:*
  (i) $\dim_{\kappa(\mathfrak{n})} \mathrm{Hom}_K(V, \Pi(\kappa(\mathfrak{n}))) = 1$,
  (ii) $\dim_{\kappa(\mathfrak{n})} \mathrm{Hom}_K(V, \Pi(R_{\mathfrak{n}}/\mathfrak{n}^2)) \le d$,

then $R/\mathfrak{a}$ is reduced, of dimension d and we have an equality of $(d-1)$-dimensional cycles

$$z_{d-1}(R/(\varpi, \mathfrak{a})) = \sum_{\sigma} m_\sigma z_{d-1}(M(\sigma)),$$

where the sum is taken over the set of isomorphism classes of smooth irreducible k-representations of K and m_σ is the multiplicity with which σ occurs as a subquotient of Θ/ϖ.

We want to specify the following criterion in our situation, which allows us to check the first two conditions of Theorem 2.1.

Theorem 2.2 (Paškūnas,[18])**.** *Suppose that R is Cohen-Macaulay and N is flat over R. If*

$$\mathrm{projdim}_{\mathcal{O}[\![P]\!]}\, k\hat{\otimes}_R N + \max_{\sigma}\{\dim_R M(\sigma)\} \le \dim R, \tag{7}$$

where the maximum is taken over all the irreducible smooth k-representations of K, then the following holds:

(o) *(7) is an equality,*
(i) *N is projective in $\mathrm{Mod}^{\mathrm{pro}}_K(\mathcal{O})$,*
(ii) *$M(\Theta)$ is a Cohen-Macaulay module,*
(iii) *$R/\mathrm{ann}_R\, M(\Theta)$ is equidimensional, and all the associated prime ideals are minimal.*

We start with the following setup. Let $\rho\colon G_{\mathbb{Q}_p} \to \mathrm{GL}_2(k)$ be a continuous representation of the form $\rho(g) = \begin{pmatrix} \chi(g) & \phi(g) \\ 0 & \chi(g) \end{pmatrix}$, as in (1). After twisting we may assume that χ is trivial so that for all $g \in G_{\mathbb{Q}_p}$

$$\rho(g) = \begin{pmatrix} 1 & \phi(g) \\ 0 & 1 \end{pmatrix}. \tag{8}$$

Let $\psi\colon \mathbb{Q}_p^\times \to \mathcal{O}^\times$ be a continuous character with $\psi\epsilon \equiv \mathbb{1} \mod \varpi$. Let R be a complete local noetherian $\mathcal{O}$-algebra and let

$$\rho_R\colon G_{\mathbb{Q}_p} \to \mathrm{GL}_2(R) \tag{9}$$

be a continuous representation with determinant $\psi\epsilon\colon G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ such that $\rho_R \equiv \rho \mod \mathfrak{m}_R$. Let $R^{\mathrm{ps},\psi}$ denote the universal deformation ring that parametrizes 2-dimensional pseudo-characters of $G_{\mathbb{Q}_p}$ lifting the trace of the trivial representation and having determinant $\psi\epsilon$. Let $T\colon G_{\mathbb{Q}_p} \to \mathcal{O}$ be the associated universal pseudo-character. Since $\mathrm{tr}\,\rho_R$ is a pseudo-character lifting $\mathrm{tr}\,\rho$, the universal property of $R^{\mathrm{ps},\psi}$ induces a morphism of $\mathcal{O}$-algebras

$$R^{\mathrm{ps},\psi} \to R. \tag{10}$$

Let from now on $G := \mathrm{GL}_2(\mathbb{Q}_p)$, P the subgroup of upper triangular matrices and $K := \mathrm{GL}_2(\mathbb{Z}_p)$. Let I_1 be the subgroup of K which consists of the matrices that are upper unipotent modulo p. In particular, I_1 is a maximal pro-p Sylow subgroup of K. We let ω be the mod p cyclotomic character, via local class field theory considered as $\omega\colon \mathbb{Q}_p^\times \to k^\times, x \mapsto x\,|x| \mod p$, and define

$$\pi := (\mathrm{Ind}_P^G \mathbb{1} \otimes \omega^{-1})_{\mathrm{sm}}. \tag{11}$$

We let $\mathrm{Mod}^{\mathrm{sm}}_{G,\psi}(\mathcal{O})$ be the full subcategory of $\mathrm{Mod}^{\mathrm{sm}}_{G}(\mathcal{O})$ that consists of smooth G-representations with central character ψ and denote by $\mathrm{Mod}^{\mathrm{lfin}}_{G,\psi}(\mathcal{O})$ its full subcategory of representations that are locally of finite length. We denote by $\mathrm{Mod}^{\mathrm{pro}}_{G,\psi}(\mathcal{O})$ resp. $\mathfrak{C}(\mathcal{O})$ the full subcategories of $\mathrm{Mod}^{\mathrm{pro}}_{G}(\mathcal{O})$ that are anti-equivalent to $\mathrm{Mod}^{\mathrm{sm}}_{G,\psi}(\mathcal{O})$ resp. $\mathrm{Mod}^{\mathrm{lfin}}_{G,\psi}(\mathcal{O})$ via Pontryagin duality. We see that π is an object of $\mathrm{Mod}^{\mathrm{lfin}}_{G,\psi}(\mathcal{O})$. Let $\tilde{P}$ be a projective envelope of $\pi^\vee$ in $\mathfrak{C}(\mathcal{O})$. We define $\tilde{E} := \mathrm{End}_{\mathfrak{C}(\mathcal{O})}(\tilde{P})$. Paškūnas has shown in [17, Cor. 9.24] that the center of $\tilde{E}$ is isomorphic to $R^{\mathrm{ps},\psi}$ and

$$\tilde{E} \cong (R^{\mathrm{ps},\psi} \hat{\otimes}_{\mathcal{O}} \mathcal{O}[\![G_{\mathbb{Q}_p}]\!])/J, \tag{12}$$

where J is the closure of the ideal generated by $g^2 - T(g)g + \psi\epsilon(g)$ for all $g \in G_{\mathbb{Q}_p}$ [17, Cor. 9.27]. The representation ρ_R induces a morphism of $\mathcal{O}$-algebras $\mathcal{O}[\![G_{\mathbb{Q}_p}]\!] \to M_2(R)$. Together with the morphism (10) we obtain a morphism of $R^{\mathrm{ps},\psi}$-algebras

$$R^{\mathrm{ps},\psi} \hat{\otimes}_{\mathcal{O}} \mathcal{O}[\![G_{\mathbb{Q}_p}]\!] \to M_2(R). \tag{13}$$

The Cayley-Hamilton theorem tells us that this morphism is trivial on J, so that we get a morphism of $R^{\mathrm{ps},\psi}$-algebras

$$\eta\colon \tilde{E} \to M_2(R). \tag{14}$$

We define

$$M^{\square}(\sigma) := \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}\left((R \oplus R)\hat{\otimes}_{\tilde{E},\eta} \tilde{P}, \sigma^\vee\right)^\vee. \tag{15}$$

Our goal is to prove the following theorem that enables us to check the condition of Paškūnas' theorem 2.2 for $N = (R \oplus R)\hat{\otimes}_{\tilde{E},\eta}\tilde{P}$ in the last section. We let $\mathrm{projdim}_{\mathcal{O}[\![I_1]\!],\psi}$ denote the length of a minimal projective resolution in $\mathrm{Mod}^{\mathrm{pro}}_{I_1,\psi}(\mathcal{O})$.

Theorem 2.3. *Let ρ and ρ_R be as before. We consider R as an $R^{\mathrm{ps},\psi}$-module via (10). Assume that* $\dim R = \dim R^{\mathrm{ps},\psi} + \dim R/\mathfrak{m}_{R^{\mathrm{ps},\psi}} R$. *Then*

$$\mathrm{projdim}_{\mathcal{O}[\![I_1]\!],\psi}\left(k\hat{\otimes}_R((R\oplus R)\hat{\otimes}_{\tilde{E},\eta}\tilde{P})\right) + \max_{\sigma}\{\dim_R M^{\square}(\sigma)\} \leq \dim R.$$

In particular, the inequality holds if R is flat over $R^{\mathrm{ps},\psi}$.

We start with computing the first summand.

Lemma 2.4.
$$\mathrm{projdim}_{\mathcal{O}[\![I_1]\!],\psi}\left(k\hat{\otimes}_R((R\oplus R)\hat{\otimes}_{\tilde{E},\eta}\tilde{P})\right) = 3.$$

Proof. We have
$$k\hat{\otimes}_R((R\oplus R)\hat{\otimes}_{\tilde{E},\eta}\tilde{P}) \cong (k\oplus k)\hat{\otimes}_{\tilde{E}}\tilde{P}.$$
Because of $k\hat{\otimes}_{\tilde{E}}\tilde{P} \cong \pi^{\vee}$, see [17, Lemma 9.1], and since $\tilde{P}$ is flat over the local ring $\tilde{E}$, $(k\oplus k)\hat{\otimes}_{\tilde{E}}\tilde{P}$ is an extension of $\pi^{\vee}$ by itself. Thus
$$\mathrm{projdim}_{\mathcal{O}[\![I_1]\!],\psi}\left(k\hat{\otimes}_R((R\oplus R)\hat{\otimes}_{\tilde{E},\eta}\tilde{P})\right) = \mathrm{projdim}_{\mathcal{O}[\![I_1]\!],\psi}\,\pi^{\vee}.$$
The rest of the proof works analogous to the proof of [18, Prop. 6.21], the respective cohomology groups are calculated in [17, Cor. 10.4]. $\square$

Lemma 2.5. *Let R, N, σ be as before,* m *a compact R-module. Then*
$$\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(\mathrm{m}\hat{\otimes}_R N, \sigma^{\vee})^{\vee} \cong \mathrm{m}\hat{\otimes}_R \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(N,\sigma^{\vee})^{\vee}.$$

Proof. Since m is compact, we can write it as an inverse limit $\mathrm{m} = \varprojlim \mathrm{m}_i$ of finitely generated R-modules. Also the completed tensor product is defined as an inverse limit, so that we obtain
$$\begin{aligned}\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(\mathrm{m}\hat{\otimes}_R N, \sigma^{\vee}) &\cong \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(\varprojlim(\mathrm{m}_i\hat{\otimes}_R N), \sigma^{\vee})\\ &\cong \mathrm{Hom}_K(\sigma, \varinjlim(\mathrm{m}_i\hat{\otimes}_R N)^{\vee}).\end{aligned}$$
The universal property of the inductive limit yields a morphism
$$\varinjlim \mathrm{Hom}_K(\sigma,(\mathrm{m}_i\hat{\otimes}_R N)^{\vee}) \to \mathrm{Hom}_K(\sigma, \varinjlim(\mathrm{m}_i\hat{\otimes}_R N)^{\vee}),$$
which is easily seen to be injective. For the surjectivity we have to show that every K-morphism from σ to $\varinjlim(\mathrm{m}_i\hat{\otimes}_R N)^{\vee}$ factors through some finite level. But this follows from the fact that σ is a finitely generated K-representation. This implies
$$\begin{aligned}\mathrm{Hom}_K(\sigma, \varinjlim(\mathrm{m}_i\hat{\otimes}_R N)^{\vee}) &\cong \varinjlim \mathrm{Hom}_K(\sigma,(\mathrm{m}_i\hat{\otimes}_R N)^{\vee})\\ &\cong \varinjlim \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(\mathrm{m}_i\hat{\otimes}_R N, \sigma^{\vee}).\end{aligned}$$
Since the statement holds for finitely generated m by [18, Prop. 2.4], taking the Pontryagin duals yields
$$\begin{aligned}\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(\mathrm{m}\hat{\otimes}_R N, \sigma^{\vee})^{\vee} &\cong \varprojlim \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(\mathrm{m}_i\hat{\otimes}_R N, \sigma^{\vee})^{\vee}\\ &\cong \varprojlim \mathrm{m}_i\hat{\otimes}_R \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(N,\sigma^{\vee})^{\vee}\\ &\cong \mathrm{m}\hat{\otimes}_R \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(N,\sigma^{\vee})^{\vee}.\end{aligned}$$
$\square$

For the rest of the section we set $N = \tilde{P}$ so that $M(\sigma) = \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(\tilde{P},\sigma^{\vee})^{\vee}$.

Lemma 2.6. *Let σ be a smooth irreducible K-representation over k. Then $M(\sigma) \neq 0$ if and only if* $\mathrm{Hom}_K(\sigma,\pi) \neq 0$. *Moreover,* $\dim_{R^{\mathrm{ps},\psi}} M(\sigma) \leq 1$.

Proof. By [17, Cor. 9.25], we know that $\tilde{E}$ is a free $R^{\mathrm{ps},\psi}$-module of rank 4. Hu-Tan have shown in [18, Prop. 2.9] that $M(\sigma)$ is a cyclic $\tilde{E}$-module, thus $M(\sigma)$ is a finitely generated $R^{\mathrm{ps},\psi}$-module. Furthermore, $M(\sigma)$ is a compact $\tilde{E}$-module, see for example [9, §IV.4, Cor.1]. The same way as in Lemma 2.5 one can show that

$$\mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(k\hat{\otimes}_{\tilde{E}}\tilde{P},\sigma^\vee)^\vee \cong k\hat{\otimes}_{\tilde{E}}M(\sigma). \tag{16}$$

By [17, Prop. 1.12], we have $k\hat{\otimes}_{\tilde{E}}\tilde{P} \cong \pi^\vee$ so that (16) implies

$$k\hat{\otimes}_{\tilde{E}}M(\sigma) \cong \mathrm{Hom}^{\mathrm{cont}}_{\mathcal{O}[\![K]\!]}(\pi^\vee,\sigma^\vee)^\vee \cong \mathrm{Hom}_K(\sigma,\pi). \tag{17}$$

Hence Nakayama lemma gives us that $M(\sigma) \neq 0$ if and only if $\mathrm{Hom}_K(\sigma,\pi) \neq 0$. If this holds, it follows again from [18, Prop. 2.4] that, if we let J denote the annihilator of $M(\sigma)$ as $\tilde{E}$-module, there is an isomorphism of rings $\tilde{E}/J \cong k[\![S]\!]$. Again by [17, Cor. 9.24], $R^{\mathrm{ps},\psi}$ is isomorphic to the center of $\tilde{E}$. If we let J^{ps} denote the annihilator of $M(\sigma)$ as $R^{\mathrm{ps},\psi}$-module, we get an inclusion

$$R^{\mathrm{ps},\psi}/J^{\mathrm{ps}} \hookrightarrow \tilde{E}/J \cong k[\![S]\!]. \tag{18}$$

Hence it suffices to show that $\dim_{R^{\mathrm{ps},\psi}} k[\![S]\!] \leq 1$, which is equivalent to the existence of an element $x \in \mathfrak{m}_{R^{\mathrm{ps},\psi}}$ that does not lie in J^{ps}. We assume that $\mathfrak{m}_{R^{\mathrm{ps},\psi}} \subset J^{\mathrm{ps}}$. Then we have a finite dimensional k-vector space $M(\sigma)/\mathfrak{m}_{R^{\mathrm{ps},\psi}}M(\sigma) \cong M(\sigma)$, on which $\tilde{E}/J \cong k[\![S]\!]$ acts faithfully, which is impossible. $\square$

The proof of the theorem is now just a combination of the above Lemmas.

Proof of Theorem 2.3. Let σ be such that $M^{\square}(\sigma) \neq 0$. Then we see from Lemma 2.5 that

$$M^{\square}(\sigma) \cong (R \oplus R)\hat{\otimes}_{\tilde{E},\eta}M(\sigma).$$

Since $\tilde{E}$ is a finite $R^{\mathrm{ps},\psi}$-module by [17, Cor. 9.17], we have

$$\begin{aligned}\dim_R M^{\square}(\sigma) &= \dim_R (R \oplus R)\hat{\otimes}_{\tilde{E},\eta}M(\sigma)\\ &\leq \dim_R (R \oplus R)\otimes_{R^{\mathrm{ps},\psi}} M(\sigma).\end{aligned}$$

By [3, A.11] we know that for a morphism of local rings $A \to B$ and non-zero finitely generated modules M, N over A resp. B, we have

$$\dim_B M \otimes_A N \leq \dim_A M + \dim_B N/\mathfrak{m}_A N. \tag{19}$$

Since we already know from Lemma 2.6 that $\dim_{R^{\mathrm{ps},\psi}} M(\sigma) = 1$, we obtain from (19) that

$$\dim_R\left((R \oplus R)\otimes_{R^{\mathrm{ps},\psi}} M(\sigma)\right) \leq 1 + \dim R/\mathfrak{m}_{R^{\mathrm{ps},\psi}}R.$$

This expression depends only on the structure of R as an $R^{\mathrm{ps},\psi}$-module and the assumption of the theorem implies

$$\dim_R\left((R \oplus R)\otimes_{R^{\mathrm{ps},\psi}} M(\sigma)\right) \leq 1 + \dim R - \dim R^{\mathrm{ps},\psi}.$$

From the explicit description of $R^{\mathrm{ps},\psi}$ in [17, Cor. 9.13] we know in particular that $R^{\mathrm{ps},\psi} \cong \mathcal{O}[\![t_1,t_2,t_3]\!]$ and thus $\dim R^{\mathrm{ps},\psi} = 4$. The statement is now an immediate consequence of Lemma 2.4. $\square$

3. Flatness

Let again $\rho \cong \begin{pmatrix} \mathbb{1} & \phi \\ 0 & \mathbb{1} \end{pmatrix}$. Our goal in this section is to show that the universal framed deformation of ρ with fixed determinant satisfies the conditions of Theorem 2.3. Let $G_{\mathbb{Q}_p}(p)$ be the maximal pro-p quotient of $G_{\mathbb{Q}_p}$. Since $p > 2$, it is a free pro-p group on 2 generators γ, δ [15, Thm. 7.5.11]. Since the image of ρ is a p-group, it factors through $G_{\mathbb{Q}_p}(p)$. We have shown in [19] that the universal framed deformation ring $R_\rho^\square$ of ρ is isomorphic to $\mathcal{O}[\![x_{11}, \hat{x}_{12}, x_{21}, t_\gamma, y_{11}, \hat{y}_{12}, y_{21}, t_\delta]\!]$ and the universal framed deformation is given by

$$\rho^\square\colon G_{\mathbb{Q}_p}(p) \to \mathrm{GL}_2(R_\rho^\square), \tag{20}$$

$$\gamma \mapsto \begin{pmatrix} 1 + t_\gamma + x_{11} & x_{12} \\ x_{21} & 1 + t_\gamma - x_{11} \end{pmatrix}, \tag{21}$$

$$\delta \mapsto \begin{pmatrix} 1 + t_\delta + y_{11} & y_{12} \\ y_{21} & 1 + t_\delta - y_{11} \end{pmatrix}, \tag{22}$$

where $x_{12} := \hat{x}_{12} + [\phi(\gamma)]$, $y_{12} := \hat{y}_{12} + [\phi(\delta)]$ and $[\phi(\gamma)], [\phi(\delta)]$ denote the Teichmüller lifts of $\phi(\gamma)$ and $\phi(\delta)$ to $\mathcal{O}$. Let $\psi\colon G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ be a continuous character with $\psi\epsilon \equiv 1 \mod \varpi$. To find the quotient $R_\rho^{\square,\psi}$ of $R_\rho^\square$ that parametrizes lifts of ρ with determinant $\psi\epsilon$, we have to impose the conditions $\det(\rho^\square(\gamma)) = \psi\epsilon(\gamma)$ and $\det(\rho^\square(\delta)) = \psi\epsilon(\delta)$. Therefore, analogous to [19], we define the ideal

$$I := \big((1+t_\gamma)^2 - x_{11}^2 - x_{12}x_{21} - \psi\epsilon(\gamma), (1+t_\delta)^2 - y_{11}^2 - y_{12}y_{21} - \psi\epsilon(\delta)\big) \subset R_\rho^{\square,\psi}$$

and obtain

$$R_\rho^{\square,\psi} := \mathcal{O}[\![x_{11}, \hat{x}_{12}, x_{21}, t_\gamma, y_{11}, \hat{y}_{12}, y_{21}, t_\delta]\!]/I. \tag{23}$$

Let again $R^{\mathrm{ps},\psi}$ denote the universal deformation ring that parametrizes 2-dimensional pseudo-characters of $G_{\mathbb{Q}_p}$ with determinant $\psi\epsilon$ that lift the trace of the trivial 2-dimensional representation. Paškūnas has shown in [17, 9.12,9.13] that $R^{\mathrm{ps},\psi}$ is isomorphic to $\mathcal{O}[\![t_1, t_2, t_3]\!]$ and the universal pseudo-character is uniquely determined by

$$\begin{aligned} T\colon G_{\mathbb{Q}_p}(p) &\to \mathcal{O}[\![t_1, t_2, t_3]\!] \\ \gamma &\mapsto 2(1 + t_1) \\ \delta &\mapsto 2(1 + t_2) \\ \gamma\delta &\mapsto 2(1 + t_3) \\ \delta\gamma &\mapsto 2(1 + t_3). \end{aligned}$$

Since the trace $T^\square$ of $\rho^\square$ is a pseudo-deformation of $2 \cdot \mathbb{1}$ to $R_\rho^\square$, we get an induced morphism

$$\phi\colon \mathcal{O}[\![t_1, t_2, t_3]\!] \to R_\rho^{\square,\psi} \tag{24}$$

$$t_1 \mapsto T^\square(\gamma) = t_\gamma \tag{25}$$

$$t_2 \mapsto T^\square(\delta) = t_\delta \tag{26}$$

$$t_3 \mapsto T^\square(\gamma\delta) = T^\square(\delta\gamma) = (1 + t_\gamma)(1 + t_\delta) + \frac{1}{2}z - 1, \tag{27}$$

where $z = x_{12}y_{21} + 2x_{11}y_{11} + x_{21}y_{12}$.

Proposition 3.1. *The map* (24) *makes $R_\rho^{\square,\psi}$ into a flat $\mathcal{O}[\![t_1, t_2, t_3]\!]$-module.*

Proof. Let $\mathfrak{m}$ denote the maximal ideal of $\mathcal{O}[\![t_1,t_2,t_3]\!]$. Since $R_\rho^{\square,\psi}$ is a regular local ring modulo a regular sequence, it is Cohen-Macaulay. Since $\mathcal{O}[\![t_1,t_2,t_3]\!]$ is regular, the statement is equivalent to

$$\dim \mathcal{O}[\![t_1,t_2,t_3]\!] + \dim R_\rho^{\square,\psi}/\mathfrak{m}R_\rho^{\square,\psi} = \dim R_\rho^{\square,\psi},$$

see for example [6, Thm. 18.16]. But since $\dim \mathcal{O}[\![t_1,t_2,t_3]\!] = 4$, $\dim R_\rho^{\square,\psi} = 7$ by (23) and

$$R_\rho^{\square,\psi}/\mathfrak{m}R_\rho^{\square,\psi} \cong k[\![x_{11},\hat{x}_{12},x_{21},y_{11},\hat{y}_{12},y_{21}]\!]/(x_{11}^2+x_{12}x_{21},y_{11}^2+y_{12}y_{21},z)$$

by (24)-(27), it just remains to prove that

$$\dim k[\![x_{11},\hat{x}_{12},x_{21},y_{11},\hat{y}_{12},y_{21}]\!]/(x_{11}^2+x_{12}x_{21},y_{11}^2+y_{12}y_{21},z) = 3.$$

We distinguish 3 cases: If $x_{12} \in (R_\rho^{\square,\psi})^\times$, we obtain

$$R_\rho^{\square,\psi}/\mathfrak{m}R_\rho^{\square,\psi} \cong k[\![x_{11},\hat{x}_{12},y_{11},\hat{y}_{12}]\!]/(y_{11}^2 - y_{12}x_{12}^{-1}(2x_{11}y_{11} - y_{12}x_{11}^2x_{12}^{-1})),$$

so that $\{x_{11},\hat{x}_{12},\hat{y}_{12}\}$ is a system of parameters for $R_\rho^{\square,\psi}/\mathfrak{m}R_\rho^{\square,\psi}$. Analogously, if $y_{12} \in (R_\rho^{\square,\psi})^\times$, then $\{y_{11},\hat{y}_{12},\hat{x}_{12}\}$ is a system of parameters. So the only case left is when $x_{12},y_{12} \notin (R_\rho^{\square,\psi})^\times$. But it is easy to see that in this case $\{x_{12},y_{21},x_{21}-y_{12}\}$ is a system of parameters for $R_\rho^{\square,\psi}/\mathfrak{m}R_\rho^{\square,\psi}$, which finishes the proof. □

4. Locally algebraic vectors

In this section we want to adapt the strategy of [18, §4] to show that part c) of Paškūnas' Theorem 2.1 holds in the following setting. Let from now on $R := R_\rho^{\square,\psi}$, $\pi \cong (\mathrm{Ind}_P^G \mathbb{1} \otimes \omega^{-1})_{\mathrm{sm}}$, $\tilde{P}$ a projective envelope of $\pi^\vee$ in $\mathfrak{C}(\mathcal{O})$. Let $N := (R \oplus R)\hat{\otimes}_{\tilde{E},\eta}\tilde{P}$, where the $\tilde{E}$-module structure on $R \oplus R$ is induced by $\rho^\square$, as in (14).

In [17, §5.6] Paškūnas defines a covariant exact functor

$$\check{\mathbf{V}}\colon \mathfrak{C}(\mathcal{O}) \to \mathrm{Mod}_{G_{\mathbb{Q}_p}}^{\mathrm{pro}}(\mathcal{O}), \tag{28}$$

which is a modification of Colmez' Montreal functor, see [4]. It satisfies

$$\check{\mathbf{V}}((\mathrm{Ind}_P^G \chi_1 \otimes \chi_2\omega^{-1})^\vee) = \chi_1, \tag{29}$$

so that in our case

$$\check{\mathbf{V}}((\mathrm{Ind}_P^G \mathbb{1} \otimes \omega^{-1})^\vee) = \mathbb{1}. \tag{30}$$

For an admissible unitary L-Banach space representation Π of G with central character ψ and an open bounded G-invariant lattice Θ in Π, we define

$$\Theta^d := \mathrm{Hom}_{\mathcal{O}}(\Theta,\mathcal{O}), \tag{31}$$

which lies in $\mathfrak{C}(\mathcal{O})$. We also define

$$\check{\mathbf{V}}(\Pi) := \check{\mathbf{V}}(\Theta^d) \otimes_{\mathcal{O}} L, \tag{32}$$

which is independent of the choice of Θ.

Lemma 4.1. *N satisfies the following three properties (see* [18, §4]*):*

(N0) *$k\hat{\otimes}_R N$ is of finite length in $\mathfrak{C}(\mathcal{O})$ and is finitely generated over $\mathcal{O}[\![K]\!]$,*
(N1) $\mathrm{Hom}_{\mathrm{SL}_2(\mathbb{Q}_p)}(1, N^\vee) = 0$,
(N2) *$\check{\mathbf{V}}(N) \cong \rho^\square$ as $R[\![G_{\mathbb{Q}_p}]\!]$-modules.*

Proof. As we have already seen in the proof of Lemma 2.4, $k\hat{\otimes}_R N$ is an extension of $\pi^\vee$ by itself. Since π is absolutely irreducible and admissible we get $(N0)$. From [17, Lemma 5.53] we obtain that

$$\check{\mathbf{V}}(\rho^\square\hat{\otimes}_{\tilde{E},\eta}\tilde{P}) \cong \rho^\square\hat{\otimes}_{\tilde{E},\eta}\check{\mathbf{V}}(\tilde{P}), \tag{33}$$

and since $\check{\mathbf{V}}(\tilde{P})$ is a free $\tilde{E}$-module of rank 1 by [17, Cor. 5.55], also $(N2)$ holds. For $(N1)$ we notice that $\pi^{\mathrm{SL}_2(\mathbb{Q}_p)} = 0$. Since $\tilde{P}$ is a projective envelope of $\pi^\vee$, $\tilde{P}^\vee$ is an injective envelope of π. Since G acts on $(\tilde{P}^\vee)^{\mathrm{SL}_2(\mathbb{Q}_p)}$ via the determinant, we must have $(\tilde{P}^\vee)^{\mathrm{SL}_2(\mathbb{Q}_p)} = 0$. $\square$

Remark 4.2. *Let* m *be a* $R[1/p]$*-module of finite length. Then Lemma 4.1 implies that*

$$\check{\mathbf{V}}(\Pi(\mathrm{m})) \cong \mathrm{m} \otimes_R \check{\mathbf{V}}(N),$$

see [18, Rmk. 4.2, Lemma 4.3]*.*

The following Proposition is analogous to [18, 4.14] and shows that condition (i) of part c) of Paškūnas' Theorem 2.1 is satisfied in our setting.

Proposition 4.3. *Let* V *be either* $\sigma(\mathbf{w},\tau)$ *or* $\sigma^{\mathrm{cr}}(\mathbf{w},\tau)$*, let* $\mathfrak{p} \in \text{m-Spec}(R[1/p])$ *and* $\kappa(\mathfrak{p}) := R[1/p]/\mathfrak{p}$*. Then*

$$\dim_{\kappa(\mathfrak{p})} \mathrm{Hom}_K(V, \Pi(\kappa(\mathfrak{p}))) \leq 1.$$

If $V = \sigma(\mathbf{w},\tau)$*, then* $\dim_{\kappa(\mathfrak{p})} \mathrm{Hom}_K(V, \Pi(\kappa(\mathfrak{p}))) = 1$ *if and only if* $\rho^\square_{\mathfrak{p}}$ *is potentially semi-stable of type* $(\mathbf{w},\tau,\psi)$*.*
If $V = \sigma^{\mathrm{cr}}(\mathbf{w},\tau)$*, then* $\dim_{\kappa(\mathfrak{p})} \mathrm{Hom}_K(V, \Pi(\kappa(\mathfrak{p}))) = 1$ *if and only if* $\rho^\square_{\mathfrak{p}}$ *is potentially crystalline of type* $(\mathbf{w},\tau,\psi)$*.*

Proof. Let $F/\kappa(\mathfrak{p})$ be a finite extension. We have

$$\dim_{\kappa(\mathfrak{p})} \mathrm{Hom}_K(V, \Pi(\kappa(\mathfrak{p}))) = \dim_F \mathrm{Hom}_K(V \otimes_{\kappa(\mathfrak{p})} F, \Pi(\kappa(\mathfrak{p})) \otimes_{\kappa(\mathfrak{p})} F),$$

see for example [17, Lemma 5.1]. Thus by replacing $\kappa(\mathfrak{p})$ by a finite extension, we can assume without loss of generality that $\rho^\square_{\mathfrak{p}}$ is either absolutely irreducible or reducible. Since $\rho^\square_{\mathfrak{p}}$ is a lift of $\rho \cong \begin{pmatrix} \mathbb{1} & * \\ 0 & \mathbb{1} \end{pmatrix}$ and N satisfies (N0), (N1) and (N2) by Lemma 4.1, the only case that is not handled in [18, 4.14] is when $\rho^\square_{\mathfrak{p}}$ is an extension

$$0 \longrightarrow \chi_1 \longrightarrow \rho^\square_{\mathfrak{p}} \longrightarrow \chi_2 \longrightarrow 0\ ,$$

where χ_1, χ_2 are two characters that have the same Hodge-Tate weight. Such a representation is clearly never of any Hodge-type with distinct Hodge-Tate weights, so it is enough to show that $\dim_{\kappa(\mathfrak{p})} \mathrm{Hom}_K(V, \Pi(\kappa(\mathfrak{p})) = 0$. It follows, for example from [7, Prop. 3.4.2], that $\Pi(\kappa(\mathfrak{p}))$ is an extension of $\Pi_2 := (\mathrm{Ind}_P^G \chi_2 \otimes \chi_1\epsilon^{-1})_{\mathrm{cont}}$ by $\Pi_1 := (\mathrm{Ind}_P^G \chi_1 \otimes \chi_2\epsilon^{-1})_{\mathrm{cont}}$. If we denote the locally algebraic vectors of Π_i by Π_i^{alg}, then [17, Prop. 12.5] tells us that $\Pi_1^{\mathrm{alg}} = \Pi_2^{\mathrm{alg}} = 0$. But this implies that also $\Pi(\kappa(\mathfrak{p}))^{\mathrm{alg}} = 0$, and since V is a locally algebraic representation, we have

$$\mathrm{Hom}_K(V, \Pi(\kappa(\mathfrak{p}))) \cong \mathrm{Hom}_K(V, \Pi(\kappa(\mathfrak{p}))^{\mathrm{alg}}) = 0.$$

$\square$

To apply Paškūnas Theorem 2.1, we have to find a set of 'good' primes of $R[1/p]$ that is dense in $\mathrm{Supp}\, M(\Theta)$.

Definition 4.4. *Let $\Sigma \subset \operatorname{Supp} M(\Theta) \cap \text{m-Spec}(R[1/p])$ consist of all primes $\mathfrak{p}$ such that either $\Pi(\kappa(\mathfrak{p}))$ is reducible but non-split or $\Pi(\kappa(\mathfrak{p}))$ is absolutely irreducible and $\Pi(\kappa(\mathfrak{p}))^{\mathrm{alg}}$ is irreducible.*

Proposition 4.5. *Σ is dense in $\operatorname{Supp} M(\Theta)$.*

Proof. We already know that $M(\Theta)$ is Cohen-Macaulay by applying Theorem 2.3 to Paškūnas' Theorem 2.2. Since R is $\mathcal{O}$-torsion free and $R[1/p]$ is Jacobson, it is enough to show that the dimension of the complement of Σ in $\operatorname{Supp} M(\Theta) \cap \text{m-Spec}(R[1/p])$ is strictly smaller than the dimension of $R[1/p]$, which is equal to 4.

Let first $\mathfrak{p} \in \text{m-Spec}\, R[1/p]$ be such that $\Pi(\kappa(\mathfrak{p}))$ is absolutely irreducible and $\Pi(\kappa(\mathfrak{p}))^{\mathrm{alg}}$ is reducible. By a result of Colmez [4, Thm. VI.6.50] we know that in this case we have $\Pi(\kappa(\mathfrak{p}))^{\mathrm{alg}} \cong \pi \otimes W$, where W is an irreducible algebraic G-representation and $\pi \cong (\operatorname{Ind}_P^G \chi\,|.| \otimes \chi\,|.|^{-1})_{\mathrm{sm}}$ for some smooth character χ. In particular, if the Hodge-Tate weights are $\mathbf{w} = (a, b)$, we have $W \cong \operatorname{Sym}^{b-a-1} L^2 \otimes \det^a$. But since $\det \rho^{\square} = \psi\epsilon$, the product of the central characters of π and W must be ψ, so that we obtain $\chi^2 \epsilon^{a+b} = \psi$, which can only be satisfied by a finite number of characters χ. By a result of Berger-Breuil [1, Cor. 5.3.2], the universal unitary completion of Π^{alg} is topologically irreducible in this case and therefore isomorphic to Π. Hence there are only finitely many absolutely irreducible Banach space representations $\Pi(\kappa(\mathfrak{p}))$ such that $\Pi(\kappa(\mathfrak{p}))^{\mathrm{alg}}$ is reducible. Moreover, all of them give rise to a point $x_{\mathfrak{p}} \in \text{m-Spec}\, R^{\mathrm{ps},\psi}[1/p]$ by taking the trace of the associated $G_{\mathbb{Q}_p}$-representation $\rho^{\square}_{\mathfrak{p}} = \check{\mathbf{V}}(\Pi(\kappa(\mathfrak{p})))$. We already know from Proposition 3.1 that R is flat over $R^{\mathrm{ps},\psi}$ and $\dim R/\mathfrak{m}_{R^{\mathrm{ps},\psi}} R = 3$. Thus, above every prime $x_{\mathfrak{p}}$ there lies only an at most 3-dimensional family of primes $\mathfrak{p} \in \text{m-Spec}\, R[1/p]$ such that $\Pi(\kappa(\mathfrak{p}))$ is absolutely irreducible and $\Pi(\kappa(\mathfrak{p}))^{\mathrm{alg}}$ is reducible.

Let now $\mathfrak{p} \in \operatorname{Supp}\ M(\Theta)$ be such that, after extending scalars if necessary, $\rho^{\square}_{\mathfrak{p}}$ is split. Hence from Proposition 4.3 we know that $\rho^{\square}_{\mathfrak{p}}$ is potentially semi-stable of a Hodge type $(\mathbf{w}, \tau, \psi)$ determined by Θ, where $\mathbf{w} = (a, b)$, $\tau = \chi_1 \oplus \chi_2$ and $\chi_i \colon I_{\mathbb{Q}_p} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ have finite image. We claim that the closed subset of $\text{m-Spec}\, R^{\square}_{\rho}[1/p]$ consisting of points of the Hodge type above, is of dimension at most 3. As before, $\rho^{\square}$ factors through the maximal pro-p quotient $G_{\mathbb{Q}_p}(p)$ of $G_{\mathbb{Q}_p}$, which is a free pro-p group of rank 2, generated by a 'cyclotomic' generator γ and an 'unramified' generator δ. From our assumptions we see that for every representation $\rho^{\square}_{\mathfrak{p}}$ of the type above there are unramified characters μ_1, μ_2 such that up to conjugation

$$\rho^{\square}_{\mathfrak{p}} \sim \begin{pmatrix} \epsilon^b \chi_1 \mu_1 & 0 \\ 0 & \epsilon^a \chi_2 \mu_2 \end{pmatrix}. \tag{34}$$

As in (20), we have $R^{\square}_{\rho} \cong \mathcal{O}[\![x_{11}, \hat{x}_{12}, x_{21}, t_\gamma, y_{11}, \hat{y}_{12}, y_{21}, t_\delta]\!]$ with the universal framed deformation determined by

$$\rho^{\square}(\gamma) = \begin{pmatrix} 1 + t_\gamma + x_{11} & x_{12} \\ x_{21} & 1 + t_\gamma - x_{11} \end{pmatrix}, \tag{35}$$

$$\rho^{\square}(\delta) = \begin{pmatrix} 1 + t_\delta + y_{11} & y_{12} \\ y_{21} & 1 + t_\delta - y_{11} \end{pmatrix}. \tag{36}$$

Since the trace is invariant under conjugation, we get the following identities from (34)-(36):

$$I_1 \colon \epsilon^b \chi_1(\gamma) + \epsilon^a \chi_2(\gamma) = 2(1+t_\gamma), \tag{37}$$

$$I_2 \colon \mu_1(\delta) + \mu_2(\delta) = 2(1+t_\delta). \tag{38}$$

We get

$$R_\rho^\square/(I_1, I_2) \cong \mathcal{O}[\![x_{11}, \hat{x}_{12}, x_{21}, y_{11}, \hat{y}_{12}, y_{21}]\!].$$

Moreover, using (37),(38), we get the following relations for the determinants:

$$I_3 \colon x_{11}^2 + x_{12}x_{21} = \frac{1}{4}(\epsilon^a \chi_1(\gamma) - \epsilon^b \chi_2(\gamma))^2, \tag{39}$$

$$I_4 \colon y_{11}^2 + y_{12}y_{21} = \frac{1}{4}(\mu_1(\delta) - \mu_2(\delta))^2. \tag{40}$$

Since we assume the representation $\rho_{\mathfrak{p}}^\square$ to be split, it is, in particular, abelian. This can be summed up in the following relations:

$$I_5 : 0 = x_{12}y_{21} - x_{21}y_{12}, \tag{41}$$

$$I_6 : 0 = x_{12}y_{11} - x_{11}y_{12}, \tag{42}$$

$$I_7 : 0 = x_{21}y_{11} - x_{11}y_{21}. \tag{43}$$

We want to find a system of parameters $\mathcal{S}$ for $R_\rho^\square/(I_1, \dots, I_7)$ of length at most 4. If $x_{12} \in (R_\rho^\square)^\times$, it is easy to check that $\mathcal{S} = \{\varpi, \hat{x}_{12}, \hat{y}_{12}, x_{11}\}$ is such a system. If $y_{12} \in (R_\rho^\square)^\times$, we can take $\mathcal{S} = \{\varpi, \hat{x}_{12}, \hat{y}_{12}, y_{11}\}$. In the last case, when $x_{12}, y_{12} \in \mathfrak{m}_{R_\rho^\square}$, which means that $\hat{x}_{12} = x_{12}, \hat{y}_{12} = y_{12}$, we can take $\mathcal{S} = \{\varpi, x_{12}, y_{21}, x_{21} - y_{12}\}$. Thus $\dim R_\rho^\square/(I_1, \dots, I_7) \leq 4$ and since R is $\mathcal{O}$-torsion free, we obtain

$$\dim R_\rho^\square[1/p]/(I_1, \dots, I_7) \leq 3, \tag{44}$$

which proves the claim. □

The next step is to prove that part c)ii) of Paškūnas' Theorem 2.1 is satisfied for all $\mathfrak{p} \in \Sigma$. The following definition is analogous to [18, 4.17].

Definition 4.6. *Let* $\mathrm{Ban}_{G,\psi}^{\mathrm{adm}}(L)$ *be the category of admissible L-Banach space representations of G with central character ψ and let Π in* $\mathrm{Ban}_{G,\psi}^{\mathrm{adm}}(L)$ *be absolutely irreducible. Let $\mathcal{E}$ be the subspace of* $\mathrm{Ext}^1_{G,\psi}(\Pi,\Pi)$ *that is generated by extensions* $0 \to \Pi \to E \to \Pi \to 0$ *such that the resulting sequence of locally algebraic vectors* $0 \to \Pi^{\mathrm{alg}} \to E^{\mathrm{alg}} \to \Pi^{\mathrm{alg}} \to 0$ *is exact. We say that Π satisfies (RED), if* $\Pi^{\mathrm{alg}} \neq 0$ *and* $\dim \mathcal{E} \leq 1$.

The following lemma is a generalization of [18, Lemma 4.18] which avoids the assumption $\dim_L \mathrm{Hom}_G(\Pi, E) = 1$.

Lemma 4.7. *Let* $\Pi \in \mathrm{Ban}_{G,\psi}^{\mathrm{adm}}(L)$ *be absolutely irreducible. Let $n \geq 1$ and let*

$$0 \to \Pi \to E \to \Pi^{\oplus n} \to 0 \tag{45}$$

be an exact sequence in $\mathrm{Ban}_{G,\psi}^{\mathrm{adm}}(L)$. *Let V be either $\sigma(\mathbf{w}, \tau)$ or $\sigma^{\mathrm{cr}}(\mathbf{w}, \tau)$. If Π^{alg} is irreducible and Π satisfies (RED), then*

$$\dim_L \mathrm{Hom}_K(V, E) \leq \dim_L \mathrm{Hom}_G(\Pi, E) + 1.$$

Proof. Since Π^{alg} is irreducible, we obtain by [18, Lemma 4.10] and [11] that $\dim_L \mathrm{Hom}_K(V,\Pi) = 1$. We apply the functors $\mathrm{Hom}_G(\Pi, \_)$ and $\mathrm{Hom}_K(V, \_)$ to the sequence (45) to obtain the following diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_G(\Pi,\Pi) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_G(\Pi,E) \arrow[r] \arrow[d, hook] & \mathrm{Hom}_G(\Pi,\Pi^{\oplus n}) \arrow[r] \arrow[d, "\cong"] & \mathrm{Ext}^1_{G,\psi}(\Pi,\Pi) \arrow[d, "\alpha"] \\
0 \arrow[r] & \mathrm{Hom}_K(V,\Pi) \arrow[r] & \mathrm{Hom}_K(V,E) \arrow[r] & \mathrm{Hom}_K(V,\Pi^{\oplus n}) \arrow[r] & \mathrm{Ext}^1_{K,\psi}(V,\Pi),
\end{tikzcd}$$

where Ext^1 means the Yoneda extensions in $\mathrm{Ban}^{\mathrm{adm}}_{G,\psi}(L)$ resp. $\mathrm{Ban}^{\mathrm{adm}}_{K,\psi}(L)$. The diagram yields an exact sequence

$$0 \longrightarrow \mathrm{Hom}_G(\Pi,E) \longrightarrow \mathrm{Hom}_K(V,E) \longrightarrow \ker(\alpha),$$

and therefore

$$\dim_L \mathrm{Hom}_K(V,\Pi) \le \dim_L \mathrm{Hom}_G(\Pi,E) + \dim_L \ker(\alpha). \tag{46}$$

The irreducibility of Π^{alg} implies that $\ker(\alpha)$ is equal to the space $\mathcal{E}$ of Definition 4.6. Since we assume that Π satisfies (RED), we are done. □

Lemma 4.8. *Let $\mathfrak{p} \in \Sigma$. If $\mathrm{End}_{G_{\mathbb{Q}_p}}(\rho_{\mathfrak{p}}^{\square}) = \kappa(\mathfrak{p})$, then*

$$\dim_{\kappa(\mathfrak{p})} \mathrm{Hom}_{G_{\mathbb{Q}_p}}\left(\rho^{\square,\psi}[1/p]/\mathfrak{p}^2, \rho_{\mathfrak{p}}^{\square}\right) = 4.$$

If $\rho_{\mathfrak{p}}^{\square}$ is reducible such that there is a non-split exact sequence

$$0 \longrightarrow \delta_2 \longrightarrow \rho_{\mathfrak{p}}^{\square} \longrightarrow \delta_1 \longrightarrow 0,$$

with $\delta_1\delta_2^{-1} \neq \mathbb{1}, \epsilon^{\pm 1}$, then

$$\dim_{\kappa(\mathfrak{p})} \mathrm{Hom}_{G_{\mathbb{Q}_p}}\left(\rho^{\square,\psi}[1/p]/\mathfrak{p}^2, \delta_1\right) = 4.$$

Proof. We start with the exact sequence

$$0 \longrightarrow \mathfrak{p}/\mathfrak{p}^2 \longrightarrow R[1/p]/\mathfrak{p}^2 \longrightarrow \kappa(\mathfrak{p}) \longrightarrow 0. \tag{47}$$

Tensoring (47) with $\rho^{\square,\psi}[1/p]$ over $R[1/p]$ and applying the functor $\mathrm{Hom}_{G_{\mathbb{Q}_p}}(\_, \rho_{\mathfrak{p}}^{\square})$ yields the exact sequence

$$\mathrm{Hom}_{G_{\mathbb{Q}_p}}\left(\rho^{\square,\psi}[1/p]/\mathfrak{p}^2, \rho_{\mathfrak{p}}^{\square}\right) \longrightarrow \mathrm{Hom}_{G_{\mathbb{Q}_p}}\left(\mathfrak{p}/\mathfrak{p}^2 \otimes_{R[1/p]} \rho^{\square,\psi}[1/p], \rho_{\mathfrak{p}}^{\square}\right) \xrightarrow{\partial} \mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\rho_{\mathfrak{p}}^{\square}, \rho_{\mathfrak{p}}^{\square}).$$

Since we assume $\mathrm{End}_{G_{\mathbb{Q}_p}}(\rho_{\mathfrak{p}}^{\square}) = \kappa(\mathfrak{p})$, we have

$$\dim_{\kappa(\mathfrak{p})} \mathrm{Hom}_{G_{\mathbb{Q}_p}}\left(\rho^{\square,\psi}[1/p]/\mathfrak{p}^2, \rho_{\mathfrak{p}}^{\square}\right) = 1 + \dim_{\kappa(\mathfrak{p})} \ker(\partial).$$

We see that

$$\ker(\partial) = \{\phi\colon R \to \kappa(\mathfrak{p})[\epsilon] \mid \rho^{\square,\psi}[1/p] \otimes_{R[1/p],\phi} \kappa(\mathfrak{p})[\epsilon] \cong \rho_{\mathfrak{p}}^{\square} \oplus \rho_{\mathfrak{p}}^{\square} \text{ as } G_{\mathbb{Q}_p}\text{-reps.}\}.$$

Let $\phi \in \ker(\partial)$ and let $\hat{R}$ be the $\mathfrak{p}$-adic completion of $R[1/p]$. Then we can identify $\hat{R}$ with the universal framed deformation ring that parametrizes lifts of $\rho_{\mathfrak{p}}^{\square}$ with determinant $\psi\epsilon$ [14, (2.3.5)] and ϕ induces a morphism $\hat{R} \to \kappa(\mathfrak{p})[\epsilon]$. If we denote the adjoint representation of $\rho_{\mathfrak{p}}^{\square}$ by $\mathrm{ad}\,\rho_{\mathfrak{p}}^{\square}$, there is a natural isomorphism

$$\mathrm{Hom}_{\kappa(\mathfrak{p})-\mathrm{Alg}}(\hat{R}, \kappa(\mathfrak{p})[\epsilon]) \cong Z^{1,\psi}(G_{\mathbb{Q}_p}, \mathrm{ad}\,\rho_{\mathfrak{p}}^{\square}), \tag{48}$$

where $Z^{1,\psi}(G_{\mathbb{Q}_p}, \operatorname{ad}\rho_{\mathfrak{p}}^{\square})$ denotes the space of cocyles that correspond to deformations with determinant $\psi\epsilon$. Here the morphism $\phi \in \operatorname{Hom}_{\kappa(\mathfrak{p})-\mathrm{Alg}}(\hat{R}, \kappa(\mathfrak{p})[\epsilon])$ that corresponds to a lift $\tilde{\rho}$ of $\rho_{\mathfrak{p}}^{\square}$ is mapped to the cocycle Φ that appears in the equality

$$\tilde{\rho}(g) = \rho_{\mathfrak{p}}^{\square}(g)(1 + \Phi(g)\epsilon).$$

Since $\operatorname{Ext}^1_{G_{\mathbb{Q}_p}}(\rho_{\mathfrak{p}}^{\square}, \rho_{\mathfrak{p}}^{\square}) \cong H^1(G_{\mathbb{Q}_p}, \operatorname{ad}\rho_{\mathfrak{p}}^{\square})$, we obtain that

$$\ker(\partial) = \{\phi \in Z^{1,\psi}(G_{\mathbb{Q}_p}, \operatorname{ad}\rho_{\mathfrak{p}}^{\square}) \mid \phi = 0 \text{ in } H^1(G_{\mathbb{Q}_p}, \operatorname{ad}\rho_{\mathfrak{p}}^{\square})\}.$$

Hence $\ker(\partial) \cong B^{1,\psi}(G_{\mathbb{Q}_p}, \operatorname{ad}\rho_{\mathfrak{p}}^{\square})$, the corresponding coboundaries. There is an exact sequence

(49)

$$0 \longrightarrow (\operatorname{ad}\rho_{\mathfrak{p}}^{\square})^{G_{\mathbb{Q}_p}} \longrightarrow \operatorname{ad}\rho_{\mathfrak{p}}^{\square} \longrightarrow Z^1(G_{\mathbb{Q}_p}, \operatorname{ad}\rho_{\mathfrak{p}}^{\square}) \longrightarrow H^1(G_{\mathbb{Q}_p}, \operatorname{ad}\rho_{\mathfrak{p}}^{\square}) \longrightarrow 0,$$

where the middle map is given by $x \mapsto (g \mapsto gx - x)$. Since by assumption $\operatorname{End}_{G_{\mathbb{Q}_p}}(\rho_{\mathfrak{p}}^{\square}) = \kappa(\mathfrak{p})$, we see from (49) that

$$\dim_{\kappa(\mathfrak{p})} B^{1,\psi}(G_{\mathbb{Q}_p}, \operatorname{ad}\rho_{\mathfrak{p}}^{\square}) = 3.$$

Let now $\rho_{\mathfrak{p}}^{\square}$ be reducible such that there is a non-split exact sequence

$$0 \longrightarrow \delta_2 \longrightarrow \rho_{\mathfrak{p}}^{\square} \longrightarrow \delta_1 \longrightarrow 0\ ,$$

with $\delta_1 \neq \delta_2$. Tensoring (47) with $\rho^{\square,\psi}[1/p]$ and applying the functor $\operatorname{Hom}_{G_{\mathbb{Q}_p}}(\_, \delta_1)$ gives us an exact sequence

(50)

$$\operatorname{Hom}_{G_{\mathbb{Q}_p}}\left(\rho^{\square}[1/p]/\mathfrak{p}^2, \delta_1\right) \longrightarrow \operatorname{Hom}_{G_{\mathbb{Q}_p}}\left(\mathfrak{p}/\mathfrak{p}^2 \otimes_{R[1/p]} \rho^{\square,\psi}[1/p], \delta_1\right) \xrightarrow{\partial'} \operatorname{Ext}^1_{G_{\mathbb{Q}_p}}(\rho_{\mathfrak{p}}^{\square}, \delta_1).$$

Since $\delta_1 \neq \delta_2$ we have $\dim_{\kappa(\mathfrak{p})} \operatorname{Hom}(\rho_{\mathfrak{p}}^{\square}, \delta_1) = 1$ and therefore

$$\dim_{\kappa(\mathfrak{p})} \operatorname{Hom}_{G_{\mathbb{Q}_p}}(\rho^{\square,\psi}[1/p]/\mathfrak{p}^2, \delta_1) = 1 + \dim_{\kappa(\mathfrak{p})} \ker(\partial'). \tag{51}$$

Moreover, we obtain isomorphisms

$$\operatorname{Hom}_{G_{\mathbb{Q}_p}}\left(\mathfrak{p}/\mathfrak{p}^2 \otimes_{R[1/p]} \rho^{\square,\psi}[1/p], \delta_1\right) \cong (\mathfrak{p}/\mathfrak{p}^2)^* \cong \operatorname{Hom}_{\kappa(\mathfrak{p})-\mathrm{Alg}}(\hat{R}^{\square}, \kappa(\mathfrak{p})[\epsilon]) \tag{52}$$

$$\cong Z^{1,\psi}(G_{\mathbb{Q}_p}, \operatorname{ad}\rho_{\mathfrak{p}}^{\square}). \tag{53}$$

From (49) we obtain again that the kernel of the natural surjection

$$Z^1(G_{\mathbb{Q}_p}, \operatorname{ad}\rho_{\mathfrak{p}}^{\square}) \twoheadrightarrow H^1(G_{\mathbb{Q}_p}, \operatorname{ad}\rho_{\mathfrak{p}}^{\square}) \cong \operatorname{Ext}^1_{G_{\mathbb{Q}_p}}(\rho_{\mathfrak{p}}^{\square}, \rho_{\mathfrak{p}}^{\square}) \tag{54}$$

is 3-dimensional. Hence (50), and (52)-(54) give us an induced map

$$\bar{\partial}' \colon \operatorname{Ext}^{1,\psi}_{G_{\mathbb{Q}_p}}(\rho_{\mathfrak{p}}^{\square}, \rho_{\mathfrak{p}}^{\square}) \to \operatorname{Ext}^1_{G_{\mathbb{Q}_p}}(\rho_{\mathfrak{p}}^{\square}, \delta_1)$$

with

$$\dim_{\kappa(\mathfrak{p})} \ker(\partial') = 3 + \dim_{\kappa(\mathfrak{p})} \ker(\bar{\partial}'). \tag{55}$$

Since $\operatorname{End}_{G_{\mathbb{Q}_p}}(\rho_{\mathfrak{p}}^{\square}) = \kappa(\mathfrak{p})$, also the universal (non-framed) deformation ring $\hat{R}^{\mathrm{un}}$ of $\rho_{\mathfrak{p}}^{\square}$ exists, that parametrizes deformations of $\rho_{\mathfrak{p}}^{\square}$ with determinant $\psi\epsilon$. Therefore we can use the same argument as in the proof of [18, Lemma 4.20.], with $\rho_{\mathfrak{p}}^{\square}$ instead of

$\rho_{\mathfrak{p}}^{\mathrm{un}}$, to obtain that $\ker(\bar{\partial}') = \mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\delta_1, \delta_2)/\mathcal{L}$, where $\mathcal{L}$ is the subspace corresponding to $\rho_{\mathfrak{p}}^{\square}$. Since we assume $\delta_1\delta_2^{-1} \neq \mathbb{1}, \epsilon^{\pm 1}$, we have $\dim_{\kappa(\mathfrak{p})} \mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\delta_1, \delta_2) = 1$ and obtain from (51) and (55) that

$$\dim_{\kappa(\mathfrak{p})} \mathrm{Hom}_{G_{\mathbb{Q}_p}}(\rho^{\square,\psi}[1/p]/\mathfrak{p}^2, \delta_1) = 4.$$

□

Corollary 4.9. *Let V be either $\sigma(\mathbf{w}, \tau)$ or $\sigma^{\mathrm{cr}}(\mathbf{w}, \tau)$ and let Θ be a K-invariant $\mathcal{O}$-lattice in V. Then for all $\mathfrak{p} \in \Sigma$,*

$$\dim_{\kappa(\mathfrak{p})} \mathrm{Hom}_K\left(V, \Pi(R[1/p]/\mathfrak{p}^2)\right) \leq 5.$$

Proof. Let $\mathfrak{p} \in \Sigma$. If $\Pi(\kappa(\mathfrak{p}))$ is absolutely irreducible, then also $\Pi(\kappa(\mathfrak{p}))^{\mathrm{alg}}$ is irreducible. By the same argument as in [18, Thm. 4.19] that uses a result of Dospinescu [5, Thm. 1.4, Prop. 1.3], we obtain that $\Pi(\kappa(\mathfrak{p}))$ satisfies (RED). From the exact sequence

$$0 \longrightarrow \mathfrak{p}/\mathfrak{p}^2 \longrightarrow R[1/p]/\mathfrak{p}^2 \longrightarrow \kappa(\mathfrak{p}) \longrightarrow 0 \tag{56}$$

we obtain an exact sequence of unitary Banach space representations

$$0 \longrightarrow \Pi(\kappa(\mathfrak{p})) \longrightarrow \Pi(R[1/p]/\mathfrak{p}^2) \longrightarrow \Pi(\kappa(\mathfrak{p}))^{\oplus n} \longrightarrow 0\ . \tag{57}$$

Thus we can apply Lemma 4.7 and obtain

$$\dim_{\kappa(\mathfrak{p})} \mathrm{Hom}_K\left(V, \Pi(R[1/p]/\mathfrak{p}^2)\right) \leq \dim_{\kappa(\mathfrak{p})} \mathrm{Hom}_G\left(\Pi(\kappa(\mathfrak{p})), \Pi(R[1/p]/\mathfrak{p}^2)\right) + 1.$$

The contravariant functor $\check{\mathbf{V}}$ induces an injection

(58)

$$\mathrm{Hom}_G\left(\Pi(\kappa(\mathfrak{p})), \Pi(R[1/p]/\mathfrak{p}^2)\right) \hookrightarrow \mathrm{Hom}_{G_{\mathbb{Q}_p}}\left(\check{\mathbf{V}}(\Pi(R[1/p]/\mathfrak{p}^2)), \check{\mathbf{V}}(\Pi(\kappa(\mathfrak{p})))\right).$$

Since the target is isomorphic to $\mathrm{Hom}_{G_{\mathbb{Q}_p}}\left(\rho^{\square}[1/p]/\mathfrak{p}^2, \rho_{\mathfrak{p}}^{\square}\right)$ by Remark 4.2, the claim follows from Lemma 4.8.

Let now $\Pi(\kappa(\mathfrak{p}))$ be reducible. Then, as in the proof of Proposition 4.3, it comes from an exact sequence

$$0 \longrightarrow \delta_2 \longrightarrow \rho_{\mathfrak{p}}^{\square} \longrightarrow \delta_1 \longrightarrow 0\ , \tag{59}$$

with $\delta_1\delta_2^{-1} \neq \mathbb{1}, \epsilon^{\pm 1}$. We obtain an associated exact sequence

$$0 \longrightarrow \Pi_1 \longrightarrow \Pi(\kappa(\mathfrak{p})) \longrightarrow \Pi_2 \longrightarrow 0\ , \tag{60}$$

where $\check{\mathbf{V}}(\Pi_i) = \delta_i$, $\Pi(\kappa(\mathfrak{p}))^{\mathrm{alg}} = \Pi_1^{\mathrm{alg}}$ and (60) splits if and only if (59) splits, see [7, Prop. 3.4.2]. Furthermore, Π_1 is irreducible and, again as in [18, Thm. 4.19], Π_1 satisfies (RED). If we let E be the closure of the locally algebraic vectors in $\Pi(R[1/p]/\mathfrak{p}^2)$, we obtain an isomorphism

$$\mathrm{Hom}_K(V, \Pi(R[1/p]/\mathfrak{p}^2)) \cong \mathrm{Hom}_K(V, E).$$

Now (57) gives rise to another exact sequences of unitary Banach space representations

$$0 \longrightarrow \Pi_1 \longrightarrow E \longrightarrow \Pi_1^{\oplus m} \longrightarrow 0. \tag{61}$$

Since Π_1 satisfies (RED), we can apply Lemma 4.7 to obtain

$$\dim_{\kappa(\mathfrak{p})} \mathrm{Hom}_K(V, E) \leq \dim_{\kappa(\mathfrak{p})} \mathrm{Hom}_G(\Pi_1, E) + 1.$$

Because of the inclusions

$$\mathrm{Hom}_G(\Pi_1, E) \hookrightarrow \mathrm{Hom}_G(\Pi_1, \Pi(R[1/p]/\mathfrak{p}^2)) \hookrightarrow \mathrm{Hom}_{G_{\mathbb{Q}_p}}(\rho^{\square,\psi}[1/p]/\mathfrak{p}^2, \delta_1)$$

we obtain

$$\dim_{\kappa(\mathfrak{p})} \mathrm{Hom}_K(V, E) \le \dim_{\kappa(\mathfrak{p})} \mathrm{Hom}_{G_{\mathbb{Q}_p}}(\rho^{\square,\psi}[1/p], \delta_1) + 1.$$

But by Lemma 4.8 $\dim_{\kappa(\mathfrak{p})} \mathrm{Hom}_{G_{\mathbb{Q}_p}}(\rho^{\square,\psi}[1/p], \delta_1) = 4$, and we are done. $\square$

Now we are finally able to prove the main theorem. We let again $\chi\colon G_{\mathbb{Q}_p} \to k^\times$ be a continuous character and let

$$\begin{aligned} \rho\colon G_{\mathbb{Q}_p} &\to \mathrm{GL}_2(k) \\ g &\mapsto \begin{pmatrix} \chi(g) & \phi(g) \\ 0 & \chi(g) \end{pmatrix}. \end{aligned}$$

Theorem 4.10. *Let $p > 2$ and let $(\mathbf{w}, \tau, \psi)$ be a Hodge type. There exists a reduced $\mathcal{O}$-torsion free quotient $R^\square_\rho(\mathbf{w}, \tau, \psi)$ (resp. $R^{\square,\mathrm{cr}}_\rho(\mathbf{w}, \tau, \psi)$) of $R^\square_\rho$ such that for all $\mathfrak{p} \in$ m-Spec$(R^\square_\rho[1/p])$, $\mathfrak{p}$ is an element of* m-Spec $\big(R^\square_\rho(\mathbf{w}, \tau, \psi)[1/p]\big)$ *(resp.* m-Spec $\big(R^{\square,\mathrm{cr}}_\rho(\mathbf{w}, \tau, \psi)[1/p]\big)$*) if and only if $\rho^\square_\mathfrak{p}$ is potentially semi-stable (resp. potentially crystalline) of p-adic Hodge type $(\mathbf{w}, \tau, \psi)$. If $R^\square_\rho(\mathbf{w}, \tau, \psi)$ (resp. $R^{\square,\mathrm{cr}}_\rho(\mathbf{w}, \tau, \psi)$) is non-zero, then it has Krull dimension* 5.

Furthermore, there exists a four-dimensional cycle $z(\rho) := z_4(M(\lambda))$ of $R^\square_\rho$, where $\lambda := \mathrm{Sym}^{p-2} k^2 \otimes \chi \circ \det$, such that there are equalities of four-dimensional cycles

$$z_4\big(R^\square_\rho(\mathbf{w}, \tau, \psi)/(\varpi)\big) = m_\lambda(\mathbf{w}, \tau) z(\rho), \tag{62}$$

$$z_4\big(R^{\square,\mathrm{cr}}_\rho(\mathbf{w}, \tau, \psi)/(\varpi)\big) = m^{\mathrm{cr}}_\lambda(\mathbf{w}, \tau) z(\rho). \tag{63}$$

Proof. We set $N := (R \oplus R)\hat{\otimes}_{\tilde{E},\eta} \tilde{P}$, as in Theorem 2.3. Hence, if we let $\mathfrak{a}$ be the R-annihilator of $M(\Theta)$, we obtain from Proposition 4.3 and [18, Prop. 2.22], analogous to [18, Thm. 4.15], that for any K-invariant $\mathcal{O}$-lattice Θ in $\sigma(\mathbf{w}, \tau)$ (resp. $\sigma^{\mathrm{cr}}(\mathbf{w}, \tau)$) $R/\sqrt{\mathfrak{a}} \cong R^\square_\rho(\mathbf{w}, \tau, \psi)$ (resp. $R/\sqrt{\mathfrak{a}} \cong R^{\square,\mathrm{cr}}_\rho(\mathbf{w}, \tau, \psi)$). Since R is Cohen-Macaulay, Proposition 3.1 shows that we can apply Theorem 2.3 in our situation. Let Z be the center of G and let $Z_1 := I_1 \cap Z$. Since $p > 2$, there exists a continuous character $\sqrt{\psi}\colon Z_1 \to \mathcal{O}^\times$ with $\sqrt{\psi}^2 = \psi$. Twisting by $\sqrt{\psi} \circ \det$ induces an equivalence of categories between $\mathrm{Mod}^{\mathrm{pro}}_{I_1,\psi}(\mathcal{O})$ and $\mathrm{Mod}^{\mathrm{pro}}_{I_1/Z_1}(\mathcal{O})$. In this way we can use Theorem 2.3 to show the inequality of Theorem 2.2 for the setup $G = \mathrm{GL}_2(\mathbb{Q}_p)/Z_1$, $K = \mathrm{GL}_2(\mathbb{Z}_p)/Z_1$ and $P = I_1/Z_1$. Hence we obtain from Paškūnas' Theorem 2.2 that the conditions a) and b) of the criterion 2.1 for the Breuil-Mézard conjecture are satisfied. We let Σ be as in Definition 4.4. Since we know from Corollary 4.5 that Σ is dense in $\mathrm{Supp}\, M(\Theta)$, condition (i) of part c) follows from Proposition 4.3. As already remarked in the proof of Proposition 4.5, we have $\dim M(\Theta) = 5$. Thus condition (ii) of part c) is the statement of Corollary 4.9. Hence Theorem 2.1 says that there are equations of the form

$$z_4\big(R^\square_\rho(\mathbf{w}, \tau, \psi)/(\varpi)\big) = \sum_\sigma m_\sigma(\mathbf{w}, \tau) z_4(M(\sigma)), \tag{64}$$

$$z_4\big(R^{\square,\mathrm{cr}}_\rho(\mathbf{w}, \tau, \psi)/(\varpi)\big) = \sum_\sigma m^{\mathrm{cr}}_\sigma(\mathbf{w}, \tau) z_4(M(\sigma)). \tag{65}$$

where the sum runs over all isomorphism classes of smooth irreducible K-representations over k. By Lemma 2.6 we have that $M(\sigma) \neq 0$ if and only if σ lies in the K-socle of π. We let K_1 denote the kernel of the projection $K \to \mathrm{GL}_2(\mathbb{F}_p)$ and let B denote the subgroup of upper triangular matrices of $\mathrm{GL}_2(\mathbb{F}_p)$. Let now σ be a smooth irreducible K-representation in the K-socle of π. Since K_1 is a normal pro-p subgroup of K, we must have $\sigma^{K_1} \neq 0$ and thus $\sigma = \sigma^{K_1}$. There are isomorphisms of K-representations

$$\pi^{K_1} \cong \left((\mathrm{Ind}_{P\cap K}^{K} \mathbb{1} \otimes \omega^{-1})_{\mathrm{sm}}\right)^{K_1} \cong \mathrm{Ind}_B^{\mathrm{GL}_2(\mathbb{F}_p)} \mathbb{1} \otimes \omega^{-1}, \tag{66}$$

and it follows from [16, Lemma 4.1.3] that the K-socle of π^{K_1} is isomorphic to $\mathrm{Sym}^{p-2} k^2 \otimes \chi \circ \det$, in particular, it is irreducible. Therefore there is only a single cycle $z(\rho) = z_4(M(\mathrm{Sym}^{p-2} k^2 \otimes \chi \circ \det))$ on the right hand side of (64) and (65). $\square$

Remark 4.11. *If $\tau = \mathbb{1} \oplus \mathbb{1}$ and $\mathbf{w} = (a, b)$ with $b - a \leq p - 1$, then the right hand side of* (63) *is non-trivial if and only if $b - a = p - 1$, in which case the Hilbert-Samuel multiplicity of $z(\rho)$ is equal to the multiplicity of $R_\rho^{\square,\mathrm{cr}}(\mathbf{w}, \mathbb{1} \oplus \mathbb{1}, \psi)/(\varpi)$. In* [19]*, we computed that this multiplicity is* 1 *if $\rho \otimes \chi^{-1}$ is ramified,* 2 *if $\rho \otimes \chi^{-1}$ is unramified and indecomposable, and* 4 *if $\rho \otimes \chi^{-1}$ is split.*

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