HARDY-STEIN IDENTITIES AND SQUARE FUNCTIONS FOR SEMIGROUPS

RODRIGO BAÑUELOS, KRZYSZTOF BOGDAN, AND TOMASZ LUKS

ABSTRACT. We prove a Hardy-Stein type identity for the semigroups of symmetric, pure-jump Lévy processes. Combined with the Burkholder-Gundy inequalities, it gives the L^p two-way boundedness, for $1 < p < \infty$, of the corresponding Littlewood-Paley square function. The square function yields a direct proof of the L^p boundedness of Fourier multipliers obtained by transforms of martingales of Lévy processes.

1. INTRODUCTION

Littlewood and Paley introduced the square functions to harmonic analysis in [21]. Many applications and intrinsic beauty of the subject brought about enormous literature, which would be impossible to review here in a reasonably complete way. For results on classical square functions we refer the reader to Zygmund [33] and Stein [30], [31]. In particular, [30] uses harmonic functions on the upper half-space and the related Gaussian and Poisson semigroups to develop Littlewood-Paley theory for the L^p spaces. In [31] Stein employs more general symmetric semigroups in a similar manner. He uses square functions defined in terms of the generalized Poisson semigroup, that is the original semigroup subordinated in the sense of Bochner by the $1/2$-stable subordinator [27]. He also proposes square functions defined in terms of time derivatives of the original semigroup. Similarly, Meyer [23] employs the generalized Poisson semigroup, and Varopoulos in [32] uses time derivatives of the original semigroup.

It may be helpful to note that Littlewood-Paley theory and square functions (including the Lusin area integral) are auxiliary for studying L^p and other function spaces, Fourier multipliers theorems, partial differential equations and boundary behavior of functions. This explains, in part, the large variety of square functions used in literature toward different goals. At the same time the multipliers and PDEs manageable by a square function depend on the semigroup employed in its definition, which motivates the study of square functions specifically related to a given semigroup. We also note that square functions usually combine the *carré du champ* corresponding to the semigroup [23] and integration against the semigroup or its Poisson subordination.

It is well-known that the probabilistic counterpart of square functions is the quadratic variation of the martingales. Similarly, the Littlewood-Paley inequalities for square functions may be considered analytic analogues of the Burkholder-Davis-Gundy inequalities, which relate the L^p integrability of the martingale and its maximal function to the L^p integrability of its quadratic variation. The probabilistic connections to Littlewood-Paley theory have been explored by countless authors for many years. For a highly incomplete list of results, we refer the reader to Stein [31], Meyer [23], [22], [24], Varopoulos [32], Bañuelos [3], Bañuelos and Moore [8], Bennett [9], Bouleau and Lamberton [14], Karli [16], Kim and Kim [18], Krylov [20], and the many references given in these papers.

In the analytic, as opposed to probabilistic, realm the L^p boundedness of the classical Littlewood-Paley square functions can be obtained from the Calderón-Zygmund theory of singular integrals, as done in Stein

Date: August 23, 2021.
2010 *Mathematics Subject Classification.* Primary 42B25; Secondary 60J75, 42B15.
Key words and phrases. square function, Littlewood-Paley theory, Hardy-Stein identity, nonlocal operator, Fourier multiplier.
R. Bañuelos was supported in part by NSF grant #1403417-DMS, K. Bogdan was supported in part by NCN grant #2012/07/B/ST1/03356, T. Luks was supported in part by Agence Nationale de la Recherche grant ANR-09-BLAN-0084-01.

[30, pp. 82-84]. The singular integral approach can also be used for a wide range of Littlewood-Paley square functions constructed from volume preserving dilations of approximations to the identity. For this (well-known) approach, we refer the reader to [8]. An alternative beautiful way to prove L^p boundedness in the classical case for $1 < p < 2$ is via the so called Hardy-Stein identities. This approach is employed in Stein [30, pp. 86-88] and, outside of some standard maximal function estimates that hold in very general settings when the Hardy-Littlewood maximal function is replaced by the semigroup maximal function, it is based on the fact that the Laplacian satisfies a special case of what in diffusion theory is often called the chain rule. That is, $\Delta u^p = p(p-1)u^{p-2}|\nabla u|^2 + pu^{p-1}\Delta u$ for $1 < p < \infty$ and suitable functions u; see [30, Lemma 1, p. 86]. Stein's proof can be easily adapted to Markovian semigroups whose generators satisfy the chain rule as discussed in [2], Formula (10). It is also explained in [2] that such chain rule requires the process to have continuous trajectories, thus ruling out the nonlocal operators.

The purpose of the present paper is to prove the two-way L^p bounds for square functions of Markovian semigroups generated by nonlocal operators. Indeed, we define an intrinsic square function $\tilde{G}(f)$ for such semigroups and prove the upper and lower boundedness in L^p. The square function thus characterizes the L^p spaces for $1 < p < \infty$. We like to note a certain asymmetry in the definition of $\tilde{G}(f)$ and the fact that the more natural and symmetric square function $G(f)$ fails to be bounded in L^p for $1 < p < 2$.

Our technique is based on new Hardy-Stein identities for the considered semigroups (which replace the chain rule for $1 < p \leq 2$) and on Burkholder-Gundy inequalities for suitable martingales driven by the stochastic processes corresponding to those semigroups (these are important for $2 \leq p < \infty$). Once the upper bound inequalities are obtained, the lower bound inequalities may be proved by polarization and duality. Our Hardy-Stein identities are inspired by those given in [12] for harmonic and conditionally harmonic functions of the Laplacian and the fractional Laplacian, but the present setting is distinctively different.

The paper may be considered as a streamlined approach from semigroups to Hardy-Stain identities to square functions to multiplier theorems. To avoid certain technical problems our present results are restricted to the (convolution) semigroups of symmetric, pure-jump Lévy processes satisfying the Hartman-Wintner condition. The results should hold in much more general setting, but the scope of the extension is unclear at this moment. As mentioned, we give applications to the L^p-boundedness of Fourier multipliers. Namely, we recover the results of [1], [5], [6], where Fourier multipliers were constructed by tampering with jumps of Lévy processes with symmetric Lévy measure. Our present approach to Fourier multipliers is simpler than in those papers because we do not use Burkholder's inequalities for martingale transforms. While the approach does not yield sharp constants in L^p comparisons, it should be of interest in applications to multipliers which do not necessarily arise from martingale transforms.

We note in passing that the approach to Fourier multiplers via square functions has been used in various settings to prove bounds for operators that arise from martingale transforms, such as Riesz transforms and other singular integrals. For some recent application of this idea, see [25, Lemma 1] and [17, proof of Theorem 1.1], where different Littlewood-Paley square functions are employed to prove L^p–boundedness for operators arising from martingale transforms. We also note that the constants in our L^p estimates of the square functions and Fourier multipliers depend only on $p \in (1, \infty)$ and in particular they do not depend on the dimension of $\mathbb{R}^d$. It is interesting to note that our applications, unlike those presented in Stein [30] for his proof of the Hörmander multiplier theorem, do not depend on pointwise comparisons of Littlewood-Paley square functions before and after applying the multiplier. Instead, it suffices to have an integral control of the quantities involved, because we can use the isometry property of the square function on L^2 and the usual pairing to define and study the multiplier. In particular, in applications we only use two square functions $\tilde{G}(f)$ and $G(f)$, rather than a whole family of square functions.

The structure of the paper is as follows. In §2 we introduce the considered semigroups and we recall their basic properties. In §3 we prove the Hardy-Stein identities. In §4 we define the square functions and give their upper and lower bounds in L^p. In §5 we present applications to Fourier multipliers.

2. Preliminaries

We use ":=" to emphasize definitions, e.g., $a \wedge b := \min\{a,b\}$ and $a \vee b := \max\{a,b\}$. For two nonnegative functions f and g on the same domain we write $f \approx g$ if there is a positive number $c \geq 1$ such that $c^{-1}\, g \leq f \leq c\, g$ (uniformly for all arguments involved). All the sets and functions considered in this work are assumed real-valued and Borel measurable, unless stated otherwise.

We consider the Euclidean space $\mathbb{R}^d$ with dimension $d \geq 1$ and the d-dimensional Lebesgue measure dx. The Euclidean scalar product and norm on $\mathbb{R}^d$ are denoted by $x \cdot y$ and $|x|$. For every $p \in [1, \infty)$ we let $L^p := L^p(\mathbb{R}^d, dx)$ be the collection of all the (real-valued Borel-measurable) functions f on $\mathbb{R}^d$ with finite norm

$$\|f\|_p := \left[\int_{\mathbb{R}^d} |f(x)|^p dx\right]^{1/p}.$$

As usual, $\|f\|_\infty$ denotes the essential supremum of $|f|$. For $p = 2$ we use the usual scalar product on L^2,

$$\langle f, g\rangle := \int_{\mathbb{R}^d} f(x)g(x)dx.$$

Let ν be a measure on $\mathbb{R}^d$ such that $\nu(\{0\}) = 0$ and

$$\int_{\mathbb{R}^d} (1 \wedge |y|^2)\nu(dy) < \infty. \tag{LM}$$

In short: ν is a Lévy measure. We assume that ν is symmetric: for all (Borel) sets $B \subset \mathbb{R}^d$,

$$\nu(B) = \nu(-B). \tag{S}$$

For later convenience we note that given of nonnegativity or absolute integrability of function k,

$$\int\int k(x,y)\nu(dy)dx = \int\int k(x,-y)\nu(dy)dx = \int\int k(x+y,-y)\nu(dy)dx. \tag{2.1}$$

Here we used the symmetry of ν, Fubini's theorem and the translation invariance of the Lebesgue measure. In effect the variables in (2.1) are changed according to $(x, y, x+y) \mapsto (x+y, -y, x)$. As a consequence,

$$\begin{aligned} &\int\int \mathbf{1}_{|k(x)|>|k(x+y)|}|k(x+y) - k(x)|\,|h(x+y) - h(x)|\nu(dy)dx \\ = \;& \frac{1}{2}\int\int |k(x) - k(x+y)|\,|h(x) - h(x+y)|\nu(dy)dx, \end{aligned} \tag{2.2}$$

where k, h are arbitrary. We define

$$\psi(\xi) = \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot x))\,\nu(dx), \quad \xi \in \mathbb{R}^d, \tag{2.3}$$

Clearly, $\psi(-\xi) = \psi(\xi)$ for all ξ. Finally, we shall assume the following Hartman-Wintner condition on ν:

$$\lim_{|\xi|\to\infty} \frac{\psi(\xi)}{\log|\xi|} = \infty. \tag{HW}$$

Below we work precisely under these three assumptions (LM), (S) (HW), except in specialized examples. We let

$$p_t(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\xi\cdot x} e^{-t\psi(\xi)} d\xi, \quad t > 0,\ x \in \mathbb{R}^d. \tag{2.4}$$

Clearly, $p_t(-x) = p_t(x)$ for all x and t, and $p_t(x) \leq p_t(0) \to 0$ as $t \to \infty$. By the characterization of the infinitely divisible distributions, i.e. the Lévy-Khintchine formula, p_t is a density function of a probability measure on $\mathbb{R}^d$ (see [11] for a direct construction),

$$\int_{\mathbb{R}^d} p_t(x)dx = 1.$$

The Fourier transform of p_t is

$$\hat{p_t}(\xi) := \int_{\mathbb{R}^d} e^{i\xi\cdot x} p_t(x)dx = e^{-t\psi(\xi)}, \quad \xi \in \mathbb{R}^d,\ t > 0. \tag{2.5}$$

By (2.4) and (HW), $p_t(x)$ is smooth in x and t. By (2.5), p_t form a convolution semigroup of functions:

$$p_t * p_s = p_{t+s}.$$

For notational convenience we let

$$p_t(x, y) = p_t(y - x), \qquad x, y \in \mathbb{R}^d,\ t > 0.$$

From the above discussion we have the following symmetry property

$$p_t(x, y) = p_t(y, x)\,, \quad x, y \in \mathbb{R}^d,\ t > 0, \tag{2.6}$$

the Chapman–Kolmogorov equations

$$\int_{\mathbb{R}^d} p_s(x, y)p_t(y, z)dy = p_{s+t}(x, z), \qquad x, z \in \mathbb{R}^d,\ s, t > 0 \tag{2.7}$$

and the Markovian property

$$\int_{\mathbb{R}^d} p_t(x, y)dy = \int_{\mathbb{R}^d} p_t(x, y)dx = 1. \tag{2.8}$$

In fact, p_t is a transition probability density of a symmetric, pure jump Lévy process $\{X_t, t \geq 0\}$ with values in $\mathbb{R}^d$ and the characteristic function given by

$$\mathbb{E}\left[e^{i\xi\cdot X_t}\right] = e^{-t\psi(\xi)}, \quad t \geq 0.$$

The function ψ is called the *characteristic* or *Lévy-Khintchine exponent* of X_t. For an initial state $x \in \mathbb{R}^d$, a Borel set $A \subset \mathbb{R}^d$ and a function f on $\mathbb{R}^d$ we let

$$\mathbb{P}_x(X_t \in A) := \mathbb{P}(X_t + x \in A), \qquad \mathbb{E}_x f(X_t) := \mathbb{E}f(X_t + x).$$

It is well-known that

$$P_t f(x) := \mathbb{E}_x f(X_t) = \int_{\mathbb{R}^d} p_t(x, y)f(y)dy$$

defines a Feller semigroup on $C_0(\mathbb{R}^d)$, the space of continuous functions on $\mathbb{R}^d$ vanishing at infinity. That is, $P_t C_0(\mathbb{R}^d) \subset C_0(\mathbb{R}^d)$ for all $t > 0$, and (P_t) is strongly continuous: $\|P_t f - f\|_\infty \to 0$ as $t \to 0$ for all $f \in C_0(\mathbb{R}^d)$. We let L be the corresponding infinitesimal generator of (P_t):

$$Lf := \lim_{t\searrow 0} \frac{P_t f - f}{t}.$$

Here the limit is taken in the supremum norm. Let $\mathcal{D}(L)$ be the domain of L. Then $C_0^2(\mathbb{R}^d) \subset \mathcal{D}(L)$, where

$$C_0^2(\mathbb{R}^d) := \left\{ f \in C^2(\mathbb{R}^d) \cap C_0(\mathbb{R}^d) : \frac{\partial f}{\partial x_i}, \frac{\partial^2 f}{\partial x_i \partial x_j} \in C_0(\mathbb{R}^d), \quad 1 \leq i, j \leq d \right\}.$$

We similarly define the spaces $C_0^k(\mathbb{R}^d)$, $k = 1, 2, 3, \ldots$, and their intersection $C_0^\infty(\mathbb{R}^d)$. By [26, Theorem 31.5] and the symmetry of ν, the generator L satisfies

$$Lf(x) = \lim_{\varepsilon\searrow 0} \int_{|y|>\varepsilon} \left(f(x + y) - f(x)\right)\nu(dy), \quad f \in C_0^2(\mathbb{R}^d),\ x \in \mathbb{R}^d. \tag{2.9}$$

By Jensen's inequality and Fubini-Tonelli, (P_t) is also a semigroup of contractions on L^p for every $1 \leq p < \infty$, that is, $\|P_t f\|_p \leq \|f\|_p$. Furthermore, (P_t) is strongly continuous on L^p for every $1 \leq p < \infty$. By [19, Theorem 2.1] we have $p_t(x, \cdot) \in C_0^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ for all $x \in \mathbb{R}^d$ and $t > 0$. In fact it follows from [19, the proof of Theorem 2.1] that for fixed $t > 0$ and $x \in \mathbb{R}^d$, $p_t(x) = \varphi * \tilde{p}(x)$, where φ is a function in the Schwarz class $\mathcal{S}(\mathbb{R}^d)$, and $\tilde{p}$ is a probability measure. Hence, if $f \in L^p(\mathbb{R}^d)$ for some $1 \leq p < \infty$, then

$P_t f \in L^p(\mathbb{R}^d) \cap C_0^\infty(\mathbb{R}^d)$. Since $p_t(x,y) \le p_t(0)$, we have that $P_t : L^2(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$ is bounded for all $t > 0$. This property is called ultracontractivity. For more on this topic, see Davies [15].

Example 1. The above assumptions are satisfied for the semigroup of many Lévy processes and in particular for the semigroup of the isotropic symmetric stable Lévy processes, associated with the fractional Laplacian. Indeed, as is well-known, the transition density of these processes for $0 < \alpha < 2$, can be written as

$$p_t^{(\alpha)}(x,y) = p_t^{(\alpha)}(x-y) = \int_0^\infty \frac{1}{(4\pi s)^{d/2}} e^{\frac{-|x-y|^2}{4s}} \eta_t^{\alpha/2}(s)\, ds, \tag{2.10}$$

where $\eta_t^{\alpha/2}(s)$ is the density for the $\alpha/2$-stable subordinator [11]. From this it follows that for each $t > 0$, $p_t^{(\alpha)}(x)$ is a radially decreasing function of x, and

$$p_t^{(\alpha)}(x) \le p_t^{(\alpha)}(0) = \frac{p_1(0)}{t^{\alpha/d}} < \infty.$$

In particular the corresponding semigroup is ultracontractive. Its Lévy measure is

$$\nu(dy) = \mathcal{A}_{d,-\alpha}|y|^{-d-\alpha} dy, \quad y \in \mathbb{R}^d,$$

where

$$\mathcal{A}_{d,-\alpha} = 2^\alpha \Gamma\big((d+\alpha)/2\big)\pi^{-d/2}/|\Gamma(-\alpha/2)|. \tag{2.11}$$

Our assumptions also hold for many other semigroups obtained by subordination of the Brownian motion [27] and for the more general unimodal Lévy processes [13], provided they satisfy the so-called weak lower scaling condition [13]. □

We shall need the following fundamental inequality of Stein [29] which holds for symmetric Markovian semigroups.

Lemma 2.1. *For $f \in L^p$, $1 < p \le \infty$, define the maximal function $f^*(x) = \sup_t |P_t f(x)|$. Then,*

$$\|f^*\|_p \le \frac{p}{p-1}\|f\|_p, \tag{2.12}$$

where the right hand side is just $\|f\|_\infty$, if $p = \infty$.

We note that Stein [29] gives an unspecified constant depending only on p for this inequality. For our applications here this is sufficient, however it is well-known that the inequality actually holds with the explicit constant given above. In fact, this is nothing more than the constant in Doob's inequality for martingales. The latter is the tool used in [31, Chapter 4] for the proof of the inequality. For a shorter argument using continuous time martingales and Doob's inequality, we refer the reader to Kim [17, Proposition 2.3]. Kim's proof is the zero-potential case of the proof given in Shigekawa [28] for Feynman–Kac semigroups. This proof (the zero-potential case of Shigekawa) has been known to experts for many years.

3. Hardy-Stein identity

The following elementary results are given in [12]. Let $1 < p < \infty$. For $a, b \in \mathbb{R}$ we set

$$F(a,b) = |b|^p - |a|^p - pa|a|^{p-2}(b-a)\,. \tag{3.1}$$

Here $F(a,b) = |b|^p$ if $a = 0$, and $F(a,b) = (p-1)|a|^p$ if $b = 0$. For instance, if $p = 2$, then $F(a,b) = (b-a)^2$. Generally, $F(a,b)$ is the second-order Taylor remainder of $\mathbb{R} \ni x \mapsto |x|^p$, therefore by convexity, $F(a,b) \ge 0$. Furthermore, for $1 < p < \infty$ and $\varepsilon \in \mathbb{R}$ we define

$$F_\varepsilon(a,b) = (b^2+\varepsilon^2)^{p/2} - (a^2+\varepsilon^2)^{p/2} - pa(a^2+\varepsilon^2)^{(p-2)/2}(b-a)\,. \tag{3.2}$$

Since $F_\varepsilon(a,b)$ is the second-order Taylor remainder of $\mathbb{R} \ni x \mapsto (x^2+\varepsilon^2)^{p/2}$, by convexity, $F_\varepsilon(a,b) \ge 0$. Of course, $F_\varepsilon(a,b) \to F_0(a,b) = F(a,b)$ as $\varepsilon \to 0$.

Lemma 3.1 ([12]). *For every $p > 1$, we have constants $0 < c_p \leq C_p < \infty$ such that*

$$c_p(b-a)^2(|b| \vee |a|)^{p-2} \leq F(a,b) \leq C_p(b-a)^2(|b| \vee |a|)^{p-2}, \qquad a, b \in \mathbb{R}. \tag{3.3}$$

If $p \in (1,2)$, then

$$0 \leq F_\varepsilon(a,b) \leq \frac{1}{p-1} F(a,b)\,, \qquad \varepsilon, a, b \in \mathbb{R}\,. \tag{3.4}$$

The main result of this section is the following Hardy-Stein identity.

Theorem 3.2. *If $1 < p < \infty$ and $f \in L^p(\mathbb{R}^d)$, then*

$$\int_{\mathbb{R}^d} |f(x)|^p dx = \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F(P_t f(x), P_t f(x+y)) \nu(dy) dx dt. \tag{3.5}$$

Proof. We first prove the theorem assuming that $f \in L^p(\mathbb{R}^d) \cap C_0^2(\mathbb{R}^d)$. If $2 \leq p < \infty$, then we proceed as follows. Let $0 \leq t \leq T$ and

$$\xi(t) = |P_{T-t} f|^p.$$

Then $\xi(t) \in C_0^2(\mathbb{R}^d) \subset \mathcal{D}(L)$ for every $t \in [0,T]$ since $P_t f \in C_0^2(\mathbb{R}^d)$ for all $t \geq 0$. Furthermore, if $u \in C_0^2(\mathbb{R}^d)$, then we have

$$\frac{\partial}{\partial x_i} |u|^p = p|u|^{p-2} u u_i\,,$$

$$\frac{\partial^2}{\partial x_j \partial x_i} |u|^p = p(p-1)|u|^{p-2} u_j u_i + p|u|^{p-2} u u_{ji}\,,$$

hence $|u|^p \in C_0^2(\mathbb{R}^d)$. Also, $[0,T] \ni t \mapsto \xi(\cdot)(x)$ is of class C^1 for every $x \in \mathbb{R}^d$ as it can be seen from the following direct differentiation where L denotes the generator of the semigroup.

$$\begin{aligned} \frac{d}{dt}\xi(t)(x) &= p P_{T-t} f(x)\, |P_{T-t} f(x)|^{p-2} \frac{d}{dt} P_{T-t} f(x) \\ &= -p P_{T-t} f(x)\, |P_{T-t} f(x)|^{p-2}\, L P_{T-t} f(x). \end{aligned} \tag{3.6}$$

We have

$$P_T |f|^p(x) - |P_T f(x)|^p = \int_0^T \frac{d}{dt}\left(P_t \xi(t)(x)\right) dt \tag{3.7}$$

$$= \int_0^T \left[P_t \xi'(t)(x) + P_t L \xi(t)(x)\right] dt \tag{3.8}$$

$$= \int_0^T P_t \left[\xi'(t) + L\xi(t)\right](x) dt. \tag{3.9}$$

The equality (3.8) requires some explanation. Following [10], we have

$$\frac{P_{t+h}\xi(t+h)(x) - P_t \xi(t)(x)}{h} = P_{t+h}\xi'(t)(x) \tag{3.10}$$

$$+ P_{t+h}\left(\frac{\xi(t+h) - \xi(t)}{h} - \xi'(t)\right)(x) + \frac{P_{t+h}\xi(t)(x) - P_t \xi(t)(x)}{h}. \tag{3.11}$$

Recall that $\xi(t) \in \mathcal{D}(L)$. By (3.6), $\xi'(t) \in C_0(\mathbb{R}^d)$ for every $t \in [0,T]$. Furthermore, since P_t is strongly continuous and $p \geq 2$, both $L P_{T-t} f$ and $P_{T-t} f\, |P_{T-t} f|^{p-2}$ are continuous mapping $[0,T]$ to $C_0(\mathbb{R}^d)$. In view of (3.6), $[0,T] \ni t \mapsto \xi'(t) \in C_0(\mathbb{R}^d)$ is also continuous. Letting $h \to 0$ in (3.10), we get (3.8). We then have

$$[\xi'(t) + L\xi(t)](x) = \int_{\mathbb{R}^d} \Big\{ |P_{T-t}f(x+y)|^p - |P_{T-t}f(x)|^p \\ -pP_{T-t}f(x)\,|P_{T-t}f(x)|^{p-2}\,[P_{T-t}f(x+y) - P_{T-t}f(x)] \Big\}\,\nu(dy) \\ = \int_{\mathbb{R}^d} F(P_{T-t}f(x), P_{T-t}f(x+y))\nu(dy). \tag{3.12}$$

Integrating (3.7) with respect to x and using (2.8) we obtain

$$\int_{\mathbb{R}^d} |f(x)|^p dx - \int_{\mathbb{R}^d} |P_T f(x)|^p dx = \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F(P_t f(x), P_t f(x+y))\nu(dy)dxdt.$$

But $\int_{\mathbb{R}^d} |P_T f(x)|^p dx \to 0$ as $T \to \infty$ because of dominated convergence theorem. Indeed, by (2.12) and $|P_T f(x)|^p \le f^*(x)^p$ for every $x \in \mathbb{R}^d$, and for $q = p/(p-1)$ by Hölder inequality we have

$$\left| \int p_T(x,y) f(y) dy \right| \le \|f\|_p \left(\int_{\mathbb{R}^d} p_T(x,y)^q dy \right)^{1/q}, \tag{3.13}$$

whereas $\int_{\mathbb{R}^d} p_T(x,y)^q dy \le \sup_{x,y\in\mathbb{R}^d} p_T(x,y)^{q-1} \to 0$ as $T \to \infty$. Thus, (3.5) follows.

Suppose $1 < p < 2$. For $0 \le t \le T$ and $\varepsilon > 0$ we define

$$\xi_\varepsilon(t) = \left((P_{T-t}f)^2 + \varepsilon^2\right)^{p/2} - \varepsilon^p.$$

As in the case $2 \le p < \infty$, we conclude that $\xi_\varepsilon(t) \in C_0^2(\mathbb{R}^d) \subset \mathcal{D}(L)$ for every $t \in [0,T]$. Indeed, for any $u \in C_0^2(\mathbb{R}^d)$ we have

$$\frac{\partial}{\partial x_i}\left(u^2 + \varepsilon^2\right)^{p/2} = p\left(u^2 + \varepsilon^2\right)^{(p-2)/2} u u_i\,,$$
$$\frac{\partial^2}{\partial x_j \partial x_i}\left(u^2 + \varepsilon^2\right)^{p/2} = p(p-2)\left(u^2 + \varepsilon^2\right)^{(p-4)/2} u^2 u_j u_i \\ + p\left(u^2 + \varepsilon^2\right)^{(p-2)/2}\left(u_j u_i + u u_{ji}\right).$$

Furthermore, $[0,T] \ni t \mapsto \xi_\varepsilon(\cdot)(x)$ is also of class C^1 for every $x \in \mathbb{R}^d$, and

$$\frac{d}{dt}\xi_\varepsilon(t)(x) = pP_{T-t}f(x)\left[(P_{T-t}f(x))^2 + \varepsilon^2\right]^{(p-2)/2} \frac{d}{dt}P_{T-t}f(x) \\ = -pP_{T-t}f(x)\left[(P_{T-t}f(x))^2 + \varepsilon^2\right]^{(p-2)/2} LP_{T-t}f(x).$$

Therefore $\xi'_\varepsilon(t) \in C_0(\mathbb{R}^d)$ for every $t \in [0,T]$, and $[0,T] \ni t \mapsto \xi'_\varepsilon(t) \in C_0(\mathbb{R}^d)$ is continuous. We have

$$P_T\left((f^2 + \varepsilon^2)^{p/2}\right)(x) - \left((P_T f(x))^2 + \varepsilon^2\right)^{p/2} = \int_0^T \frac{d}{dt}\left(P_t \xi_\varepsilon(t)(x)\right) dt \tag{3.14}$$

$$= \int_0^T \left[P_t \xi'_\varepsilon(t)(x) + P_t L\xi_\varepsilon(t)(x)\right] dt = \int_0^T P_t\left[\xi'_\varepsilon(t) + L\xi_\varepsilon(t)\right](x) dt. \tag{3.15}$$

Consequently,

$$[\xi'_\varepsilon(t) + L\xi_\varepsilon(t)](x) = \int_{\mathbb{R}^d} \Big\{ \left((P_{T-t}f(x+y))^2 + \varepsilon^2\right)^{p/2} - \left((P_{T-t}f(x))^2 + \varepsilon^2\right)^{p/2} \\ -pP_{T-t}f(x)\left((P_{T-t}f(x))^2 + \varepsilon^2\right)^{(p-2)/2}\left[P_{T-t}f(x+y) - P_{T-t}f(x)\right] \Big\}\,\nu(dy) \\ = \int_{\mathbb{R}^d} F_\varepsilon(P_{T-t}f(x), P_{T-t}f(x+y))\nu(dy).$$

Integrating (3.14) with respect to x we obtain

$$\begin{aligned}&\int_{\mathbb{R}^d}\left[P_T\left((f^2+\varepsilon^2)^{p/2}\right)(x)-\left((P_Tf(x))^2+\varepsilon^2\right)^{p/2}\right]dx\\&=\int_{\mathbb{R}^d}\left((f(x)^2+\varepsilon^2)^{p/2}-\varepsilon^p\right)dx-\int_{\mathbb{R}^d}\left[\left((P_Tf(x))^2+\varepsilon^2\right)^{p/2}-\varepsilon^p\right]dx\\&=\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}F_\varepsilon(P_tf(x),P_tf(x+y))\nu(dy)dxdt.\end{aligned}$$

Note that the expression above is finite and uniformly bounded with respect to T and ε. Indeed, since $0<p/2<1$, the function $x\mapsto x^{p/2}$ is $p/2$-Hölder continuous on $[0,\infty)$, we have

$$(f(x)^2+\varepsilon^2)^{p/2}-\varepsilon^p\le c_p|f(x)|^p,$$

and

$$\left((P_Tf(x))^2+\varepsilon^2\right)^{p/2}-\varepsilon^p\le c_p|P_Tf(x)|^p.$$

Let $\varepsilon\to0$. In view of (3.4) and dominated convergence (see also [12, Remark 7]) we get

$$\int_{\mathbb{R}^d}|f(x)|^pdx-\int_{\mathbb{R}^d}|P_Tf(x)|^pdx=\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}F(P_tf(x),P_tf(x+y))\nu(dy)dxdt.$$

Using the same argument as in the previous part we get $\int_{\mathbb{R}^d}|P_Tf(x)|^pdx\to0$ as $T\to\infty$. This together with the previous case gives (3.5) for all $1<p<\infty$ and $f\in L^p(\mathbb{R}^d)\cap C_0^2(\mathbb{R}^d)$.

We next relax the assumption that $f\in L^p(\mathbb{R}^d)\cap C_0^2(\mathbb{R}^d)$. For $1<p<\infty$ and general $f\in L^p(\mathbb{R}^d)$ we let $s>0$. Then $P_sf\in L^p(\mathbb{R}^d)\cap C_0^\infty(\mathbb{R}^d)$, and so by the preceding, discussion

$$\int_{\mathbb{R}^d}|P_sf(x)|^pdx=\int_s^\infty\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}F(P_tf(x),P_tf(x+y))\nu(dy)dxdt. \tag{3.16}$$

By the strong continuity of P_t in $L^p(\mathbb{R}^d)$, the left-hand side of (3.16) tends to $\|f\|_p^p$ as $s\to0$. The right-hand side also converges as $s\to0$. The theorem follows. □

4. Square functions

For $f\in L^1(\mathbb{R}^d)\cup L^\infty(\mathbb{R}^d)$ we let

$$G(f)(x):=\left(\int_0^\infty\int_{\mathbb{R}^d}(P_tf(x+y)-P_tf(x))^2\nu(dy)dt\right)^{1/2},$$

and

$$\widetilde{G}(f)(x):=\left(\int_0^\infty\int_{\{|P_tf(x)|>|P_tf(x+y)|\}}(P_tf(x+y)-P_tf(x))^2\nu(dy)dt\right)^{1/2}.$$

Clearly, $0\le\widetilde{G}(f)(x)\le G(f)(x)$ for every x. By (3.5) and the symmetry,

$$\|f\|_2^2=\|G(f)\|_2^2=2\|\widetilde{G}(f)\|_2^2. \tag{4.1}$$

By polarization, for $f,g\in L^2(\mathbb{R}^d)$ we have

$$\langle f,g\rangle=\int_{\mathbb{R}^d}\int_0^\infty\int_{\mathbb{R}^d}\left[P_tf(x+y)-P_tf(x)\right]\left[P_tg(x+y)-P_tg(x)\right]\nu(dy)dtdx. \tag{4.2}$$

The main result of this section is the following theorem.

Theorem 4.1. *Let $1<p<\infty$. There is a constant C depending only on p such that*

$$C^{-1}\|f\|_p\le\|\widetilde{G}(f)\|_p\le C\|f\|_p,\qquad f\in L^p(\mathbb{R}^d). \tag{4.3}$$

The result is proved below after a sequence of partial results. In another direction, at the end of this section we show in Example 2 that G is too large to give a characterization of $L^p(\mathbb{R}^d)$ for $1 < p < 2$. Nevertheless, $\|G(f)\|_p \leq C_p\|f\|_p$, for $2 \leq p < \infty$, as we now prove by using the Burkholder-Gundy inequalities.

We start by introducing the Littlewood-Paley function G_* which is the conditional expectation of the quadratic variation of a martingale. For classical harmonic functions in the upper half-space of $\mathbb{R}^d$, such objects have appeared many times in the literature, see for example [3], [9]. The construction for the generalized Poisson semigroups is presented in [32]. Here we simply fix $f \in C_c^\infty(\mathbb{R}^d)$, $T > 0$, and let

$$M_t = P_{T-t}f(X_t) - P_T f(z), \quad 0 < t < T.$$

When the process X_t starts at $z \in \mathbb{R}^d$, M_t is a martingale starting at 0. Such space-time (*parabolic*) martingales were first used for the Brownian motion in Bañuelos and Méndez-Hernández [7] to study martingale transforms that lead to Fourier multipliers related to the Beurling-Ahlfors operator. They were then applied in [6, 1] to more general Lévy processes. We recall the properties of M_t here to clarify the use of the Burkholder-Gundy inequality and to elucidate the origins of our Littlewood-Paley square functions. For full details, we refer the reader to [1].

Applying the Itô formula (see [1, p. 1118], where this is done for general Lévy processes) we have that

$$M_t = \int_0^t \int_{\mathbb{R}^d} \left[P_{T-s}f(X_{s-} + y) - P_{T-s}f(X_{s-})\right] \tilde{N}(ds, dy), \quad 0 < t < T. \tag{4.4}$$

Here

$$\tilde{N}(t, A) = N(t, A) - t\nu(A),$$

and N is a Poisson random measure on $\mathbb{R}^+ \times \mathbb{R}^d$ with intensity measure $dt \times d\nu$. In fact we take

$$N(t, A) = \#\{0 \leq s \leq t, \Delta X_s \in A\}, \qquad t \geq 0,\ A \subset \mathbb{R}^d,$$

where $\Delta X_s = X_s - X_{s^-}$ denotes the jump of the process at time $s > 0$. The quadratic variation of M_t is

$$[M]_t = \int_0^t \int_{\mathbb{R}^d} |P_{T-s}f(X_{s-} + y) - P_{T-s}f(X_{s-})|^2 d\nu(y)\, ds.$$

For a slightly different representation of (4.4) without using the process N, and for references to Itô's formula for processes with jumps, see [4, p. 847].

We now define

$$G_*(f)(x) = \left(\int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |P_t f(z + y) - P_t f(z)|^2 p_t(x, z) dz \nu(dy) dt\right)^{1/2},$$

and

$$G_{*,T}(f)(x) = \left(\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |P_t f(z + y) - P_t f(z)|^2 p_t(x, z) dz \nu(dy) dt\right)^{1/2}.$$

Notice that $G_{*,T}(f)(x) \nearrow G_*(f)(x)$ as $T \to \infty$. We claim that

$$G^2_{*,T}(f)(x) = \int_{\mathbb{R}^d} \mathbb{E}^x_z \left(\int_0^T \int_{\mathbb{R}^d} |P_{T-s}f(X_s + y) - P_{T-s}f(X_s)|^2 \nu(dy) ds\right) p_T(z, x) dz, \tag{4.5}$$

where

$$\begin{aligned} &\mathbb{E}^x_z \left(\int_0^T \int_{\mathbb{R}^d} |P_{T-s}f(X_s + y) - P_{T-s}f(X_s)|^2 \nu(dy) ds\right) \\ &:= \mathbb{E}_z \left(\int_0^T \int_{\mathbb{R}^d} |P_{T-s}f(X_s + y) - P_{T-s}f(X_s)|^2 \nu(dy) ds \,\Big|\, X_T = x\right), \end{aligned}$$

cf. below. Thus,

$$G^2_{*,T}(f)(x) = \int_{\mathbb{R}^d} \mathbb{E}_z \left([M]_T \,\middle|\, X_T = x\right) p_T(z,x)dz = \int_{\mathbb{R}^d} \left(\mathbb{E}^x_z[M]_T\right)\, p_T(z,x)dz.$$

The proof of (4.5) is exactly the same as the proof for harmonic functions in the upper half-space of $\mathbb{R}^d$ given in [3, p. 663]. (See [32] for the more general construction for Poisson semigroups.) Indeed, by the definition of the conditional distribution of X_s under $\mathbb{P}_z$ given $X_T = x$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathbb{E}^x_z \left(\int_0^T \int_{\mathbb{R}^d} |P_{T-s}f(X_s+y) - P_{T-s}f(X_s)|^2 \nu(dy)ds \right) p_T(z,x)dz \\
=&\int_{\mathbb{R}^d} \left(\int_0^T \int_{\mathbb{R}^d} \frac{p_s(z,w)p_{T-s}(w,x)}{p_T(z,x)} \int_{\mathbb{R}^d} |P_{T-s}f(w+y) - P_{T-s}f(w)|^2 \nu(dy)dwds \right) p_T(z,x)dz \\
=&\int_0^T \int_{\mathbb{R}^d} p_{T-s}(w,x) \int_{\mathbb{R}^d} |P_{T-s}f(w+y) - P_{T-s}f(w)|^2 \nu(dy)dwds \\
=&\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |P_s f(w+y) - P_s f(w)|^2 p_s(x,w)dw\nu(dy)ds = G^2_{*,T}(f)(x).
\end{aligned}$$

With (4.5) established, we now apply the martingale inequalities to prove that $\|G_*(f)\|_p \le C_p\|f\|_p$ for $2 \le p < \infty$, which also yields the same result for $G(f)$.

Lemma 4.2. *Let $2 \le p < \infty$. There is a constant C depending only on p such that $\|G(f)\|_p \le C\|f\|_p$ for every $f \in L^p(\mathbb{R}^d)$.*

Proof. Since $p \ge 2$, by Jensen's inequality we get

$$\begin{aligned}
&\int_{\mathbb{R}^d} G^p_{*,T}(f)(x)dx \\
\le &\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbb{E}^x_z \left(\int_0^T \int_{\mathbb{R}^d} |P_{T-s}f(X_s+y) - P_{T-s}f(X_s)|^2 \nu(dy)ds \right)^{p/2} p_T(z,x)dzdx \\
= &\int_{\mathbb{R}^d} \mathbb{E}_z \left(\int_0^T \int_{\mathbb{R}^d} |P_{T-s}f(X_s+y) - P_{T-s}f(X_s)|^2 \nu(dy)ds \right)^{p/2} dz.
\end{aligned}$$

By the Burkholder-Gundy inequality the last term above is less than

$$\begin{aligned}
&C_p \int_{\mathbb{R}^d} \mathbb{E}_z\, |f(X_T) - P_T f(z)|^p\, dz \le C_p \int_{\mathbb{R}^d} (\mathbb{E}_z|f(X_T)|^p + P_T|f|^p(z))dz \\
=&C_p \int_{\mathbb{R}^d} P_T|f(z)|^p dz = C_p\|f\|^p_p.
\end{aligned}$$

By the monotone convergence,

$$\int_{\mathbb{R}^d} G^p_*(f)(x)dx = \lim_{T\to\infty} \int_{\mathbb{R}^d} G^p_{*,T}(f)(x)dx \le C_p\|f\|^p_p.$$

We claim that $G(f)(x) \leq \sqrt{2}G_*(f)(x)$. Indeed, by the semigroup property and Jensen's inequality,

$$\begin{aligned} G^2(f)(x) &= \int_0^\infty \int_{\mathbb{R}^d} |P_t f(x+y) - P_t f(x)|^2 \nu(dy)dt \\ &= \int_0^\infty \int_{\mathbb{R}^d} |P_{t/2}P_{t/2} f(x+y) - P_{t/2}P_{t/2} f(x)|^2 \nu(dy)dt \\ &\leq \int_0^\infty \int_{\mathbb{R}^d} P_{t/2}|P_{t/2} f(x+y) - P_{t/2} f(x)|^2 \nu(dy)dt \\ &= \int_0^\infty \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |P_{t/2} f(z+y) - P_{t/2} f(z)|^2 p_{t/2}(x,z) dz \nu(dy)dt \\ &= 2\int_0^\infty \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |P_t f(z+y) - P_t f(z)|^2 p_t(x,z) dz \nu(dy)dt. \end{aligned}$$

This completes the proof of the lemma for $f \in C_c^\infty(\mathbb{R}^d)$. For arbitrary $f \in L^p(\mathbb{R}^d)$, we choose $f_n \in C_c^\infty(\mathbb{R}^d)$ such that $f_n \to f$ in L^p. The inequality $\|G(f)\|_p \leq C\|f\|_p$ follows from Fatou's lemma. $\square$

For every $2 \leq p < \infty$ and $f \in L^p(\mathbb{R}^d)$ we have by (3.5), (3.3) and (2.2),

$$\begin{aligned} \|f\|_p^p &\asymp \int_{\mathbb{R}^d}\int_0^\infty \int_{\mathbb{R}^d} (P_t f(x+y) - P_t f(x))^2 (|P_t f(x+y)| \vee |P_t f(x)|)^{p-2} \nu(dy)dtdx \\ &= 2\int_{\mathbb{R}^d}\int_0^\infty \int_{\{|P_t f(x)|>|P_t f(x+y)|\}} (P_t f(x+y) - P_t f(x))^2 |P_t f(x)|^{p-2} \nu(dy)dtdx \\ &\leq 2\int_{\mathbb{R}^d} f^*(x)^{p-2} \widetilde{G}(f)(x)^2 dx. \end{aligned} \tag{4.6}$$

Lemma 4.3. *Suppose $2 \leq p < \infty$. There is a constant C depending only on p such that*

$$C^{-1}\|f\|_p \leq \|\widetilde{G}(f)\|_p \leq C\|f\|_p, \qquad f \in L^p(\mathbb{R}^d). \tag{4.7}$$

Proof. Since $\widetilde{G}(f)(x) \leq G(f)(x)$, the right-hand side of (4.7) follows immediately from Lemma 4.2. By Hölder's inequality,

$$\begin{aligned} \int_{\mathbb{R}^d} f^*(x)^{p-2}\widetilde{G}(f)(x)^2 dx &\leq \left[\int_{\mathbb{R}^d} (f^*(x)^{p-2})^{\frac{p}{p-2}} dx\right]^{\frac{p-2}{p}} \left[\int_{\mathbb{R}^d} (\widetilde{G}(f)(x)^2)^{\frac{p}{2}} dx\right]^{\frac{2}{p}} \\ &= \|f^*\|_p^{p-2}\|\widetilde{G}(f)\|_p^2 \leq C\|f\|_p^{p-2}\|\widetilde{G}(f)\|_p^2. \end{aligned}$$

By (4.6) and (2.12) we get $\|f\|_p^p \leq C\|f\|_p^{p-2}\|\widetilde{G}(f)\|_p^2$, which yields the result. $\square$

Combining Lemma 4.2 and Lemma 4.3 we obtain the following.

Corollary 4.4. *Suppose $2 \leq p < \infty$. There is a constant C depending only on p such that*

$$C^{-1}\|f\|_p \leq \|G(f)\|_p \leq C\|f\|_p, \qquad f \in L^p(\mathbb{R}^d). \tag{4.8}$$

We now discuss the regime $1 < p < 2$.

Lemma 4.5. *Suppose $1 < p < 2$. There is a constant C depending only on p such that*

$$C^{-1}\|f\|_p \leq \|\widetilde{G}(f)\|_p \leq C\|f\|_p, \qquad f \in L^p(\mathbb{R}^d). \tag{4.9}$$

Proof. We first consider the right-hand inequality. Our proof proceeds exactly as the proof in [30, pp. 87-88] for the boundedness of the Littlewood-Paley square function g in the range $1 < p < 2$. Here, however, instead of using the Hardy-Littlewood maximal function and its boundedness on $L^p(\mathbb{R}^d)$, we use the maximal function of the semigroup and Lemma 2.1. Also, in place of the identity Lemma 2 of [30, p.88], we use our Hardy-Stein identity. More precisely, setting

$$I(x)=\int_0^\infty\int_{\{|P_tf(x)|>|P_tf(x+y)|\}}(P_tf(x+y)-P_tf(x))^2|P_tf(x)|^{p-2}\nu(dy)dt$$

we have, by (3.5) of Theorem 3.2 and (3.3) of Lemma 3.1, that there exists a constant C_p depending only on p such that

$$\int_{\mathbb{R}^d} I(x)dx\le C_p\int_{\mathbb{R}^d}|f(x)|^p dx. \tag{4.10}$$

Now observe that

$$\begin{aligned}\widetilde{G}(f)(x)^2 &= \int_0^\infty\int_{\{|P_tf(x)|>|P_tf(x+y)|\}}(P_tf(x+y)-P_tf(x))^2\nu(dy)dt\\ &= \int_0^\infty\int_{\{|P_tf(x)|>|P_tf(x+y)|\}}(P_tf(x+y)-P_tf(x))^2|P_tf(x)|^{p-2}|P_tf(x)|^{2-p}\nu(dy)dt\\ &\le f^*(x)^{2-p}I(x),\end{aligned}$$

where we used the fact that $1<p<2$. With $r=2/(2-p)$ and $r'=2/p$ so that $1<r,r'<\infty$ and $1/r+1/r'=1$, we can integrate both sides of this inequality and apply Hölder's inequality to obtain

$$\begin{aligned}\int_{\mathbb{R}^d}\widetilde{G}(f)(x)^p dx &\le \int_{\mathbb{R}^d} f^*(x)^{\frac{p(2-p)}{2}} I(x)^{p/2}\,dx\\ &\le \left(\int_{\mathbb{R}^d} f^*(x)^p dx\right)^{(2-p)/2}\left(\int_{\mathbb{R}^d} I(x)dx\right)^{p/2}\\ &\le \left(\frac{p}{p-1}\right)^{\frac{p(2-p)}{2}} C_p^{p/2}\left(\int_{\mathbb{R}^d}|f(x)|^p\,dx\right)^{(2-p)/2}\left(\int_{\mathbb{R}^d}|f(x)|^p\,dx\right)^{p/2}\\ &= \left(\frac{p}{p-1}\right)^{\frac{p(2-p)}{2}} C_p^{p/2}\int_{\mathbb{R}^d}|f(x)|^p\,dx,\end{aligned}$$

where in the last inequality we used Lemma 2.1 and the Hardy-Stein bound (4.10). This gives

$$\|\widetilde{G}(f)\|_p\le\left(\frac{p}{p-1}\right)^{\frac{(2-p)}{2}} C_p^{1/2}\|f\|_p,\qquad 1<p<2. \tag{4.11}$$

In order to prove the left-hand side of (4.9), we fix nonzero $f\in L^p(\mathbb{R}^d)$ and let $s>0$. Define $f_s:=P_sf$ and $g_s:=|f_s|^{(p-1)}\operatorname{sgn} f_s$. By ultracontractivity, $f_s\in L^2(\mathbb{R}^d)$ and $g_s\in L^\infty(\mathbb{R}^d)$. Furthermore, for $q=p/(p-1)$ we have $\|g_s\|_q=\|f_s\|_p^{p-1}$ and

$$\|f_s\|_p^p=\int_{\mathbb{R}^d} f_s(x)g_s(x)dx.$$

Let $\varphi_n := \mathbf{1}_{B(0,n)} g_s$. Since g_s is bounded, $\varphi_n \in L^2(\mathbb{R}^d)$ for all $n \geq 1$. By (4.1) and (2.2),

$$\int_{\mathbb{R}^d} f_s(x)\varphi_n(x)dx = \frac{1}{4}(\|f_s+\varphi_n\|_2^2 - \|f_s-\varphi_n\|_2^2) = \frac{1}{4}(\|G(f_s+\varphi_n)\|_2^2 - \|G(f_s-\varphi_n)\|_2^2)$$
$$= \int_{\mathbb{R}^d}\int_0^\infty\int_{\mathbb{R}^d} (P_tf_s(x+y)-P_tf_s(x))(P_t\varphi_n(x+y)-P_t\varphi_n(x))\nu(dy)dtdx$$
$$=2\int_{\mathbb{R}^d}\int_0^\infty\int_{\{|P_tf_s(x)|>|P_tf_s(x+y)|\}} (P_tf_s(x+y)-P_tf_s(x))(P_t\varphi_n(x+y)-P_t\varphi_n(x))\nu(dy)dtdx$$
$$\leq 2\int_{\mathbb{R}^d} \widetilde{G}(f_s)(x)G(\varphi_n)(x)dx \leq 2\|\widetilde{G}(f_s)\|_p\|G(\varphi_n)\|_q.$$

In the last line we used the Cauchy-Schwarz inequality and Hölder's inequality. Finally, since $q > 2$, by Lemma 4.2 we have $\|G(\varphi_n)\|_q \leq C\|\varphi_n\|_q$ and so

$$\int_{\mathbb{R}^d} f_s(x)\varphi_n(x)dx \leq C\|\widetilde{G}(f_s)\|_p\|\varphi_n\|_q. \tag{4.12}$$

By the monotone convergence, $\|\varphi_n\|_q \to \|g_s\|_q$ as $n\to\infty$, and the left-hand side of (4.12) converges to $\|f_s\|_p^p$. This gives

$$\|f_s\|_p^p \leq C\|\widetilde{G}(f_s)\|_p\|g_s\|_q = C\|\widetilde{G}(f_s)\|_p\|f_s\|_p^{p-1}.$$

Dividing by $\|f_s\|_p^{p-1}$ we obtain $\|f_s\|_p \leq C\|\widetilde{G}(f_s)\|_p$. We let $s\to 0$ in

$$\widetilde{G}(f_s) = \left(\int_s^\infty\int_{\{|P_tf(x)|>|P_tf(x+y)|\}} (P_tf(x+y)-P_tf(x))^2\nu(dy)dt\right)^{1/2}.$$

The monotone convergence and strong continuity of P_t in $L^p(\mathbb{R}^d)$ yield (4.9). □

Proof of Theorem 4.1. The result combines Lemma 4.3 and Lemma 4.5. □

It is well-known that the classical Littlewood-Paley operator G_* constructed from harmonic functions is not bounded on L^p, if $1<p<2$. An explicit example for this failure is presented in [9]. Inspired by [9] we show that the square operator G also fails to be bounded on L^p, if $1<p<2$. Thus, $\tilde{G}$ and G differ significantly.

Example 2. For $d\geq 2$ and $x\in\mathbb{R}^d$ we let $h(x) = |x|^{-(d+1)/2}$ and $f(x) = h(x)\mathbf{1}_{|x|\leq 1}$. We have that $f\in L^p(\mathbb{R}^d)$ for $1<p<2d/(d+1)$. Let P_t be the rotationally invariant Cauchy (Poisson) semigroup on $\mathbb{R}^d$. That is, the semigroup of the α-stable processes with $\alpha = 1$ with transition density

$$p_t(x,y) = \mathcal{C}_d\frac{t}{(t^2+|x-y|^2)^{\frac{d+1}{2}}},$$

where $\mathcal{C}_d = \Gamma((d+1)/2)\pi^{-(d+1)/2}$. Since h is locally integrable on $\mathbb{R}^d$ and vanishes at infinity, the function

$$v(x,t) := \begin{cases} P_th(x), & x\in\mathbb{R}^d,\ t>0,\\ h(x), & x\in\mathbb{R}^d,\ t=0,\end{cases}$$

is well defined and continuous except at $(x,t) = (0,0)$. We see that v is the classical harmonic extension of h to the upper half-space in $\mathbb{R}^{d+1}$. For $x\in\mathbb{R}^d$ and $s,t>0$ we let

$$v_s(x,t) = \int_{B(0,1/s)} p_t(x,y)h(y)dy.$$

From scaling it follows that

$$\begin{aligned} P_t f(x) &= \mathcal{C}_d \int_{B(0,1)} \frac{t}{(t^2 + |x-y|^2)^{\frac{d+1}{2}}} h(y) dy \\ &= t^{-(d+1)/2} \mathcal{C}_d \int_{B(0,1)} \frac{1}{t^d \left(1 + |x/t - y/t|^2\right)^{\frac{d+1}{2}}} \frac{1}{|y/t|^{\frac{d+1}{2}}} dy \\ &= t^{-(d+1)/2} v_t(x/t, 1), \end{aligned}$$

and that

$$v(x,t) = t^{-(d+1)/2} v(x/t, 1), \qquad x \in \mathbb{R}^d,\ t > 0.$$

We have

$$\begin{aligned} G(f)(x)^2 &= \mathcal{A}_{d,-1} \int_0^\infty \int_{\mathbb{R}^d} \frac{(P_t f(y) - P_t f(x))^2}{|x-y|^{d+1}} dy dt \\ &= \mathcal{A}_{d,-1} \int_0^\infty \int_{\mathbb{R}^d} \frac{(v_t(y/t,1) - v_t(x/t,1))^2}{t^{d+1}|x-y|^{d+1}} dy dt \\ &= \mathcal{A}_{d,-1} \int_0^\infty \int_{\mathbb{R}^d} \frac{(v_t(z,1) - v_t(x/t,1))^2}{t|x-tz|^{d+1}} dz dt, \end{aligned} \tag{4.13}$$

where $\mathcal{A}_{d,-1}$ is the constant in (2.11). Observe that $v_t(z,1) \nearrow v(z,1) > 0$, for all $z \in \mathbb{R}^d$, as $t \searrow 0$. Furthermore,

$$v_t(x/t,1) \leq v(x/t,1) = t^{\frac{d+1}{2}} v(x,t) \to 0 \quad \text{for } t \searrow 0,\ x \neq 0.$$

Applying Fubini's theorem in (4.13) we see that $G(f) \equiv \infty$. On the other hand, $\tilde{G}(f) \in L^p$ for every $1 < p < 2d/(d+1)$, as follows from Theorem 4.1. $\square$

5. Application to Lévy multipliers

Among the many applications of classical square functions are those to Fourier multipliers. Accordingly, in this section we prove L^p boundedness for a class of Fourier multipliers that arise in connection to Lévy processes. The multipliers were first studied in [6] and subsequently in [5] and [1] where explicit L^p bounds were proved by using Burkholder's sharp inequalities for martingale transforms. These multipliers include the differences of second order Riesz transforms, $R_1^2 - R_2^2$, for which the bounds given in [5] and [1] were already known to be best possible. Below we derive L^p boundedness of the operators in a different way by using our square function inequalities and the representation of Fourier multipliers from [1].

As previously, we consider a symmetric pure-jump Lévy process $\{X_t, t \geq 0\}$ on $\mathbb{R}^d$ with the semi-group (P_t) and (symmetric) Lévy measure ν satisfying (HW). Recall from (2.3) that the Lévy-Khintchine exponent is

$$\psi(\xi) = \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot x))\, \nu(dx), \quad \xi \in \mathbb{R}^d.$$

Let $\phi(t,y)$ be a bounded function on $(0,\infty) \times \mathbb{R}^d$. Let $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. For $f \in L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and $h \in L^q(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, we consider

$$\Lambda(f,h) = \int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} [P_t f(x+y) - P_t f(x)][P_t h(x+y) - P_t h(x)] \phi(t,y) \nu(dy) dx dt. \tag{5.1}$$

Although not needed for our argument here, it should be pointed out that this quantity arises in [1] from the Itô isometry after taking inner products of the martingale transform of f by the function ϕ and the martingale corresponding to h (see [1, Theorem 3.4] for more details on this pairing). Here we just observe that the integral is absolutely convergent, by (4.1) and Cauchy-Schwarz inequality. By (2.2) and Cauchy-Schwarz,

$$\begin{aligned}|\Lambda(f,h)| &\leq \|\phi\|_\infty \int_0^\infty \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |P_tf(x+y)-P_tf(x)|\,|P_th(x+y)-P_th(x)|\nu(dy)\,dxdt \\ &= 2\|\phi\|_\infty \int_{\mathbb{R}^d}\int_0^\infty \int_{\{|P_tf(x)|>|P_tf(x+y)|\}} |P_tf(x+y)-P_tf(x)||P_th(x+y)-P_th(x)|\nu(dy)dtdx \\ &\leq 2\|\phi\|_\infty \int_{\mathbb{R}^d} \widetilde{G}(f)(x)G(h)(x)\,dx.\end{aligned}$$

Assuming $1<p\leq 2$, we have $2\leq q<\infty$, and by Hölder inequality and Theorem 4.1 we get

$$|\Lambda(f,h)| \leq 2\|\phi\|_\infty\|\widetilde{G}(f)\|_p\|G(h)\|_q \leq C_p\|\phi\|_\infty\|f\|_p\|h\|_q.$$

If $2<p<\infty$, then $1<q<2$, and we similarly have

$$|\Lambda(f,h)| \leq 2\|\phi\|_\infty\|G(f)\|_p\|\widetilde{G}(h)\|_q \leq C_p\|\phi\|_\infty\|f\|_p\|h\|_q.$$

By the Riesz representation theorem, there is a unique linear operator S_ϕ on $L^p(\mathbb{R}^d)$ such that $\Lambda(f,g) = (S_\phi f, g)$, and $\|S_\phi\|\leq C_p\|\phi\|_\infty$.

The computation of the symbol of the multiplier is now exactly as in [1, p.1134] where it is done for arbitrary Lévy measures. In our case, Plancherel's identity yields

$$\begin{aligned}\Lambda(f,h) &= (2\pi)^{-d}\int_{\mathbb{R}^d}\left\{\int_{\mathbb{R}^d}\int_0^\infty e^{-2t\psi(\xi)}|e^{-i\xi\cdot y}-1|^2\phi(t,y)dt\nu(dy)\right\}\hat{f}(\xi)\overline{\hat{h}}(\xi)d\xi \\ &= (2\pi)^{-d}\int_{\mathbb{R}^d}\left\{2\int_{\mathbb{R}^d}\int_0^\infty e^{-2t\psi(\xi)}\left(1-\cos(\xi\cdot y)\right)\phi(t,y)dt\nu(dy)\right\}\hat{f}(\xi)\overline{\hat{h}}(\xi)d\xi \\ &= (2\pi)^{-d}\int_{\mathbb{R}^d} m(\xi)\hat{f}(\xi)\overline{\hat{h}}(\xi)d\xi,\end{aligned} \tag{5.2}$$

where

$$m(\xi) = 2\int_{\mathbb{R}^d}(1-\cos(\xi\cdot y))\left(\int_0^\infty e^{-2t\psi(\xi)}\phi(t,y)dt\right)\nu(dy). \tag{5.3}$$

Thus, S_ϕ is an L^p–Fourier multiplier with $\widehat{S_\phi f}(\xi) = m(\xi)\hat{f}(\xi)$, $f\in L^2\cap L^p$, and $\|m\|_\infty\leq\|\phi\|_\infty$. If ϕ is independent of t, then we further get

$$m(\xi) = \frac{\int_{\mathbb{R}^d}(1-\cos(\xi\cdot y))\,\phi(y)\nu(dy)}{\int_{\mathbb{R}^d}(1-\cos(\xi\cdot y))\,\nu(dy)},$$

the symbols of [5]. Typical examples obtained in this way are the Marcinkiewicz multipliers [6] given by

$$m(\xi_1,\ldots,\xi_d) = \frac{|\xi_j|^\alpha}{|\xi_1|^\alpha+\ldots+|\xi_d|^\alpha},$$

where $0<\alpha<2$ and $j=1,\ldots,d$.

Taking $\phi\equiv 1$, the above calculations give

Corollary 5.1. *If $f\in L^p(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$, $1<p\leq 2$, $h\in L^q(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$, $q=\frac{p}{p-1}$, then*

$$\left|\int_{\mathbb{R}^d} f(x)h(x)dx\right| \leq 2\int_{\mathbb{R}^d}\widetilde{G}(f)G(h)dx \leq 2\|\widetilde{G}(f)\|_p\|G(h)\|_q. \tag{5.4}$$

Similarly, if $f\in L^p(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$, $2<p\leq\infty$, $h\in L^q(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$ and $q=\frac{p}{p-1}$, then

$$\left|\int_{\mathbb{R}^d} f(x)h(x)dx\right| \leq 2\int_{\mathbb{R}^d}\widetilde{G}(f)G(h)dx \leq 2\|G(f)\|_p\|\widetilde{G}(h)\|_q, \tag{5.5}$$

Acknowledgements.
Krzysztof Bogdan thanks the Department of Statistics of Stanford University for its hospitality during this work. We thank Fabrice Baudoin and Krzysztof Michalik for discussions on square functions and Jacek Zienkiewicz for a conversation on the proof of the Hardy-Stein identity. We thank Elias Stein for discussions on the results of the paper and on the history of the subject.

REFERENCES

[1] David Applebaum and Rodrigo Bañuelos. Martingale transform and Lévy processes on Lie groups. *Indiana Univ. Math. J.*, 63(4):1109–1138, 2014.
[2] Dominique Bakry. Functional inequalities for Markov semigroups. In *Probability measures on groups: Recent directions and trends*, pages 91–147. Tata Inst. Fund. Res., Mumbai, 2006.
[3] Rodrigo Bañuelos. Brownian motion and area functions. *Indiana Univ. Math. J.*, 35(3):643–668, 1986.
[4] Rodrigo Bañuelos. The foundational inequalities of D. L. Burkholder and some of their ramifications. *Illinois J. Math.*, 54(3):789–868 (2012), 2010.
[5] Rodrigo Bañuelos, Adam Bielaszewski, and Krzysztof Bogdan. Fourier multipliers for non-symmetric Lévy processes. In *Marcinkiewicz centenary volume*, volume 95 of *Banach Center Publ.*, pages 9–25. Polish Acad. Sci. Inst. Math., Warsaw, 2011.
[6] Rodrigo Bañuelos and Krzysztof Bogdan. Lévy processes and Fourier multipliers. *J. Funct. Anal.*, 250(1):197–213, 2007.
[7] Rodrigo Bañuelos and Pedro J. Méndez-Hernández. Space-time Brownian motion and the Beurling-Ahlfors transform. *Indiana Univ. Math. J.*, 52(4):981–990, 2003.
[8] Rodrigo Bañuelos and Charles N. Moore. *Probabilistic behavior of harmonic functions*, volume 175 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1999.
[9] Andrew G. Bennett. Probabilistic square functions and a priori estimates. *Trans. Amer. Math. Soc.*, 291(1):159–166, 1985.
[10] Krzysztof Bogdan, Yana Butko, and Karol Szczypkowski. Majorization, 4G Theorem and Schrödinger perturbations. *ArXiv e-prints*, November 2014.
[11] Krzysztof Bogdan, Tomasz Byczkowski, Tadeusz Kulczycki, Michal Ryznar, Renming Song, and Zoran Vondraček. *Potential analysis of stable processes and its extensions*, volume 1980 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009. Edited by Piotr Graczyk and Andrzej Stos.
[12] Krzysztof Bogdan, Bartłomiej Dyda, and Tomasz Luks. On Hardy spaces of local and nonlocal operators. *Hiroshima Math. J.*, 44(2):193–215, 2014.
[13] Krzysztof Bogdan, Bartłomiej Siudeja, and Andrzej Stós. Martin kernel for fractional Laplacian in narrow cones. *Potential Anal.*, 42(4):839–859, 2015.
[14] Nicolas Bouleau and Damien Lamberton. Théorie de Littlewood-Paley-Stein et processus stables. In *Séminaire de Probabilités, XX*, volume 1204 of *Lecture Notes in Mathematics*, pages 162–185. Springer-Verlag, 1986.
[15] Edward B. Davies. *Heat kernels and spectral theory*, volume 92 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1989.
[16] Deniz Karli. Harnack inequality and regularity for a product of symmetric stable process and Brownian motion. *Potential Anal.*, 38(1):95–117, 2013.
[17] Daesung Kim. Martingale transforms and the Hardy-Littlewood-Sobolev inequality for semigroups. *ArXiv e-prints*, June 2015.
[18] Ildoo Kim and Kyeong-Hun Kim. A generalization of the Littlewood-Paley inequality for the fractional Laplacian $(-\Delta)^{\alpha/2}$. *J. Math. Anal. Appl.*, 388(1):175–190, 2012.
[19] Victoria Knopova and René L. Schilling. A note on the existence of transition probability densities of Lévy processes. *Forum Math.*, 25(1):125–149, 2013.
[20] Nicolai V. Krylov. A generalization of the Littlewood-Paley inequality and some other results related to stochastic partial differential equations. *Ulam Quart.*, 2(4):16–26, 1994.
[21] John E. Littlewood and Raymond E. A. C. Paley. Theorems on Fourier Series and Power Series, II. *J. London Math. Soc.*, 42(1):57–89, 1937.
[22] Paul-André Meyer. Demonstration probabiliste de certaines inégalités de Littlewood-Paley. In *Séminaire de Probabilités, X*, volume 511 of *Lecture Notes in Mathematics*. Springer-Verlag.
[23] Paul-André Meyer. Retour sur la thèorie de Littlewood-Paley. In *Séminaire de Probabilités, XV*, volume 850 of *Lecture Notes in Mathematics*, pages 151–166. Springer-Verlag, 1979.
[24] Paul-André Meyer. Transformations de Riesz pour les lois gaussiennes. In *Séminaire de Probabilités, XVIII*, volume 1059 of *Lecture Notes in Mathematics*, pages 179–193. Springer-Verlag, 1984.
[25] Michael Perlmutter. On a class of Calderón-Zygmund operators arising from projections of martingale transforms. *Potential Anal.*, 42(2):383–401, 2015.

[26] Ken-Iti Sato. *Lévy processes and infinitely divisible distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2013. Translated from the 1990 Japanese original, Revised edition of the 1999 English translation.
[27] René L. Schilling, Renming Song, and Zoran Vondraček. *Bernstein functions*, volume 37 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, second edition, 2012. Theory and applications.
[28] Ichiro Shigekawa. Littlewood-Paley inequality for a diffusion satisfying the logarithmic Sobolev inequality and for the Brownian motion on a Riemannian manifold with boundary. *Osaka J. Math.*, 39(4):897–930, 2002.
[29] Elias M. Stein. On the maximal ergodic theorem. *Proc. Nat. Acad. Sci. U.S.A.*, 47:1894–1897, 1961.
[30] Elias M. Stein. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.
[31] Elias M. Stein. *Topics in harmonic analysis related to the Littlewood-Paley theory*. Annals of Mathematics Studies, No. 63. Princeton University Press, Princeton, N.J., 1970.
[32] Nicolas Th. Varopoulos. Aspects of probabilistic Littlewood-Paley theory. *J. Funct. Anal.*, 38(1):25–60, 1980.
[33] Antoni Zygmund. *Trigonometrical Series*. Annals of Mathematics Studies, No. 63. Cambridge University Press, Cambridge, 1959.

DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, 150 N. UNIVERSITY STREET, WEST LAFAYETTE, IN 47907-2067, USA
E-mail address: banuelos@math.purdue.edu

DEPARTMENT OF MATHEMATICS, WROCŁAW UNIVERSITY OF TECHNOLOGY, WYB. WYSPIAŃSKIEGO 27, 50-370 WROCŁAW, POLAND
E-mail address: Krzysztof.Bogdan@pwr.edu.pl

ECOLE CENTRALE DE MARSEILLE, I2M, 38 RUE FRÉDÉRIC JOLIOT CURIE, 13013 MARSEILLE, FRANCE
E-mail address: tomasz.luks@centrale-marseille.fr