REPRESENTATION SPACES FOR CENTRAL EXTENSIONS AND ALMOST COMMUTING UNITARY MATRICES

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ABSTRACT. Let Γ denote a central extension of the form $1 \to \mathbb{Z}^r \to \Gamma \to \mathbb{Z}^n \to 1$. In this paper we enumerate and describe the structure of the connected components of the spaces of homomorphisms $\operatorname{Hom}(\Gamma, U(m))$ and the associated moduli spaces $\operatorname{Rep}(\Gamma, U(m))$, where U(m) is the group of $m \times m$ unitary matrices.

1. INTRODUCTION

The space of ordered commuting n-tuples in a Lie group G can be analyzed using a variety of methods from algebraic topology and representation theory (see [1]); in particular these spaces can be identified with $\operatorname{Hom}(\mathbb{Z}^n, G) \subset G^n$. In this paper our goal is to consider a more complicated source group, namely the space of homomorphisms $\operatorname{Hom}(\Gamma, G)$ where Γ is no longer abelian, but rather a central extension of the form $1 \to \mathbb{Z}^r \to \Gamma \to \mathbb{Z}^n \to 1$. A key ingredient we will use is the notion of spaces of almost commuting elements (see [7] and [3]). We focus our attention on the unitary groups, for which we obtain complete descriptions. These in turn are used to shed light on the structure of $\operatorname{Hom}(\Gamma, U(m))$ and the associated spaces of representations $\operatorname{Rep}(\Gamma, U(m))$. An important motivation for this is the fact that they arise as moduli spaces of isomorphism classes of flat connections on principal U(m)-bundles over compact manifolds M which can be described as r-torus bundles over the n-torus.

Our results are rather intricate, as they expose a very rich structure encoding the components of these spaces of representations. For clarity of exposition we will focus here on the case when r = 1; the more cumbersome general case is described in Section 6. Let $B_n(U(m))$ denote the space of almost commuting n-tuples in U(m) i.e. the space of ordered n-tuples (A_1, \ldots, A_n) such that the pairwise commutators $[A_i, A_j]$ are all central in U(m). The characteristic polynomial defines a map $\chi: U(m) \to \mathbb{C}[z]$; for a central extension $1 \to \mathbb{Z} \to \Gamma \to \mathbb{Z}^n \to 1$ we have a natural restriction $\operatorname{Hom}(\Gamma, U(m)) \to U(m)$. These two maps can be composed to yield a function $\operatorname{Hom}(\Gamma, U(m)) \to \mathbb{C}[z]$. Given a polynomial p(z)we denote its inverse image in U(m) by $U(m)_{p(z)}$ and its inverse image in $\operatorname{Hom}(\Gamma, U(m))$ by $\operatorname{Hom}(\Gamma, U(m))_{p(z)}$. The extension Γ is defined by a k-invariant $\omega \in H^2(\mathbb{Z}^n, \mathbb{Z})$; for our purposes we write it in the following form (see Proposition 4.1): there exists a basis e_1, \ldots, e_n of \mathbb{Z}^n and an integer $t \leq n/2$ such that $\omega = c_1 e_1^* \wedge e_{t+1}^* + \cdots + c_t e_t^* \wedge e_{2t}^*$ where c_1, \ldots, c_t are positive integers such that c_i divides c_{i+1} for $i = 1, \ldots, t - 1$. We now define $\mathbb{C}[z]_{\Gamma}^m \subset \mathbb{C}[z]$ as the set of degree m complex polynomials p(z) such that (1) all roots of p(z) are roots of unity; and (2) if a root λ of p(z) is a primitive k-th root of unity, then the multiplicity of λ

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in p(z) is divisible by $\mu_k = \prod_{i=1}^t k/(k, c_i)$, where (k, c_i) is the greatest common divisor of k and c_i . We now state our main theorems, which summarize the results in §3 and §4.

Theorem A. Let $1 \to \mathbb{Z} \to \Gamma \to \mathbb{Z}^n \to 1$ with non-trivial k-invariant ω . Then there is a decomposition into connected components $Hom(\Gamma, U(m)) = \coprod_{p(z) \in \mathbb{C}[z]_{\Gamma}^m} Hom(\Gamma, U(m))_{p(z)}$, where the number of components is given by the coefficient of x^m in the generating function $\prod_{k\geq 1} \frac{1}{(1-x^{\mu_k})^{\Phi(k)}}$, where Φ is Euler's phi function.

For example, if Γ_1 denotes the integral Heisenberg group, then Hom $(\Gamma_1, U(m))$ has 1, 2, 4, 7, 13 components for m = 1, 2, 3, 4, 5 respectively (see Example 4.7). The number of components can be explicitly determined for any m using Theorem A.

Next we describe the structure of the components. As explained in Section 4, there is a map $B_n(U(m)) \to T(n, \mathbb{Q}/\mathbb{Z})$ defined using commutators, where $T(n, \mathbb{Q}/\mathbb{Z})$ denotes the set of all $n \times n$ skew-symmetric matrices with entries in \mathbb{Q}/\mathbb{Z} . These matrices can be used to count the components of the space of $n \times n$ almost commuting matrices. Given $D \in T(n, \mathbb{Q}/\mathbb{Z})$ we let $B_n(U(M))_D$ denote its inverse image under the map above. For $2t \leq n$ and $d_1, d_2, \ldots, d_t \neq 0 \in \mathbb{Q}/\mathbb{Z}$, let $D_n(d_1, d_2, \ldots, d_t) = (d_{ij}) \in T(n, \mathbb{Q}/\mathbb{Z})$ be the skew-symmetric matrix with

$$d_{ij} = \begin{cases} d_k & \text{if } (i,j) = (k+t,k), 1 \le k \le t; \\ -d_k & \text{if } (i,j) = (k,k+t), 1 \le k \le t; \\ 0 & \text{otherwise.} \end{cases}$$

We show that $B_n(U(m))_D$ is non-empty if and only if m is divisible by $\sigma(D) := \prod |d_k|$. If $m = l\sigma(D)$ for some positive integer l, then there is a map

$$\phi_D: \left[\left(U(m)/\mathbb{T}^l \right) \times (\mathbb{T}^n)^l \middle/ (\prod_{i=1}^t \mathbb{Z}/|d_k|)^l \right] \middle/ \Sigma_l \to B_n(U(m))_D$$

which is a rational homology equivalence for $l \geq 1$ and is a homeomorphism for l = 1. Moreover, ϕ_D induces a homeomorphism $\bar{B}_n(U(m))_D \cong (\mathbb{T}^n)^l / \Sigma_l$ after passing to quotients by the action of U(m).

Theorem B. Let $p(z) = \prod_{j=1}^{s} (z - \lambda_j)^{m_j}$, where $\lambda_1, \ldots, \lambda_s \in \mathbb{C}$ are distinct roots which are primitive k_j -th roots of unity. If Γ is a central extension of \mathbb{Z}^n by \mathbb{Z} with k-invariant $\omega = c_1 e_1^* \wedge e_{t+1}^* + \cdots + c_t e_t^* \wedge e_{2t}^*$ where c_1, \ldots, c_t are positive integers such that c_i divides c_{i+1} , then for every non-empty component there is a U(m)-equivariant homeomorphism

$$Hom(\Gamma, U(m))_{p(z)} \cong U(m) \times_{\prod_{j=1}^{s} U(m_j)} \prod_{j=1}^{s} B_n(U(m_j))_{D_n(-c_1q_j,...,-c_tq_j)}$$

where $q_j = \frac{1}{2\pi\sqrt{-1}} \log \lambda_j$. Moreover the orbit space under the action of U(m) is homeomorphic to a product of symmetric products of tori $\prod_{j=1}^{s} (\mathbb{T}^n)^{l_j} / \Sigma_{l_j}$ where $l_j = m_j / (\prod_{i=1}^{t} k_j / (k_j, c_i))$ for $j = 1, \ldots, s$.

These results give complete descriptions of the moduli spaces $\operatorname{Rep}(\Gamma, U(m))$, extending the techniques and results in [2], [3]. This paper was motivated by the results obtained in [10] for the case G = SU(2). As Γ is nilpotent, by [6] there are homotopy equivalences

 $\operatorname{Hom}(\Gamma, U(m)) \simeq \operatorname{Hom}(\Gamma, \operatorname{GL}(n, \mathbb{C}))$ and $\operatorname{Rep}(\Gamma, U(m)) \simeq \operatorname{Rep}(\Gamma, \operatorname{GL}(n, \mathbb{C}))$, and thus our results also provide descriptions for these a priori more complicated spaces.

This paper is organized as follows: in §2 we provide preliminaries and background; in §3 we discuss the spaces of almost commuting elements in the unitary groups; in §4 and §5 we analyze the spaces $\operatorname{Hom}(\Gamma, U(m))$ where Γ is a central extension $1 \to \mathbb{Z}^r \to \Gamma \to \mathbb{Z}^n \to 1$.

2. Preliminaries and background

Let $X_1, \ldots, X_n \in U(m)$ be commuting unitary matrices. We say that $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{T}^n := (\mathbb{S}^1)^n$ is an *n*-tuple of eigenvalues of (X_1, \ldots, X_n) if there exists a non-zero vector v such that $X_i v = \lambda_i v$ for all $1 \leq I \leq n$. Let $E_{\lambda} = \bigcap_{i=1}^n E_{\lambda_i}(X_i)$, where $E_{\lambda_i}(X_i)$ denotes the eigenspace of X_i associated to the eigenvalue λ_i . It is well known that we can simultaneously diagonalize all of the matrices X_i . Thus there is a direct sum decomposition

(1)
$$\mathbb{C}^m = E_{\lambda^1} \oplus \ldots \oplus E_{\lambda^s}$$

where $\lambda^1, \ldots, \lambda^s \in \mathbb{T}^n$ are the distinct *n*-tuples of eigenvalues of (X_1, \ldots, X_n) . The decomposition is unique up to the order of the eigenvalues.

The space of ordered commuting *n*-tuples of $m \times m$ unitary matrices can be identified with $\operatorname{Hom}(\mathbb{Z}^n, U(m))$. Let $T = \mathbb{T}^m$ be the maximal torus of diagonal matrices in U(m). Then $Z := \operatorname{Hom}(\mathbb{Z}^n, U(m))^T \cong (\mathbb{T}^n)^m$ is the subspace of $\operatorname{Hom}(\mathbb{Z}^n, U(m))$ consisting of ordered *n*-tuples of diagonal unitary matrices. Let U(m) act on $U(m) \times Z$ by left multiplication on the first factor and on $\operatorname{Hom}(\mathbb{Z}^n, U(m))$ by conjugation. Consider the U(m)-equvariant map

(2)
$$U(m) \times Z \to \operatorname{Hom}(\mathbb{Z}^n, U(m))$$

given by the conjugation action $(M, (X_i)) \mapsto (MX_iM^{-1})$. Let $N = N_{U(m)}(\mathbb{T}^m)$ be the normalizer and $W = N/\mathbb{T}^m \cong \Sigma_m$ be the Weyl group. The map (2) factors through $U(m) \times_N Z \cong (U(m)/\mathbb{T}^m) \times_{\Sigma_m} (\mathbb{T}^n)^m$ and descends to

(3)
$$\phi: (U(m)/\mathbb{T}^m) \times_{\Sigma_m} (\mathbb{T}^n)^m \to \operatorname{Hom}(\mathbb{Z}^n, U(m))$$

The map ϕ has a geometric interpretation. Note that $U(m)/\mathbb{T}^m$ is the space of ordered m-tuples of pairwisely orthogonal complex lines (L_1, \ldots, L_m) in \mathbb{C}^m . Hence, each element of the domain of ϕ can be regarded as an unordered m-tuple $[(L_1, \alpha^1), \ldots, (L_m, \alpha^m)]$ with $\alpha^1, \ldots, \alpha^m \in \mathbb{T}^n$. The map ϕ sends such an element to the almost commuting tuple $(X_1, X_2, \ldots, X_n) \in B(U(m))_D$ such that each α^j is an n-tuple of eigenvalues of the matrices X_i and each complex line L_j lies in the common eigenspaces E_{α^j} .

The map ϕ is surjective since commuting unitary matrices can be simultaneously diagonalized. It is not injective in general, but for $(X_1, \ldots, X_n) \in \text{Hom}(\mathbb{Z}^n, U(m))$ with eigenspace decomposition (1) and dim $E_{\lambda j} = m_j$, the preimage $\phi^{-1}(X_1, \ldots, X_n) \cong \prod_{j=1}^s (U(m_j)/\mathbb{T}^{m_j})/\Sigma_{m_j}$ and is Q-acyclic [5]. The map ϕ is a special case of the action map

$$(4) G/T \times_W X^T \to X$$

described in [5] and [4], where G is a connected compact Lie group with a maximal torus T acting on a space X with maximal rank isotropy subgroups and W is the Weyl group associated to T. As observed in [5], for any field \mathbb{F} with characteristic relative prime to |W|, the preimage of any point in X under the map (4) is \mathbb{F} -acyclic. By the Vietoris-Begle mapping theorem, it follows that $H^*(X;\mathbb{F}) \cong H^*(G/T \times_W X^T;\mathbb{F}) \cong H^*(G/T \times X^T;\mathbb{F})^W$. In particular, ϕ induces an isomorphism in rational cohomology. Passing to the U(m)-quotients,

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 ϕ becomes a homeomorphism and hence $\operatorname{Rep}(\mathbb{Z}^n, U(m)) \cong (\mathbb{T}^n)^m / \Sigma_m = \operatorname{Sym}_m \mathbb{T}^n$. In the following sections, we will generalize ϕ and the results above to almost commuting *n*-tuples of unitary matrices.

3. The space of almost commuting unitary matrices

Let G be a Lie group and K be a closed subgroup contained in the center Z(G) of G. The space of K-almost commuting n-tuples in G, denoted by $B_n(G, K)$, was studied in [3]. It consists of all ordered n-tuples $(A_1, A_2, \ldots, A_n) \in G^n$ such that the commutators $[A_i, A_j] \in K$ for all $1 \leq i, j \leq n$.

An equivalent formulation for K-almost commuting n-tuples is in terms of group homomorphisms. Let F_n be the free group on n generators a_1, \ldots, a_n . A homomorphism $f: (F_n, [F_n, F_n]) \to (G, K)$ is a group homomorphism $f: F_n \to G$ whose image $f([F_n, F_n])$ of the commutator subgroup $[F_n, F_n] \subset F_n$ is contained in K. It is clear that there is a bijection $f \mapsto (f(a_1), \ldots, f(a_n))$ between the sets of such homomorphisms and K-almost commuting n-tuples. We sometimes write $f \in B_n(G, K)$ to represent the corresponding almost commuting n-tuple.

In this section, we will study almost commuting tuples of unitary matrices. For notational simplicity, $B_n(U(m), Z(U(m)))$ will be abbreviated as $B_n(U(m))$.

Lemma 3.1. Let A, B be $m \times m$ unitary matrices with $[A, B] = \gamma I_m$. Then $\gamma^m = 1$. *Proof.* $\gamma^m = \det(\gamma I_m) = \det[A, B] = \det(ABA^{-1}B^{-1}) = 1$.

Suppose that $f: F_n \to U(m)$ is in $B_n(U(m))$. For any $u, v \in F_n$, $[f(u), f(v)] = \gamma I_m$ for some *m*-th root of unity γ by Lemma 3.1. The exponential function $z \mapsto e^{2\pi\sqrt{-1}z}$ establishes a group isomorphism between \mathbb{R}/\mathbb{Z} and $\mathbb{S}^1 \subset \mathbb{C}$ with inverse $w \mapsto \frac{1}{2\pi\sqrt{-1}} \log w$. The multiplicative group of *m*-th roots of unity and all roots of unity corresponds to the subgroup $\mathbb{Z}[\frac{1}{m}]/\mathbb{Z} \cong \mathbb{Z}/m$ and \mathbb{Q}/\mathbb{Z} of \mathbb{R}/\mathbb{Z} respectively under this isomorphism. Hence there is a map $F_n \times F_n \to \mathbb{Z}/m \subset \mathbb{Q}/\mathbb{Z}$ defined by $(u, v) \mapsto \frac{1}{2\pi\sqrt{-1}} \log \gamma$. Since $f([F_n, F_n]) \subset$ $\mathbb{Z}(U(m))$, the map factors through the abelianization of $F_n \times F_n$ and thus gives rise to a (\mathbb{Q}/\mathbb{Z}) -valued skew-symmetric bilinear form $\omega_f : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Q}/\mathbb{Z}$.

For an abelian group A, let T(n, A) be the set of all $n \times n$ skew-symmetric matrices with entries in A. Define a map

(5)
$$\rho: B_n(U(m)) \to T(n, \mathbb{Q}/\mathbb{Z})$$

by $\rho(f) = (d_{ij})$, where $[f(a_i), f(a_j)] = e^{2d_{ij}\pi\sqrt{-1}}I_m$. For $D \in T(n, \mathbb{Q}/\mathbb{Z})$, let $B_n(U(m))_D = \rho^{-1}(D)$. For $f \in B_n(U(m))_D$, the ordered *n*-tuple $(f(a_1), \ldots, f(a_n))$ is said to be *D*-commuting. Note that $\rho(f)$ is the skew-symmetric matrix associated to the bilinear form ω_f . The \mathbb{Z} -module structure on \mathbb{Q}/\mathbb{Z} induces a bi-module structure on $T(n, \mathbb{Q}/\mathbb{Z})$ over the ring of $n \times n$ matrices with integral entries. The following proposition is about the effect of change of basis of \mathbb{Z}^n on ρ .

Proposition 3.2. Suppose that $D, D' \in T(n, \mathbb{Q}/\mathbb{Z})$ and $D' = A^T D A$ for some $A \in GL(n, \mathbb{Z})$. Then there is an automorphism $\alpha : F_n \to F_n$ such that $f \in B_n(U(m))_D$ if and only if $f \circ \alpha \in B_n(U(m))_{D'}$. Hence α induces a homeomorphism $\alpha^* : B_n(U(m))_D \cong B_n(U(m))_{D'}$. Proof. Let F_n be the free product on generators a_1, \ldots, a_n . Since every $A \in GL(n, \mathbb{Z})$ can be written as a product of finitely many elementary matrices, it suffices to prove the proposition when A is an elementary matrix. If A is the elementary matrix obtained from I_n by adding k times the *i*-th column to the *j*-th one, then α can be taken as the automorphism with $\alpha(a_j) = a_i^k a_j$ and $\alpha(a_l) = a_l$ for $l \neq j$. For the cases where A is obtained from I_n by swapping two columns or by multiplying a column by -1, α can also be chosen in the obvious way. \Box

For $0 \leq t \leq n/2$ and $d_1, d_2, \ldots, d_t \neq 0 \in \mathbb{Q}/\mathbb{Z}$, let $D_n(d_1, d_2, \ldots, d_t) = (d_{ij}) \in T(n, \mathbb{Q}/\mathbb{Z})$ be the skew-symmetric matrix with

$$d_{ij} = \begin{cases} d_k & \text{if } (i,j) = (k+t,k), 1 \le k \le t; \\ -d_k & \text{if } (i,j) = (k,k+t), 1 \le k \le t; \\ 0 & \text{otherwise.} \end{cases}$$

Let $D = (d_{ij}) \in T(n, \mathbb{Q}/\mathbb{Z})$. By taking a common denominator, all the d_{ij} are contained in the subgroup $\langle [\frac{1}{q}] \rangle \cong \mathbb{Z}/q \subset \mathbb{Q}/\mathbb{Z}$ for some large enough integer q. By [9, Proposition 4.1], D is congruent to some matrix $D_n(d_1, \ldots, d_t)$ with the orders of $d_i \in \mathbb{Z}/q \subset \mathbb{Q}/\mathbb{Z}$ satisfying $|d_{i+1}|$ divides $|d_i|$ for all $1 \leq i \leq t-1$. Hence, by Proposition 3.2, understanding $B_n(U(m))_{D_n(d_1,\ldots,d_t)}$ is fundamental to the study of $B_n(U(m))$.

For $D = D_n(d_1, \ldots, d_t)$, define

(6)
$$\sigma(D) = \prod |d_i|.$$

Theorem 3.3. Let $2t \leq n$, $D = D_n(d_1, \ldots, d_t) \in T(n, \mathbb{Q}/\mathbb{Z})$ and $\gamma_i = e^{2d_i\pi\sqrt{-1}}$ for $1 \leq i \leq t$. Then $B_n(U(m))_D$ is non-empty if and only if $\sigma(D)$ divides m. In that case, suppose $l = m/\sigma(D) \in \mathbb{N}$ and $(A_1, \ldots, A_n) \in B_n(U(m))_D$. Then there exist orthonormal vectors $v_1, \ldots, v_l \in \mathbb{C}^m$ and $\alpha_{i'j}, \beta_{ij} \in \mathbb{S}^1 \subset \mathbb{C}$ for $1 \leq i \leq t < i' \leq n$ and $1 \leq j \leq l$ such that

- (1) $\{A_1^{p_1}A_2^{p_2}\dots A_t^{p_t}v_j | 0 \le p_i < |d_i| \text{ for } 1 \le i \le t, 1 \le j \le l\}$ is an orthonormal basis of \mathbb{C}^m ;
- (2) $A_1^{p_1} A_2^{p_2} \dots A_t^{p_t} v_j$ is an eigenvector of A_i with eigenvalue $\gamma_{i-t}^{p_{i-t}} \alpha_{ij}$ for $t+1 \leq i \leq 2t$ and an eigenvector of A_i with eigenvalue α_{ij} for $2t+1 \leq i \leq n$;
- (3) $A_i^{|d_i|}v_j = \beta_{ij}v_j \text{ for } 1 \le i \le t.$

Remark 3.4. (1) The theorem reduces to the well-known result of commuting unitary matrices when t = 0. In that case, $\sigma(D) = 1$ and l = m.

(2) The characteristic polynomial $\chi_i(z)$ of A_i can be expressed in terms of α_{ij} and β_{ij} as follows.

$$\chi_i(z) = \begin{cases} \prod_j (z^{|d_i|} - \beta_{ij})^{m/(|d_i|l)} & \text{if } 1 \le i \le t; \\ \prod_j (z^{|d_{i-t}|} - \alpha_{ij}^{|d_{i-t}|})^{m/(|d_{i-t}|l)} & \text{if } t+1 \le i \le 2t; \\ \prod_j (z - \alpha_{ij})^{m/l} & \text{if } 2t+1 \le i \le n. \end{cases}$$

Proof. First, suppose that $B_n(U(m))_D$ is non-empty and $(A_1, \ldots, A_n) \in B_n(U(m))_D$. Note that A_{t+1}, \ldots, A_n are pairwisely commuting unitary matrices and thus can be simultaneous diagonalized. In particular, there exist eigenvalues $\alpha_{i1} \in \mathbb{S}^1$ of $A_i, t+1 \leq i \leq n$, such that the intersection $W \subset \mathbb{C}^m$ of the corresponding eigenspaces is non-zero. Suppose $v \in W$.

Then for any $t+1 \leq i \leq 2r$,

$$A_i A_1^{p_1} A_2^{p_2} \dots A_t^{p_t} v = \gamma_{i-r}^{p_{i-r}} A_1^{p_1} A_2^{p_2} \dots A_t^{p_t} A_i v = \gamma_{i-t}^{p_{i-t}} A_1^{p_1} A_2^{p_2} \dots A_t^{p_t} \alpha_{i1} v$$
$$= \gamma_{i-t}^{p_{i-t}} \alpha_{i1} A_1^{p_1} A_2^{p_2} \dots A_t^{p_t} v.$$

Similarly, for any $2t + 1 \le i \le n$,

$$A_i A_1^{p_1} A_2^{p_2} \dots A_t^{p_t} v = A_1^{p_1} A_2^{p_2} \dots A_t^{p_r} A_i v = A_1^{p_1} A_2^{p_2} \dots A_t^{p_r} \alpha_{i1} v$$
$$= \alpha_{i1} A_1^{p_1} A_2^{p_2} \dots A_t^{p_t} v.$$

It follows that $A_1^{|d_1|}, A_2^{|d_2|}, \ldots, A_t^{|d_t|}$ restrict to unitary automorphisms on W. Since these $A_i^{|d_i|}$ commute and are unitary, there exists an unit vector $v_1 \in W$ which is a common eigenvector for $A_1^{|d_1|}, A_2^{|d_2|}, \ldots, A_t^{|d_t|}$. Let $A_i^{|d_i|}v_1 = \beta_{i1}v_1$. Then v_1 satisfies the properties (2) and (3) in the theorem. Let

$$V_1 = \operatorname{span} \{ A_1^{p_1} A_2^{p_2} \dots A_t^{p_t} v_1 | 0 \le p_i < |d_i| \text{ for } 1 \le i \le t \}.$$

Any two vectors in $\{A_1^{p_1}A_2^{p_2}\ldots A_t^{p_t}v_1| 0 \leq p_i < |d_i|$ for $1 \leq i \leq t\}$ are eigenvectors of different eigenvalues of A_i for some $t + 1 \leq i \leq n$. Hence the spanning set is a basis of V_1 and $\dim(V_1) = \prod |d_i| = \sigma(D)$. Let V_1^{\perp} be the orthogonal complement of V_1 in \mathbb{C}^m . Then $\dim(V_1^{\perp}) = m - \sigma(D)$. For any $1 \leq i \leq n$, since V_1 is invariant under the unitary matrix A_i , so is V_1^{\perp} . By repeating the argument above to the restrictions of A_1, \ldots, A_n on V_1^{\perp} , we conclude by induction that $\sigma(D)$ divides m and there exist vectors v_1, v_2, \ldots, v_l with the desired properties.

If *m* is divisible by $\sigma(D)$, one can choose arbitrary complex numbers $\alpha_{ij}, \beta_{ij} \in \mathbb{S}^1$ and vectors $A_1^{p_1}A_2^{p_2}\ldots A_t^{p_t}v_j$ for $0 \leq p_i < |d_i|, 1 \leq j \leq l$, which form an orthonormal basis of \mathbb{C}^n . Any such choice uniquely determines an ordered *n*-tuple $(A_1,\ldots,A_n) \in B_n(U(m))_D$ satisfying the properties stated in the theorem. This shows that $B_n(U(m))_D$ is non-empty if *m* is divisible by $\sigma(D)$.

Let $D = D_n(d_1, \ldots, d_t)$ and $m = l\sigma(D)$ for some positive integer l. By Proposition 3.3, $B_n(U(m))_D$ is non-empty. We will describe a subspace $Z_D \subset B_n(U(m))_D$ which is homeomorphic to a torus. In light of Theorem 3.3, it is more convenient for us to index the rows and columns of a $m \times m$ matrix by the indexing set

(7)
$$I = \{(p_1, p_2, \dots, p_t, j) | 0 \le p_i < |d_i|, 1 \le j \le l\}.$$

Definition 3.5. Let $D = D_n(d_1, \ldots, d_t) \in T(n, \mathbb{Q}/\mathbb{Z})$ and $\gamma_i = e^{2d_i\pi\sqrt{-1}}$ for $1 \leq i \leq t$. Suppose that $m = l\sigma(D)$ for some positive integer l. Index the rows and columns of $m \times m$ matrices by the index set I in (7). Define $Z_D \subset B_n(U(m))_D$ be the subset consisting of all ordered n-tuples (A_1, \ldots, A_n) of the following forms, parametrized by $\alpha_{i'j}, \beta_{ij} \in \mathbb{S}^1 \subset \mathbb{C}$ for $1 \leq i \leq t < i' \leq n$ and $1 \leq j \leq l$:

(1) For $1 \leq i \leq t$,

$$(A_i)_{\mu\nu} = \begin{cases} 1 & \text{if } \mu = (p_1, \dots, p_i, \dots, p_t, j), \\ \nu = (p_1, \dots, p_i - 1, \dots, p_t, j); \\ \beta_{ij} & \text{if } \mu = (p_1, \dots, p_{i-1}, 1, p_{i+1}, \dots, p_t, j), \\ \nu = (p_1, \dots, p_{i-1}, |d_i| - 1, p_{i+1}, \dots, p_t, j); \\ 0 & \text{otherwise.} \end{cases}$$

(2) For $t+1 \le i \le 2t$, $(A_i)_{\mu\nu} = \begin{cases} \gamma_{i-t}^{p_{i-t}} \alpha_{ij} & \text{if } \mu = \nu = (p_1, \dots, p_t, j); \\ 0 & \text{otherwise.} \end{cases}$

(3) For $2t + 1 \le i \le n$,

$$(A_i)_{\mu\nu} = \begin{cases} \alpha_{ij} & \text{if } \mu = \nu = (p_1, \dots, p_t, j); \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.6. If m = 6, n = 5 and

$$D = D_5(1/2, 1/3) = \begin{bmatrix} 0 & 0 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & -1/3 & 0 \\ 1/2 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

then $\gamma_1 = -1, \gamma_2 = e^{2\pi\sqrt{-1/3}}, \sigma(D) = 6$ and l = 1. The subspace $Z_D \cong (\mathbb{S}^1)^5$ consists of tuples (A_1, \ldots, A_5) of the following forms

$$A_{1} = \begin{bmatrix} \beta_{11} & & & \\ 1 & & \beta_{11} \\ & 1 & & \\ & & 1 \end{bmatrix}, A_{2} = \begin{bmatrix} \alpha_{21} & & & \\ \beta_{21} \\ & & \beta_{21} \\ & & & \\ 1 & & & \\ & & & & \\ & & & \\ & & & \\ & & &$$

In general for any positive integer l and $m = l\sigma(D)$, each matrix A_i in $(A_1, \ldots, A_n) \in Z_D$ is a block sum of l matrices of the form above.

By Theorem 3.3, any $(A_1, \ldots, A_n) \in B_n(U(m))_D$ is conjugate to an element in Z_D . In other words, the map

(8)
$$U(m) \times Z_D \to B_n(U(m))_D$$

given by the conjugation action $(M, (A_i)) \mapsto (MA_iM^{-1})$ is surjective. The map is equivariant with respect to the U(m)-action given by left multiplication on the factor U(m) of the domain and conjugation action on the target. This map (8) is invariant under a few obvious actions on the domain corresponding to different choices of the vectors v_j in Theorem 3.3.

(a) For each $1 \leq j \leq l$, an S¹-action on the domain is given by

$$(M, (A_i)) \mapsto (MI(j, \theta), (I(j, \theta)^{-1}A_iI(j, \theta))) = (MI(j, \theta), (A_i))$$

where $\theta \in \mathbb{S}^1$ and $I(j, \theta) \in U(m)$ is the matrix

(9)
$$I(j,\theta)_{\mu\nu} = \begin{cases} 1 & \text{if } \mu = \nu = (p_1, \dots, p_t, j'), j' \neq j \\ \theta & \text{if } \mu = \nu = (p_1, \dots, p_t, j) \\ 0 & \text{otherwise,} \end{cases}$$

obtained from the $m \times m$ identity matrix by multiplying all the (p_1, \ldots, p_t, j) -th columns, $0 \le p_i < |d_i|$, by $\theta \in \mathbb{S}^1$. This action corresponds to replacing v_j by θv_j in Theorem 3.3. (b) For each $1 \le k \le t, 1 \le j \le l$, an action on the domain is given by

$$(M, (A_i)) \mapsto (MA_k(j), (A_k(j)^{-1}A_iA_k(j))),$$

where $A_k(j) \in U(m)$ is defined by

$$A_{k}(j)_{\mu\nu} = \begin{cases} 1 & \text{if } \mu = (p_{1}, \dots, p_{k}, \dots, p_{t}, j), \nu = (p_{1}, \dots, p_{k} - 1, \dots, p_{t}, j) \\ & \text{or } \mu = \nu = (p_{1}, \dots, p_{k}, \dots, p_{t}, j'), j' \neq j; \\ \beta_{kj} & \text{if } \mu = (p_{1}, \dots, p_{k-1}, 1, p_{k+1}, \dots, p_{t}, j), \\ & \nu = (p_{1}, \dots, p_{k-1}, |d_{k}| - 1, p_{k+1}, \dots, p_{t}, j); \\ 0 & \text{otherwise.} \end{cases}$$

This action corresponds to replacing v_j by $A_k v_j$ in Theorem 3.3. Note that $A_k(j)^{|d_k|}$ is equal to $I(j, \beta_{kj})$ in (9) above. Also, under the parametrization $Z_D \cong (\mathbb{T}^n)^l$, this action multiplies $\alpha_{k+t,j}$ by γ_k and keeps the other $\alpha_{i'j}$ and β_{ij} fixed.

(c) A Σ_l -action on the domain is given by

$$(M, (A_i)) \mapsto (MP_{\tau}^{-1}, (P_{\tau}A_iP_{\tau}^{-1})),$$

where $\tau \in \Sigma_l$ and P_{τ} is the matrix

$$P_{\tau} = \begin{cases} 1 & \text{if } \mu = (p_1, \dots, p_t, j), \nu = (p_1, \dots, p_t, \tau(j)) \\ 0 & \text{otherwise.} \end{cases}$$

obtained by applying the permutation τ to the columns of the $m \times m$ identity matrix. This action corresponds to replacing v_j by $v_{\tau(j)}$ in Theorem 3.3.

Let H be the group consisting of self homeomorphisms of $U(m) \times Z_D$ generated by the three types of actions above. Then the action induced by $\{I(j,\theta) : 1 \leq j \leq l, \theta \in \mathbb{S}^1\}$ in (a) generates a normal subgroup of H isomorphic to \mathbb{T}^l with quotient H/\mathbb{T}^l the wreath product $\mathbb{Z}_D \wr \Sigma_l \cong (\mathbb{Z}_D)^l \rtimes \Sigma_l$ generated by the actions in (b) and (c). Here \mathbb{Z}_D is the abelian group $\prod_{i=1}^t \mathbb{Z}/|d_i|$. Hence, the map (8) factors through

$$(U(m) \times Z_D)/H \cong \left(U(m)/\mathbb{T}^l \right) \times (\mathbb{T}^n)^l / (\mathbb{Z}_D \wr \Sigma_l) \cong \left[\left(U(m)/\mathbb{T}^l \right) \times (\mathbb{T}^n)^l / (\mathbb{Z}_D)^l \right] / \Sigma_l.$$

Our next theorem states that the induced factor map

(10)
$$\phi_D : \left[\left(U(m)/\mathbb{T}^l \right) \times (\mathbb{T}^n)^l \middle/ (\mathbb{Z}_D)^l \right] \middle/ \Sigma_l \to B_n(U(m))_D$$

of (8) is a good approximation to $B_n(U(m))_D$.

Theorem 3.7. Let $2t \leq n$ and $D = D_n(d_1, \ldots, d_t) \in T(n, \mathbb{Q}/\mathbb{Z})$. Suppose that $\sigma(D) = \prod |d_i|$ divides m and $l = m/\sigma(D)$. Then the map ϕ_D in (10) is a rational homology equivalence for any $l \geq 1$ and a homeomorphism for l = 1. The space $B_n(U(m))_D$ is path-connected and has rational cohomology

(11)
$$H^*(B_n(U(m))_D; \mathbb{Q}) \cong H^*((U(m)/\mathbb{T}^l) \times (\mathbb{T}^n)^l; \mathbb{Q})^{\Sigma_l}.$$

Also, ϕ_D induces a homeomorphism $\overline{B}_n(U(m))_D := B_n(U(m))_D/U(m) \cong (\mathbb{T}^n)^l/\Sigma_l$ after passing to the U(m)-quotients.

Proof. Since the map (8) is surjective by Theorem 3.3, so is ϕ_D . Let $(A_1, \ldots, A_n) \in B_n(U(m))_D$. Since $A_{t+1}, A_{t+2}, \ldots, A_n$ are pairwisely commuting, they can be simultaneously diagonalized. Let Λ be the set of all (n-t)-tuples $\lambda = (\lambda_{t+1}, \ldots, \lambda_n) \in \mathbb{T}^{n-t}$ of eigenvalues of $(A_{t+1}, A_{t+2}, \ldots, A_n)$ and $\Lambda_0 = \{\lambda = (\lambda_{t+1}, \ldots, \lambda_n) \in \Lambda \mid 0 \leq \frac{1}{2\pi\sqrt{-1}} \log \lambda_{t+i} < \frac{1}{|d_i|}, \forall 1 \leq i \leq t\}$. Suppose $\lambda^1, \ldots, \lambda^s$ are all the distinct elements in Λ_0 . Let $l_j = \dim E_{\lambda^j}$. Then $\sum_{j=1}^s l_j = l$. One can show that the preimage $\phi_D^{-1}(A_1, \ldots, A_n) \cong \prod_{j=1}^s U(l_j)/(T^{l_j} \rtimes \Sigma_{l_j})$, which is Q-acyclic in general and a single point if l = 1. Hence, ϕ_D is a rational homology equivalence for any $l \geq 1$ by the Vietoris-Begle mapping theorem and a homeomorphism for l = 1. In particular, $B_n(U(m))_D$ is path-connected. Note that the self-homeomorphisms of $(U(m)/\mathbb{T}^l) \times (\mathbb{T}^n)^l$ given by the action of \mathbb{Z}_D in the domain of ϕ_D are homotopic to the identity map. Thus we have the isomorphism (11). Also, each preimage of ϕ_D is U(m)-transitive and so ϕ_D becomes a homeomorphism after passing to the U(m)-quotients. Hence,

$$\bar{B}_{n}(U(m))_{D} \cong \left[\left(\mathbb{T}^{n} \right)^{l} \middle/ \left(\mathbb{Z}_{D} \right)^{l} \right] \middle/ \Sigma_{l} \cong \left[\left(\mathbb{T}^{n} \middle/ \mathbb{Z}_{D} \right)^{l} \middle/ \Sigma_{l} \right]^{l} \\
\cong \left[\left(\mathbb{S}^{1} \right)^{n} \middle/ \left(\prod_{i=1}^{t} \mathbb{Z} \middle/ \left| d_{i} \right| \right) \right]^{l} \middle/ \Sigma_{l} \\
\cong \left[\left(\prod_{i=1}^{t} \mathbb{S}^{1} \middle/ \left(\mathbb{Z} \middle/ \left| d_{i} \right| \right) \right) \times \left(\mathbb{S}^{1} \right)^{n-t} \right]^{l} \middle/ \Sigma_{l} \\
\cong \left[\left(\prod_{i=1}^{t} \mathbb{S}^{1} \right) \times \left(\mathbb{S}^{1} \right)^{n-t} \right]^{l} \middle/ \Sigma_{l} = \left(\mathbb{T}^{n} \right)^{l} \middle/ \Sigma_{l}.$$

By Theorem 3.3 and 3.7, we know that for $D = D_n(d_1, \ldots, d_t)$, the space $B_n(U(m))_D$ is non-empty and path connected if $\sigma(D)$ divides m and empty otherwise. We will extend the definition of σ in such a way that the same statement holds for any $D \in T(n, \mathbb{Q}/\mathbb{Z})$. This allows us to identify the path connected components of $B_n(U(m))$ and compute the number of them.

Definition 3.8. For any $n \times n$ matrix $A \in M_{n \times n}(\mathbb{Q}/\mathbb{Z})$, the row space R(A) of A is the sub-module of $(\mathbb{Q}/\mathbb{Z})^n$ generated by the rows of A over \mathbb{Z} . Define

(12)
$$\sigma(A) = \sqrt{|R(A)|},$$

where |R(A)| is the cardinality of R(A).

Lemma 3.9. (a) For $D = D_n(d_1, \ldots, d_t) \in T(n, \mathbb{Q}/\mathbb{Z}), \sqrt{|R(D)|} = \prod_i |d_i|$. (b) If $A, A' \in M_{n \times n}(\mathbb{Q}/\mathbb{Z})$ with A' = BAC for some $B, C \in GL(n, \mathbb{Z})$, then $\sigma(A) = \sigma(A')$.

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(c) For $D \in T(n, \mathbb{Q}/\mathbb{Z})$, $\sigma(D)$ is an integer. If $\sigma(D)$ divides m, then $D \in T(n, \mathbb{Z}/m)$.

Proof. Part (a) is obvious. For part (b), since BA can be obtained from A by elementary row operations, R(BA) = R(A). Also, note that the multiplication map $v \mapsto vC$ for any row vectors $v \in (\mathbb{Q}/\mathbb{Z})^n$ establishes an automorphism of $(\mathbb{Q}/\mathbb{Z})^n$. Hence, $R(A') = R(BAC) \cong$ R(BA) = R(A) as \mathbb{Z} -modules. This proves part (b). For part (c), note that $D \in T(n, \mathbb{Z}/q)$ for some positive integer q. By [9, Proposition 4.1], D is congruent to some matrix D' = $D_n(d_1, \ldots, d_t) \in T(n, \mathbb{Z}/q)$. It follows from part (a) and (b) that $\sigma(D) = \sigma(D') = \prod_i |d_i|$ is an integer. Finally, if $D \in T(n, \mathbb{Q}/\mathbb{Z})$ has an entry $d_{ij} \notin \mathbb{Z}/m$, then *i*-th and *j*-th rows of D generate a \mathbb{Z} -submodule of R(D) with cardinality a multiple of $|d_{ij}|^2$. This implies $|d_{ij}|$ divides $\sigma(D)$ and so $\sigma(D)$ does not divide m.

Corollary 3.10. For $D \in T(n, \mathbb{Q}/\mathbb{Z})$, the space $B_n(U(m))_D$ is non-empty and path connected if $\sigma(D)$ divides m, and is empty otherwise. Hence, $B_n(U(m))$ can be expressed as the disjoint union $B_n(U(m)) = \coprod_{\substack{D \in T(n,\mathbb{Z}/m) \\ \sigma(D)|m}} B_n((U(m))_D$ of path connected components.

Proof. By [9, Proposition 4.1], D is congruent to some matrix $D' = D_n(d_1, \ldots, d_t) \in T(n, \mathbb{Q}/\mathbb{Z})$. Since $B_n(U(m))_D \cong B_n(U(m))_{D'}$ by Proposition 3.2 and $\sigma(D) = \sigma(D')$ by Lemma 3.9, the first statement of the corollary follows from the corresponding statement for D' in Theorem 3.3 and 3.7. Since the map ρ in (5) is continuous, $B_n(U(m))$ can be expressed as the disjoint union of path connected components above.

By Corollary 3.10, the number of path-connected components of $B_n(U(m))$ is equal to

(13)
$$N(n,m) = |\{D \in T(n,\mathbb{Z}/m) : \sigma(D) \text{ divides } m\}|.$$

We will derive formulas for N(n, m). To do this, we need to introduce some notation. For positive integers $k, n \ge 1$, let J'(n, k) be the set of all partitions of k with at most n/2 parts and $J(n, k) = \bigcup_{1 \le k' \le k} J'(n, k')$. Every element in J(n, k) is a partition which can be written uniquely in the form

$$\underbrace{a_1 + \ldots + a_1}_{t_1 a_1's} + \underbrace{a_2 + \ldots + a_2}_{t_2 a_2's} + \ldots + \underbrace{a_j + \ldots + a_j}_{t_j a_j's}$$

with $a_1 > a_2 > \ldots > a_j > 0$, $\sum t_i a_i \le k$ and $\sum t_i \le n/2$. For instance, J(5, 4) consists of the eight partitions 1, 2, 1 + 1, 3, 2 + 1, 4, 3 + 1 and 2 + 2.

For the rest of this section, we will denote the elements of \mathbb{Z}/m by $0, 1, 2, \ldots, m-1$ rather than $[0], [\frac{1}{m}], [\frac{2}{m}], \ldots, [\frac{m-1}{m}]$. Suppose that $q = p^k$ is a power of a prime p with $k \ge 1$ and $\alpha \in J(n,k)$ is the partition $\sum_{i=1}^{j} t_i a_i$. Let $b_i = k - a_i$. Define $D_{\alpha} \in T(n, \mathbb{Z}/q)$ to be the matrix

$$D_{\alpha} = D_{n}(\underbrace{p^{b_{1}}, \dots, p^{b_{1}}}_{t_{1} p^{b_{1}'s}}, \underbrace{p^{b_{2}}, \dots, p^{b_{2}}}_{t_{2} p^{b_{2}'s}}, \dots, \underbrace{p^{b_{j}}, \dots, p^{b_{j}}}_{t_{j} p^{b_{j}'s}})$$

Note that $\sigma(D_{\alpha}) = \prod (p^{a_i})^{t_i} = p^{\sum t_i a_i}$ divides q.

Proposition 3.11. Fix $k, n \ge 1$. Let p be a prime, $q = p^k$ and $\alpha \in J(n,k)$. Then the number of matrices in $T(n, \mathbb{Z}/q)$ congruent to D_{α} is given by

$$N_p(\alpha) = \frac{p \sum_{i=1}^{j} t_i a_i (s_{i-1}+s_i-1)}{\prod_{i=1}^{j} \prod_{l=1}^{t_i} \prod_{l=1}^{n} \left(1 - \frac{1}{p^l}\right)}$$

where $s_i = n - 2\sum_{l=1}^{i} t_l$. The number of matrices $D \in T(n, \mathbb{Z}/q)$ with $\sigma(D)$ divides q is equal to

(14)
$$N(n,q) = 1 + \sum_{\alpha \in J(n,k)} N_p(\alpha).$$

Remark 3.12. The summand 1 in (14) accounts for the zero matrix $\mathbf{0} \in T(n, \mathbb{Z}/q)$ in the definition of N(n,q) in (13). This corresponds to the path connected component $B_n(U(m))_{\mathbf{0}}$ of $B_n(U(m))$ consisting of commuting matrices in Corollary 3.10.

Proof. Let $G = GL(n, \mathbb{Z}/q)$ and $G_{D_{\alpha}}$ be the stabilizer of D_{α} . Then the number of matrices in $T(n, \mathbb{Z}/q)$ congruent to D_{α} is equal to $|G|/|G_{D_{\alpha}}|$. Recall the well-known result that

(15)
$$|G| = q^{n^2} \prod_{l=1}^{n} \left(1 - \frac{1}{p^l}\right)$$

To count $|G_{D_{\alpha}}|$, consider the skew-symmetric bilinear form $\omega : (\mathbb{Z}/q)^n \times (\mathbb{Z}/q)^n \to \mathbb{Z}/q$ associated to $D_{\alpha} = (d_{ij})$. Let $v_1, \ldots v_n$ be the columns of a matrix $A \in GL(n, \mathbb{Z}/q)$. Then $A^T D_{\alpha} A = D_{\alpha}$ if and only if $\omega(v_i, v_j) = d_{ij}$ for all $1 \leq i, j \leq n$. Hence, $|G_{D_{\alpha}}|$ is equal to the number of ordered bases $v_1, v_2, \ldots v_n$ of $(\mathbb{Z}/q)^n$ satisfying $\omega(v_i, v_j) = d_{ij}$. We will count them by picking the basis vectors in the order $v_1, v_{t+1}, v_2, v_{t+2}, \ldots, v_t, v_{2t}, v_{2t+1}, v_{2t+2}, \ldots, v_n$. To pick v_1 , there are $\left(q^{2t_1} - \left(\frac{q}{p}\right)^{2t_1}\right)q^{s_1} = q^n\left(1 - \frac{1}{p^{2t_1}}\right)$ choices. Without loss of generality, we may assume $v_1 = e_1$. Then it is obvious that there are $p^{b_1}q^{n-1}$ choices for v_{t+1} . Again we may assume $v_{t+1} = e_{t+1}$. Similar arguments show that there are $(p^{b_1})^2\left(q^{2t_1-2} - \left(\frac{q}{p}\right)^{2t_1-2}\right)q^{s_1} = (p^{b_1})^2q^{n-2}\left(1 - \frac{1}{p^{2t_1-2}}\right)$ choices for v_2 , then $(p^{b_1})^3q^{n-3}$ choices for v_{t+2} and so on. Hence, in total there are

$$(p^{b_1})^{1+2+\ldots+(2t_1-1)}q^{n+(n-1)+\ldots(n-2t_1+1)}\prod_{l=1}^{t_1}\left(1-\frac{1}{p^{2l}}\right) = (p^{b_1})^{t_1(2t_1-1)}q^{t_1(s_0+s_1+1)}\prod_{l=1}^{t_1}\left(1-\frac{1}{p^{2l}}\right)$$

choices for the vectors $v_1, v_{t+1}, \ldots, v_{t_1}, v_{t+t_1}$. Similarly, for the next $2t_2$ vectors $v_{t_1+1}, v_{t+t_1+1}, \ldots, v_{t_1+t_2}, v_{t+t_1+t_2}$, there are $(p^{b_1})^{4t_1t_2}(p^{b_2})^{t_2(2t_2-1)}q^{t_2(s_1+s_2+1)}\prod_{l=1}^{t_2} \left(1-\frac{1}{p^{2l}}\right)$ choices. Proceeding in this way, after the vectors $v_a, v_{t+a}, 1 \le a \le t_1 + t_2 + \ldots + t_{i-1}$ are chosen, there are

(16)
$$(p^{b_1})^{4t_1t_i} \dots (p^{b_{i-1}})^{4t_{i-1}t_i} (p^{b_i})^{t_i(2t_i-1)} q^{t_i(s_{i-1}+s_i+1)} \prod_{l=1}^{t_i} \left(1 - \frac{1}{p^{2l}}\right)$$

choices for the next $2t_i$ vectors v_a, v_{t+a} , where $t_1 + t_2 + \ldots + t_{i-1} + 1 \le a \le t_1 + t_2 + \ldots + t_i$. Finally, after the first $2(t_1 + \ldots + t_j)$ vectors are chosen, in total there are

(17)
$$(p^{b_1})^{2t_1s_j}\dots(p^{b_j})^{2t_js_j}q^{s_j^2}\prod_{l=1}^{s_j}\left(1-\frac{1}{p^l}\right)$$

choices for the remaining s_j vectors $v_{t_1+\ldots+t_j+1},\ldots,v_n$. Hence, taking the product of (16) for $i = 1, \ldots, j$ and (17), we get

(18)
$$|G_{D_{\alpha}}| = q^{e} \left(\prod_{i=1}^{j} (p^{b_{i}})^{f_{i}}\right) \left(\prod_{i=1}^{j} \prod_{l=1}^{t_{i}} \left(1 - \frac{1}{p^{2l}}\right)\right) \left(\prod_{l=1}^{s_{j}} \left(1 - \frac{1}{p^{l}}\right)\right)$$

where $e = s_j^2 + \sum_{i=1}^j t_i (s_{i-1} + s_i + 1)$ and

$$f_{i} = t_{i} \left(2s_{j} + 2t_{i} - 1 + 4\sum_{l=i+1}^{j} t_{l} \right)$$

= $t_{i} \left(2n - 4\sum_{l=1}^{j} t_{l} + 2t_{i} - 1 + 4\sum_{l=i+1}^{j} t_{l} \right)$
= $t_{i} \left(2n - 1 - 2\sum_{l=1}^{i-1} t_{l} - 2\sum_{l=1}^{i} t_{l} \right)$
= $t_{i} (s_{i-1} + s_{i} - 1).$

Note that

$$e + \sum_{i=1}^{j} f_i = s_j^2 + \sum_{i=1}^{j} t_i (s_{i-1} + s_i + 1) + \sum_{i=1}^{j} t_i (s_{i-1} + s_i - 1)$$

$$= s_j^2 + \sum_{i=1}^{j} 2t_i (s_{i-1} + s_i)$$

$$= s_j^2 + \sum_{i=1}^{j} (s_{i-1} - s_i) (s_{i-1} + s_i)$$

$$= s_j^2 + \sum_{i=1}^{j} (s_{i-1}^2 - s_i^2)$$

$$= s_0^2$$

$$= n^2.$$

Hence, dividing (15) by (18), we get

$$N_{p}(\alpha) = \frac{|G|}{|G_{D_{\alpha}}|} = \frac{\left(\prod_{i=1}^{j} q^{f_{i}}\right) \left(\prod_{l=s_{j}+1}^{n} \left(1 - \frac{1}{p^{l}}\right)\right)}{\left(\prod_{i=1}^{j} (p^{b_{i}})^{f_{i}}\right) \left(\prod_{i=1}^{j} \prod_{l=1}^{t_{i}} \left(1 - \frac{1}{p^{2l}}\right)\right)}$$
$$= \frac{\left(\prod_{i=1}^{j} (p^{a_{i}})^{f_{i}}\right) \left(\prod_{l=s_{j}+1}^{n} \left(1 - \frac{1}{p^{l}}\right)\right)}{\prod_{i=1}^{j} \prod_{l=1}^{t_{i}} \left(1 - \frac{1}{p^{2l}}\right)}.$$

The last statement of the proposition follows easily from the fact that any non-zero $D \in T(n, \mathbb{Z}/q)$ with $\sigma(D)$ divides q is congruent to D_{α} for a unique $\alpha \in J(n, k)$.

Example 3.13. Let α, β, γ be the partition 1, 2, 1 + 1 respectively. For the partitions α, β , $j = 1, t_1 = 1, s_0 = n$ and $s_1 = n - 2$.

$$N_p(\alpha) = \frac{p^{2n-3} \left(1 - \frac{1}{p^{n-1}}\right) \left(1 - \frac{1}{p^n}\right)}{1 - \frac{1}{p^2}} = \frac{(p^{n-1} - 1)(p^n - 1)}{p^2 - 1}$$

$$N_p(\beta) = \frac{p^{2(2n-3)} \left(1 - \frac{1}{p^{n-1}}\right) \left(1 - \frac{1}{p^n}\right)}{1 - \frac{1}{p^2}} = \frac{p^{2n-3} (p^{n-1} - 1)(p^n - 1)}{p^2 - 1}$$

For the partition γ , $j = 1, t_1 = 2, a_1 = 1, s_0 = n$ and $s_1 = n - 4$.

$$N_p(\gamma) = \frac{p^2(p^{n-3}-1)(p^{n-2}-1)(p^{n-1}-1)(p^n-1)}{(p^2-1)(p^4-1)}$$

Hence, for a prime number p,

$$N(n,p) = 1 + N_p(\alpha) = 1 + \frac{(p^{n-1} - 1)(p^n - 1)}{p^2 - 1}$$

This agrees with [2, Theorem 1]. For $n \ge 4$,

$$N(n, p^2) = 1 + N_p(\alpha) + N_p(\beta) + N_p(\gamma)$$

= 1 + $\frac{(p^{n-1} - 1)(p^n - 1)(p^{2n+1} - p^n - p^{n-1} + p^4 + p^2 - 1)}{(p^2 - 1)(p^4 - 1)}$

We will now look at N(n,m) in the general case where m is not necessarily a prime power. Let m_1, m_2 be relatively prime positive integers and $m = m_1 m_2$. The inclusions $\mathbb{Z}/m_i \cong \mathbb{Z}[\frac{1}{m_i}]/\mathbb{Z} \subset \mathbb{Z}[\frac{1}{m_i}]/\mathbb{Z} \cong \mathbb{Z}/m$ establish direct sum decompositions $\mathbb{Z}/m_1 \oplus \mathbb{Z}/m_2 \cong \mathbb{Z}/m$ and

(19)
$$T(n,\mathbb{Z}/m_1)\oplus T(n,\mathbb{Z}/m_2) \xrightarrow{\cong} T(n,\mathbb{Z}/m).$$

Proposition 3.14. Let $m = m_1m_2$ with m_1, m_2 be relatively prime positive integers. Suppose $D_i \in T(n, \mathbb{Z}/m_i)$ for i = 1, 2. Then $D_1 + D_2 \in T(n, \mathbb{Z}/m)$ satisfies $\sigma(D_1 + D_2) = \sigma(D_1)\sigma(D_2)$. Moreover, $N(n, m) = N(n, m_1)N(n, m_2)$.

Proof. We will show that $R(D_1) \oplus R(D_2) \cong R(D_1 + D_2)$ as Z-submodules of $(\mathbb{Z}/m_1)^n \oplus (\mathbb{Z}/m_2)^n \cong (\mathbb{Z}/m)^n$. For $1 \leq j \leq n$, let $u_j \in (\mathbb{Z}/m_1)^n, v_j \in (\mathbb{Z}/m_2)^n$ be the *j*-th row of D_1, D_2 respectively. Then each row $u_i + v_i$ of $D_1 + D_2$ is in $R(D_1) \oplus R(D_2)$ and hence $R(D_1 + D_2) \subset R(D_1) \oplus R(D_2)$. For the other inclusion, pick integers a, b so that $am_1 \equiv 1 \pmod{m_2}$ and $bm_2 \equiv 1 \pmod{m_1}$. Then $am_1(u_i + v_i) = v_i$ since $m_1u_i = m_2v_i = \vec{0} \in (\mathbb{Z}/m)^n$. Similarly $bm_2(u_i + v_i) = u_i$. It follows that $R(D_1) \oplus R(D_2) \subset R(D_1 + D_2)$. Hence $R(D_1) \oplus R(D_2) \cong R(D_1 + D_2)$ and $\sigma(D_1 + D_2) = \sigma(D_1)\sigma(D_2)$. To prove the last part of the proposition, note that since $R(D_i) \subset (\mathbb{Z}/m_i)^n$, any prime divisor of $\sigma(D_i)$ divides m_i . Therefore, $\sigma(D_1 + D_2) = \sigma(D_1)\sigma(D_2)$ divides $m = m_1m_2$ if and only if $\sigma(D_i)$ divides m_i for each i = 1, 2. We conclude from (19) that $N(n, m) = N(n, m_1)N(n, m_2)$.

Corollary 3.15. Let p_1, \ldots, p_r be r distinct prime numbers and $m = p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r}$. Then $B_n(U(m))$ has

$$N(n,m) = \prod_{i=1}^{r} \left(1 + \sum_{\alpha \in J(n,k_i)} N_{p_i}(\alpha) \right)$$

path-connected components.

We would like to close this section by giving the map ϕ_D in (10) a geometric interpretation analogous to that of (3) and a generalization to any $D \in T(n, \mathbb{Q}/\mathbb{Z})$ such that $B_n(U(m))_D$ is non-empty. Define $B_D = B_n(U(\sigma(D)))$ for any $D \in T(n, \mathbb{Q}/\mathbb{Z})$. Suppose that D = $D_n(d_1,\ldots,d_t)$. By Theorem 3.7, $B_D \cong ((U(\sigma(D))/\mathbb{T}^1) \times \mathbb{T}^n)/\mathbb{Z}_D$. The domain of ϕ_D can thus be expressed in terms of B_D as

(20)

$$\begin{bmatrix} \left(U(m)/\mathbb{T}^{l}\right) \times (\mathbb{T}^{n})^{l} / (\mathbb{Z}_{D})^{l} \end{bmatrix} / \Sigma_{l} \\
\cong \begin{bmatrix} U(m) \times_{U(\sigma(D))^{l}} (U(\sigma(D))/\mathbb{T}^{1})^{l} \times (\mathbb{T}^{n})^{l} / (\mathbb{Z}_{D})^{l} \end{bmatrix} / \Sigma_{l} \\
\cong \begin{bmatrix} U(m) \times_{U(\sigma(D))^{l}} \left((U(\sigma(D))/\mathbb{T}^{1}) \times \mathbb{T}^{n} / \mathbb{Z}_{D} \right)^{l} \end{bmatrix} / \Sigma_{l} \\
\cong U(m) \times_{U(\sigma(D))^{l}} B_{D}^{l} / \Sigma_{l}.$$

In this form, each point of the domain of ϕ_D is an unordered *l*-tuple of pairwisely orthogonal $\sigma(D)$ -dimensional complex subspaces of \mathbb{C}^m with a *D*-commuting tuple of unitary automorphism defined on each of these subspaces. The map ϕ_D simply glues the automorphisms using direct sum to produce a *D*-commuting tuple of unitary matrices. With this interpretation one can readily generalize ϕ_D to any $D \in T(n, \mathbb{Q}/\mathbb{Z})$ with $m = l\sigma(D)$ and obtain the following corollary of Theorem 3.7.

Corollary 3.16. Suppose that $D \in T(n, \mathbb{Q}/\mathbb{Z})$ and $m = l\sigma(D)$. The map

(21)
$$\phi_D: U(m) \times_{U(\sigma(D))^l} B_D^{l} / \Sigma_l \to B_n(U(m))_D.$$

induces an isomorphism in rational cohomology. Moreover, ϕ_D induces a homeomorphism $\bar{B}_n(U(m))_D \cong \bar{B}_n(\sigma(D))_D{}^l / \Sigma_l$ after passing to quotients by the action of U(m).

Proof. Suppose $D \in T(n, \mathbb{Q}/\mathbb{Z})$. By Proposition 3.2 and [9, Proposition 4.1], we can pick an automorphism of F_n which gives canonical homeomorphisms $B_n(U(m)_D) \cong B_n(U(m)_{D'})$ and $B_D \cong B_{D'}$ for some $D' = D_n(d_1, \ldots, d_t)$. These homeomorphisms induce the vertical homeomorphisms in the commutative diagram

Hence, ϕ_D satisfies the properties stated in the corollary because $\phi_{D'}$ does by (20) and Theorem 3.7. This proves the corollary.

4. The rank one case

Let Γ be the central extension

(22)
$$1 \to \mathbb{Z}^r \to \Gamma \to \mathbb{Z}^n \to 1.$$

with k-invariant $\omega = (\omega_1, \ldots, \omega_r) \in H^2(\mathbb{Z}^n; \mathbb{Z}^r) \cong (H^2(\mathbb{Z}^n; \mathbb{Z}))^r$, where

(23)
$$\omega_l = \sum_{1 \le i < j \le n} \omega_{i,j}^l e_i^* \wedge e_j^* \in H^2(\mathbb{Z}^n; \mathbb{Z}).$$

In this description, $\{e_i\}_{i=1}^n$ are the standard generators of \mathbb{Z}^n , $\{e_i^*\}_{i=1}^n$ are the dual generators of the cohomology ring $H^*(\mathbb{Z}^n;\mathbb{Z})$ and $\omega_{i,j}^l \in \mathbb{Z}$. The group Γ corresponding to this kinvariant ω is given in terms of generators and relations by

$$\Gamma = \langle a_1, \dots, a_n, x_1, \dots, x_r : [a_i, a_j] = \prod_{l=1}^r x_l^{\omega_{i,j}^l}, x_i \text{ is central} \rangle$$

A group homomorphism from Γ to U(m) is determined by the images of the generators of Γ . By abuse of notation, we sometimes write $(X_1, \ldots, X_r, A_1, \ldots, A_n) \in Hom(\Gamma, U(m))$ to represent the homomorphism $f : \Gamma \to U(m)$ with $f(a_i) = A_i$ and $f(x_j) = X_j$ for $1 \leq i \leq n, 1 \leq j \leq r$.

For the rest of the paper, we will study the space $\operatorname{Hom}(\Gamma, U(m))$. In this section, our focus will be on the special case of rank r = 1. The first step to analyze this space is to simplify the expression of the k-invariant $\omega \in H^2(\mathbb{Z}^n; \mathbb{Z})$ of the central extension. The following proposition allows us to work with central extensions with a particularly simple form of k-invariants. The proof is exactly analogous to that of [9, Proposition 4.1] for the case of (\mathbb{Z}/m) -valued skew-symmetric forms on \mathbb{Z}^n and we will leave it to the readers.

Proposition 4.1. Let $\omega : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ be a skew-symmetric bilinear form on \mathbb{Z}^n . Then there exist $t \leq n/2$ and a \mathbb{Z} -module basis e_1, e_2, \ldots, e_n of \mathbb{Z}^n such that

$$\omega = c_1 e_1^* \wedge e_{t+1}^* + \ldots + c_t e_t^* \wedge e_{2t}^*,$$

where c_1, \ldots, c_t are positive integers with $c_i | c_{i+1}$ for $i = 1, \ldots, t-1$.

Thanks to Proposition 4.1, we can assume throughout this section that Γ is the central extension

$$(24) 1 \to \mathbb{Z} \to \Gamma \to \mathbb{Z}^n \to 1$$

with k-invariant $\omega = \sum_{i=1}^{t} c_i e_i^* \wedge e_{t+i}^*$, where $0 \leq 2t \leq n$ and $c_i > 0$ for $1 \leq i \leq t$. In this case, an (n+1)-tuple of $m \times m$ unitary matrices $(X, A_1, \ldots, A_n) \in \text{Hom}(\Gamma, U(m))$ if and only if for $1 \leq i \leq i' \leq n$,

$$(25) [X, A_i] = I_m;$$

(26)
$$[A_i, A_{i'}] = \begin{cases} X^{c_i} & \text{if } i' = i+t, 1 \le i \le t; \\ I_m & \text{otherwise.} \end{cases}$$

Lemma 4.2. Let $1 \to \mathbb{Z} \to \Gamma \to \mathbb{Z}^n \to 1$ be a central extension with non-zero k-invariant ω . Then there exists a positive integer L such that for any $(X, A_1, \ldots, A_n) \in Hom(\Gamma, U(m))$, each eigenvalue of X is a L-th root of unity.

Proof. Since $\omega \neq 0$, $[A_i, A_j] = X^{\omega_{ij}}$ with $\omega_{ij} > 0$ for some $1 \leq i, j \leq n$. Suppose that λ is an eigenvalue of X and the associated eigenspace E_{λ} is k dimensional. Since A_i, A_j commutes with X, they restrict to unitary automorphisms on E_{λ} . $[A_i|_{E_{\lambda}}, A_j|_{E_{\lambda}}] = X^{\omega_{ij}}|_{E_{\lambda}} = \lambda^{\omega_{ij}}Id_{E_{\lambda}}$. By taking determinant, we have $\lambda^{k\omega_{ij}} = 1$. Therefore, we can take $L = m! \omega_{ij}$.

Consider the map $\chi : U(m) \to \mathbb{C}[z]$ which sends a unitary matrix to its characteristic polynomial and the restriction map res : $\operatorname{Hom}(\Gamma, U(m)) \to \operatorname{Hom}(\mathbb{Z}, U(m)) \cong U(m)$ which

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sends (X, A_1, \ldots, A_n) to X. For a complex polynomial p(z), define $U(m)_{p(z)} = \chi^{-1}(p(z))$, and $\operatorname{Hom}(\Gamma, U(m))_{p(z)} = (\chi \circ res)^{-1}(p(z))$. The map res restricts to a map

$$\operatorname{res}_{p(z)} : \operatorname{Hom}(\Gamma, U(m))_{p(z)} \to U(m)_{p(z)}.$$

Note that if Γ is non-abelian, the range of the map $\chi \circ \text{res}$ is a discrete set by Lemma 4.2. Hence, by continuity, each $\text{Hom}(\Gamma, U(m))_{p(z)}$ is both an open and closed subset of $\text{Hom}(\Gamma, U(m))$.

Let $p(z) = \prod_{j=1}^{s} (z - \lambda_j)^{m_j} \in \mathbb{C}[z]$ with distinct roots $\lambda_1, \ldots, \lambda_s \in \mathbb{S}^1$. Let $X \in U(m)_{p(z)}$ and $(X, A_1, \ldots, A_n) \in (\operatorname{res}_{p(z)})^{-1}(X)$. There is an orthogonal decomposition

$$\mathbb{C}^m = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \ldots \oplus E_{\lambda_n}$$

of \mathbb{C}^m into eigenspaces of X. X restricts to complex multiplication by λ_j on E_{λ_j} and $X|_{E_{\lambda_j}}$ has characteristic polynomial $(z - \lambda_j)^{m_j}$. Since A_i commutes with X, A_i also preserves each of these eigenspaces E_{λ_j} . All these n + 1 matrices X, A_1, \ldots, A_n restrict to unitary automorphisms on the eigenspace E_{λ_j} and their restrictions on E_{λ_j} satisfy commutation relations similar to those in (25) and (26). Hence, for $1 \leq i \leq i' \leq n$ and $1 \leq j \leq s$,

(27)
$$X|_{E_{\lambda_i}} = \lambda_j I d_{E_{\lambda_i}} \text{ is central};$$

(28)
$$[A_i|_{E_{\lambda_j}}, A_{i'}|_{E_{\lambda_j}}] = \begin{cases} \lambda_j^{c_i} Id_{E_{\lambda_j}} & \text{if } i' = i+t, 1 \le i \le t; \\ Id_{E_{\lambda_j}} & \text{otherwise.} \end{cases}$$

Under an unitary isomorphism $E_{\lambda_j} \cong \mathbb{C}^{m_j}$, these relations between the restrictions of X, A_1, \ldots, A_n on E_{λ_j} are the same as those among an (n+1)-tuple in $\operatorname{Hom}(\Gamma, U(m_j))_{(z-\lambda_j)^{m_j}}$. Since A_1, \ldots, A_n are uniquely determined by their restrictions on the eigenspaces E_{λ_j} , this implies

(29)
$$(\operatorname{res}_{p(z)})^{-1}(X) \cong \prod_{j=1}^{s} \operatorname{Hom}(\Gamma, U(m_j))_{(z-\lambda_j)^{m_j}}$$

For instance, if we take X to be the matrix

(30)
$$X_{0} = \begin{bmatrix} \lambda_{1}I_{m_{1}} & 0 & \dots & 0 \\ 0 & \lambda_{2}I_{m_{2}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_{j}I_{m_{j}} \end{bmatrix} \in U(m)_{p(z)},$$

then $(\operatorname{res}_{p(z)})^{-1}(X_0)$ consists of all (n+1)-tuples (X_0, A_1, \ldots, A_n) such that each A_i has the form

$$A_{i} = \begin{bmatrix} A_{i1} & 0 & \dots & 0 \\ 0 & A_{i2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_{is} \end{bmatrix} \in U(m)$$

where $(\lambda_j I_{m_j}, A_{1j}, A_{2j}, \dots, A_{nj}) \in \operatorname{Hom}(\Gamma, U(m_j))_{(z-\lambda_j)^{m_j}}$ for each j.

Our next theorem says that $\operatorname{res}_{p(z)} : \operatorname{Hom}(\Gamma, U(m))_{p(z)} \to U(m)_{p(z)}$ is a fiber bundle with fiber $\operatorname{res}^{-1}(X_0)$ homeomorphic to product of spaces of almost commuting tuples of unitary matrices.

Theorem 4.3. Let Γ be a central extension as in (24) and $p(z) = \prod_{j=1}^{s} (z - \lambda_j)^{m_j}$, where $\lambda_1, \ldots, \lambda_s \in \mathbb{S}^1$ are its distinct roots. Then there are U(m)-equivariant homeomorphisms

$$Hom(\Gamma, U(m))_{p(z)} \cong U(m) \times_{\prod_{j=1}^{s} U(m_j)} \left(\prod_{j=1}^{s} Hom(\Gamma, U(m_j))_{(z-\lambda_j)^{m_j}} \right)$$
$$\cong U(m) \times_{\prod_{j=1}^{s} U(m_j)} \left(\prod_{j=1}^{s} B_n(U(m_j))_{D_n(-c_1q_j,\dots,-c_tq_j)} \right)$$

where $q_j = \frac{1}{2\pi\sqrt{-1}}\log\lambda_j$. Also, $U(m)_{p(z)} \cong U(m)/(\prod_{j=1}^s U(m_j))$ is a flag manifold and $\operatorname{res}_{p(z)}$: $\operatorname{Hom}(\Gamma, U(m))_{p(z)} \to U(m)_{p(z)}$ is a fiber bundle induced by the collapse map of $\prod_{j=1}^s \operatorname{Hom}(\Gamma, U(m_j))_{(z-\lambda_j)}^{m_j}$.

The proof of Theorem 4.3 makes use of the following lemma in [8, Proposition 2.3.2].

Lemma 4.4. Let G be a compact Lie group and $f: Y \to Z$ be a G-map between compact G-spaces. Suppose that the G-action on Z is transitive with stabilizer subgroup $H \subset G$ at a point $z_0 \in Z$. Then $Y_0 = f^{-1}(z_0)$ is an H-space, $Y \cong G \times_H Y_0$ as G-spaces and f is the fiber bundle $G \times_H Y_0 \to G/H \cong Z$ induced by the collapse map of Y_0 .

of Theorem 4.3. It is clear that $\operatorname{res}_{p(z)}$: $\operatorname{Hom}(\Gamma, U(m))_{p(z)} \to U(m)_{p(z)}$ is equivariant with respect to the conjugation action of U(m). Also, note that this conjugation action on $U(m)_{p(z)}$ is transitive with the stabilizer subgroup of X_0 in (30) equal to $\prod_{j=1}^s U(m_j)$. By Lemma 4.4 and (29), we obtain the first homeomorphism and prove that $\operatorname{res}_{p(z)}$ is induced by the collapse map of $\prod_{j=1}^s \operatorname{Hom}(\Gamma, U(m_j))_{(z-\lambda_j)^{m_j}}$. The second homeomorphism follows immediately from (27), (28) and the definition of $B_n(U(m_j))_{D_n(-c_1q_j,\dots,-c_tq_j)}$.

Finally, using results from the last section, we can write down the image of the map res and the connected components of $\text{Hom}(\Gamma, U(m))$. To do so, we need to introduce the following definition.

Definition 4.5. Let Γ be a central extension as in (24) with non-zero k-invariant. Let $\mathbb{C}[z]_{\Gamma}^m \subset \mathbb{C}[z]$ be the set of all monic complex polynomials p(z) of degree m such that

- (1) all the roots of p(z) are roots of unity;
- (2) If a root λ of p(z) is a primitive k-th root of unity, then the multiplicity of λ in p(z) is divisible by $\mu_k := \prod_{i=1}^t k/(k, c_i)$, where (k, c_i) denotes the greatest common divisor of k and c_i .

Corollary 4.6. Suppose that Γ is a central extension as in (24) and res : $Hom(\Gamma, U(m)) \rightarrow Hom(\mathbb{Z}, U(m)) \cong U(m)$ is the restriction map. If Γ is abelian, then res is surjective. If Γ is non-abelian, then the image of res is $\coprod_{p(z)\in\mathbb{C}[z]_{\Gamma}^{m}} U(m)_{p(z)}$ and $Hom(\Gamma, U(m))$ can be expressed as the union of its path-connected components as

(31)
$$Hom(\Gamma, U(m)) = \prod_{p(z) \in \mathbb{C}[z]_{\Gamma}^{m}} Hom(\Gamma, U(m))_{p(z)}.$$

The number of path-connected components of $Hom(\Gamma, U(m))$ is given by the coefficient of x^m in the generating function $\prod_{k\geq 1} \frac{1}{(1-x^{\mu_k})^{\Phi(k)}}$, where μ_k is defined as in Definition 4.5 and $\Phi(k)$ is Euler's phi function.

Proof. If Γ is abelian, then for any $X \in U(m)$, $(X, I_m, I_m, \ldots, I_m) \in \text{Hom}(\Gamma, U(m))$ and so res is surjective. If Γ is non-abelian, then for a degree m complex polynomial p(z), $\text{Hom}(\Gamma, U(m))_{p(z)}$ is non-empty only if all the roots of p(z) are roots of unity by Lemma 4.2. Suppose that $p(z) = \prod_{j=1}^{s} (z - \lambda_j)^{m_j}$ with distinct roots λ_j 's and each λ_j is a primitive k_j -th root of unity. Let $q_j = \frac{1}{2\pi\sqrt{-1}} \log \lambda_j$ and $D_j = D_n(-c_1q_j, \ldots, -c_tq_j)$. Then by Theorem 4.3 and 3.3, $\text{Hom}(\Gamma, U(m))_{p(z)}$ is non-empty if and only if each $B_n(U(m_j))_{D_j}$ is non-empty, which is true if and only if each $\sigma(D_j) = \prod_{i=1}^{t} |c_iq_j| = \prod_{i=1}^{t} k_j/(k_j, c_i)$ divides m_j , or equivalently, $p(z) \in \mathbb{C}[z]_{\Gamma}^m$. Theorem 4.3 and Corollary 3.10 also imply that for $p(z) \in \mathbb{C}[z]_{\Gamma}^m$, $\text{Hom}(\Gamma, U(m))_{p(z)}$ is path-connected and $\operatorname{res}_{p(z)}$ is surjective. Since $\mathbb{C}[z]_{\Gamma}^m$ is a finite discrete set in $\mathbb{C}[z]$ and $\operatorname{res}, \chi : U(m) \to \mathbb{C}[z]$ are continuous, we conclude that $\operatorname{Hom}(\Gamma, U(m))$ can be written as the disjoint union of path-connected components as in (31). Hence, the number of path-connected components of $\operatorname{Hom}(\Gamma, U(m))$ is equal to $|\mathbb{C}[z]_{\Gamma}^m|$, which is obviously the coefficient of x^m in $\prod_{k\geq 1} \frac{1}{(1-x^{\mu_k})^{\Phi(k)}}$ from Definition 4.5.

Example 4.7. Let Γ_1 be the integral Heisenberg group, which can be described as the central extension $1 \to \mathbb{Z} \to \Gamma_1 \to \mathbb{Z}^2 \to 1$ with k-invariant $\omega = e_1^* \wedge e_2^*$. Thus $t = 1, c_1 = 1$ and $\mu_k = k$. The generating function in Corollary 4.6 is given by

$$\prod_{k\geq 1} \frac{1}{(1-x^{\mu_k})^{\Phi(k)}} = \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{(1-x^3)^2} \cdot \frac{1}{(1-x^4)^2} \cdot \frac{1}{(1-x^5)^4} \cdots$$
$$= 1+x+2x^2+4x^3+7x^4+13x^5+higher\ terms$$

Hence the number of components of $Hom(\Gamma_1, U(m))$ is 1, 2, 4, 7, 13 for $m = 1, \ldots, 5$ respectively. We can also consider the generalized version of Γ_1 given by the central extension $1 \to \mathbb{Z} \to \Gamma_t \to \mathbb{Z}^{2t} \to 1$ with k-invariant $\omega = \sum_{i=1}^t e_i^* \wedge e_{t+i}^*$ where $t \ge 2$. In this case, $\mu_k = k^t$ and the generating function

$$\prod_{k\geq 1} \frac{1}{(1-x^{\mu_k})^{\Phi(k)}} = (1+x+x^2+\ldots)(1+x^{2^t}+x^{2^{t+1}}+\ldots)(1+x^{3^t}+(x^{3^t})^2+\ldots)^2\cdots$$

From the coefficients it follows that $Hom(\Gamma_t, U(m))$ is connected for $1 \le m \le 2^t - 1$, has two components for $2^t \le m \le 2^{t+1} - 1$ and at least three components for $m \ge 2^{t+1}$.

5. Rank r > 1 case

In this section we will study the space $\operatorname{Hom}(\Gamma, U(m))$ for a central extension Γ of the form (22) with rank r > 1. We will decompose the restriction map $\operatorname{Hom}(\Gamma, U(m)) \to \operatorname{Hom}(\mathbb{Z}^r, U(m))$ into maps of fiber bundles over flag manifolds and relate the spaces to almost commuting tuples of unitary matrices. Results about the components, rational homology type and the associated representation space of $\operatorname{Hom}(\Gamma, U(m))$ can then be obtained. We first determine the image of the restriction map $\operatorname{Hom}(\Gamma, U(m)) \to \operatorname{Hom}(\mathbb{Z}^r, U(m))$.

Definition 5.1. Given a central extension Γ of the form (22) with ω_{ij}^l as in (23), define $\omega : \mathbb{T}^r \to T(n, \mathbb{R}/\mathbb{Z})$ by

(32)
$$\omega(\lambda)_{ij} = \frac{1}{2\pi\sqrt{-1}} \log\left(\lambda_1^{\omega_{ij}^1} \lambda_2^{\omega_{ij}^2} \dots \lambda_r^{\omega_{ij}^r}\right)$$

for $1 \leq i < j \leq n$. Let $Hom_{\Gamma}(\mathbb{Z}^r, U(m)) \subset Hom(\mathbb{Z}^r, U(m))$ be the subset consisting of those commuting tuples (X_1, X_2, \ldots, X_r) which have all their r-tuples of eigenvalues $\lambda \in \mathbb{T}^r$ satisfying $\omega(\lambda) \in T(n, \mathbb{Q}/\mathbb{Z})$ and $\sigma(\omega(\lambda))$ divides dim E_{λ} .

Proposition 5.2. The image of the restriction map res : $Hom(\Gamma, U(m)) \to Hom(\mathbb{Z}^r, U(m))$ is $Hom_{\Gamma}(\mathbb{Z}^r, U(m))$.

Proof. Suppose $(X_1, \ldots, X_r, A_1, \ldots, A_n) \in \text{Hom}(\Gamma, U(m))$. Let $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{T}^r$ be a *r*-tuple of eigenvalues of (X_1, \ldots, X_r) . Since each pair of A_i and X_j commutes, E_{λ} is invariant under each A_i . The restrictions of A_i, A_j on E_{λ} satisfy the relation

(33)
$$[A_i|_{E_{\lambda}}, A_j|_{E_{\lambda}}] = (X_1^{\omega_{ij}^1} X_2^{\omega_{ij}^2} \dots X_r^{\omega_{ij}^r})|_{E_{\lambda}} = \lambda_1^{\omega_{ij}^1} \lambda_2^{\omega_{ij}^2} \dots \lambda_r^{\omega_{ij}^r} Id_{E_{\lambda}} = e^{2\omega(\lambda)_{ij}\pi\sqrt{-1}} Id_{E_{\lambda}}.$$

By Lemma 3.1 and Corollary 3.10, $\omega(\lambda) \in T(n, \mathbb{Q}/\mathbb{Z})$ and $\sigma(\omega(\lambda))$ divides dim E_{λ} . Hence $(X_1, \ldots, X_r) \in \operatorname{Hom}_{\Gamma}(\mathbb{Z}^r, U(m))$.

On the other hand, suppose $(X_1, \ldots, X_r) \in \operatorname{Hom}_{\Gamma}(\mathbb{Z}^r, U(m))$ is given. Using Corollary 3.10, one can find matrices $A_1, \ldots, A_n \in U(m)$ such that $[A_i, A_j] = X_1^{\omega_{ij}^1} X_2^{\omega_{ij}^2} \ldots X_r^{\omega_{ij}^r}$ by constructing $A_1|_{E_{\lambda}}, \ldots, A_n|_{E_{\lambda}}$ satisfying (33) for each *r*-tuple of eigenvalues $\lambda \in \mathbb{T}^r$ of (X_1, \ldots, X_r) and taking direct sum. It shows that (X_1, \ldots, X_r) lies in the image of res. \Box

Let \mathcal{F}_n be the free commutative monoid on $T(n, \mathbb{Q}/\mathbb{Z})$. We can extend $\sigma : T(n, \mathbb{Q}/\mathbb{Z}) \to \mathbb{Z}$ as defined in (12) to a function on \mathcal{F}_n by linearity. Therefore for $\mathcal{D} = \sum_{j=1}^s l_j D_j, l_j > 0$, $\sigma(\mathcal{D}) = \sum_{j=1}^s l_j \sigma(D_j)$. Let $\mathcal{F}_{n,m}$ be the preimage $\sigma^{-1}(m)$ of m under $\sigma : \mathcal{F}_n \to \mathbb{Z}$. Define $f : \operatorname{Hom}_{\Gamma}(\mathbb{Z}^r, U(m)) \to \mathcal{F}_{n,m}$ by

$$f(X_1,\ldots,X_r) = \sum_{j=1}^s \frac{\dim E_{\lambda^j}}{\sigma(\omega(\lambda^j))} \omega(\lambda^j),$$

where $\lambda^1, \ldots, \lambda^s$ are all the distinct *r*-tuples of eigenvalues of (X_1, \ldots, X_r) . For $\mathcal{D} \in \mathcal{F}_{n,m}$, let $\operatorname{Hom}(\mathbb{Z}^r, U(m))_{\mathcal{D}} = f^{-1}(\mathcal{D})$, $\operatorname{Hom}(\Gamma, U(m))_{\mathcal{D}} = \operatorname{res}^{-1}(\operatorname{Hom}(\mathbb{Z}^r, U(m))_{\mathcal{D}})$ and $\operatorname{res}_{\mathcal{D}} =$ $\operatorname{res}|_{\operatorname{Hom}(\Gamma, U(m))_{\mathcal{D}}}$, where $\operatorname{res} : \operatorname{Hom}(\Gamma, U(m)) \to \operatorname{Hom}_{\Gamma}(\mathbb{Z}^r, U(m))$ is the restriction map. Since f is continuous if $\mathcal{F}_{n,m}$ is given the discrete topology, we obtain the following theorem.

Theorem 5.3. For a central extension Γ of the form (22), $Hom_{\Gamma}(\mathbb{Z}^r, U(m))$, $Hom(\Gamma, U(m))$ and res can be expressed as disjoint unions indexed by \mathcal{F} and there is a commutative diagram

Because of Theorem 5.3, we can focus on $\operatorname{Hom}(\Gamma, U(m))_{\mathcal{D}}$ for each $\mathcal{D} \in \mathcal{F}_{n,m}$. The next two theorems break down these spaces and relate them to $B_n(U(m))$.

Theorem 5.4. Suppose that $\mathcal{D} = \sum_{j=1}^{s} l_j D_j \in \mathcal{F}_{n,m}$ with $l_j > 0$ and $D_i \neq D_j$ for $i \neq j$. Let $m_j = l_j \sigma(D_j)$. Then there are U(m)-equivariant homeomorphisms

$$Hom(\Gamma, U(m))_{\mathcal{D}} \cong U(m) \times_{\prod_{j=1}^{s} U(m_j)} \left(\prod_{j=1}^{s} Hom(\Gamma, U(m_j))_{l_j D_j} \right),$$

and

$$Hom(\mathbb{Z}^r, U(m))_{\mathcal{D}} \cong U(m) \times_{\prod_{j=1}^s U(m_j)} \left(\prod_{j=1}^s Hom(\mathbb{Z}^r, U(m_j))_{l_j D_j} \right)$$

Hence, both $Hom(\Gamma, U(m))_{\mathcal{D}}$ and $Hom(\mathbb{Z}^r, U(m))_{\mathcal{D}}$ are fiber bundles over $U(m)/(\prod_{j=1}^s U(m_j))$. The map res_{\mathcal{D}} is a map of fiber bundles which fits into the following commutative diagram.

Proof. The space $U(m)/(\prod_{j=1}^{s} U(m_j))$ can be considered as a flag manifold with points represented by ordered tuples (V_1, \ldots, V_s) , where each $V_j \subset \mathbb{C}^m$ is a m_j -dimensional complex subspace and $\mathbb{C}^m = \bigoplus_{j=1}^{s} V_j$ is an orthogonal decomposition of \mathbb{C}^m . Let

$$g: \operatorname{Hom}(\mathbb{Z}^r, U(m))_{\mathcal{D}} \to U(m)/(\prod_{j=1}^s U(m_j))$$

be defined by $g(X_1, \ldots, X_r) = (V_1, \ldots, V_s)$ with $V_j = \bigoplus_{\lambda \in \omega^{-1}(D_j)} E_{\lambda}$. Then $g, g \circ \operatorname{res}_{\mathcal{D}}$ are continuous and U(m)-equivariant. Since the U(m)-action on $U(m)/(\prod_{j=1}^s U(m_j))$ is transitive, we obtain the U(m)-homeomorphisms in the theorem by Lemma 4.4. It is obvious that $\operatorname{res}_{\mathcal{D}}$ can be realized as a map of fiber bundles induced by $\prod_{j=1}^s \operatorname{res}_{l_j D_j}$ and hence the given diagram commutes.

Recall that we defined $B_D = B_n(U(\sigma(D)))$ for $D \in T(n, \mathbb{Q}/\mathbb{Z})$. In Corollary 3.16 we showed that it can be used as the building blocks for approximating $B(U(m))_D$ rationally. We can do it for $\operatorname{Hom}(\Gamma, U(m))_{lD}$ too.

Theorem 5.5. For $\mathcal{D} = lD$ and $m = \sigma(\mathcal{D}) = l\sigma(D)$, there is a commutative diagram

where the horizontal maps are rational homology equivalences for any $l \ge 1$ and homeomorphisms for l = 1. Here $\omega : \mathbb{T}^r \to T(n, \mathbb{R}/\mathbb{Z})$ is the function as defined in (32).

Remark 5.6. The horizontal maps are defined in the same way as ϕ_D in (21). The domain of η_D can be considered as the space of unordered l-tuples of pairwisely orthogonal $\sigma(D)$ -dimensional complex subspaces of \mathbb{C}^m with a label $\lambda \in \omega^{-1}(D)$ attached to each of the subspaces. A commuting tuple $(X_1, \ldots, X_r) \in Hom(\mathbb{Z}^r, U(m))_D$ can be uniquely determined by specifying these λ as the r-tuples of eigenvalues of the subspaces they label.

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This defines $\eta_{\mathcal{D}}$. The map $\phi_{\mathcal{D}}$ is defined similarly: Each point of its domain carries the additional data of a D-commuting tuple of unitary automorphisms of each of those complex subspaces. The additional data is used for constructing the matrices A_i in its image $(X_1, \ldots, X_r, A_1, \ldots, A_n) \in Hom(\Gamma, U(m))_{\mathcal{D}}$ under $\phi_{\mathcal{D}}$.

Proof. It is clear that the diagram commutes. Suppose $(X_1, \ldots, X_r) \in \text{Hom}(\mathbb{Z}^r, U(m))_{\mathcal{D}}$. Let $\lambda^1, \ldots, \lambda^s \in \mathbb{T}^r$ be all its distinct *r*-tuples of eigenvalues. Note that each $\lambda^j \in \omega^{-1}(D)$ and dim $E_{\lambda^j} = m_j = l_j \sigma(D)$ for some $l_j \in \mathbb{Z}$. It is easy to see that

$$\eta_{\mathcal{D}}^{-1}(X_1,\ldots,X_r) \cong \prod_{j=1}^s U(m_j)/U(\sigma(D))^{l_j} / \sum_{l_j},$$

which is Q-acyclic in general and is a point if l = 1. Thus $\eta_{\mathcal{D}}$ satisfies the properties stated in the theorem. For $\phi_{\mathcal{D}}$, it is clear that there exist homeomorphisms

$$\operatorname{res}_{\mathcal{D}}^{-1}(X_1,\ldots,X_r) \cong \prod_{j=1}^s B_n(U(m_j))_D$$
$$\operatorname{res}_{\mathcal{D}} \circ \phi_{\mathcal{D}})^{-1}(X_1,\ldots,X_r) \cong \prod_{j=1}^s U(m_j) \times_{U(\sigma(D))^{l_j}} B_D^{l_j} / \sum_{l_j}$$

such that the restriction of $\phi_{\mathcal{D}}$ on $(\operatorname{res}_{\mathcal{D}} \circ \phi_{\mathcal{D}})^{-1}(X_1, \ldots, X_r)$ can be regarded as the product of

$$\phi_D : U(m_j) \times_{U(\sigma(D))^{l_j}} B_D^{l_j} / \Sigma_{l_j} \to B_n(U(m_j))_D$$

under these homeomorphisms. It follows from Theorem 3.7 that $\phi_{\mathcal{D}}$ is a rational homology equivalence for any $l \geq 1$ and a homeomorphism for l = 1.

The space $\omega^{-1}(D)$ can be studied using linear systems over \mathbb{R}/\mathbb{Z} .

Definition 5.7. Suppose that Γ is a central extension of the form (22) and ω_{ij}^l are the coefficients of its k-invariant as in (23).

- (1) Let Ω be the $C_2^n \times r$ matrix with rows indexed by $(i, j), 1 \leq i < j \leq n$, columns indexed by $1 \leq l \leq r$ and the *l*-th entry on the (i, j)-th row equals to ω_{ij}^l .
- (2) Let Q be a row echelon form of Ω over Z. Define B to be the absolute value of the product of the pivot entries of Q. The number B is independent of the choice of Q.
- (3) Let R be the reduced row echelon form of Ω over \mathbb{Q} . The columns of R can be regarded as vectors in $\mathbb{Q}^{\mathbb{C}_2^n}$. Let C(R) be the \mathbb{Z} -submodule of $(\mathbb{Q}/\mathbb{Z})^{\mathbb{C}_2^n}$ generated by the images of the columns of R under the quotient map $\mathbb{Q}^{\mathbb{C}_2^n} \to (\mathbb{Q}/\mathbb{Z})^{\mathbb{C}_2^n}$. Define C to be the number of elements in C(R).
- (4) Define $P(\Omega) = B/C$.

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Example 5.8. Let r = 4, n = 3,

$$Q = \begin{pmatrix} 4 & -1 & 1 & -6 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and so } R = \begin{pmatrix} 1 & 0 & 1/3 & -4/3 \\ 0 & 1 & 1/3 & 2/3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then B = (4)(3) = 12 and C(R) is the Z-submodule of $(\mathbb{Q}/\mathbb{Z})^4$ generated by

$$\begin{bmatrix} 1\\0\\0 \end{bmatrix} \equiv \begin{bmatrix} 0\\1\\0 \end{bmatrix} \equiv \begin{bmatrix} 0\\0\\0 \end{bmatrix}, \begin{bmatrix} 1/3\\1/3\\0 \end{bmatrix} and \begin{bmatrix} -4/3\\2/3\\0 \end{bmatrix} \equiv 2 \begin{bmatrix} 1/3\\1/3\\0 \end{bmatrix}.$$

Hence C = 3 *and* $P(\Omega) = 12/3 = 4$.

Lemma 5.9. Let $\omega : \mathbb{T}^r \to T(n, \mathbb{R}/\mathbb{Z})$ be the map (32) and $\Omega, P(\Omega)$ be as in Definition 5.7. Then for any $D \in T(n, \mathbb{Q}/\mathbb{Z}), \omega^{-1}(D)$ is either empty or homeomorphic to a disjoint union of $P(\Omega)$ copies of $\mathbb{T}^{nul(\Omega)}$, where $nul(\Omega)$ is the nullity of Ω .

Proof. Suppose $\omega^{-1}(D)$ is non-empty. A *r*-tuple $\lambda = (\lambda_1, \ldots, \lambda_r) \in \omega^{-1}(D)$ if and only if

$$\sum_{k=1}^{r} \omega_{ij}^{k} \frac{\log \lambda_k}{2\pi\sqrt{-1}} = d_{ij}$$

for all $1 \leq i < j \leq n$. Let $D' \in (\mathbb{Q}/\mathbb{Z})^{C_2^n}$, with entries indexed by $(i, j), 1 \leq i < j \leq n$, be obtained from D by rewriting the entries in a column vector. Suppose Q, R, B, C are as in Definition 5.7 and $x_k = \frac{\log \lambda_k}{2\pi\sqrt{-1}}$. Then $\omega^{-1}(D)$ is homeomorphic to the solution space S of the equivalent linear systems $\Omega \vec{x} = D' \iff Q \vec{x} = D''$ over \mathbb{Q}/\mathbb{Z} . Here D'' is obtained from D' by performing the same elementary operations used to obtain Q from Ω . Without loss of generality, assume x_1, \ldots, x_p are the basic variables and x_{p+1}, \ldots, x_r are the free variables in the system $Q\vec{x} = D''$. For $1 \leq s \leq p$, let b_s be the pivot entry in the s-th column of Q. Then for points in S, $b_1 x_1, \ldots, b_p x_p \in \mathbb{R}/\mathbb{Z}$ are uniquely determined by any $(x_{p+1},\ldots,x_r) \in T := (\mathbb{R}/\mathbb{Z})^{r-p} \cong \mathbb{T}^{r-p}$. Hence, S is a B-sheeted covering space over T. Note that all entries below the p-th row of R are zero. Let $\vec{v_s} \in \mathbb{R}^p$ be the first p entries of the s-th column of R and $\vec{w_s}$ be its image under the quotient map $\mathbb{R}^p \to (\mathbb{R}/\mathbb{Z})^p$. Then a loop in T parametrized by $x_s \in \mathbb{S}^1$ with other coordinates fixed lifts to a path connecting \vec{x} and $\vec{x} + \begin{vmatrix} \vec{w_s} \\ \vec{0} \end{vmatrix}$ in S. From this it can be deduced that S, and hence $\omega^{-1}(D)$, is homeomorphic to a disjoint union of $B/C = P(\Omega)$ copies of \mathbb{T}^{r-p} , where $r-p = \operatorname{nul}(\Omega)$ is the nullity of Ω . This proves the lemma.

Combining Theorems 3.7, 5.5 and Lemma 5.9, we obtain the following result about the number of components, rational cohomology and the associated representation space for $\operatorname{Hom}(\Gamma, U(m))_{\mathcal{D}}$.

Theorem 5.10. Suppose that $\mathcal{D} = \sum_{j=1}^{s} l_j D_j \in \mathcal{F}_{n,m}$ with $l_j > 0$ and $D_i \neq D_j$ for $i \neq j$. Let $l = \sum_{j=1}^{s} l_j$ and $m_j = l_j \sigma(D_j)$. Then $Hom(\Gamma, U(m))_{\mathcal{D}}$ is non-empty if and only if $\omega^{-1}(D_j)$ is non-empty for all $j = 1, \ldots, s$. In that case, $Hom(\Gamma, U(m))_{\mathcal{D}}$ has

$$\prod_{j=1}^{s} \binom{P(\Omega) + l_j - 1}{l_j}$$

components and there is a rational homology equivalence

$$\left[\left(U(m)/\mathbb{T}^l \right) \times \left(\coprod_{P(\Omega)} \mathbb{T}^{n+nul(\Omega)} \right)^l / \mathbb{Z}_{\mathcal{D}} \right] / \prod_{j=1}^s \Sigma_{l_j} \to Hom(\Gamma, U(m))_{\mathcal{D}}$$

where the action of the finite abelian group $\mathbb{Z}_{\mathcal{D}} := \prod_{j=1}^{s} (\mathbb{Z}_{D_{j}})^{l_{j}}$ on the space $(U(m)/\mathbb{T}^{l}) \times (\prod_{P(\Omega)} \mathbb{T}^{n+nul(\Omega)})^{l}$ is trivial on rational cohomology. Moreover the map induces a homeomorphism

$$Hom(\Gamma, U(m))_{\mathcal{D}} / U(m) \cong \prod_{j=1}^{s} \left[\left(\coprod_{P(\Omega)} \mathbb{T}^{n+nul(\Omega)} \right)^{l_j} / \Sigma_{l_j} \right]$$

after passing to quotients by the action of U(m).

Finally we note that the rational cohomology of the spaces $B_n(U(m))$ and $\operatorname{Hom}(\Gamma, U(m))$ can be computed using standard spectral sequence arguments and invariant theory. Details are left to the interested (and highly motivated) reader.

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