

Bifurcation of Solutions to the Allen-Cahn Equation

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Abstract: We use Morse Homology to study bifurcation of the solution sets of the Allen-Cahn Equation.

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1 - Introduction.

1.1 - Main Results. Let $M := (M^n, g)$ be a compact, n -dimensional Riemannian manifold. For $\epsilon > 0$, we define the **Allen-Cahn Operator** over $C^\infty(M)$ with parameter ϵ by:

$$\text{AC}_{\epsilon, g}(u) := -\epsilon \Delta_g u + u^3 - u,$$

where Δ_g is the Laplacian operator of g . The Allen-Cahn Operator appears in mathematical physics to describe the process of phase separation in metal alloys (c.f. [2]), and its interesting properties have already made it the object of various mathematical studies (c.f., for example, [4], [5] and [7]). In particular, the Allen-Cahn Operator is variational, arising as the Euler-Lagrange Equations (that is, the L^2 -gradient) of the Ginzburg-Landau-Wilson Free Energy Functional:

$$\mathcal{E}_{\epsilon, g}(u) = \int_M \epsilon \|\nabla^g u\|^2 + \frac{1}{4}(u^2 - 1)^2 d\text{Vol},$$

and for this reason naturally lends itself to analysis by Morse theoretical techniques. In this note, we use Morse Homology, we show how the space of solutions to the Allen-Cahn Equation bifurcates as ϵ becomes small. Indeed, the Morse Homology yields a lower bound for the number of solutions, which increases discretely as ϵ^{-1} crosses points of the spectrum of $-\Delta_g$, tending to infinity as ϵ tends to zero.

We denote by \mathcal{M} the space of smooth Riemannian metrics over M , which we furnish with the topology of C^∞ convergence. For any smooth metric g , we define the **solution space** $\mathcal{Z}_g \subseteq]0, \infty[\times C^\infty(M)$ by:

$$\mathcal{Z}_g = \{(\epsilon, u) \mid \text{AC}_{\epsilon, g}(u) = 0\},$$

so that \mathcal{Z}_g is the union of all solution sets for the metric g and all parameters $\epsilon \in]0, \infty[$. We denote by $e_g : \mathcal{Z}_g \rightarrow]0, \infty[$ the projection onto the second factor, and we will see presently (c.f. Proposition 2.1.3) that this is a proper map. For all $\epsilon \in]0, \infty[$, we denote:

$$\mathcal{Z}_{\epsilon, g} = e_g^{-1}(\{\epsilon\}) = \{u \mid \text{AC}_{\epsilon, g}(u) = 0\},$$

so that $\mathcal{Z}_{\epsilon, g}$ is the solution set for the metric g and the parameter ϵ . We aim to study the manner in which $\mathcal{Z}_{\epsilon, g}$ bifurcates as ϵ tends to 0, and the simplest way to do so is to study the geometry of (\mathcal{Z}_g, e_g) . We first show that, upon perturbing the metric by an arbitrarily small amount, we may suppose that this geometry is relatively straightforward but for a countable set of singularities determined by the Laplacian of g . Indeed, we denote by $\text{Spec}(-\Delta_g)$ the set of all eigenvalues of $-\Delta_g$ (which, by convention, is non-negative), and we define the **singular set**, $\text{Sing}_g \subseteq]0, \infty[\times C^\infty(M)$ by:

$$\text{Sing}_g = \{(\epsilon, 0) \mid \epsilon^{-1} \in \text{Spec}(-\Delta_g)\}.$$

We recall that a subset X of \mathcal{M} is said to be **generic** (or equivalently, in the second category in the sense of Baire), whenever it contains a countable intersection of dense open sets. A given property is then said to hold for generic elements whenever it holds for all elements of some generic set. We recall that, by the Baire Category Theorem, any property that holds for generic elements of \mathcal{M} in particular holds over a dense subset of \mathcal{M} . Using transversality techniques, we to show:

Theorem 1.1.1

For generic $g \in \mathcal{M}$, $\mathcal{Z}_g \setminus \text{Sing}_g$ is a smooth, 1-dimensional submanifold of $]0, \infty[\times C^\infty(M)$. Moreover, if $\text{Dim}(M) \geq 3$, then we may assume in addition that all critical points of e_g are non-degenerate. In particular:

- (1) if $\epsilon^{-1} \notin \text{Spec}(-\Delta_g)$, then $\mathcal{Z}_{\epsilon,g}$ is finite; and
- (2) there exists a discrete subset X of the complement of $\text{Spec}(-\Delta_g)$ such that if $\epsilon^{-1} \notin X \cup \text{Spec}_g$, then $\mathcal{Z}_{\epsilon,g}$ only consists of non-degenerate solutions of the Allen-Cahn Equation.

Remark: We recall that a solution to an elliptic partial differential equation is said to be non-degenerate whenever the linearisation of the operator about that solution is invertible. \square

When u is a solution of the Allen-Cahn Equation with parameter ϵ , we denote by $LAC_{\epsilon,g}(u)$ the linearisation of the Allen-Cahn Operator about u . When u is non-degenerate, we denote by $\text{Index}(u)$ its **Morse Index**, which we recall is defined to be equal to the number of strictly negative eigenvalues of $LAC_{\epsilon,g}(u)$ counted with geometric multiplicity. Observe that the constant function $u = 0$ is a solution of $AC_{\epsilon,g}$ for all g and for all ϵ . Moreover, as we will see presently (c.f. Proposition 2.2.1), $u = 0$ is non-degenerate if and only if $\epsilon^{-1} \notin \text{Spec}(-\Delta_g)$, and the index of this solution is given by:

$$\text{Index}(0) = \# \{ \lambda \in \text{Spec}(-\Delta_g) \mid \lambda < \epsilon^{-1} \}.$$

Observe that $\text{Index}(0)$ tends to infinity as ϵ tends to zero. Our second result now describes in terms of the number of solutions of a given Morse Index how $\mathcal{Z}_{\epsilon,g}$ bifurcates as ϵ tends to 0:

Theorem 1.1.2

For generic $g \in \mathcal{M}$, if $\epsilon^{-1} \notin \text{Spec}(-\Delta_g)$, then for all $0 \leq k < \text{Index}(0)$, there exist at least two non-degenerate solutions u and $-u$ of the Allen-Cahn Equation such that $\text{Index}(u) = \text{Index}(-u) = k$.

Remark: In particular, we do not require Part 2 of Theorem 1.1.1. For countably many values of ϵ , there may exist finitely many degenerate solutions. We simply ignore them. \square

Theorem 1.1.2 follows from Theorem 1.1.1 in a straightforward manner from standard Morse homological techniques. Indeed, for all g and for all ϵ , the constant functions $u = \pm 1$ are also solutions of the Allen-Cahn Equation, this time with Morse Index equal to 0. Denoting $l = \text{Index}(0)$, it follows that the chain groups C_0 and C_l of the Morse-Complex of the Ginzburg-Landau-Wilson Free Energy Functional are at least 2- and 1-dimensional respectively. Since the underlying space (that is, $C^\infty(M)$) is contractible, the Morse-Homology, H_k , is non-trivial only for $k = 0$. Finally, as the Allen-Cahn Operator is an odd operator, all the intermediate chain groups C_1, C_2, \dots, C_{l-1} have even dimension, and this fact, used together with the algebraic relations of Morse Homology allows us to deduce that they are non-trivial, thus proving the theorem.

Theorem 1.1.1 is proven using the Sard-Smale Theorem, and this paper is therefore mostly devoted to obtaining the requisite surjectivity results. We draw the reader's attention

to the fact that our usage of the Sard-Smale Theorem differs from standard approaches in one subtle but interesting respect. Indeed, whilst any application of the Sard-Smale Theorem is generally considered to require the separability of the function spaces used, we replace this condition by one that we call “paraproperness”, which is to properness as paracompactness is to compactness. In Proposition 2.2.2, we make paraproperness into a useful concept by showing that it is preserved by restriction to both closed and open subsets, and in Theorem 2.2.4, we reprove the Sard-Smale Theorem in this new context. This will be of particular use in the forthcoming paper [14] where it makes possible the construction of a working Morse Homology theory in the Hölder space framework.

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2 - The Solution Space.

2.1 - Preliminaries and Compactness. For $\lambda \in [0, \infty] \setminus \mathbb{N}$, that is, for $\lambda = +\infty$, or for $\lambda = k + \alpha$, where $k \in \mathbb{N}$ and $\alpha \in]0, 1[$, we denote by $C^\lambda := C^\lambda(M)$ the space of λ -times Hölder differentiable functions over M , and when $\lambda < \infty$, we denote by $\|\cdot\|_\lambda$ the corresponding C^λ -Hölder norm. For $\mu \in [0, \infty] \setminus \mathbb{N}$, we likewise denote by \mathcal{M}^μ the space of C^μ -Riemannian metrics over M . It is well known that these spaces are non-separable, but as indicated in the introduction, this is of no consequence to us, and is satisfactorily treated by the concept of paraproperness (c.f. Section 2.2, below).

We consider a slightly more general problem than that discussed in the introduction. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function such that f is not linear over any interval, both f and f' have non-degenerate zeroes, and:

$$\text{LimSup}_{t \rightarrow -\infty} f(t) < 0, \quad \text{LimInf}_{t \rightarrow +\infty} f(t) > 0. \quad (\text{A})$$

As we will see presently (c.f. Proposition 2.1.2, below) our theory only depends on the restriction of f to the smallest interval containing all its zeroes. We therefore modify f outside this interval, and replace (A) with the following technically more convenient property:

$$\text{Lim}_{t \rightarrow \pm\infty} f(t)/t|t| = +\infty. \quad (\text{B})$$

For $\mu > \lambda \in [0, \infty] \setminus \mathbb{N}$, we define the **Allen-Cahn Operator**, $\text{AC} :]0, \infty[\times \mathcal{M}^{\mu+1} \times C^{\lambda+2} \rightarrow C^\lambda$ by:

$$\text{AC}_{\epsilon, g}(u) := \text{AC}(\epsilon, g, u) = \epsilon \Delta_g u - f(u), \quad (\text{C})$$

where Δ_g is the Laplacian operator of g . Since AC is constructed via a finite combination of multiplication, addition, differentiation and post-composition by smooth functions, it defines a smooth mapping between Banach manifolds. Importantly, the Allen-Cahn Operator arises as the L^2 -gradient of the Ginzburg-Landau-Wilson Free Energy Functional:

$$\mathcal{E}_{\epsilon, g} = \int_M \epsilon \|\nabla^g u\|^2 + F(u) d\text{Vol},$$

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where F is any primitive of u . In particular, solutions of the Allen-Cahn Equation are critical points of $\mathcal{E}_{\epsilon, g}$.

We define the **solution space** $\mathcal{Z} \subseteq]0, \infty[\times \mathcal{M}^{\mu+1} \times C^{\lambda+2}$ by:

$$\mathcal{Z} = \text{AC}^{-1}(0).$$

Let $\Pi : \mathcal{Z} \rightarrow]0, \infty[\times \mathcal{M}^{\mu+1}$ be the projection onto the first two factors and let $\Pi_g : \mathcal{Z} \rightarrow \mathcal{M}^{\mu+1}$ and $\Pi_u : \mathcal{Z} \rightarrow C^{\lambda+2}$ be the projection onto the second and third factor, respectively. For all $(\epsilon, g) \in]0, \infty[\times \mathcal{M}^{\mu+1}$, we define $\mathcal{Z}_{\epsilon, g} \subseteq \mathcal{Z}$, the **solution space** for the data (ϵ, g) by:

$$\mathcal{Z}_{\epsilon, g} = \Pi^{-1}((\epsilon, g)).$$

We study the bifurcations of $\mathcal{Z}_{\epsilon, g}$ as ϵ varies. For this reason, we prefer to study all values of ϵ simultaneously and thus define $\mathcal{Z}_g \subseteq \mathcal{Z}$ by:

$$\mathcal{Z}_g = \Pi_g^{-1}(g).$$

The main results of this paper follow from the differential topological properties of \mathcal{Z} , Π and Π_g , which we now proceed to study.

We first review the analytic properties of the Allen-Cahn Operator. Elements of \mathcal{Z} have the following regularity properties:

Proposition 2.1.1

Given $\mu > \lambda \in [0, \infty] \setminus \mathbb{N}$, if $(\epsilon, g, u) \in \mathcal{Z}$, then $u \in C^{\mu+2}$.

Proof: Observe that $\epsilon \Delta_g$ is a second-order elliptic partial differential operator with coefficients in C^μ . Thus, if u lies in $C^{\mu+2(1-k)}$ for some positive integer k with $\mu + 2(1-k) > 0$, then, since f is smooth:

$$\epsilon \Delta_g u = f \circ u \in C^{\mu+2(1-k)},$$

and by elliptic regularity (c.f. [6]), $u \in C^{\mu+2(2-k)}$. Observe that since $u \in C^{\lambda+2}$, there exists k such that $u \in C^{\mu+2(1-k)}$, and it follows by induction that $u \in C^{\mu+2}$, as desired. \square

In order to obtain a-priori estimates, we define $T_0 > 0$ by:

$$T_0 = \text{Sup} \{ |t| \mid f(t) = 0 \}.$$

It follows from (B) that T_0 is finite. We have:

Proposition 2.1.2

For all $(\epsilon, g, u) \in \mathcal{Z}$:

$$\|u\|_{L^\infty} \leq T_0.$$

Proof: Suppose the contrary, that is, $\|u\|_{L^\infty} > T_0$. Since M is compact, there exists $p \in M$ such that $|u(p)| = \|u\|_{L^\infty}$. If $u(p) \geq 0$, then $u(p) = \|u\|_{L^\infty}$, and since p is a maximum of u , $(\Delta_g u)(p) \leq 0$, so that:

$$f(\|u\|_{L^\infty}) = \epsilon (\Delta_g u)(p) \leq 0.$$

On the other hand, if $u(p) < 0$, then $u(p) = -\|u\|_{L^\infty}$ and $(\Delta_g u)(p) \geq 0$ so that:

$$f(-\|u\|_{L^\infty}) = \epsilon (\Delta_g u)(p) \geq 0.$$

In each case, this is absurd by definition of T_0 and Property (B) of f , and the result follows. \square

Proposition 2.1.3

Π defines a proper map from \mathcal{Z} into $]0, \infty[\times \mathcal{M}^{\mu+1}$.

Proof: Let $(\epsilon_m, g_m, u_m)_{m \in \mathbb{N}}$ be a sequence in \mathcal{Z} and suppose that $(\epsilon_m, g_m)_{m \in \mathbb{N}}$ converges to $(\epsilon_\infty, g_\infty) \in]0, \infty[\times \mathcal{M}^{\mu+1}$, say. By Proposition 2.1.1, $u_m \in C^{\mu+2}(M)$ for all M . By the Schauder estimates (c.f. [6]), there exists $B_1 > 0$ such that for all m :

$$\begin{aligned} \|u_m\|_{\mu+2} &\leq B_1(\|u_m\|_{L^\infty} + \|\epsilon_m \Delta_{g_m} u_m\|_\mu) \\ &= B_1(\|u_m\|_{L^\infty} + \|f \circ u_m\|_\mu) \end{aligned}$$

By Proposition 2.1.2, for all m , u_m takes values in the compact set $[-T_0, T_0]$. Since f is smooth, it follows from the chain rule and Gagliardo-Nirenberg-Moser type interpolation estimates (c.f. [15]) that there exists $B_2 > 0$ such that for all m :

$$\|f \circ u_m\|_\mu \leq B_2(\|u_m\|_{L^\infty} + \|u_m\|_\mu).$$

By standard interpolation inequalities (c.f. [6]), there exists $B_3 \geq 0$ such that for all m :

$$\|u_m\|_\mu \leq B_3 \|u_m\|_{L^\infty} + \frac{1}{2B_1 B_2} \|u_m\|_{\mu+2}.$$

Combining these estimates yields, for all m :

$$\begin{aligned} \|u_m\|_{\mu+2} &\leq 2B_1(1 + B_2)(1 + B_3) \|u_m\|_{L^\infty} \\ &\leq 2B_1(1 + B_2)(1 + B_3) T_0. \end{aligned}$$

It now follows by the Arzela-Ascoli Theorem that there exists $u_\infty \in C^{\lambda+2}(M)$ towards which $(u_m)_{m \in \mathbb{N}}$ subconverges, and this completes the proof. \square

2.2 - The Regular Solution Space and Paraproperness. For all $(\epsilon, g, u) \in \mathcal{Z}$, we denote by LAC the linearisation of $AC_{\epsilon, g}$ about u . By definition, $LAC = D_3 AC(\epsilon, g, u)$, where $D_3 AC$ denotes the partial derivative of AC with respect to the third component. In particular, for all $(\epsilon, g, u) \in \mathcal{Z}$ and for all $\varphi \in C^{\lambda+2}(M)$:

$$LAC\varphi = \epsilon \Delta_g \varphi - f'(u)\varphi, \tag{D}$$

so that LAC is a self-adjoint second-order elliptic linear operator. In particular, it is Fredholm of index zero and by classical spectral theory, its spectrum is discrete, real and bounded above, and all of its eigenvalues have finite multiplicity. We say that the solution u is **non-degenerate** whenever LAC is invertible, and we define the **Morse Index** of u , which we denote by $\text{Index}(u)$ by:

$$\text{Index}(u) := \text{Index}(LAC) = \sum_{\lambda \in \text{Spec}(LAC), \lambda > 0} \text{Mult}(\lambda),$$

where, for all $\lambda \in \text{Spec}(LAC)$, $\text{Mult}(\lambda)$ is its multiplicity.

Proposition 2.2.1

For all $(\epsilon, g) \in]0, \infty[\times \mathcal{M}^{\mu+1}$, the constant function $u = c$ is a solution to the Allen-Cahn Equation $AC_{\epsilon, g}(u) = 0$ if and only if $f(c) = 0$. Moreover, this solution is non-degenerate if and only if:

$$\epsilon^{-1} f'(c) \notin \text{Spec}(\Delta_g),$$

in which case its Morse Index is given by:

$$\text{Index}(c) = \sum_{\lambda \in \text{Spec}(\Delta_g), \lambda > \epsilon^{-1} f'(c)} \text{Mult}(\lambda).$$

Proof: The first assertion follows immediately from (C). By definition, u is non-degenerate if and only if $0 \notin \text{Spec}(LAC)$. By (D), this holds if and only if $\epsilon^{-1} f'(c) \notin \text{Spec}(LAC)$ and the second assertion follows. Finally, by (D), $\lambda > 0$ is an eigenvalue of LAC at c if and only if $\mu := \lambda + \epsilon^{-1} f'(c)$ is an eigenvalue of Δ_g . The third assertion then follows, and this completes the proof. \square

Since f has non-degenerate zeroes, if $f(c) = 0$, $f'(c)$ is either positive or negative. When $f'(c)$ is positive, it follows from Proposition 2.2.1 that $u = c$ is always non-degenerate with Morse Index zero. On the other hand, if $f'(c)$ is negative, then $u = c$ is degenerate for countably many values of c and its Morse Index tends to $+\infty$ as ϵ tends to 0. It follows that zeroes of f with negative derivative behave qualitatively differently from zeroes of f with positive derivative. In fact, they yield singularities which are fundamental in the sense that they cannot be removed by perturbations of the metric, and this will be key to the bifurcation theory that follows. We therefore define the **singular set**, $\text{Sing} \subseteq]0, \infty[\times \mathcal{M}^{\mu+1} \times \mathcal{C}^{\lambda+2}$ by:

$$\text{Sing} = \{(\epsilon, g, c) \mid f(c) = 0, \epsilon^{-1} f'(c) \in \text{Spec}(\Delta_g)\},$$

and we define the **regular solution space**, $\mathcal{Z}^* \subseteq \mathcal{Z}$ by:

$$\mathcal{Z}^* = \mathcal{Z} \setminus \text{Sing}.$$

We now construct a countable exhaustion of \mathcal{Z}^* by closed sets. For all $g \in \mathcal{M}^{\mu+1}$, we define:

$$\text{Sing}_g = \{(\epsilon, c) \mid f(c) = 0, \epsilon^{-1} f'(c) \in \text{Spec}(\Delta_g)\}.$$

By classical perturbation theory (c.f. [8]), Sing_g varies continuously with g in the Hausdorff sense. For all $m \in \mathbb{N}$, we define $\mathcal{Z}_m \subseteq \mathcal{Z}$ by:

$$\mathcal{Z}_m = \{(\epsilon, g, u) \in \mathcal{Z} \mid 1/m \leq \epsilon \leq n, d((\epsilon, u), \text{Sing}_g) \geq 1/m\},$$

and it follows from the continuous dependence of Sing_g on g that \mathcal{Z}_m is closed. Moreover:

$$\mathcal{Z}^* = \bigcup_{m \in \mathbb{N}} \mathcal{Z}_m.$$

We now say that a continuous mapping $\Phi : X \rightarrow Y$ between two topological spaces is **paraproper** whenever there exists a countable exhaustion $(X_m)_{m \in \mathbb{N}}$ of X by closed sets such that for all m , the restriction of Φ to X_m is proper. Paraproperness is made workable as a concept by the following restriction property:

Proposition 2.2.2

Let X and Y be topological spaces and let $\Phi : X \rightarrow Y$ be paraproper.

- (1) if $K \subseteq X$ is closed, then the restriction of Φ to K is paraproper; and
- (2) if X is metrisable and if $\Omega \subseteq X$ is open, then the restriction of Φ to Ω is paraproper.

Proof: Indeed, $(X_m \cap K)_{m \in \mathbb{N}}$ is a countable exhaustion of K by closed sets and for all m , the restriction of Φ to $X_m \cap K$ is proper, which proves (1). Now let d be a distance function over X . For all $n \in \mathbb{N}$, we define $\Omega_n \subseteq X$ by:

$$\Omega_n = \{x \in X \mid d(x, \Omega^c) \geq (1/n)\}.$$

For all n , Ω_n is closed, and since Ω is open:

$$\Omega = \bigcup_{n \in \mathbb{N}} \Omega_n.$$

$(\Omega_n \cap X_m)_{m, n \in \mathbb{N}}$ therefore constitutes a covering of Ω by closed sets. Moreover, for all $m, n \in \mathbb{N}$, since $\Omega_n \cap X_m$ is a closed subset of X_m , the restriction of Φ to this set is proper, and the restriction of Φ to Ω is therefore paraproper, which proves (2). \square

In particular Π_g defines a para-proper map from \mathcal{Z} into $\mathcal{M}^{\mu+1}$:

Proposition 2.2.3

For all n , Π_g defines a proper map from \mathcal{Z}_n into $\mathcal{M}^{\mu+1}$.

Proof: Let $(\epsilon_m, g_m, u_m)_{m \in \mathbb{N}}$ be a sequence in \mathcal{Z}_n and suppose that $(g_m)_{m \in \mathbb{N}}$ converges to $g_\infty \in \mathcal{M}^{\mu+1}$. Since $\epsilon_m \in [1/n, n]$ for all n , by the Heine-Borel Theorem, we may suppose that there exists $\epsilon_\infty \in [1/n, n]$ towards which $(\epsilon_m)_{m \in \mathbb{N}}$ converges. By Proposition 2.1.3, there exists $u_\infty \in C^{\lambda+2}(M)$ towards which $(u_m)_{m \in \mathbb{N}}$ subconverges. Since \mathcal{Z}_n is closed, $(\epsilon_\infty, g_\infty, u_\infty) \in \mathcal{Z}_n$, and the result follows. \square

Paraproperness now substitutes separability in our version of the Sard-Smale Theorem (c.f. [13]):

Theorem 2.2.4, Sard-Smale

If X and Y are smooth Banach manifolds, and if $\Phi : X \rightarrow Y$ is a smooth, paraproper Fredholm map, then the set of regular values of Φ is generic in Y .

Remark: As Smale's result often mystifies, it is worth underlining the straightforward idea behind it. Using Fredholm Theory and the Implicit Function Theorem for Banach manifolds we reduce the problem to one of smooth maps between finite dimensional manifolds, and the result then follows by the classical Sard Theorem. \square

Proof: Let $(X_n)_{n \in \mathbb{N}}$ be a countable exhaustion of X by closed sets such that for all n , the restriction of Φ to X_n is proper. For all n , we denote the restriction of Φ to X_n by Φ_n , and we denote the set of regular values of Φ_n in Y by Y_n . Since Φ_n is proper, and since surjectivity of Fredholm maps is an open property, Y_n is open for all n .

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We now show that Y_n is dense in Y . Indeed, choose $y \in Y$. Since we are only concerned with a neighbourhood of y in Y , without loss of generality, we may suppose that Y is a Banach space and that $y = 0$. Define $\Psi : X \times Y \rightarrow Y$ by $\Psi(\tilde{x}, \tilde{y}) = \Phi(\tilde{x}) + \tilde{y}$. Now choose $x \in \Phi_n^{-1}(0)$. Since Φ is Fredholm, $D\Phi(x)$ is closed and has finite dimensional cokernel, which we denote by E_x . In particular, the restriction of $D\Psi(x, 0)$ to $T_x X \times E_x$ is surjective, and since surjectivity of Fredholm maps is an open property, there exists a neighbourhood U_x of x in X_n such that the restriction of $D\Psi(\tilde{x}, 0)$ to $T_{\tilde{x}} X \times E_x$ is surjective for all $\tilde{x} \in U_x$. Since $\Phi_n^{-1}(0)$ is compact, it may be covered by finitely many such open sets, and there therefore exists a finite-dimensional subspace $E \subseteq Y$ such that the restriction of $D\Psi(\tilde{x}, 0)$ to $T_{\tilde{x}} X \times E$ is surjective for all $\tilde{x} \in \Phi_n^{-1}(y)$. We now consider the restriction of Ψ to $X \times E$ and we denote $Z = \Psi^{-1}(0)$. By the Implicit Function Theorem for Banach manifolds, there exists a neighbourhood Ω of $\Phi_n^{-1}(0) \times \{0\}$ in Z which is a smooth finite-dimensional submanifold of $X \times E$. Moreover, since $\Phi_n^{-1}(0)$ is compact, upon reducing Ω is necessary, we may suppose that this submanifold is separable. Let $\pi : \Omega \rightarrow E$ be the projection onto the first factor. Observe that if $\tilde{y} \in E$ is a regular value of π , then it is also a regular value of Φ_n . However, by Sard's Theorem, regular values of π are dense in E . It follows that $y = 0$ is a concentration point of regular values of Φ_n , and Y_n is therefore a dense subset of Y as asserted.

Since the set of regular values of Φ coincides with $\bigcap_{n \in \mathbb{N}} Y_n$, it follows that this set is generic, which completes the proof. \square

2.3 - The Regular Solution Space. In this section, we prove the following:

Proposition 2.3.1

If $\dim(M) \geq 2$, then for all $\mu > \lambda \in [0, \infty[\setminus \mathbb{N}$, \mathcal{Z}^ is a smooth Banach manifold modelled on $\mathbb{R} \times \mathcal{M}^{\mu+1}$. Moreover, Π_g defines a smooth, paraproper Fredholm map from \mathcal{Z}^* into \mathcal{M}^μ of Fredholm index equal to 1.*

We prove this result using the Implicit Function Theorem for Banach manifolds. It is thus necessary to show that the derivative of AC is surjective at every point of \mathcal{Z}^* . We denote by $D_1\text{AC}$, $D_2\text{AC}$ and $D_3\text{AC}$ the partial derivatives of AC with respect to the first, second and third components in $]0, \infty[\times \mathcal{M}^{\mu+1} \times C^{\lambda+2}$ respectively. We are interested in particular in $D_2\text{AC}$. The tangent space of $\mathcal{M}^{\mu+1}$ at any point canonically identifies with the space of $C^{\mu+1}$ sections of $\text{Symm}(TM)$. We denote this space by $\Gamma^{\mu+1} := \Gamma^{\mu+1}(\text{Symm}(TM))$ and we refer to elements therein as **first order perturbations** of the metric. We then identify $C^{\mu+1}$ with a subspace of $\Gamma^{\mu+1}$ by identifying every $C^{\mu+1}$ function f with the $C^{\mu+1}$ section fg , and this induces the orthogonal splitting, $\Gamma^{\mu+1} = \Gamma_0^{\mu+1} \oplus C^{\mu+1}$, where $\Gamma_0^{\mu+1} := \Gamma^{\mu+1}(\text{Symm}_0(TM))$ is the space of trace-free sections of $\text{Symm}(TM)$. The first order perturbations arising from sections of $C^{\mu+1}$ are precisely the conformal perturbations of the metric. However, it turns out that the useful perturbations for us are those whose trace vanishes. Indeed, for $g \in \mathcal{M}^{\mu+1}$, and for any first order perturbation A of g , denoting by $\delta_A \Delta_g$ the resulting first order perturbation of Δ_g , we obtain:

Proposition 2.3.2

If $\text{Tr}(A) = 0$, then, viewing A as a section of $\text{End}(TM)$, for all $\varphi \in C^{\lambda+2}$:

$$(\delta_A \Delta_g)\varphi = -\nabla \cdot (A\nabla\varphi),$$

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where ∇ and $\nabla \cdot$ are the gradient and divergence operators of g respectively.

Proof: We denote respectively by $\delta_A \Omega_g$ and $\delta_A \text{Hess}_g$ the first order perturbations resulting from A of the Levi-Civita covariant derivative and the Hessian operator of g . The Koszul formula yields:

$$(\delta_A \Omega_g)^k{}_{;ij} = \frac{1}{2}(A^k{}_{i;j} + A^k{}_{j;i} - A_{ij}{}^{;k}),$$

where indices are raised and lowered with respect to g . Thus:

$$\begin{aligned} \delta_A \text{Hess}_g(u)_{ij} &= -\frac{1}{2}(A^k{}_{i;j} + A^k{}_{j;i} - A_{ij}{}^{;k})u_{;k} \\ \Rightarrow \delta_A \Delta_g(u) &= -A_j{}^i u_{;i}{}^{;j} - A_j{}^{i;j} u_{;i} + \frac{1}{2} \text{Tr}(A)^{;k} u_{;k} \\ &= -(A_j{}^i u_{;i})^{;j} + \frac{1}{2} \text{Tr}(A)^{;k} u_{;k}, \end{aligned}$$

and since $\text{Tr}(A) = 0$, the result follows. \square

We recall the following straightforward result:

Proposition 2.3.3

Let X be a set consisting of at least n distinct points. Let E be an n -dimensional subset of the space of real-valued functions over X . Then there exist n points $p_1, \dots, p_n \in X$ such that the mapping $\text{Eval} : E \rightarrow \mathbb{R}^n$ given by:

$$\text{Eval}(f)_k = f(p_k),$$

is a linear isomorphism.

This allows us to prove the required surjectivity result:

Proposition 2.3.4

If $\text{Dim}(M) \geq 2$, if $(u, g, \epsilon) \in \mathcal{Z}$ and if u is non-constant, then DAC is surjective at (u, g, ϵ) .

Proof: Since $D_3AC = LAC$ is elliptic, it has finite-dimensional cokernel, which we denote by E . Since D_3AC is self-adjoint with respect to the L^2 -inner-product of g , for all $\varphi \in E$:

$$D_3AC(\varphi) = -\epsilon \Delta_g \varphi + f'(u)\varphi = 0.$$

Since u is non-constant, there exists $p \in M$ such that $\nabla u(p) \neq 0$. Let Ω be a neighbourhood of p in M diffeomorphic to the unit ball in Euclidean space over which ∇u doesn't vanish. By Aronszajn's unique continuation theorem (c.f. [3]), no non-trivial element of E vanishes over Ω . Furthermore, since f' has non-degenerate zeroes, $f'(u)$ does not vanish identically over Ω , and therefore no non-zero element of E restricts to a constant map over this set. Thus, by Proposition 2.3.3, there exist $p_1, \dots, p_m \in \Omega \setminus \{p\}$ such that the mapping $\alpha : C^\lambda \rightarrow \mathbb{R}^m$ given by:

$$\alpha(\varphi)_k = \varphi(p) - \varphi(p_k),$$

restricts to a bijection on E .

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For any vector $\xi := (\xi_0, \dots, \xi_m)$ of functions in $C_0^\infty(\Omega)$ we define $\alpha_\xi : C^\lambda \rightarrow \mathbb{R}^m$ by:

$$\alpha_\xi(\varphi)_i = \int_M (\xi_0 - \xi_i)\varphi d\text{Vol}_g.$$

If $\xi_0 - \xi_k$ is sufficiently close to $\delta_p - \delta_{p_k}$ in the weak sense for all k , where δ_p and δ_{p_k} are the Dirac delta functions supported at p and p_k respectively, then α_ξ is close to α and, in particular, is invertible. It follows that if F is the linear span of $(\xi_0 - \xi_k)_{1 \leq k \leq m}$, then the L^2 -inner-product restricts to a non-degenerate bilinear form over $E \times F$. In particular, $\text{Dim}(F) = \text{Dim}(E)$ and:

$$F \cap \text{Im}(D_3\text{AC}) = F \cap E^\perp = \{0\}.$$

F is therefore complementary to $\text{Im}(D_3\text{AC})$ in C^λ . That is:

$$C^\lambda = F \oplus \text{Im}(D_3\text{AC}).$$

However, we may suppose in addition that for all $1 \leq k \leq m$:

$$\int_M (\xi_0 - \xi_k) d\text{Vol}_g = 0$$

It then follows from classical de-Rham cohomology theory that for all k there exists a smooth vector field X_k supported in Ω such that:

$$\nabla \cdot X_k = \xi_0 - \xi_k.$$

By Proposition 2.1.1, ∇u is of class $C^{\mu+1}$, and thus, since it is non-vanishing over Ω , there exists for all k a $C^{\mu+1}$ field A_k of symmetric matrices such that $A_k \nabla u = X_k$. In addition, since M has dimension at least 2, we may assume moreover that $\text{Tr}(A_k) = 0$ for all k , and it follows from Proposition 2.3.2 that:

$$D_2\text{AC} \cdot A_k = -\epsilon \nabla \cdot (A_k \nabla u) = \epsilon \nabla \cdot X_k = \epsilon(\xi_0 - \xi_k).$$

It follows that $F \subseteq \text{Im}(D_2\text{AC})$ and so $C^\lambda \subseteq \text{Im}(D\text{AC})$ and surjectivity follows. \square

Proposition 2.3.1 follows readily:

Proof of Proposition 2.3.1: By Propositions 2.2.1 and 2.3.4, $D\text{AC}$ is surjective at every point of \mathcal{Z}^* . Since $D_3\text{AC}$ is self-adjoint and elliptic, it is Fredholm of index zero, and it follows from the Implicit Function Theorem for Banach manifolds that \mathcal{Z}^* is a smooth Banach manifold modelled on $\mathbb{R} \times \mathcal{M}^{\mu+1}$ and Π_g is a smooth Fredholm map of Fredholm index equal to 1. Finally, by Proposition 2.2.3, Π_g is para-proper, and this completes the proof. \square

Applying the Sard/Smale Theorem, now yields:

Proposition 2.3.5

If $\dim(M) \geq 2$, then for generic $g \in \mathcal{M}^{\mu+1}$, \mathcal{Z}_g^* is a smooth, 1-dimensional submanifold of $]0, \infty[\times C^{\lambda+2}$. Moreover, if we denote by $\epsilon_g : \mathcal{Z}_g^* \rightarrow]0, \infty[$ the projection onto the first factor, then ϵ_g is proper.

Remark: Observe that paraproperness allows us to show that \mathcal{Z}_g^* is separable even though \mathcal{Z}^* isn't. \square

Proof: By the Sard-Smale Theorem, the set of regular values of Π_g is generic in $\mathcal{M}^{\mu+1}$. Let $g \in \mathcal{M}^{\mu+1}$ be a regular value of Π_g . By definition, $D\Pi_g(\epsilon, u, g)$ is surjective for all $(\epsilon, u) \in \mathcal{Z}_g^*$. Since Π_g is a smooth Fredholm map of Fredholm index equal to 1, it follows from the Implicit Function Theorem for Banach manifolds that \mathcal{Z}_g^* is a (not necessarily separable) smooth, 1-dimensional submanifold of $]0, \infty[\times C^{\lambda+2}$. By Proposition 2.1.3, $\epsilon_g : \mathcal{Z}_g \rightarrow]0, \infty[$ is proper, and since $]0, \infty[$ has a compact exhaustion, so too does \mathcal{Z}_g . In particular, \mathcal{Z}_g is separable, and therefore so too is \mathcal{Z}_g^* , which completes the proof. \square

2.4 - Non-Degeneracy of Critical Points. Let $\epsilon :]0, \infty[\times \mathcal{M}^{\mu+1} \times C^{\lambda+2} \rightarrow]0, \infty[$ be the projection onto the first factor, and denote its restriction to \mathcal{Z}_g^* by ϵ_g . We now aim to show that for generic $g \in \mathcal{M}^{\mu+1}$, every critical point ϵ_g is non-degenerate. We first characterise those points where $d\epsilon_g$ vanishes:

Proposition 2.4.1

At every point of \mathcal{Z}^* :

$$\text{Ker}(D\epsilon) \cap \text{Ker}(D\Pi_g) \cap T\mathcal{Z}^* = \{0\} \times \{0\} \times \text{Ker}(LAC).$$

Proof: Indeed, by definition:

$$\begin{aligned} T\mathcal{Z}^* &= \text{Ker}(DAC) \\ &= \text{Ker}(D_1AC \circ D\epsilon + D_2AC \circ D\Pi_g + D_3AC \circ D\Pi_u), \end{aligned}$$

where D_1AC , D_2AC and D_3AC represent the partial derivatives of AC with respect to the first, second and third components respectively. Thus:

$$\begin{aligned} \text{Ker}(D\epsilon) \cap \text{Ker}(D\Pi_g) \cap T\mathcal{Z}^* &= \text{Ker}(D\epsilon) \cap \text{Ker}(D\Pi_g) \cap \text{Ker}(D_3AC \circ D\Pi_u) \\ &= \{0\} \times \{0\} \times \text{Ker}(LAC), \end{aligned}$$

as desired. \square

Proposition 2.4.2

If g is a regular value of Π_g , then at every point of \mathcal{Z}_g^* :

$$\text{Ker}(d\epsilon_g) = \text{Ker}(D\epsilon) \cap T\mathcal{Z}_g^* = \{0\} \times \{0\} \times \text{Ker}(LAC).$$

In particular, LAC has nullity at most 1.

Proof: If g is a regular value of Π_g , then:

$$\text{Ker}(D\Pi_g) \cap T\mathcal{Z}^* = T\mathcal{Z}_g^*,$$

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and the result now follows by Proposition 2.4.1. \square

By Proposition 2.4.2, if g is a regular value of Π_g and if $p \in \mathcal{Z}_g$ is such that $d\epsilon_g = 0$, then $\text{Ker}(LAC)$ is 1-dimensional. In particular, we may split $C^{\lambda+2}$ as the direct sum of $\text{Ker}(LAC)$ and $\text{Ker}(LAC)^\perp$ where $\text{Ker}(LAC)^\perp$ is the orthogonal complement of $\text{Ker}(LAC)$ in $C^{\lambda+2}$ with respect to the L^2 -inner-product.

Proposition 2.4.3

If g is a regular value of Π_g , and if $p \in \mathcal{Z}_g$ is such that $d\epsilon_g(p) = 0$, then there is a neighbourhood Ω of p in \mathcal{Z}^ which is a graph over $\mathcal{M}^{\mu+1} \times \text{Ker}(LAC)$.*

Proof: Let $\pi : C^{\lambda+2} \rightarrow \text{Ker}(LAC)$ be the orthogonal projection. Consider the restriction of the mapping $(\Pi_g, \pi \circ \Pi_u)$ to \mathcal{Z}^* . Since $d\epsilon_g(p) = 0$, bearing in mind Proposition 2.4.2, at p :

$$\text{Ker}(D\Pi_g) \cap T\mathcal{Z}^* = T\mathcal{Z}_g^* = \text{Ker}(d\epsilon_g) \cap T\mathcal{Z}_g^* = \{0\} \times \{0\} \times \text{Ker}(LAC).$$

In particular, the restriction of $\pi \circ \Pi_u$ to $\text{Ker}(D\Pi_g)$ is a linear isomorphism. The restriction of $(\Pi_g, \pi \circ \Pi_u)$ to $T\mathcal{Z}^*$ is therefore also a linear isomorphism at p and the result now follows by the Inverse Function Theorem for smooth maps between Banach manifolds. \square

Let $\Omega \subseteq \mathcal{Z}^*$ be as in Proposition 2.4.3. We construct a non-vanishing vector field, X over Ω which is always tangent to \mathcal{Z}_g as follows. Choose $\varphi_0 \in \text{Ker}(LAC)$ such that $\|\varphi_0\|_{L^2}^2 = 1$. Let X be the unique, smooth vector field over Ω which projects down to φ . There exist smooth functions $s : \Omega \rightarrow \mathbb{R}$ and $\varphi : \Omega \rightarrow \varphi_0 + \text{Ker}(LAC)^\perp$ such that, throughout Ω :

$$X = (s, 0, \varphi).$$

Trivially:

$$D\Pi_g \cdot X = 0,$$

so that X is always tangent to \mathcal{Z}_g , as desired.

We now recall the following formula for the variation of a non-degenerate eigenvalue. Let $E \subseteq F \subseteq L^2(M)$ be Banach spaces and let $i : E \rightarrow F$ be a continuous embedding with dense image. It is normal to suppress i and identify elements of E with their image in F . Let $A \in \text{Lin}(E, F)$ be a bounded, linear map. We recall that E is said to be **self-adjoint** if and only if for all $u, v \in E$:

$$\langle u, A(v) \rangle = \langle A(u), v \rangle.$$

The Implicit Function Theorem for Banach manifolds readily yields:

Proposition 2.4.4

Let X, E and F be Banach spaces. Let $A : X \rightarrow \text{Lin}(E, F)$ be a smooth mapping such that for all $x \in X$, $A_x := A(x)$ is self-adjoint and Fredholm of index zero. Suppose that $\text{Null}(A_0) = 1$ and let φ_0 be a non-zero element of $\text{Ker}(A_0)$. Then there exists a neighbourhood U of 0 in X and smooth maps $\lambda : X \rightarrow \mathbb{R}$ and $\varphi : X \rightarrow \varphi_0 + \text{Ker}(A_0)^\perp$ such that $\lambda(0) = 0$, $\varphi(0) = \varphi_0$ and for all $x \in X$:

$$A(x)\varphi(x) = \lambda(x)\varphi(x).$$

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Moreover, for any tangent vector ξ to X at 0:

$$d\lambda(\xi) = \langle DA_0(\xi)\varphi_0, \varphi_0 \rangle.$$

By Proposition 2.4.4, upon reducing Ω if necessary, there exist smooth functions $\lambda : \Omega \rightarrow \mathbb{R}$ and $\tilde{\varphi} : \Omega \rightarrow \varphi_0 + \text{Ker}(LAC(p_0))^\perp$ such that $\lambda(p_0) = 0$, $\tilde{\varphi}(p_0) = \varphi_0$ and throughout Ω :

$$LAC\tilde{\varphi} = \lambda\tilde{\varphi}.$$

The role played by λ is revealed by the following quantitative analogue of Proposition 2.4.2:

Proposition 2.4.5

Let $g \in \mathcal{M}^{\mu+1}$ be a regular value of Π_g . If $p \in \mathcal{Z}_g$ is such that $d\epsilon_g(p) = 0$, then:

$$\langle D_1AC \cdot D^2\epsilon_g(X_p, X_p), \varphi_0 \rangle = -d\lambda(X_p).$$

In particular, if p is a non-degenerate zero of λ , then it is also a non-degenerate zero of $d\epsilon_g$.

Proof: By definition, AC vanishes over \mathcal{Z}^* and so:

$$D_1AC \circ D\epsilon(X) = -D_2AC \circ D\Pi_g(X) - D_3AC \circ D\Pi_u(X) = -LAC \circ D\Pi_u(X).$$

Since $D\epsilon(X_p) = 0$, differentiating a second time yields:

$$D_1AC \circ D^2\epsilon(X_p, X_p) = -D_{X_p}LAC \circ D\Pi_u(X_p) - LAC \circ D^2\Pi_u(X_p, X_p).$$

Observe that $D^2\Pi_u(X_p, X_p)$ takes values in $\text{Ker}(LAC)^\perp$. Moreover, $\varphi_0 \in \text{Ker}(LAC)$, and since LAC preserves both $\text{Ker}(LAC)$ and $\text{Ker}(LAC)^\perp$, taking the inner-product with φ_0 yields:

$$\begin{aligned} \langle D_1AC \circ D^2\epsilon(X_p, X_p), \varphi_0 \rangle &= -\langle D_{X_p}LAC \circ D\Pi_u(X_p), \varphi_0 \rangle \\ &= -\langle D_{X_p}LAC\varphi_0, \varphi_0 \rangle. \end{aligned}$$

Thus, by Proposition 2.4.4:

$$\langle D_1AC \circ D^2\epsilon_g(X_p, X_p), \varphi_0 \rangle = -d\lambda(X_p),$$

as desired. \square

The above discussion is most usefully summarised as follows:

Proposition 2.4.6

There exists an open subset $\Omega \subseteq \mathcal{Z}^*$ and a smooth function $\lambda : \Omega \rightarrow \mathbb{R}$ with the following properties:

- (1) if $p \in \mathcal{Z}^*$ is such that $D\Pi_g(p)$ is surjective and $d\epsilon_g(p) = 0$, then $p \in \Omega$ and $\lambda = 0$; and
- (2) for all $p \in \Omega$, $\lambda(p)$ is an eigenvalue of $LAC(p)$.

Proof: Let $p \in \mathcal{Z}^*$ be such that $D\Pi_g(p)$ is surjective and $D\epsilon_g(p) = 0$. Let Ω_p and $\lambda : \Omega_p \rightarrow \mathbb{R}$ be as in the preceding discussion. Upon reducing Ω_p if necessary, we may assume that λ is the eigenvalue of $LAC(p)$ with least absolute value. It follows then that λ is uniquely defined, and taking the union over all such Ω_p yields the desired open set and smooth function. \square

Proposition 2.4.7

Suppose that $\text{Dim}(M) \geq 3$. Choose $p \in \Omega$ and let $\varphi \in C^{\lambda+2}$ be an element of $\text{Ker}(LAC(p))$. If there exists a point $x \in M$ such that $du(x)$ and $d\varphi(x)$ are both non-vanishing and non-colinear, then $d\lambda$ is non-zero at p .

Proof: Let A be a trace-free first order perturbation of g and let $\delta_A d\text{Vol}_g$, $\delta_A LAC$ and $\delta_A \lambda$ denote the resulting first order perturbations of $d\text{Vol}_g$, LAC and λ respectively. Then:

$$\delta_A d\text{Vol}_g = \text{Tr}(A)d\text{Vol}_g = 0.$$

Thus, by Proposition 2.4.4:

$$\delta_A \lambda = \int \varphi(\delta_A LAC)\varphi d\text{Vol}_g.$$

However, by Proposition 2.3.2:

$$\delta_A LAC\varphi = -\epsilon \nabla \cdot (A \nabla \varphi),$$

and so:

$$\begin{aligned} \delta_A \lambda &= -\epsilon \int \varphi \nabla \cdot (A \nabla \varphi) d\text{Vol}_g \\ &= \epsilon \int \langle A, \nabla \varphi \otimes \nabla \varphi \rangle d\text{Vol}_g. \end{aligned}$$

Let $p \in M$ be such that $du(p)$ and $d\varphi(p)$ are non-vanishing and non-colinear. Since M is 3-dimensional, there exists a first-order perturbation A of g , supported near p such that:

$$\text{Tr}(A) = 0, \quad (\delta_A \Delta_g)u = 0, \quad \delta_A \lambda \neq 0.$$

It follows from the first two relations that the vector $(0, A, 0)$ is tangent to \mathcal{Z}^* , and it follows from the third relation that $d\lambda(0, A, 0) \neq 0$, which completes the proof. \square

Applying the Sard/Smale Theorem, we now obtain:

Proposition 2.4.8

If $\text{Dim}(M) \geq 3$, then for generic $g \in \mathcal{M}^{\mu+1}$ and for $p \in \mathcal{Z}_g$, if $d\epsilon_g(p) = 0$, then either:

- (1) $D^2\epsilon_g(p) \neq 0$; or
- (2) if $\varphi \in \text{Ker}(LAC(p))$, then for all $Y \in TM$, if $du(Y) = 0$, then $d\varphi(Y) = 0$.

Proof: Let $X \subseteq \Omega$ be the set of all points p such that $d\epsilon_g(p) = 0$ and (2) is satisfied. Observe that (2) implies that du and $d\varphi$ are everywhere colinear. Since this is a closed condition, it follows that X is a closed subset of Ω , and $\tilde{\Omega} := \Omega \setminus X$ is therefore open. Let $Y \subseteq \tilde{\Omega}$ be the set of all points where λ vanishes. By Proposition 2.4.7 and the Implicit Function Theorem for Banach manifolds, Y is a smooth codimension-1 Banach submanifold of $\tilde{\Omega}$. Observe that the restriction of Π_g to Y is a smooth Fredholm map of Fredholm index 0. Moreover, by Proposition 2.2.2, this restriction is paraproper. It therefore follows from Theorem 2.2.4 that for generic $g \in \mathcal{M}^{\mu+1}$, g is a regular value of this restriction. Moreover, since the intersection of two generic sets is also generic, we may assume that g is also a regular value of Π_g . For such a g , \mathcal{Z}_g is a smooth 1-dimensional manifold and the restriction of λ to \mathcal{Z}_g has non-degenerate zeroes at all points where (2) is satisfied, and the result now follows by Proposition 2.4.5. \square

2.5 - The Degenerate Case. We now eliminate Case (2) of Proposition 2.4.8. We begin by characterising its geometry:

Proposition 2.5.1

Let $u, \varphi \in C^{\lambda+2}(M)$ be such that u is non-constant, φ is non-zero, and:

$$\epsilon \Delta_g u = f(u), \quad \epsilon \Delta_g \varphi = f'(u)\varphi.$$

If for all vectors $X \in TM$ such that $du(X) = 0$ we have $d\varphi(X) = 0$, then $\|du\|_g$ is constant over each connected component of every level set of u .

Remark: In fact, we can prove more: the complement of the vanishing set of du in M is foliated by compact hypersurfaces of constant mean curvature. This property is interesting, as it is independent of the parameter ϵ . \square

Proof: Observe that if u is constant over any non-trivial neighbourhood, then it is equal to a zero of f , c say over this neighbourhood. Since $u = c$ is also a solution of $AC(u) = 0$, it follows from Aronszajn's unique continuation theorem (c.f. [3]) that $f = c$ over the whole of M , which is absurd, and it follows that du is almost everywhere non-vanishing.

Now choose $p \in M$ such that $du(p) \neq 0$. Let Ω be a neighbourhood of p over which du does not vanish. Observe that the image of the restriction of u to Ω is an open interval, I , say. Moreover, by Aronszajn's unique continuation theorem again, the restriction of φ to Ω is non-zero. Let \mathcal{F} denote the foliation of Ω by level hypersurfaces of u . By hypothesis, φ is constant over each leaf of \mathcal{F} . Thus, upon reducing Ω if necessary, there exists a non-zero $C^{\lambda+2}$ -function $\Phi : I \rightarrow \mathbb{R}$ such that, over Ω , $\varphi = \Phi(u)$. Taking the Laplacian of both sides of this relation yields:

$$f'(u)\Phi(u) = \epsilon \Phi''(u)\|du\|_g^2 + f(u)\Phi'(u).$$

We claim that Φ'' is almost everywhere non-vanishing. Indeed, otherwise, upon reducing Ω further if necessary, we may suppose that Φ is linear and that $f'\Phi - f\Phi' = 0$. The restriction of f to I is therefore also linear, which is absurd by the hypothesis on f , and Φ'' is therefore almost everywhere non-vanishing, as asserted. However, whenever $\Phi''(u) \neq 0$, we have:

$$\|du\|_g^2 = \frac{1}{\epsilon \Phi''(u)}(f'(u)\Phi(u) - f(u)\Phi'(u)),$$

from which it follows that $\|du\|_g^2$ is constant over every leaf of \mathcal{F} where $\Phi''(u)$ does not vanish. Since the set of all such leaves is dense, it follows that $\|du\|_g$ is constant over every leaf of \mathcal{F} .

Choose $t \in \mathbb{R}$ and denote $X = u^{-1}(t)$. Let $X_0, X_1 \subseteq X$ be respectively the subset of X consisting of those points where du vanishes, and the subset of X consisting of those points where it does not vanish. Trivially, $\|du\|_g$ is constant over X_0 . Observe that X_1 is a submanifold of M . Moreover, by the above discussion, $\|du\|_g$ is constant over every connected component of X_1 . Every connected component of X_1 is therefore a closed submanifold, and, in particular, is disjoint from X_0 . It follows that if X' is a connected component of X , then X' is either contained wholly in X_0 or wholly in X_1 . In either case, $\|du\|_g$ is constant over X' , and this completes the proof. \square

The following refinement of Proposition 2.5.1 is easier to work with:

Proposition 2.5.2

Under the same hypotheses as Proposition 2.5.1, if $X_p \in TM$ is such that $du(X_p) = 0$, then:

$$\text{Hess}^g(u)(\nabla^g u, X_p) = 0.$$

Proof: If $du(p) = 0$, then $\nabla^g u(p) = 0$, and the result follows trivially. Otherwise, $du(p) \neq 0$, and, by Proposition 2.5.1, $\|\nabla^g u\| = \|du\|_g$ is constant over the level hypersurface of u passing through p . Since $du(X_p) = 0$, X_p is tangent to S , and so:

$$\text{Hess}(u)(\nabla^g u, X_p) = \langle \nabla_{X_p} \nabla^g u, \nabla^g u \rangle = \frac{1}{2} X_p \|\nabla^g u\|^2 = 0,$$

as desired. \square

Proposition 2.5.3

Suppose that $\text{Dim}(M) \geq 2$ and let $u \in C^{\lambda+2}$ be a non-constant function such that $\epsilon \Delta_g u = f(u)$. Choose $p \in M$ such that $\nabla^g u(p) \neq 0$ and $Y \in T_p M$ such that $du(Y) = 0$. There exists a $C^{\mu+1}$ first order perturbation A of the metric supported in an arbitrarily small neighbourhood of p such that:

- (1) $A(p) = 0$;
- (2) $(\delta_A \Delta_g)u = 0$; and
- (3) $\delta_A \text{Hess}(u)(\nabla u, Y)(p) \neq 0$.

Proof: Let Ω be a neighbourhood of p diffeomorphic to the unit ball. Let X be a smooth divergence-free vector field supported in Ω such that $X(p) = 0$. Since M is at least two dimensional, and since ∇u does not vanish over Ω , there exists a $C^{\mu+2}$ section A of $\text{Symm}(TM)$ supported in Ω such that $A \cdot \nabla u = X$ and $\text{Tr}(A) = 0$. In particular, we may suppose that $A(p) = 0$. By Proposition 2.3.2, for any such A :

$$(\delta_A \Delta_g)u = 0.$$

As in the proof of Proposition 2.3.2, the first order perturbation of the Hessian of u is given by:

$$\delta_A \text{Hess}_g(u)(X, Y) = \frac{1}{2} (A_{ij}{}^k - A^k{}_{ij} - A^k{}_{j;i}) u_k X^i Y^j.$$

Thus, bearing in mind that $(\delta_A \nabla^g)u(p) = X(p) = 0$:

$$\begin{aligned} \delta_A \text{Hess}_g(u)(\nabla^g u, Y) &= \frac{1}{2} (A_{ij}{}^k - A^k{}_{ij} - A^k{}_{j;i}) u_k u^i Y^j \\ &= -\frac{1}{2} A_{kij} u^k u^i Y^j \\ &= -\frac{1}{2} Y \langle A \nabla^g u, \nabla^g u \rangle + \langle A \nabla_Y^g \nabla^g u, \nabla^g u \rangle \\ &= -\frac{1}{2} Y \langle A \nabla^g u, \nabla^g u \rangle + \text{Hess}_g(u)(Y, X) \\ &= -\frac{1}{2} Y \langle X, \nabla^g u \rangle \\ &= -\frac{1}{2} Y du(X). \end{aligned}$$

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Since X is divergence free and compactly supported in Ω , it follows from classical de-Rham cohomology theory that there exists a 2-form Z supported in Ω such that:

$$X = \nabla^g \cdot Z.$$

We choose exponential coordinates about p , and write Z as:

$$Z = \sum_{i < j} Z^{ij} \partial_i \wedge \partial_j,$$

so that:

$$X^k = \sum_{i > k} \partial_i Z^{ik} - \sum_{i < k} \partial_i Z^{ki}.$$

We choose the basis at p such that $\nabla^g u$ and Y are colinear with ∂_1 and ∂_2 respectively. Thus, at the origin, bearing in mind that $X(p) = 0$:

$$-\frac{1}{2} Y du(X) = \frac{1}{2} \|\nabla^g u\|_g \|Y\|_g \sum_{i > 1} \partial_2 \partial_i Z^{1i}.$$

We choose Z such that $\partial_i Z^{jk} = 0$ for all i, j and k , $\partial_2 \partial_1 Z^{11} = 1$ and $\partial_2 \partial_i Z^{1i} = 0$ for all $i > 1$. Then, if $X = \nabla^g \cdot Z$:

$$X(p) = 0, \quad -\frac{1}{2} Y du(X) = \frac{1}{2} \|\nabla^g u\|_g \|Y\|_g.$$

$X = \nabla^g \cdot Z$ is the desired vector field, and this completes the proof. \square

Proposition 2.5.4

If $\dim(M) \geq 2$, then for generic $g \in \mathcal{M}^{\mu+1}$, if $(\epsilon, g, u) \in \mathcal{Z}_g$, if u is non-constant and if $\varphi \in \text{Ker}(LAC(u))$ is non-zero, then there exists a point $p \in M$ such that $du(p)$ and $d\varphi(p)$ are both non-zero and non-colinear.

Proof: Let X_p and Y_q be unit vectors over distinct points of M . Let $\Omega := \Omega(X_p, Y_q) \subseteq \mathcal{Z}^*$ be the open set of all (ϵ, g, u) such that $\nabla^g u(p)$ and $\nabla^g u(q)$ are both non-zero and non-colinear with X_p and Y_q respectively. We define the functions $\Phi_p, \Phi_q : \Omega \rightarrow \mathbb{R}$ by:

$$\Phi_p(\epsilon, g, u) = \text{Hess}_g(u)(X_p^\perp, \nabla^g u(p)), \quad \Phi_q(\epsilon, g, u) = \text{Hess}_g(u)(Y_q^\perp, \nabla^g u(q)),$$

where X_p^\perp and Y_q^\perp are the orthogonal projections of X_p and Y_q respectively onto the normal hyperplanes to $\nabla^g u(p)$ and $\nabla^g u(q)$ respectively. Observe that both Φ_p and Φ_q define smooth functions over Ω . Moreover, it follows from Proposition 2.5.3 that $D(\Phi_p, \Phi_q)$ is surjective at every point of Ω . Thus, if $Z := Z(X_p, Y_q)$ is the zero set of this functional then it is a smooth, codimension 2 submanifold of Ω . In particular, the restriction of Π_g to Z is a smooth Fredholm map of index -1 . Thus, if $g \in \mathcal{M}^{-1}$ is a regular value of the restriction of Π_g to Z , then $\Pi_g^{-1}(g) \cap Z$ is a smooth submanifold of Z of dimension equal to -1 , that is, it is empty. However, by Proposition 2.2.2, the restriction of Π_g to Ω , and

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therefore also to Z , is para-proper, and it follows by Theorem 2.2.4 that the set of regular values of this restriction is generic in $\mathcal{M}^{\mu+1}$.

Let $\mathcal{X} \subseteq (UM \times UM) \setminus \pi^{-1}(\text{Diag})$ be a countable dense family of pairs (X_p, Y_q) of unit vectors above distinct points of M . Since the intersection of a countable family of generic sets is generic, it follows that for generic $g \in \mathcal{M}^{\mu+1}$, and for all $(X_p, Y_q) \in \mathcal{X}$, $\Pi_g^{-1}(g) \cap Z(X_p, Y_q)$ is empty. For such a g , choose $(\epsilon, g, u) \in \mathcal{Z}_g$. Let $\tilde{p}, \tilde{q} \in M$ be distinct points such that both $du(\tilde{p})$ and $du(\tilde{q})$ are non-zero, and let $\tilde{X}_{\tilde{p}}$ and $\tilde{Y}_{\tilde{q}}$ be unit vectors in UM normal to $\nabla^g u(\tilde{p})$ and $\nabla^g u(\tilde{q})$ respectively. Since \mathcal{X} is dense, there exists a pair $(X_p, Y_q) \in \mathcal{X}$ such that $du(p)$ and $du(q)$ are non-zero and X_p and Y_q are non-colinear with $\nabla^g u(p)$ and $\nabla^g u(q)$ respectively. However, by definition of g , $(\epsilon, g, u) \notin Z(X_p, Y_q)$, from which it follows that one of $\Phi_p(\epsilon, g, u)$ and $\Phi_q(\epsilon, g, u)$ is non-zero. In other words, without loss of generality:

$$\text{Hess}_g(u)(X_p^\perp, \nabla^g u(p)) \neq 0,$$

and it now follows from Proposition 2.5.2 that there exists at least one point in M where du and $d\varphi$ are non-zero and non-colinear, as desired. \square

Combining these relations, we obtain Theorem 1.1.1:

Proof of Theorem 1.1.1: Since the intersection of finitely many generic sets is generic, this follows from Propositions 2.3.5, 2.4.8 and 2.5.4. \square

2.6 - The Solution Space at Infinity. Now fix $g \in \mathcal{M}^{\mu+1}$. We show that for ϵ sufficiently large, the only elements of $\mathcal{Z}_{\epsilon, g}$ are the constant solutions. We recall that for all g , $\text{Ker}(\Delta_g)^\perp$ coincides with the space of functions whose integral with respect to the volume form of g vanishes.

Proposition 2.6.1

Let $c \in \mathbb{R}$ be such that $f(c) = 0$. There exist $B > 0$ and $\delta > 0$ such that if $\epsilon > B$, if $v \in C^{\lambda+2}$ and $t \in \mathbb{R}$ are such that:

$$\int_M v d\text{Vol}_g = 0, \quad \|v\|_{\lambda+2} < \delta \epsilon^{-1}, \quad |t| < \delta,$$

and if $AC(\epsilon, g, c + v + t) = 0$, then $(v, t) = (0, 0)$.

Proof: Define $\mathcal{F} : \text{Ker}(\Delta_g)^\perp \times \mathbb{R}^2 \rightarrow C^\lambda$ by:

$$\mathcal{F}(v, t, \eta) = \Delta_g v - f(c + \eta v + t).$$

Observe that \mathcal{F} is a smooth function between Banach manifolds. Moreover, if we denote by $D_1\mathcal{F}$ and $D_2\mathcal{F}$ its partial derivatives with respect to the first and second factors respectively, then since $f'(c) \neq 0$, $D_1\mathcal{F} + D_2\mathcal{F}$ is surjective at $(0, 0, 0)$. It follows from the Implicit Function Theorem for Banach manifolds that there exists $b > 0$ and a neighbourhood W of $(0, 0)$ in $\text{Ker}(\Delta_g)^\perp \times \mathbb{R}$ such that if $\eta < b$ then there exists a unique point $(v_\eta, t_\eta) \in W$ such that $\mathcal{F}(v_\eta, t_\eta, \eta) = 0$. Since, in particular, $\mathcal{F}(0, 0, \eta) = 0$ for all η , it follows that if

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$(v, t) \in W$ is such that $\mathcal{F}(v, t, \eta) = 0$, then $(v, t) = (0, 0)$. Let $B = 1/b$ and let $\delta > 0$ be such that:

$$\{(v, t) \mid \|v\|_{\lambda+2} < \delta, |t| < \delta\} \subseteq W.$$

We claim that B and δ have the desired properties. Indeed, let $\epsilon > B$, $v \in C^{\lambda+2}$ and $t \in \mathbb{R}$ be such that $v \in \text{Ker}(\Delta_g)^\perp$, $\|v\|_{\lambda+2} < \delta\epsilon^{-1}$, $|t| < \delta$ and $\text{AC}(\epsilon, g, c + v + t) = 0$. Then, denoting $\eta = 1/\epsilon$:

$$\mathcal{F}(\epsilon v, t, \eta) = \Delta_g(\epsilon v) - f(c + \eta(\epsilon v) + t) = 0.$$

Since $\|\epsilon v\|_{\lambda+2}, |t| < \delta$, it follows from the preceding discussion that $(v, t) = (0, 0)$, as desired. \square

Proposition 2.6.2

There exists $B > 0$ such that if $\epsilon > B$, then $\mathcal{Z}_{\epsilon, g}$ only consists of constant solutions.

Proof: Suppose the contrary. There exists a sequence $(u_n, t_n, \epsilon_n)_{n \in \mathbb{N}} \in \text{Ker}(\Delta_g)^\perp \times \mathbb{R}^2$ such that $(\epsilon_n)_{n \in \mathbb{N}}$ tends to $+\infty$, u_n is non-zero, and for all n :

$$\text{AC}(\epsilon_n, g, u_n + t_n) = 0.$$

For all n , denote $v_n = u_n + t_n$. Observe that the argument of Proposition 2.1.3 is uniform in ϵ as ϵ tends to $+\infty$, and there therefore exists $v_\infty \in C^{\lambda+2}$ towards which $(v_n)_{n \in \mathbb{N}}$ subconverges. For all n :

$$\Delta_g v_n - \epsilon_n^{-1} f(v_n) = 0.$$

Upon taking limits, it follows that $\Delta_g v_\infty = 0$, and so v_∞ is equal to a constant, c , say. On the other hand, for all n :

$$\int f(v_n) d\text{Vol} = \int \epsilon_n \Delta_g v_n d\text{Vol} = 0,$$

and upon taking limits, it follows that:

$$f(c) \text{Vol}(M) = \int f(c) d\text{Vol} = 0,$$

and so c is a zero of f . In particular, $\Delta_g(\epsilon_n u_n) = (f(v_n))_{n \in \mathbb{N}}$ converges to 0 in the C^λ -topology. However, by the Closed Graph Theorem, the restriction of Δ_g to $\text{Ker}(\Delta_g)^\perp$ is a linear isomorphism onto its image, and it follows that $(\epsilon_n \|u_n\|_{\lambda+2})_{n \in \mathbb{N}}$ converges to 0. Finally, for all n :

$$\text{Vol}(M) t_n = \int v_n d\text{Vol},$$

from which it follows that $(|t_n - c|)_{n \in \mathbb{N}}$ converges to 0. It now follows from Proposition 2.6.1 that for sufficiently large n , $u_n = 0$. This is absurd by hypothesis, and the result follows. \square

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2.7 - Morse Homology. We now study the Morse Homology of the Allen Cahn Equation. The construction is fairly standard, and we refer the reader to our forthcoming paper [14] for a detailed outline in the Hölder space framework. We assume henceforth that $\text{Dim}(M) \geq 3$. Let g be as in Theorem 1.1.1 and let ϵ be such that $\epsilon^{-1} \notin \text{Spec}(-\Delta_g)$.

For all $k \in \mathbb{N}$, we define $\mathcal{Z}_{\epsilon,g,k} \subseteq \mathcal{Z}_{\epsilon,g}$ by:

$$\mathcal{Z}_{\epsilon,g,k} = \{u \in \mathcal{Z}_{\epsilon,g} \mid \text{Index}(u) = k\},$$

and for all $k \in \mathbb{N}$, we define the **chain group** C_k by:

$$C_k = \mathbb{Z}_2[\mathcal{Z}_{\epsilon,g,k}] = \{f : \mathcal{Z}_{\epsilon,g,k} \rightarrow \mathbb{Z}_2\}.$$

Morse Homology theory defines a canonical chain mapping $\partial_k : C_k \rightarrow C_{k-1}$ in terms of solutions to the parabolic Allen-Cahn Equation, $\text{pAC}_{\epsilon,g} := \partial_t - \text{AC}_{\epsilon,g}$, over the space $\mathbb{R} \times M$. The Morse Homology of the Allen-Cahn Equation is then defined to be the homology of the chain complex (C_*, ∂_*) . That is, for all k :

$$\text{HAC}_k = \frac{\text{Ker}(\partial_k)}{\text{Im}(\partial_{k+1})}.$$

Importantly, HAC_* is independant, up to isomorphism, of the pair (ϵ, g) used to define it. In actual fact, the preceding construction would require that all elements of $\mathcal{Z}_{\epsilon,g}$ be non-degenerate. However, since all critical points of e_g are themselves non-degenerate, we use a perturbation argument to show that degenerate elements of $\mathcal{Z}_{\epsilon,g}$ do not contribute to the homology: in other words, we simply ignore them. The justification is analogous to the manner in which the function $F_\epsilon(t) := t^3 + \epsilon t$ has a degenerate critical point at 0 when $\epsilon = 0$, and no critical points for $\epsilon > 0$, in contrast to the function $G_\epsilon(t) = t^4 + \epsilon t^2$, which has a critical point at 0 for all ϵ .

In order to calculate the Morse Homology, we suppose that $\epsilon \gg 0$. By Proposition 2.6.2, we may suppose that $\mathcal{Z}_{\epsilon,g}$ only consists of constant solutions, and furthermore, by Proposition 2.2.1, we may suppose that the Morse Index of the constant solution $u = c$ is equal to 0 or 1 according as $f'(c)$ is positive or negative respectively. Let F be any primitive of f , let c_\pm be zeroes of f , and let $w : \mathbb{R} \rightarrow \mathbb{R}$ be such that:

$$\partial_t w = -f \circ w, \quad \text{Lim}_{t \rightarrow \pm\infty} w = c_\pm.$$

That is, w is a gradient flow of F from c_- to c_+ . We extend w to a function from $\mathbb{R} \times M$ into \mathbb{R} by setting it to be constant in the x direction. Observe that w is then a bounded solution to the parabolic Allen-Cahn Equation. That is:

$$\text{pAC}_{\epsilon,g} w = (\partial_t - \text{AC}_{\epsilon,g})w = 0.$$

We therefore refer to such a function w as a **space-constant trajectory**. As in the elliptic case, we say that w is **non-degenerate** whenever the linearisation of $\text{pAC}_{\epsilon,g}$ around w defines a surjective mapping from the inhomogeneous Sobolev space $H^{1,2}(\mathbb{R} \times \mathbb{M})$ into $L^2(\mathbb{R} \times \mathbb{M})$. In order to correctly calculate the Morse Homology, we have to show that all trajectories that we study are non-degenerate. However:

Proposition 2.7.1

For all $g \in \mathcal{M}^{\mu+1}$, there exists $B > 0$ such that for $\epsilon > B$, every space-constant trajectory is non-degenerate.

Proof: Let $w : \mathbb{R} \times M \rightarrow \mathbb{R}$ be a space constant trajectory, and let L be the linearisation of $\text{pAC}_{\epsilon,g}$ about w . For all $\varphi : \mathbb{R} \times M \rightarrow M$:

$$L\varphi = (\partial_t - \epsilon\Delta_g)\varphi - (f' \circ w)(t)\varphi.$$

By the Sturm-Liouville Theorem, there exists an orthonormal basis $(\psi_n)_{n \in \mathbb{N}}$ of $L^2(M)$ consisting of eigenfunctions of $-\Delta_g$. Let $0 = \lambda_0 < \lambda_1 \leq \dots$ be the corresponding eigenvalues. Define $B > 0$ such that $B > \|f'\|_{L^\infty}/\lambda_1$. We claim that B has the desired properties. Indeed, choose $\epsilon > B$. By Proposition 2.2.1, both c_- and c_+ are non-degenerate with Morse Indices equal either to 0 or 1. Observe, moreover, that $\text{Index}(c_-) = 1$ and $\text{Index}(c_+) = 0$. By the Atiyah-Patodi-Singer Index Theorem (c.f. [10]), L defines a Fredholm mapping from $H^{1,2}(\mathbb{R} \times M)$ into $L^2(\mathbb{R} \times M)$ of Fredholm index equal to 1. Thus, in order to show that w is non-degenerate, it suffices to show that $\text{Dim}(\text{Ker}(L)) \leq 1$. However, choose $\varphi \in \text{Ker}(L)$. For $k \geq 1$, define $\varphi_k : \mathbb{R} \rightarrow \mathbb{R}$ by $\varphi_k(t) = \langle \varphi_t, \psi_n \rangle$. Observe that $\varphi_k \in L^2(\mathbb{R})$. However:

$$\dot{\varphi}_k = (\epsilon\lambda_k + (f' \circ w)(t))\varphi_k.$$

Since $\epsilon > B$, there exists $\delta > 0$ such that $(\epsilon\lambda_k + (f' \circ w)(t)) > \delta$. Thus, over any interval in which φ_n is non-vanishing, we have:

$$\partial_t(\text{Log}(|\varphi_n|)) \geq \delta,$$

and since $\psi_k \in L^2(\mathbb{R})$, it must therefore vanish identically. φ_t therefore lies in the linear span of ψ_0 for all t . That is, it is constant in space. However, since the space of solutions to a first order ODE is at most 1-dimensional, it follows that $\text{Ker}(L)$ is also at most one-dimensional, and we conclude that w is non-degenerate, as desired. \square

This allows us to calculate the Morse-Homology:

Proposition 2.7.2

The Morse Homology of the Allen-Cahn Operator is given by:

$$HAC_k = \begin{cases} \mathbb{Z}_2 & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof: Let B be as in Proposition 2.7.1 and choose $\epsilon > B$. Upon increasing B is necessary, it follows from Propositions 2.2.1 and 2.6.2 that $\mathcal{Z}_{\epsilon,g}$ only consists of constant solutions and, moreover, that if $u = c$ is a constant solution, then it is non-degenerate and its Morse Index is equal to 0 or 1 according as $f'(c)$ is positive or negative respectively. By Property (B) of f , f has an odd number of zeroes, $c_1 < \dots < c_{2n+1}$. Moreover, if k is odd, then $f'(c_k) > 0$, and if k is even, then $f'(c_k) < 0$. Consequently:

$$\mathcal{Z}_{d,0} = \{c_1, c_3, \dots, c_{2n+1}\}, \quad \mathcal{Z}_{d,1} = \{c_2, c_4, \dots, c_{2n}\},$$

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and $\mathcal{Z}_{d,p}$ is empty for all $p \geq 2$. In particular, for all $p \geq 2$, $C_p = 0$ and so $HAC_p = 0$. For $1 \leq k \leq n$, there are two space-constant trajectories leaving c_{2k} , terminating in c_{2k-1} and c_{2k+1} respectively. Moreover, by Proposition 2.7.1, these space-constant trajectories are non-degenerate, and by the unstable manifold theorem (c.f. [16]), up to reparametrisation in time, there are no other bounded solutions $w_t(\cdot) := w(t, \cdot)$ to the parabolic Allen-Cahn Equation which converge to c_{2k} as t tends to minus infinity. It follows from the definition of the chain map (c.f. [14]) that:

$$\partial_1 c_{2k} = c_{2k-1} + c_{2k+1}.$$

In particular, $\{\partial_1 c \mid c \in \mathcal{Z}_{d,1}\}$ is a linearly independent subset of C_0 , and so:

$$\text{Dim}(HAC_1) = \text{Dim}(\text{Ker}(\partial_1)) = 0.$$

Finally, by the Rank-Nullity Theorem, $\text{Dim}(\text{Im}(\partial_1)) = n$, and so $\text{Dim}(HAC_0) = 1$, and the result now follows. \square

We now return to the specific case studied in the introduction where $f(u) = u^3 - u$, and we prove Theorem 1.1.2:

Proof of Theorem 1.1.2: For all k , we define $X_k \subseteq \mathcal{Z}_{\epsilon,g}$ to be the set of all stationary solutions of Morse-Index equal to k , and we define C_k and ∂_k as outlined above. Denote $l = \text{Index}(0)$. Since f is odd, multiplication by -1 maps $\mathcal{Z}_{\epsilon,g}$ to itself, and all solutions of $AC_{\epsilon,g}u = 0$ which are different to 0 therefore exist in pairs. The set X_k therefore has even cardinality for all $k \neq l$ and odd cardinality when $k = l$. In other words, C_k has odd dimension for $k \neq l$ and even dimension for $k = l$. For all k , let K_k be the kernel of ∂_k . We claim that K_k is odd-dimensional for all $0 < k < l$. Indeed, choose $0 < k < l - 1$ and suppose that K_k is odd-dimensional. Then, since $HAC_k = 0$, it follows that the image of ∂_{k+1} is also odd-dimensional, and since C_{k+1} is even-dimensional, it follows by the Rank-Nullity Theorem that K_{k+1} is odd-dimensional. However, since $K_0 = C_0$ and since $HAC_0 = \mathbb{Z}_2$, $\text{Im}(\partial_1)$ is also odd-dimensional, and it follows by the Rank-Nullity Theorem that K_1 is also odd-dimensional. We conclude by induction that K_k is odd dimensional for all $0 < k < l$ as asserted. In particular, for all $0 < k < l$, K_k is non-trivial, and thus so too is C_k , from which it follows that X_k is non-empty, as desired. \square

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