

THE ARC LENGTH OF A RANDOM LEMNISCATE

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ABSTRACT. A polynomial lemniscate is a curve in the complex plane defined by $\{z \in \mathbb{C} : |p(z)| = t\}$. Erdős, Herzog, and Piranian posed the extremal problem of determining the maximum length of a lemniscate $\Lambda = \{z \in \mathbb{C} : |p(z)| = 1\}$ when p is a monic polynomial of degree n . In this paper, we study the length and topology of a random lemniscate whose defining polynomial has independent Gaussian coefficients. In the special case of the Kac ensemble we show that the length approaches a nonzero constant as $n \rightarrow \infty$. We also show that the average number of connected components is asymptotically n , and we observe a positive probability (independent of n) of a giant component occurring.

1. INTRODUCTION

A (polynomial) lemniscate is a curve defined in the complex plane by the equation $|p(z)| = t$, where p is a polynomial. If the degree of p is n , then from the conjugation-invariant equation $p(z)\overline{p(z)} = t^2$, it is apparent that the lemniscate is a real algebraic curve of degree $2n$. Calculating the length of a lemniscate is a problem of classical Mathematics that played a role in the development of elliptic integrals. Namely, the length of Bernoulli's lemniscate $|z^2 - 1| = 1$ is an elliptic integral of the second kind (the same type of integral that appears in classical mechanics, as the period of a pendulum, and in classical statics, as the length of an elastica).

1.1. The Erdős lemniscate problem. Erdős, Herzog, and Piranian [5] posed the extremal problem of determining the maximum length of a lemniscate

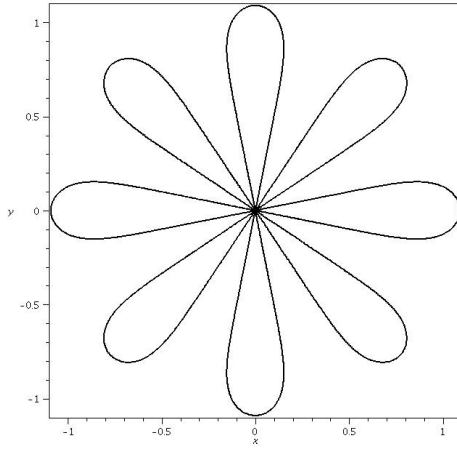
$$\Lambda = \{z \in \mathbb{C} : |p(z)| = 1\}$$

when p is a monic polynomial of degree n . The problem was restated by Erdős several times (e.g., see [6]) and is often referred to as the *Erdős lemniscate problem*. Taking p monic guarantees that the length of the lemniscate is bounded, for instance by $2\pi n$ [3]. The maximum was conjectured [5] to occur for the so-called *Erdős lemniscate*, i.e, when $p(z) = z^n - 1$. This conjecture remains open but has seen positive results [2, 7, 12], and Fryntov and Nazarov [8] have proved that Erdős lemniscate is indeed a *local* maximum and that as $n \rightarrow \infty$ the maximum length is $2n + o(n)$ which is asymptotic to the conjectured extremal.

1.2. The arc length of a random lemniscate. A random variable X has the standard complex Gaussian distribution if it has density $\frac{1}{\pi} \exp(-|z|^2)$ on \mathbb{C} . We denote this by $X \sim N_{\mathbb{C}}(0, 1)$.

Motivated by seeking a broad point of view on the Erdős lemniscate problem, we give a probabilistic treatment of the length, by studying the *average* outcome for a random polynomial lemniscate. We select $\Lambda = \Lambda_n$ randomly by taking $p_n(z)$ to be a random polynomial from the Kac ensemble,

$$(1) \quad p_n(z) = \sum_{k=0}^n a_k z^k,$$

FIGURE 1. The Erdős lemniscate for $n = 8$.

where $a_k \sim N_{\mathbb{C}}(0, 1)$ are independent, identically distributed complex Gaussians. The resulting distribution for the random curve Λ is invariant under rotation of the angular coordinate. Indeed, we have:

$$|p_n(e^{i\theta} z)| = \left| \sum_{k=0}^n a_k e^{ik\theta} z^k \right|,$$

and invariance follows from the observation that $b_k = a_k e^{ik\theta}$ are i.i.d and distributed as $N_{\mathbb{C}}(0, 1)$.

We now state our main result.

Theorem 1. *Consider a sequence of random polynomials $p_n(z) = \sum_{k=0}^n a_k z^k$, where the a_k are i.i.d $N_{\mathbb{C}}(0, 1)$. Let $\Lambda_n = \{z \in \mathbb{C} : |p_n(z)| = 1\}$. Then,*

$$\lim_{n \rightarrow \infty} \mathbb{E}|\Lambda_n| = C,$$

where the constant $C \approx 8.3882$ is given by the integral (11) below.

1.3. The Erdős lemniscate is an outlier. The following Corollary of Theorem 1 provides weak concentration of measure around lemniscates having length of constant order.

Corollary 2. *Let L_n be any sequence with $L_n \rightarrow \infty$ as $n \rightarrow \infty$. The probability that $|\Lambda_n| \geq L_n$ converges to zero.*

Proof. Since the length $|\Lambda_n|$ is a positive random variable, we can apply Markov's inequality:

$$\mathbb{P}\{|\Lambda_n| \geq L_n\} \leq \frac{\mathbb{E}|\Lambda_n|}{L_n} = O(L_n^{-1}), \quad \text{as } n \rightarrow \infty,$$

by Theorem 1. □

In particular, the probability that the length has the same order as the extremal case (i.e., exceeding some fixed portion of n) converges to zero.

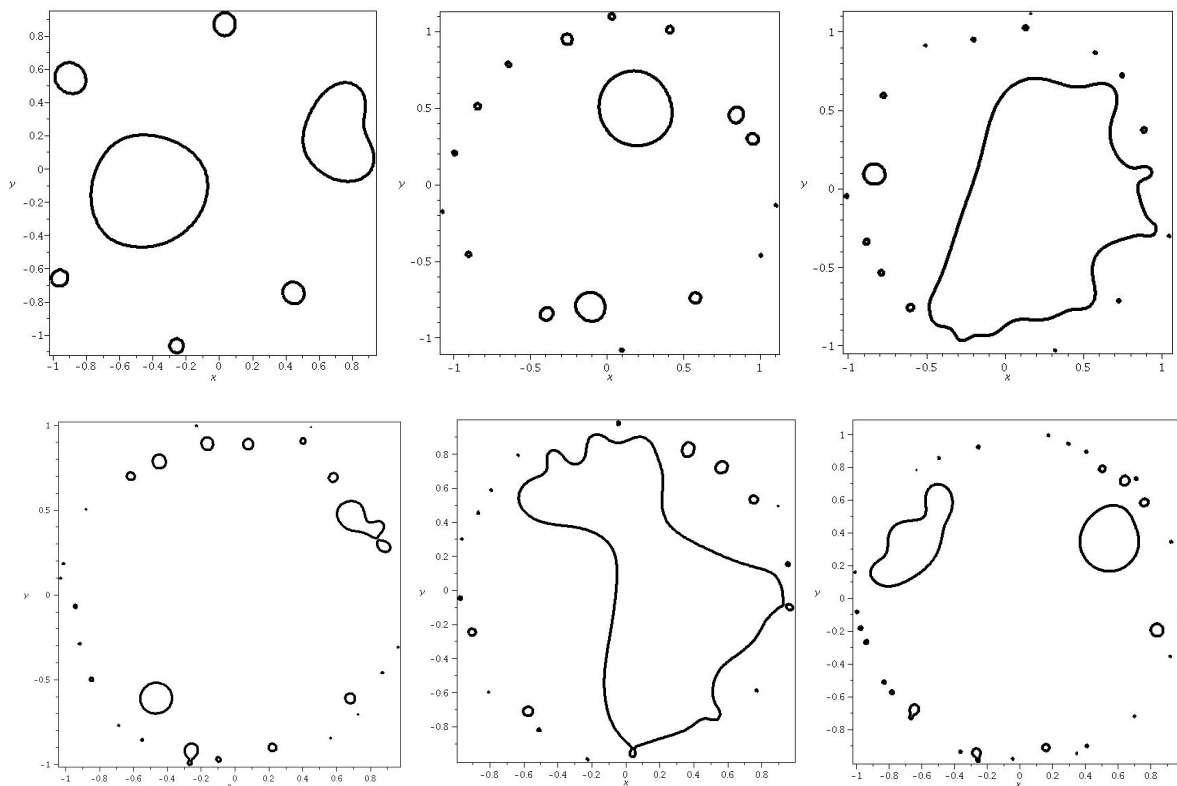


FIGURE 2. Random lemniscates using Kac polynomials of degree $n = 10, 20, 30, 40, 50, 60$ (from left to right).

1.4. The connected components of a random lemniscate. How many connected components does a random lemniscate have? This question was addressed in [13] in the setting of rational lemniscates. The next theorem answers this question for a random polynomial lemniscate based on the Kac model. The notation $b_0(\Lambda_n)$ denotes the zeroth Betti number, which is the number of connected components.

Theorem 3. *The number $b_0(\Lambda_n)$ of connected components of a random Kac lemniscate satisfies*

$$\mathbb{E}b_0(\Lambda_n) \sim n, \quad \text{as } n \rightarrow \infty.$$

Along with Theorem 1, this indicates a prevalence of small components. In fact, the idea of the proof of Theorem 3 is to check in the vicinity of a zero for a component to appear within a disk of radius $n^{-1-\alpha}$ where $0 < \alpha < 1/2$. This suggests that relatively few components account for most of the length. It seems natural to further investigate the distribution of lengths of components, and we begin to do this with the next Theorem that establishes, with some positive probability independent of n , the presence of at least one “giant component” (compare with the samples plotted in Figure 2).

Theorem 4. *Fix $r \in (0, 1)$ and let Λ_n be a random Kac lemniscate. There is a positive probability (depending on r but independent of n) that Λ_n has a component with length at least $2\pi r$.*

1.5. Remarks. The Erdős lemniscate is extremely singular and symmetric (see Figure 1), and its length appears to diminish rapidly under perturbations. Naively, this suggests that it occupies a rather far corner of the parameter space. The probabilistic approach taken here provides a

framework for making this notion precise as we have done in Section 1.3. The authors expect that the rate of decay in Corollary 2 can be improved, and it would be interesting to investigate this topic from the point of view of large deviations.

The outcome for the average length of a random lemniscate depends on the definition of “random”. The Kac ensemble is one of the most well-studied instances, and it seems especially appropriate in the context of the Erdős lemniscate problem, since the zeros of p_n resemble those of the defining polynomial of the Erdős lemniscate in that they are approximately equidistributed on the unit circle [14, 15]. We consider several models in the sections below, including the case that the variances have binomial coefficient weights and also the case in which they have reciprocal binomial coefficient weights. In each of these cases, the expected length has order $O(n^{-1/2})$.

Another extremal problem, to find the maximal spherical length of a rational lemniscate, was posed and solved by Eremenko and Hayman [7]. Lerario and the first author considered random rational lemniscates on the Riemann sphere [13] and computed the average spherical length. They also studied the connected components while giving special attention to nesting of components, which can occur for rational lemniscates, but is not possible for polynomial lemniscates (the latter statement follows from the maximum principle).

1.6. Outline of the paper. Theorem 1 will follow from a more general result proved in Section 2, namely, Theorem 6 provides the expected length while allowing the coefficients appearing in (1) to be independent centered Gaussians with different variances. The methods in proving Theorem 6 are based on planar integral geometry combined with the Kac-Rice formula. In Section 3, we then derive Theorem 1 as a consequence of Theorem 6. We also apply Theorem 6 to three other models: lemniscates generated by Kostlan polynomials are treated in Section 4.1, Weyl polynomials in Section 4.2, and a model that we call the “reciprocal binomial” model is considered in Section 4.3. Returning to the Kac model in Section 5, we study the connected components of a random lemniscate; we prove Theorem 3 in Section 5.1 and Theorem 4 in Section 5.2.

2. A LENGTH FORMULA FOR GAUSSIAN POLYNOMIALS

In this section we assume that the coefficients appearing in $p_n(z)$ are centered, independent, but not necessarily identically distributed complex Gaussians.

2.1. Length and integral geometry. Applying the integral geometry formula as in [7], we have:

$$|\Lambda_n| = \frac{1}{2} \int_0^\pi \int_{-\infty}^\infty N_n(\theta, y) d\theta dy,$$

where $N_n(\theta, y)$ is the number of intersections of Λ_n with the line $L(\theta, y) := \{z \in \mathbb{C} : \Im(e^{-i\theta} z) = y\}$. Taking the expectation of both sides and using the rotational invariance of Λ_n , we have:

$$(2) \quad \mathbb{E}|\Lambda_n| = \frac{1}{2} \int_0^\pi \int_{-\infty}^\infty \mathbb{E}N_n(\theta, y) d\theta dy = \frac{\pi}{2} \int_{-\infty}^\infty \mathbb{E}N_n(0, y) dy.$$

2.2. The Kac-Rice formula. We use the Kac-Rice formula to compute $\mathbb{E}N_n(0, y)$ which equals the average number of real zeros of the function

$$p_n(z)\overline{p_n(z)} - 1,$$

restricted to the line $L(0, y)$. We have:

$$\frac{\partial}{\partial x}(p_n(z)\overline{p_n(z)} - 1) = p'_n(z)\overline{p_n(z)} + p_n(z)\overline{p'_n(z)}.$$

Applying the Kac-Rice formula, we have:

$$(3) \quad \mathbb{E}N_n(0, y) = \int_{-\infty}^{\infty} \mathbb{E}\delta(|p_n(z)|^2 - 1)|p'_n(z)\overline{p_n(z)} + p_n(z)\overline{p'_n(z)}|dx.$$

For the sake of notational clarity we will henceforth suppress the dependence on n . So for instance Λ_n will be denoted by Λ , p_n by p etc. We can rewrite (3) in terms of the Gaussian random complex vector $(U, V) = (p(z), p'(z))$ whose joint probability density function is:

$$\rho(u, v; x + iy) = \frac{1}{\pi^2 |\Sigma|} \exp\{-(u, v)^* \Sigma^{-1} (u, v)\},$$

where Σ is the covariance matrix of $(U, V) = (p(z), p'(z))$, which can be computed explicitly using the covariance kernel $K(z, w)$:

$$K(z, w) = \mathbb{E}p(z)\overline{p(w)}.$$

Namely, we have:

$$\Sigma = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix},$$

where

$$(4) \quad a = K(z, z), b = \partial_z K(z, z) \text{ and } c = \partial_z \partial_{\bar{z}} K(z, z).$$

In terms of this joint density, the expectation inside (3) can be expressed as:

$$\begin{aligned} \mathbb{E}\delta(|p(z)|^2 - 1)|p'(z)\overline{p(z)} + p(z)\overline{p'(z)}| &= \int_{\mathbb{C}} \int_{\mathbb{C}} \delta(|u|^2 - 1)|v\bar{u} + u\bar{v}|\rho(u, v; x + iy)dA(v)dA(u) \\ &= \int_{|u|=1} \int_{\mathbb{C}} \frac{1}{2|u|}|v\bar{u} + u\bar{v}|\rho(u, v; x + iy)dA(v)dA(u) \\ &= \frac{1}{2} \int_{|u|=1} \int_{\mathbb{C}} |v\bar{u} + u\bar{v}|\rho(u, v; x + iy)dA(v)dA(u), \end{aligned}$$

where we have used the composition property of the δ -function ([9], Chapter 6) allowing integration against $\delta(|u|^2 - 1)$ to be replaced by an integration along the set $|u|^2 = 1$.

For $|u| = 1$, we notice that

$$\begin{aligned} \rho(u, v; z) &= \frac{1}{\pi^2 |\Sigma|} \exp\{-u\bar{u}(1, \bar{u}v)^* \Sigma^{-1} (1, \bar{u}v)\} \\ &= \frac{1}{\pi^2 |\Sigma|} \exp\{-(1, \bar{u}v)^* \Sigma^{-1} (1, \bar{u}v)\} \\ &= \rho(1, \bar{u}v; z). \end{aligned}$$

Making the change of variables $t = \bar{u}v$, $dA(t) = dA(v)$, the integral above becomes

$$\frac{1}{2} \int_{|u|=1} \int_{\mathbb{C}} |t + \bar{t}|\rho(1, t; x + iy)dA(t)du = \pi \int_{\mathbb{C}} |t + \bar{t}|\rho(1, t; x + iy)dA(t).$$

Thus, we have:

$$\mathbb{E}N(0, y) = 2\pi \int_{-\infty}^{\infty} \int_{\mathbb{C}} |\Re\{t\}|\rho(1, t; x + iy)dA(t)dx.$$

Inserting this into the integral geometry formula (2) gives:

$$(5) \quad \mathbb{E}|\Lambda| = \pi^2 \int_{\mathbb{C}} \int_{\mathbb{C}} |\Re\{t\}|\rho(1, t; z)dA(t)dA(z).$$

Observe that the density ρ can be factored:

$$\begin{aligned}\rho(1, t; z) &= \frac{\exp\{-\frac{1}{a}\}}{\pi a} \frac{a}{\pi|\Sigma|} \exp\left\{-\frac{a}{|\Sigma|} \left|t - \frac{b}{a}\right|^2\right\} \\ &= \frac{\exp\{-\frac{1}{a}\}}{\pi a} \hat{\rho}(t),\end{aligned}$$

where $\hat{\rho}$ is the probability density function for a complex Gaussian $N_{\mathbb{C}}(\mu, \sigma^2)$ with mean $\mu = b/a$ and variance $\sigma^2 = \frac{|\Sigma|}{a}$. Thus, the following lemma applies.

Lemma 5. *Let $\zeta \sim N_{\mathbb{C}}(\mu, \sigma^2)$ be a complex Gaussian with mean $\mu = \mu_1 + i\mu_2$. Then the absolute moment $\mathbb{E}|\zeta_1|$ of the real part of $\zeta = \zeta_1 + i\zeta_2$ is given by*

$$\mathbb{E}|\zeta_1| = \frac{\sigma}{\sqrt{\pi}} \exp\{-\mu_1^2/\sigma^2\} + |\mu_1| \operatorname{erf}(|\mu_1|/\sigma).$$

Proof of Lemma 5. We have

$$\begin{aligned}\mathbb{E}|\zeta_1| &= \frac{1}{\pi\sigma^2} \int_{\mathbb{C}} |\zeta_1| \exp\left\{-\frac{|\zeta - \mu|^2}{\sigma^2}\right\} dA(\zeta) \\ &= \frac{1}{\pi} \int_{\mathbb{C}} |\sigma w_1 + \mu_1| \exp\{-|w|^2\} dA(w),\end{aligned}$$

where we have made the change of variables $w = \frac{\zeta - \mu}{\sigma}$, $dA(w) = \frac{1}{\sigma^2} dA(\zeta)$.

Letting $H := \{w \in \mathbb{C} : \sigma w_1 + \mu_1 > 0\}$, we can rewrite the above integral as:

$$\frac{1}{\pi} \left(\int_H (\sigma w_1 + \mu_1) \exp\{-|w|^2\} dw_1 dw_2 - \int_{\mathbb{C} \setminus H} (\sigma w_1 + \mu_1) \exp\{-|w|^2\} dw_1 dw_2 \right).$$

Since σw_1 is odd and μ_1 is even (with respect to w_1) this can be rewritten as:

$$(6) \quad \frac{1}{\pi} \left(\int_R |\mu_1| \exp\{-|w|^2\} dw_1 dw_2 + \sigma \int_{\mathbb{C} \setminus R} |w_1| \exp\{-|w|^2\} dw_1 dw_2 \right),$$

where $R := \{w \in \mathbb{C} : |w_1| < \frac{|\mu_1|}{\sigma}\}$. The first integral can be computed in terms of the error function, erf :

$$(7) \quad \int_R |\mu_1| \exp\{-|w|^2\} dw_1 dw_2 = \pi |\mu_1| \operatorname{erf}(|\mu_1|/\sigma),$$

and the second integral is elementary:

$$(8) \quad \int_{\mathbb{C} \setminus R} |w_1| \exp\{-|w|^2\} dw_1 dw_2 = \sqrt{\pi} \exp\{-\mu_1^2/\sigma^2\}.$$

Collecting (6), (7), and (8), we arrive at the formula stated in the lemma. \square

Applying Lemma 5 to (5), we obtain the following main result of this section:

Theorem 6. *Let $p(z)$ be a random polynomial whose coefficients are independent centered Complex Gaussians. Then the expected length of its lemniscate $\Lambda := \{z \in \mathbb{C} : |p(z)| = 1\}$ is given by*

$$(9) \quad \mathbb{E}|\Lambda| = \sqrt{\pi} \int_{\mathbb{C}} \frac{\exp\{-\frac{1}{a}\}}{a} \left[\sqrt{\frac{|\Sigma|}{a}} \exp\left\{-\frac{|\Re b|^2}{a|\Sigma|}\right\} + \sqrt{\pi} \frac{|\Re b|}{a} \operatorname{erf}\left\{|\Re b|/\sqrt{a|\Sigma|}\right\} \right] dA(z).$$

where as above $|\Sigma|$ denotes the determinant of the covariance matrix Σ and, the terms a, b, c are the entries of Σ given by (4).

3. KAC POLYNOMIALS: PROOF OF THEOREM 1

In the case $p(z)$ is a random Kac polynomial, for the entries in the covariance matrix,

$$\Sigma = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix},$$

we have

$$\begin{aligned} a &= K(z, z) = \sum_{k=0}^n |z|^{2k}, \\ b &= \partial_z K(z, z) = \bar{z} \sum_{k=1}^n k |z|^{2k-2}, \\ c &= \partial_z \partial_{\bar{z}} K(z, z) = \sum_{k=1}^n k^2 |z|^{2k-2}. \end{aligned}$$

We will show that the pointwise limit of the integrand appearing in (9) as $n \rightarrow \infty$ is:

$$(10) \quad \begin{cases} \exp \{-(1 - |z|^2)\} \left[\frac{\exp \{-x^2(1 - |z|^2)\}}{(1 - |z|^2)^{1/2}} + \sqrt{\pi} x \operatorname{erf} \left\{ x \sqrt{1 - |z|^2} \right\} \right], & |z| < 1, \\ 0, & |z| \geq 1. \end{cases}$$

We will also show that the dominated convergence theorem applies, so that the integral in Theorem 6 has a limit as $n \rightarrow \infty$ given by the integral of (10). After changing to polar coordinates, this becomes:

$$\begin{aligned} (11) \quad C &:= \lim_{n \rightarrow \infty} \mathbb{E}|\Lambda| \\ &= \sqrt{\pi} \int_{|z| < 1} \exp \{-(1 - |z|^2)\} \left[\frac{\exp \{-x^2(1 - |z|^2)\}}{(1 - |z|^2)^{1/2}} + \sqrt{\pi} x \operatorname{erf} \left\{ x \sqrt{1 - |z|^2} \right\} \right] dA(z) \\ &\approx 8.3882, \end{aligned}$$

which proves Theorem 1. It remains to compute the pointwise limit and to show dominated convergence.

First, we derive certain formulas from the covariance kernel $K(z, w)$ of the Kac polynomial,

$$K(z, w) = \mathbb{E} p(z) \overline{p(w)} = \sum_{k=0}^n (z \bar{w})^k = \frac{1 - (z \bar{w})^{n+1}}{1 - z \bar{w}}.$$

Notice that

$$a = K(z, z) = \sum_{k=0}^n |z|^{2k} = \frac{1}{1 - |z|^2} - \frac{|z|^{2n+2}}{1 - |z|^2}.$$

We have

$$\frac{\Re\{b\}}{a} = \Re\{\partial_z \log K(z, z)\} = \frac{x}{(1 - |z|^2)} - \frac{(n+1)x|z|^{2n}}{1 - |z|^{2n+2}},$$

and from this we observe that for $|z| < 1$,

$$\begin{aligned} \frac{\Re\{b\}}{a^2} &= x \frac{\frac{1}{1-|z|^2} - \frac{(n+1)|z|^{2n}}{1-|z|^{2n+2}}}{\frac{1}{1-|z|^2} - \frac{|z|^{2n+2}}{1-|z|^2}} \\ &= x \frac{1 - (n+1) \frac{|z|^{2n}}{\sum_{k=0}^n |z|^{2k}}}{1 - |z|^{2n+2}} \\ &\leq x, \end{aligned}$$

and as $n \rightarrow \infty$, $\frac{\Re\{b\}}{a^2} \rightarrow x$.

On the other hand for $|z| > 1$, we note that

$$\begin{aligned} \frac{\Re\{b\}}{a^2} &= x \frac{\frac{(n+1)|z|^{2n}}{|z|^{2n+2}-1} - \frac{1}{|z|^2-1}}{\frac{|z|^{2n+2}}{|z|^2-1} - \frac{1}{|z|^2-1}} \\ &= x \frac{\frac{(n+1)|z|^{2n}}{\sum_{k=0}^n |z|^{2k}} - 1}{|z|^{2n+2} - 1} \\ &\leq x, \end{aligned}$$

and as $n \rightarrow \infty$, $\frac{\Re\{b\}}{a^2} \rightarrow 0$. Keeping in mind to apply the dominated convergence theorem for $|z| > 1$, we estimate as follows.

$$\begin{aligned} \left| \frac{\Re\{b\}}{a^2} \right| &\leq |x|, \quad 1 < |z| < 2 \\ \left| \frac{\Re\{b\}}{a^2} \right| &\leq |x| \frac{2n}{|z|^{2n+2}} \leq \frac{2}{|z|^3}, \quad |z| > 2 \end{aligned}$$

From

$$\frac{|\Sigma|}{a^2} = \partial_{\bar{z}} \partial_z \log K(z, z) = \frac{1}{(1-|z|^2)^2} - \frac{(n+1)^2 |z|^{2n}}{(1-|z|^{2n+2})^2},$$

we notice that for $|z| < 1$,

$$\begin{aligned} \frac{|\Sigma|}{a^3} &= \frac{\partial_{\bar{z}} \partial_z \log K(z, z)}{K(z, z)} \\ &= \frac{1}{1-|z|^2} \left(\frac{1 - \frac{(n+1)^2 |z|^{2n}}{(\sum_{k=0}^n |z|^{2k})^2}}{1 - |z|^{2n+2}} \right) \\ &\leq \frac{1}{1-|z|^2}, \end{aligned}$$

and $\frac{|\Sigma|}{a^3} \rightarrow \frac{1}{1-|z|^2}$ as $n \rightarrow \infty$.

A similar computation for $|z| > 1$, yields

$$\frac{|\Sigma|}{a^3} = \frac{1}{|z|^2 - 1} \left(\frac{1 - \frac{(n+1)^2 |z|^{2n}}{(\sum_{k=0}^n |z|^{2k})^2}}{|z|^{2n+2} - 1} \right) \leq \frac{1}{|z|^2 - 1},$$

and as $n \rightarrow \infty$, $\frac{|\Sigma|}{a^3} \rightarrow 0$. To apply dominated convergence, we use the following bounds which follow immediately from the above expression

$$\left| \frac{|\Sigma|}{a^3} \right| \leq \frac{1}{|z|^2 - 1}, \quad 1 < |z| < 2.$$

$$\left| \frac{|\Sigma|}{a^3} \right| \leq \frac{2}{|z|^{2n+2}} \leq \frac{2}{|z|^6}, \quad |z| > 2, n \geq 2.$$

Letting $F_n(z)$ denote the integrand in (9), we have:

$$\begin{aligned} F_n(z) &= \frac{\exp\{-1/a\}}{a} \left[\sqrt{\frac{|\Sigma|}{a}} \exp\left\{-\frac{|\Re b|^2}{a|\Sigma|}\right\} + \sqrt{\pi} \frac{|\Re b|}{a} \operatorname{erf}\left\{|\Re b|/\sqrt{a|\Sigma|}\right\} \right] \\ &\leq \exp\{-1/a\} \left[\sqrt{\frac{|\Sigma|}{a^3}} + \sqrt{\pi} \frac{|\Re b|}{a^2} \right]. \end{aligned}$$

For $|z| < 1$ we have:

$$F_n(z) \leq \exp\{-(1 - |z|^2)\} \left[\frac{1}{\sqrt{1 - |z|^2}} + \sqrt{\pi} x \right],$$

which is integrable. If $|z| > 1$ and n is large enough, we split the integral into regions $1 < |z| < 2$ and $|z| > 2$ and use the appropriate bounds from before. This justifies the use of the dominated convergence theorem.

In order to see the pointwise limit (10) of $F_n(z)$, we notice that for $|z| < 1$, we have (as $n \rightarrow \infty$):

$$\sqrt{\frac{|\Sigma|}{a^3}} \rightarrow \frac{1}{\sqrt{1 - |z|^2}},$$

$$a \rightarrow \frac{1}{1 - |z|^2},$$

$$\frac{\Re\{b\}}{a^2} \rightarrow x,$$

and

$$\frac{\Re\{b\}}{\sqrt{a|\Sigma|}} \rightarrow x\sqrt{1 - |z|^2}.$$

As pointed earlier, for $|z| > 1$, we have:

$$F_n(z) \rightarrow 0.$$

Combining these pointwise limits, we arrive at (10), and applying the dominated convergence theorem proves the formula (11) for the asymptotic expected length of a lemniscate generated by the Kac model.

4. THE EXPECTED LENGTH FOR OTHER MODELS

4.1. Kostlan Polynomials. In this section we compare the average length of the lemniscate for different ensembles of random polynomials, starting with the Kostlan ensemble.

Consider a sequence of random polynomials whose coefficients are Kostlan random variables. Namely

$$P_n(z) = \sum_{k=0}^n a_{kn} z^k,$$

where a_{kn} are independent $N_{\mathbb{C}}(0, \binom{n}{k})$. Applying Theorem 6

$$(12) \quad \mathbb{E}|\Lambda| = \sqrt{\pi} \int_{\mathbb{C}} \frac{\exp\{-\frac{1}{a}\}}{a} \left[\sqrt{\frac{|\Sigma|}{a}} \exp\left\{-\frac{|\Re b|^2}{a|\Sigma|}\right\} + \sqrt{\pi} \frac{|\Re b|}{a} \operatorname{erf}\left\{|\Re b|/\sqrt{a|\Sigma|}\right\} \right] dA(z).$$

where now for the Kostlan ensemble,

$$\Sigma = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix},$$

with

$$\begin{aligned} a &= K(z, z) = (1 + |z|^2)^n, \\ b &= n\bar{z}(1 + |z|^2)^{n-1}, \\ c &= n(n|z|^2 + 1)(1 + |z|^2)^{n-2}. \end{aligned}$$

This implies that

$$\begin{aligned} |\Sigma| &= ac - |b|^2 = n(1 + |z|^2)^{2n-2}, \\ \frac{|\Sigma|}{a^3} &= \frac{n}{(1 + |z|^2)^{n+2}}, \\ \frac{\Re\{b\}}{a^2} &= \frac{nx}{(1 + |z|^2)^{n+1}}, \\ \frac{|\Re b|}{\sqrt{a|\Sigma|}} &= \frac{nx^2}{(1 + |z|^2)^n}. \end{aligned}$$

Substituting these expressions into (12), we obtain

$$\mathbb{E}|\Lambda_n| = \sqrt{\pi} \int_{\mathbb{C}} \exp\left(-\frac{1}{(1 + |z|^2)^n}\right) [I_{1n}(z) + I_{2n}(z)] dA(z)$$

where $I_{1n}(z) = \sqrt{\frac{n}{(1 + |z|^2)^{n+2}}} \exp\left(-\frac{nx^2}{(1 + |z|^2)^n}\right)$ and $I_{2n}(z) = \sqrt{\pi} \frac{nx}{1 + |z|^2} \operatorname{erf}\left\{\sqrt{n}x/(1 + |z|^2)^{n/2}\right\}$

Converting the above integral into polar coordinates (r, θ) , followed by the substitution $r = \sqrt{\frac{t}{n}}$ leads us to

$$\mathbb{E}|\Lambda_n| = \sqrt{\pi} \int_0^{2\pi} \int_0^\infty \exp\left(-\frac{1}{(1 + t/n)^n}\right) [J_{1n}(t, \theta) + J_{2n}(t, \theta)] dt d\theta,$$

$$J_{1n}(t, \theta) = \sqrt{\frac{1}{n(1+t/n)^{n+2}}} \exp\left(-\frac{t \cos^2(\theta)}{(1+t/n)^n}\right)$$

$$J_{2n}(t, \theta) = \sqrt{\pi} \frac{\sqrt{t} \cos(\theta)}{\sqrt{n}(1+t/n)} \operatorname{erf}\left\{\sqrt{t} \cos(\theta)/(1+t/n)^{n/2}\right\}.$$

Removing a factor of $1/\sqrt{n}$ from the J_{in} , we see that the resulting integral has a limit as $n \rightarrow \infty$. Namely, we have the following result

$$\sqrt{n} \mathbb{E}|\Lambda_n| \rightarrow I \quad \text{as } n \rightarrow \infty,$$

where I is the constant given by

$$I = \sqrt{\pi} \int_0^{2\pi} \int_0^\infty \exp\left(-\frac{1}{e^t}\right) \left[\sqrt{\frac{1}{e^t}} \exp\left(-\frac{t \cos^2(\theta)}{e^t}\right) + \sqrt{\pi} \sqrt{t} \cos(\theta) \operatorname{erf}\left\{\sqrt{t} \cos(\theta)/e^{t/2}\right\} \right] dt d\theta.$$

4.2. Weyl Polynomials. We now consider Weyl polynomials defined by $P_n(z) = \sum_{k=0}^n a_k z^k$ where a_k are independent random variables with $a_k \sim N_{\mathbb{C}}(0, \frac{1}{k!})$.

One can check easily that now the covariance matrix has entries given by

$$\Sigma = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix},$$

with

$$a = \sum_{k=0}^n |z|^{2k} / k!,$$

$$b = \bar{z} \sum_{k=1}^n |z|^{2k-2} / (k-1)!,$$

$$c = \sum_{k=1}^n \frac{k^2}{k!} |z|^{2k-2}.$$

Applying Theorem 6, we obtain

$$(13) \quad \mathbb{E}|\Lambda_n| = \sqrt{\pi} \int_{\mathbb{C}} \frac{\exp\{-\frac{1}{a}\}}{a} \left[\sqrt{\frac{|\Sigma|}{a}} \exp\left\{-\frac{|\Re b|^2}{a|\Sigma|}\right\} + \sqrt{\pi} \frac{|\Re b|}{a} \operatorname{erf}\left\{|\Re b|/\sqrt{a|\Sigma|}\right\} \right] dA(z).$$

All the quantities above have finite limits as $n \rightarrow \infty$. For instance $a \rightarrow \exp(|z|^2)$, $b \rightarrow \bar{z} \exp(|z|^2)$, and $c \rightarrow (1 + |z|^2) \exp(|z|^2)$. Also, dominated convergence is easy to verify here. Taking the limit as $n \rightarrow \infty$ in (13), we obtain

$$\mathbb{E}|\Lambda_n| \rightarrow L,$$

where

$$L = \sqrt{\pi} \int_{\mathbb{C}} \frac{\exp\{-\frac{1}{e^{|z|^2}}\}}{e^{|z|^2}} \left[\sqrt{e^{|z|^2}} \exp\left\{-\frac{x^2}{e^{|z|^2}}\right\} + \sqrt{\pi} x \operatorname{erf}\left\{x/e^{|z|^2/2}\right\} \right] dA(z).$$

4.3. Reciprocal binomial distribution. Consider a random polynomial of the form

$$p_n(z) = \sum_{k=0}^n a_{nk} z^k,$$

where a_{nk} are independent random variables with $a_{nk} \sim N_{\mathbb{C}}\left(0, \frac{1}{\binom{n}{k}}\right)$.

In this case, the entries a, b and c the entries of the covariance matrix Σ are given as follows.

$$\begin{aligned} a &= \sum_{k=0}^n \frac{|z|^{2k}}{\binom{n}{k}}, \\ b &= \bar{z} \sum_{k=1}^n \frac{k|z|^{2k-2}}{\binom{n}{k}}, \\ c &= \sum_{k=1}^n \frac{k^2}{\binom{n}{k}} |z|^{2k-2}. \end{aligned}$$

Theorem 6 gives,

$$\mathbb{E}|\Lambda_n| = \sqrt{\pi} \int_{\mathbb{C}} \frac{\exp\{-\frac{1}{a}\}}{a} \left[\sqrt{\frac{|\Sigma|}{a}} \exp\left\{-\frac{|\Re b|^2}{a|\Sigma|}\right\} + \sqrt{\pi} \frac{|\Re b|}{a} \operatorname{erf}\left\{|\Re b|/\sqrt{a|\Sigma|}\right\} \right] dA(z).$$

We consider now asymptotically (with n) the contribution of this integral from $|z| < 1$ and $|z| > 1$. If $|z| < 1$, then we observe from the expressions for a, b and c that

$$\begin{aligned} a &= 1 + \frac{|z|^2}{n} + o(1), \\ b &= \frac{\bar{z}}{n} \left(1 + \frac{4}{n-1} |z|^2 + o(1) \right), \\ c &= \frac{1}{n} \left(1 + \frac{8}{n-1} |z|^2 + o(1) \right). \end{aligned}$$

This yields $|\Sigma| = ac - |b|^2 = \frac{1}{n} (1 + o(1))$, $\frac{|\Re b|}{a} = \frac{x}{n} (1 + o(1))$ and finally $\frac{|\Re b|^2}{a|\Sigma|} = \frac{x^2}{n} (1 + o(1))$. This implies that the integral for $|z| < 1$ is of order $\sqrt{\frac{1}{n}} (1 + o(1))$. So $\sqrt{n} \mathbb{E}|\Lambda_n|$ has a finite limit for z in the unit disc.

We next claim that asymptotically, the integral over $|z| > 1$ goes to 0 (after a scaling by \sqrt{n}). Indeed, notice then that

$$\begin{aligned} a &= |z|^{2n} \left(1 + \frac{1}{n|z|^2} + o(1) \right), \\ b &= n\bar{z}|z|^{2n-2} \left(1 + \frac{n-1}{n^2|z|^2} + o(1) \right), \\ c &= n^2|z|^{2n-2} \left(1 + \frac{(n-1)^2}{n^3|z|^2} + o(1) \right). \end{aligned}$$

From here we can deduce that $|\Sigma| = \frac{|z|^{4n-4}}{n}(1 + o(1))$. This gives that

$$\sqrt{\frac{|\Sigma|}{a^3}} = \sqrt{\frac{1}{n}} \frac{1}{|z|^{n+2}}(1 + o(1)),$$

$$\frac{|\Re b|}{a^2} = \frac{n|x|}{|z|^{2n+2}}(1 + o(1)).$$

The pointwise limit of the integrand (even if we scale it by \sqrt{n}) is clearly 0 and because of power decay, dominated convergence holds. So the contribution from the exterior of the unit disc to the integral is negligible. Ultimately, as $n \rightarrow \infty$, we get that $\sqrt{n}\mathbb{E}|\Lambda_n|$ approaches a positive constant given by an integral over $|z| < 1$ independent of n .

5. THE CONNECTED COMPONENTS OF A RANDOM LEMNISCATE

In this section, we prove asymptotics for the expected number of connected components $\mathbb{E}(b_0(\Lambda_n))$ of a lemniscate $\Lambda_n = \{z : |p_n(z)| = 1\}$, where p_n is a random Kac polynomial, i.e., $p_n(z) = \sum_{k=0}^n a_k z^k$, with i.i.d. coefficients $a_k \sim N_{\mathbb{C}}(0, 1)$.

Consider the set:

$$(14) \quad U_n = \{z : |p_n(z)| < 1\}.$$

Then U_n is a bounded open set and it is a well-known fact that the number of connected components of U_n is at most n . This can be seen from noticing that each component of U_n must contain a zero of p . Otherwise the maximum principle may be applied to conclude that the harmonic function $\log |p|$ is constant. It also follows from the maximum principle that each component of U_n is simply-connected. The boundary of U_n is the lemniscate Λ_n , which is smooth with probability one. We conclude that the connected components of Λ_n are in one-to-one correspondence with those of U_n .

5.1. The expectation of the number of connected components: proof of Theorem 3.

Since the number of connected components $b_0(\Lambda_n)$ is at most n , in order to show that $\mathbb{E}b_0(\Lambda_n) \sim n$ it suffices to prove the lower bound $\mathbb{E}b_0(\Lambda_n) \geq n - o(n)$.

Fix $0 < \beta < \alpha < 1/2$ with $\alpha - \beta > \frac{1}{2} - \alpha$, and suppose n is large enough that

$$n^{\beta + \frac{1}{2} - 2\alpha} \exp\{n^{-\alpha}\} < 1.$$

As a certificate for the appearance of a localized component we will use the following conditions related to the Taylor expansion of $p(z)$ centered at ζ .

$$(15) \quad \begin{cases} p(\zeta) = 0 \\ |p'(\zeta)| > 2 \cdot n^{1+\alpha} \\ |p^{(k)}(\zeta)| < n^{k+\frac{1}{2}+\beta}, \quad \text{for } k = 2, 3, \dots, n \end{cases}$$

These conditions imply that, for any z on the circle defined by $|z - \zeta| = n^{-1-\alpha}$, we have

$$\begin{aligned}
|p(z)| &= \left| p'(\zeta)(z - \zeta) + \sum_{k=2}^n \frac{p^{(k)}(\zeta)}{k!} (z - \zeta)^k \right| \\
&\geq |p'(\zeta)(z - \zeta)| - \left| \sum_{k=2}^n \frac{p^{(k)}(\zeta)}{k!} (z - \zeta)^k \right| \\
&\geq |p'(\zeta)| n^{-1-\alpha} - \sum_{k=2}^n \frac{|p^{(k)}(\zeta)|}{k!} (n^{-1-\alpha})^k \\
&> 2 - \sum_{k=2}^n \frac{n^{(k+\frac{1}{2}+\beta)}}{k!} (n^{-1-\alpha})^k \\
&> 2 - n^{\beta+\frac{1}{2}-2\alpha} \sum_{k=2}^n \frac{n^{-\alpha(k-2)}}{k!} \\
&> 2 - n^{\beta+\frac{1}{2}-2\alpha} \exp\{n^{-\alpha}\} \\
&> 1,
\end{aligned}$$

so that $p(\zeta) = 0$ and $|p(z)| > 1$ on the circle $|z - \zeta| = n^{-1-\alpha}$. This ensures that there is a connected component of Λ_n contained in the disk $|z - \zeta| < n^{-1-\alpha}$.

In order to estimate the average number of zeros for which the conditions (15) are all satisfied, we will use a modified version of the Kac-Rice formula. First recall that the Kac-Rice formula for the expectation $\mathbb{E}N_p(U)$ of the number of complex zeros of p in a region U states

$$\begin{aligned}
(16) \quad \mathbb{E}N_p(U) &= \frac{1}{\pi} \int_U \mathbb{E}|p'(z)|^2 \delta(p(z)) dA(z) \\
&= \frac{1}{\pi} \int_U \mathbb{E} [|p'(z)|^2 \mid p(z) = 0] \rho_{p(z)}(0) dA(z),
\end{aligned}$$

where $\rho_{p(z)}(0)$ is the marginal probability density of $p(z)$ evaluated at 0.

We would like to modify (16) to obtain a lower bound for the expected number \hat{N}_p of zeros satisfying the conditions (15). Our approach is based on [1], Theorem 5.1.1. Let I_1 be the indicator function of the interval $(2n^{1+\alpha}, \infty)$ and I_k be the indicator function of the interval $[0, n^{k+\frac{1}{2}+\beta})$. Let $T_n(s) := \{z \in \mathbb{C} : e^{-s/n} < |z| < e^{s/n}\}$ and $\hat{N}_p(T_n(s))$ denote the number of zeros satisfying (15) which lie in the annulus $T_n(s)$. Then we have

$$\begin{aligned}
\mathbb{E}\hat{N}_p &\geq \mathbb{E}\hat{N}_p(T_n(s)) \\
&= \frac{1}{\pi} \int_{T_n(s)} \mathbb{E}|p'(z)|^2 \delta(p(z)) \prod_{k=1}^n I_k(|p^{(k)}(z)|) dA(z) \\
&= \frac{1}{\pi} \int_{T_n(s)} \mathbb{E} \left[|p'(z)|^2 \prod_{k=1}^n I_k(|p^{(k)}(z)|) \mid p(z) = 0 \right] \rho_{p(z)}(0) dA(z).
\end{aligned}$$

In the above chain, Theorem 5.1.1 from [1] was used to go from the first line to the second. Next, for each fixed s the above provides a lower bound on the average number of connected

components

$$(17) \quad \mathbb{E} b_0(\Lambda_n) \geq \frac{1}{\pi} \int_{T_n(s)} \mathbb{E} \left[|p'(z)|^2 \prod_{k=1}^n I_k(|p^{(k)}(z)|) \mid p(z) = 0 \right] \rho_{p(z)}(0) dA(z).$$

The remainder of the proof will establish that the right hand side of (17) is asymptotic to a standard Kac-Rice integral of the form (16).

Letting \tilde{I}_k denote the indicator function of $[n^{k+\frac{1}{2}+\beta}, \infty)$, we will use the union-type bound,

$$(18) \quad \prod_{k=2}^n I_k(|p^{(k)}(z)|) \geq 1 - \sum_{k=2}^n \tilde{I}_k(|p^{(k)}(z)|),$$

in order to prove that

$$(19) \quad \mathbb{E} \left[|p'(z)|^2 \prod_{k=1}^n I_k(|p^{(k)}(z)|) \mid p(z) = 0 \right] \geq \mathbb{E} [|p'(z)|^2 I_1(|p'(z)|) \mid p(z) = 0] - O \left(\exp \left\{ -n^\beta \right\} \right).$$

First, we use the simple estimate:

$$(20) \quad \mathbb{E} \left[|p'(z)|^2 I_1(|p'(z)|) \sum_{k=2}^n \tilde{I}_k(|p^{(k)}(z)|) \mid p(z) = 0 \right] \leq \sum_{k=2}^n \mathbb{E} \left[|p'(z)|^2 \tilde{I}_k(|p^{(k)}(z)|) \mid p(z) = 0 \right].$$

We estimate each summand above using the Cauchy-Schwarz inequality.

$$(21) \quad \begin{aligned} \mathbb{E} \left[|p'(z)|^2 \tilde{I}_k(|p^{(k)}(z)|) \mid p(z) = 0 \right] &\leq \sqrt{\mathbb{E} [|p'(z)|^4 \mid p(z) = 0]} \sqrt{P(|p^{(k)}(z)| \geq n^{k+\frac{1}{2}+\beta} \mid p(z) = 0)} \\ &\leq \sqrt{\mathbb{E} [|p'(z)|^4 \mid p(z) = 0]} \exp \left\{ -\frac{n^{2\beta}}{2C_1(s)} \right\}, \end{aligned}$$

where we have used the estimates

$$P(|p^{(k)}(z)| \geq n^{k+\frac{1}{2}+\beta} \mid p(z) = 0) \leq \exp \left\{ -\frac{n^{2k+1+2\beta}}{C_1(s)n^{2k+1}} \right\} = \exp \left\{ -\frac{n^{2\beta}}{C_1(s)} \right\},$$

which follow from Lemmas 7 and 8 below. By the same lemmas, we have $\sqrt{\mathbb{E} [|p'(z)|^4 \mid p(z) = 0]} = O(n^3)$.

Applying (21) to (20) and relaxing the expression appearing in the exponent to $-n^\beta$, we can neglect the polynomially growing factor $\sqrt{\mathbb{E} [|p'(z)|^4 \mid p(z) = 0]} = O(n^3)$ as well as the number of terms $(n-1)$ in the sum. We thus obtain the bound

$$\mathbb{E} \left[|p'(z)|^2 I_1(|p'(z)|) \sum_{k=2}^n \tilde{I}_k(|p^{(k)}(z)|) \mid p(z) = 0 \right] = O \left(\exp \left\{ -n^\beta \right\} \right),$$

which establishes (19) by way of the union bound stated in (18).

The random variable $p'(z)$ conditioned on $p(z) = 0$ is distributed as a centered complex Gaussian with variance $\frac{ac-|b|^2}{a}$, and this implies that $|p'(z)|^2$ conditioned on $p(z) = 0$ is distributed as an exponential random variable with parameter $\lambda = \left(\frac{ac-|b|^2}{a} \right)^{-1}$, so we have

$$(22) \quad \mathbb{E} [|p'(z)|^2 \mid p(z) = 0] = \frac{1}{\lambda} = \frac{ac-|b|^2}{a},$$

and

$$\begin{aligned}
\mathbb{E} [|p'(z)|^2 I_1(|p'(z)|) \mid p(z) = 0] &= \int_{4n^{2+2\alpha}}^{\infty} x \lambda e^{-\lambda x} dx \\
&= \exp \{-n^{2\alpha+2} \lambda\} \left(\frac{1}{\lambda} + 4n^{2+2\alpha} \right) \\
&\geq \frac{1}{\lambda} \exp \{-n^{2\alpha+2} \lambda\} \\
&= \mathbb{E} [|p'(z)|^2 \mid p(z) = 0] (1 - O(n^{2\alpha-1}))
\end{aligned}$$

where we used (22) in the last line.

Combining this with (19) in order to reassess (17) we finally conclude the lower bound

$$\begin{aligned}
\mathbb{E} b_0(\Lambda_n) &\geq (1 - O(n^{2\alpha-1})) \frac{1}{\pi} \int_{T_n(s)} \mathbb{E} [|p'(z)|^2 \mid p(z) = 0] \rho_{p(z)}(0) dA(z), \\
&= (1 - O(n^{2\alpha-1})) \mathbb{E} N_p(T_n(s)),
\end{aligned}$$

where, as in (16) $N_p(T_n(s))$ denotes the number of zeros of p in $T_n(s)$. We recall [10] that

$$\mathbb{E} N_p(T_n(s)) \sim n \left(\frac{1 + e^{2s}}{1 - e^{2s}} - \frac{1}{s} \right),$$

which implies

$$\liminf_{n \rightarrow \infty} \frac{\mathbb{E} b_0(\Lambda_n)}{n} \geq \left(1 - \frac{1}{s} \right).$$

This lower bound can be made arbitrarily close to 1 (by increasing s), and along with the deterministic upper bound $b_0(\Lambda_n) \leq n$ this shows that the limit

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E} b_0(\Lambda_n)}{n} = 1$$

exists, i.e., $\mathbb{E} b_0(\Lambda_n) \sim n$. This proves Theorem 3.

Lemma 7. Fix $z \in \mathbb{C}$. The random variable $p^{(k)}(z)$ conditioned on $p(z) = 0$ is distributed as a centered complex Gaussian, $N_{\mathbb{C}}(0, \sigma^2)$, with variance

$$\sigma^2 = \frac{ac_k - |b_k|^2}{a},$$

where

$$a = K(z, z), \quad b_k = \partial_z^k K(z, z), \quad c_k = \partial_z^k \partial_{\bar{z}}^k K(z, z).$$

Proof of Lemma 7. Let $\rho(u, v)$ denote the joint density of $(U, V) = (p(\zeta), p^{(k)}(\zeta))$. The conditional density $\rho_{V|U=0}$ of V given $U = 0$ is given by:

$$(23) \quad \rho_{V|U=0}(v) = \frac{\rho(0, v)}{\rho_U(0)},$$

where $\rho_U(u) = \frac{1}{\pi a} \exp \left\{ -\frac{|u|^2}{a} \right\}$ is the marginal density of U .

We have

$$\rho(u, v) = \frac{1}{\pi^2 |\Sigma_k|} \exp \{ -(u, v)^* \Sigma_k^{-1} (u, v) \},$$

where Σ_k is the covariance matrix of (U, V) , which can be computed explicitly using the covariance kernel $K(z, w)$:

$$K(z, w) = \mathbb{E}p(z)\overline{p(w)}.$$

Namely, we have:

$$\Sigma_k = \begin{pmatrix} a & b_k \\ \bar{b}_k & c_k \end{pmatrix},$$

where

$$a = K(z, z), \quad b_k = \partial_z^k K(z, z), \quad c_k = \partial_z^k \partial_{\bar{z}}^k K(z, z).$$

Applying this to (23) we obtain:

$$\rho_{V|U=0}(v) = \frac{\rho(0, v)}{\rho_U(0)} = \frac{a}{\pi|\Sigma_k|} \exp\left\{-\frac{a|v|^2}{|\Sigma_k|}\right\},$$

as desired. \square

Lemma 8. *There exists a positive constant $C_1(s)$ depending on s but independent of n , such that*

$$\frac{ac_k - |b_k|^2}{a} \leq C_1(s)n^{2k+1},$$

for all $z \in T_n(s) = \{z \in \mathbb{C} : e^{-s/n} < |z| < e^{s/n}\}$ and $k = 1, 2, \dots, n$. Furthermore, there exists $C_2(s) > 0$ such that for $z \in T_n(s)$, and for all large enough $n \geq N(s)$, we have

$$\frac{ac_1 - |b_1|^2}{a} \geq C_2(s)n^3.$$

Proof of Lemma 8. For the first estimate, we note that $\frac{ac_k - |b_k|^2}{a} \leq c_k$ and so it is enough to find an upper bound for c_k . We have

$$c_k = \mathbb{E} \left(p^{(k)}(z) \overline{p^{(k)}(z)} \right) = \sum_{j=k}^n [j(j-1)(j-2)\dots(j-(k-1))]^2 |z|^{2j-2k}$$

Using the above expression, we observe that for $z \in T_n(s)$,

$$c_k \leq e^{2s} \sum_{j=k}^n [j(j-1)(j-2)\dots(j-(k-1))]^2 \leq e^{2s} \sum_{j=k}^n n^{2k} \leq e^{2s} n^{2k+1}.$$

Before proving the second inequality we recall that

$$a = \sum_{k=0}^n |z|^{2k}, \quad b_1 = \bar{z} \sum_{k=1}^n k |z|^{2k-2}$$

$$c_1 = \sum_{k=1}^n k^2 |z|^{2k-2}.$$

For $z \in T_n(s)$, we now estimate as follows:

$$a = \sum_{k=0}^n |z|^{2k} \geq (e^{-s/n})^{2n}(n+1) = e^{-2s}(n+1).$$

A similar reasoning gives $a \leq (n+1)e^{2s}$. We next proceed to bound c_1 and $|b_1|^2$.

$$c_1 = \sum_{k=1}^n k^2 |z|^{2k-2} \geq (e^{-s/n})^{2n-2} \sum_{k=1}^n k^2 = (e^{-s/n})^{2n-2} n(n+1)(2n+1)/6$$

$$\begin{aligned} |b_1|^2 &= |z|^2 \left(\sum_{k=1}^n k |z|^{2k-2} \right)^2 \\ &\leq e^{2s/n} e^{4s} \left(\sum_{k=1}^n k \right)^2 \\ &= e^{2s/n} e^{4s} \frac{n^2(n+1)^2}{4}. \end{aligned}$$

Combining all the above estimates, we obtain that for large n

$$\frac{ac_1 - |b|^2}{a} \geq C_2(s)n^3.$$

This proves the second estimate and concludes the proof of the lemma. \square

5.2. Existence of a giant component: proof of Theorem 4. We now show that a giant component exists with positive probability (independent of n).

Lemma 9. *Consider a sequence of random polynomials $p_n(z) = \sum_{k=0}^n a_k z^k$, where a_k are i.i.d $\sim N_{\mathbb{C}}(0,1)$. Let U_n be as in (14) and let $r \in (0,1)$ be given. Then, there exist $N = N(r) \in \mathbb{N}$ and $c_r > 0$ such that for all $n \geq N$*

$$(27) \quad \mathbb{P}(B(0,r) \text{ is contained in a component of } U_n) > c_r.$$

Proof. For each $r \in (0,1)$, consider $g(r) = \sum_{k=0}^{\infty} |a_k| r^k$. Then g is a random function and $\mathbb{E}(g(r)) < \infty$. Therefore, there exist $a_r, b_r > 0$ such that

$$\mathbb{P}(g(r) < a_r) > b_r > 0.$$

For a given $r \in (0,1)$ choose N so that $r^N < \frac{1}{2a_r}$. Then, for $n \geq N$

$$\begin{aligned} \mathbb{P}\left(\sup_{\partial B_r} |p_n| < 1\right) &\geq \mathbb{P}\left(|a_0| + |a_1|r + \dots |a_{N-1}|r^{N-1} < \frac{1}{2}; r^N \sum_{j=N}^n |a_j| r^{j-N} < \frac{1}{2}\right), \\ &\geq \mathbb{P}\left(|a_0| + |a_1|r + \dots |a_{N-1}|r^{N-1} < \frac{1}{2}\right) \mathbb{P}\left(r^N g(r) < \frac{1}{2}\right) \\ &\geq \eta_r \mathbb{P}(g(r) < a_r) \\ &= \eta_r b_r, \end{aligned}$$

where $\eta_r = \mathbb{P}(|a_0| + |a_1|r + \dots |a_{N-1}|r^{N-1} < \frac{1}{2}) > 0$ follows from the Gaussian nature of the coefficients. Note that we have used the independence of a_k 's to go from the first line to the second. This finishes the proof of the Lemma. \square

In the case the event in (27) occurs, by the isoperimetric inequality, the associated connected component of Λ_n has length at least $2\pi r$. This proves Theorem 4.

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