Parabolic equations with singular divergence-free drift vector fields

Zhongmin Qian* and Guangyu Xi†

Abstract

In this paper, we study an elliptic operator in divergence-form but not necessarily symmetric. In particular, our results can be applied to elliptic operator $L = \nu\Delta + u(x,t)\cdot\nabla$, where $u(\cdot,t)$ is a time-dependent vector field in $\mathbb{R}^n$, which is divergence-free in distribution sense, i.e. $\nabla\cdot u = 0$. Suppose $u \in L_t^\infty(\mathrm{BMO}_x^{-1})$. We show the existence of the fundamental solution $\Gamma(x,t;\xi,\tau)$ of the parabolic operator $L-\partial_t$, and show that Γ satisfies the Aronson estimate with a constant depending only on the dimension n, the elliptic constant ν and the norm $\|u\|_{L^\infty(\mathrm{BMO}^{-1})}$. Therefore the existence and uniqueness of the parabolic equation $(L-\partial_t)v = 0$ are established for initial data in L^2-space, and their regularity is obtained too. In fact, we establish these results for a general non-symmetric elliptic operator in divergence form.

1 Introduction

The analysis of the Navier-Stokes equations, which are non-linear partial differential equations describing the motion of incompressible fluids confined in certain spaces, has inspired the large portion of the mathematical analysis of non-linear partial differential equations (see for example [15, 16, 21, 30] and etc.) due to the fundamental work J. Leray [17]. The Navier-Stokes equations are partial differential equations of second-order

$$\frac{\partial}{\partial t}u + u\cdot\nabla u = \nu\Delta u - \nabla p, \tag{1.1}$$

$$\nabla\cdot u = 0, \tag{1.2}$$

subject to the no-slip boundary condition if the domain of fluid is finite, where $u = (u^1,u^2,u^3)$ is the velocity vector field of the fluid flow, $p(x,t)$ is the pressure at the instance t and location x. J. Leray [17] demonstrated the existence of a weak solution u which belongs to the space $L^\infty\left([0,\infty), L^2(\mathbb{R}^n)\right)$ and also to the space $L^2\left([0,\infty), H^1(\mathbb{R}^n)\right)$. The vorticity ω exists in L^2 space and formally, by differentiating the Navier-Stokes equations, solves the vorticity equation

$$\frac{\partial}{\partial t}\omega + u\cdot\nabla\omega = \nu\Delta\omega + \omega\cdot\nabla u \tag{1.3}$$

where the velocity u and the vorticity ω, which is too a time dependent vector field $\omega = (\omega^1,\omega^2,\omega^3)$, are related by the definition that $\omega = \nabla\times u$. The resolution of the Navier-Stokes equations remains

*Research supported partly by an ERC grant. Mathematical Institute, University of Oxford, OX2 6GG, England. Email: qianz@maths.ox.ac.uk

†Research supported partly by Doctoral Training Center, EPSRC. Mathematical Institute, University of Oxford, OX2 6GG, England. Email: guangyu.xi@maths.ox.ac.uk

to be an open mathematical problem (see [16, 36] for example), the current research has thus concentrated on the understanding of the related partial differential equations and on developing numerical approaches.

Observe that both the Navier-Stokes equations and the vorticity equations may be put into the following form

$$\left(\frac{\partial}{\partial t}-\nu\Delta+u\cdot\nabla\right)u=-\nabla p \tag{1.4}$$

and

$$\left(\frac{\partial}{\partial t}-\nu\Delta+u\cdot\nabla\right)\omega=\omega\cdot\nabla u \tag{1.5}$$

where the diffusion part is the same and is defined by the parabolic operator

$$L=\frac{\partial}{\partial t}-\nu\Delta+u\cdot\nabla. \tag{1.6}$$

The elliptic operator $\nu\Delta-u\cdot\nabla$ is the generator of the so-called Taylor diffusion (see J. T. Taylor [34, 35]) of the flow of fluids. There are two non-linear terms appearing in the Navier-Stokes equations and the vorticity equations, which determine the turbulent nature of flows of fluids (see for example [19, 20]). The parabolic operator L has the capability of covering the so-called non-linear convection mechanism – the rate-of-strain (for the Navier-Stokes equations [36, 18]) and the vorticity (in the case of the vorticity equations) can be amplified even more rapidly by an increase of the velocity. It is therefore important to study the parabolic equations associated with the elliptic operator $A=\nu\Delta-u\cdot\nabla$, where u is a weak solution of the Navier-Stokes equations. The main feature here is that $u(x,t)$ is a time-dependent vector field with little regularity, which however is solenoidal, that is, for every t, $\nabla\cdot u(\cdot,t)=0$ in distribution sense, so that the formal adjoint $A^{*}=\nu\Delta+u\cdot\nabla$ is also a diffusion generator. These special features have significance, and have been explored in several recent articles [13, 28, 29] etc. for example. In this paper we give a through study for a class of such parabolic equations.

Recall that in dimension three, a vector field $u=\left(u^{i}\right)$ is divergence-free, i.e. $\nabla\cdot u=0$, implies that its corresponding two form (with respect to the Hodge star operation) $\star u$ is closed, that is $d\star u=0$. In fact the divergence operator $\nabla\cdot$ coincides with the Hodge dual $\star d\star$ up to a sign, where d is the exterior differentiation. Therefore, according to the Poincaré lemma, $\star u$ is exact, that is, there is a vector field $b=\left(b^{i}\right)$ so that $\star u=db$. Hence u coincides with $\star db$ up to a sign. $\star b$ is a two form with components $b^{ij}=\sum_{k}\varepsilon^{ijk}b^{k}$, where ε^{ijk} is the usual Kronercker symbols of three elements. $\left(b^{ij}\right)$ is skew-symmetric, and

$$u\cdot\nabla=\sum_{i,j=1}^{3}\frac{\partial b^{ij}}{\partial x^{i}}\frac{\partial}{\partial x^{j}}.$$

The elliptic operator $\nu\Delta-u\cdot\nabla$ can thus be put into a divergence form

$$\sum_{i,j}\frac{\partial}{\partial x^{i}}\left(\nu\delta^{ij}-b^{ij}\right)\frac{\partial}{\partial x^{j}}\equiv\sum_{i,j}\frac{\partial}{\partial x^{i}}\left(A^{ij}\frac{\partial}{\partial x^{j}}\right)$$

where $A=\left(A^{ij}\right)$ is not necessarily symmetric. The symmetric part $\left(\nu\delta^{ij}\right)$ is uniformly elliptic, and the skew-symmetric part $\left(b^{ji}\right)$ determines the divergence-free drift vector field u.

In the present paper we develop a theory for the linear parabolic equation

$$\sum_{i,j}\frac{\partial}{\partial x^{i}}\left(A^{ij}\frac{\partial u}{\partial x^{j}}\right)-\frac{\partial}{\partial t}u=0$$

under very weak assumptions that $\left(A^{ij}\right)$ is uniformly elliptic and its anti-symmetric part only belongs to the BMO space.

The paper is organized as following. In Section 2, we describe the main result, that is the Aronson estimate which depends only on the elliptic constant and the BMO norm of the anti-symmetric part of A, which is the key tool of studying weak solutions. In Section 3, we provide several results which will be used to prove the Aronson estimate in our setting. These results are interesting by their own, including several versions of the compensated compactness, and a density result of the BMO space which seems new. In Section 4, we give the details of the proof of the Aronson estimate, and in the final Section 5, we study the weak solutions to the linear parabolic equations in divergence form (but not necessarily symmetric) under weak assumptions. In particular, we prove the existence and uniqueness of weak solutions to the parabolic equation associated with a non-symmetric diffusion matrix $A=\left(A^{ij}\right)$.

2 Aronson's estimate for non-symmetric parabolic equations

Let us begin with the description of our framework. We consider the following type of linear parabolic equations of second order

$$\frac{\partial}{\partial t}u(x,t)-\left[A_t u\right](x,t)=0 \quad \text{ on } \mathbb{R}^n\times[0,\infty) \tag{2.1}$$

where

$$A_t=\sum_{i,j=1}^{n}\frac{\partial}{\partial x^i}\left(A^{ij}(\cdot,t)\frac{\partial}{\partial x^j}\right) \tag{2.2}$$

is in divergence form but *not necessarily symmetric*, and their associated diffusion processes in terms of fundamental solutions defined by (2.1). There is a unique decomposition $A(x,t)=a(x,t)+b(x,t)$ such that $a(x,t)=\left(a^{ij}(x,t)\right)$ is symmetric, while $b(x,t)=\left(b^{ij}(x,t)\right)$ is skew-symmetric. We assume that A is uniformly elliptic in the following sense: there exists a constant $\lambda>0$ such that

$$\lambda|\xi|^2\leq\sum_{i,j=1}^{n}a^{ij}(x,t)\xi_i\xi_j=\sum_{i,j=1}^{n}A^{ij}(x,t)\xi_i\xi_j\leq\frac{1}{\lambda}|\xi|^2 \tag{2.3}$$

for any $\xi=(\xi_i)\in\mathbb{R}^n$, $x\in\mathbb{R}^n$ and $t\geq 0$.

Let us first consider the regular case where A^{ij} are smooth, bounded and possess bounded derivatives of all orders on $\mathbb{R}^n\times[0,\infty)$.

Let $L=A_t-\frac{\partial}{\partial t}$ be the parabolic linear operator associated with $\left(A^{ij}(x,t)\right)$. The formal adjoint of L is again a parabolic operator (with vanished zero order term) given by

$$L^{\star}=\sum_{i,j=1}^{n}\frac{\partial}{\partial x^i}\left(A^{ji}(\cdot,t)\frac{\partial}{\partial x^j}\right)+\frac{\partial}{\partial t}. \tag{2.4}$$

where $A^{ji}=a^{ij}-b^{ij}$ is the transpose of $\left(A^{ij}\right)$. It is known that (see A. Friedman [10], Theorem 11 and 12, Chapter 1), under the elliptic condition and smoothness assumptions on $A^{ij}(x,t)$, there is a unique positive *fundamental solution* $\Gamma(x,t;\xi,\tau)$ of the parabolic operator L, and it is smooth in (x,t,ξ,τ) on $0\leq\tau<t<\infty$ and $(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n$. Recall that the following properties are satisfied.

1) $\Gamma(x,t;\xi,\tau)>0$ for any $0\leq\tau<t$ and $x,\xi\in\mathbb{R}^n$.

2) For every $\xi\in\mathbb{R}^n$ and $\tau\in[0,\infty)$, as a function of $(x,t)\in\mathbb{R}^n\times(\tau,\infty)$, $u(x,t)\equiv\Gamma(x,t;\xi,\tau)$ solves the parabolic equation $Lu=0$ on $(\tau,\infty)\times\mathbb{R}^n$:

$$\sum_{i,j=1}^{n}\frac{\partial}{\partial x^i}\left(A^{ij}(x,t)\frac{\partial}{\partial x^j}\Gamma(x,t;\xi,\tau)\right)-\frac{\partial}{\partial t}\Gamma(x,t;\xi,\tau)=0 \qquad \text{ on } \mathbb{R}^n\times(\tau,\infty). \tag{2.5}$$

3) Chapman-Kolmogorov's equation holds

$$\Gamma(x,t;\xi,\tau) = \int_{\mathbb{R}^n} \Gamma(x,t;z,s)\Gamma(z,s;\xi,\tau)dz. \tag{2.6}$$

4) For any bounded continuous function f and $\tau \in [0,\infty)$, it holds that

$$\lim_{t\downarrow\tau} \int_{\mathbb{R}^n} f(\xi)\Gamma(x,t;\xi,\tau)d\xi = f(x) \tag{2.7}$$

for every $x \in \mathbb{R}^n$.

For $0 \le \tau < t$, let $\Gamma_{\tau,t}$ denote the corresponding linear operator defined by

$$\Gamma_{\tau,t} f(x) = \int_{\mathbb{R}^n} f(\xi)\Gamma(x,t;\xi,\tau)d\xi \tag{2.8}$$

where f is Borel measurable, either non-negative, or/and bounded. By (2.6)

$$\Gamma_{s,t} \circ \Gamma_{\tau,s} = \Gamma_{\tau,t} \tag{2.9}$$

for any $0 \le \tau < s < t$.

Define $\Gamma^\star(x,s;y,t) = \Gamma(y,t;x,s)$ for $t > s \ge 0$. Then $\Gamma^\star$ is the fundamental solution to $L^\star v = 0$ in the sense that for every fixed (y,t), as a function of (x,s), $\Gamma^\star$ solves the *backward* parabolic equation

$$\sum_{i,j=1}^n \frac{\partial}{\partial x^i}\left(A^{ji}(x,s)\frac{\partial}{\partial x^j}\Gamma^\star(x,s;y,t)\right) + \frac{\partial}{\partial s}\Gamma^\star(x,s;y,t) = 0 \tag{2.10}$$

on $(x,s) \in \mathbb{R}^n \times [0,t)$. It follows that the fundamental solution Γ also solves the backward parabolic equation:

$$\sum_{i,j=1}^n \frac{\partial}{\partial \xi^i}\left(A^{ji}(\xi,\tau)\frac{\partial}{\partial \xi^j}\Gamma(x,t;\xi,\tau)\right) + \frac{\partial}{\partial \tau}\Gamma(x,t;\xi,\tau) = 0 \tag{2.11}$$

which holds for any $\xi, x \in \mathbb{R}^n$ and $0 < \tau < t$.

We are now in a position to state the key result of the present paper.

Theorem 2.1 *There is a constant $M > 0$ depending only on the dimension n, the elliptic constant $\lambda > 0$, and the $L^\infty([0,\infty), BMO)$ norm of the skew-symmetric part $b^{ij} = \frac{1}{2}\left(A^{ij} - A^{ji}\right)$ such that*

$$\frac{1}{M(t-\tau)^{n/2}} \exp\left[-\frac{M|x-\xi|^2}{t-\tau}\right] \le \Gamma(x,t;\xi,\tau) \le \frac{M}{(t-\tau)^{n/2}} \exp\left[-\frac{|x-\xi|^2}{M(t-\tau)}\right] \tag{2.12}$$

for any $0 \le \tau < t < \infty$ and $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$, where the $L^\infty([0,\infty), BMO)$ norm of b^{ij} is defined by

$$\|b\|_{L^\infty(BMO)} = \sup_{t\ge 0} \sqrt{\sum_{i<j} \|b^{ij}(\cdot,t)\|^2_{BMO}}.$$

The fundamental heat kernel estimate (2.12) for parabolic equations has a long history. Two side estimate (2.12) was first established in D. G. Aronson [2, 1] for uniformly elliptic operators in divergence form where A^{ij} is symmetric (so that $b^{ij} \equiv 0$), his constant M depends only on the elliptic constant λ and the dimension n. The estimate (2.12) is therefore referred to as the Aronson estimate. A weaker but global estimate similar to (2.12) under the same assumption as in D. G. Aronson [2] already appeared in the Appendix of J. Nash [26]. D. G. Aronson [2, 3] indicated that his estimate can be established for a general elliptic operator, and a written proof is available in E. B. Fabes and D. W. Stroock [9], D. W. Stroock [31] and J. R. Norris and D. W. Stroock [27] too. In these papers, the Aronson estimate (2.12) was established for the following type of uniformly elliptic operator

$$\sum_{i,j=1}^{n}\frac{\partial}{\partial x^{i}}a^{ij}(x,t)\frac{\partial}{\partial x^{j}}+\sum_{i,j=1}^{n}a^{ij}(x,t)b_{j}(x,t)\frac{\partial}{\partial x^{i}}-\frac{\partial}{\partial x^{i}}\left(a^{ij}(x,t)\hat{b}_{j}(x,t)\right)+c(x,t)$$

where $\left(a^{ij}\right)$ is symmetric and uniformly elliptic. For this case, their constant M depends on the dimension, the elliptic constant λ and the L^{∞}_{tx}-norms of b, $\hat{b}$ and c.

A related topic to the Aronson estimate is the regularity of solutions to the parabolic equation $Lu=0$ (see for a complete survey of classical results [15]). If the elliptic operator is symmetric and is in divergence form, it was J. Nash [26] who proved the Hölder continuity of bounded solutions and also proved that the Hölder exponent depends only on the dimension and the elliptic constant λ. Under the same setting as that of J. Nash [26], in 1964, J. Moser [23] established the Harnack inequality for positive solutions of the parabolic equation $Lu=0$, based on which G. Aronson was able to derive his estimate (2.12). E. B. Fabes and D. W. Stroock [9] showed that J. Moser's Harnack inequality can be derived from Aronson estimate together with J. Nash's idea, and D. W. Stroock [31] further demonstrated that both the Hölder continuity of classical solutions and Moser's Harnack inequality for positive solutions can be established by utilizing the two side Aronson estimate (2.12). J. Nash's idea in [26] and the techniques in J. Moser [23, 24, 25] have been investigated intensively during the past decades. Many excellent results have been obtained in more general settings, but mainly under the symmetric setting of Dirichlet forms [11]. See for example A. A. Grigor'yan [12], E. B. Davies [7] and D. W. Stroock [32] for a small sample of references, and see also the literature therein.

The case that $\left(A^{ij}\right)$ is non-symmetric has received intensive study only recently, due to the connection with the Navier-Stokes equations and the blow-up behavior of their solutions. In H. Osada [28], the Aronson estimate (2.12) was obtained for an elliptic operator in divergence form as ours, where $\left(A^{ij}\right)$ may not be symmetric, his constant M in (2.12) however depends on the dimension n, the elliptic constant λ and the L^{∞}_{tx}-norm of the skew-symmetric part (b^{ij}). In a recent work by G. Seregin, L. Silvestre, V. Šverák, and A. Zlatoš [29], who noticed that a large portion of Nash's arguments also work for an elliptic operator with divergence-free drifts, i.e. where the elliptic operator has a form $\Delta+u(x,t)\cdot\nabla$ such that $\nabla\cdot u(\cdot,t)=0$. In particular, they mentioned that the fundamental solution Γ of the heat operator

$$\Delta+u(x,t)\cdot\nabla-\frac{\partial}{\partial t}$$

satisfies the diagonal decay estimate

$$\Gamma(x,t;x,\tau)\leq\frac{C}{(t-\tau)^{\frac{n}{2}}}$$

for all $t>\tau\geq 0$. They further proved the Harnack inequality in this case, and their constants depend on the dimension and the $L^{\infty}_{t}\mathrm{BMO}^{-1}_{x}$ of the vector field.

Our work was motivated by the observation made by G. Seregin and etc. [29], and the approach put forward by E. Davies [6], E. B. Fabes and D. Stroock [9], and D. W. Stroock [31]. We follow the approach in E. B. Davies and D. W. Stroock to the non-symmetric case, and adopt their arguments to our case by overcoming the difficulties arising from the singularities of the skew-symmetric part (b^{ij}). In a sense, the present work is to complete the program initiated by G. Seregin and etc. [29] by bringing in the techniques developed over years by various authors.

As in D. W. Stroock [31], as consequences of the Aronson estimate, we have the following continuity theorem and the Harnack inequality.

Theorem 2.2 *There exist $C > 0$ and $\alpha \in (0,1)$ depending only on the dimension n, the elliptic constant λ and the $L_t^\infty BMO_x$-norm of the skew-symmetric part (b^{ij}), such that for every $\delta > 0$*

$$|\Gamma(x,t;\xi,\tau) - \Gamma(x',t';\xi',\tau)| \leq \frac{C}{\delta^n}(|t'-t| \vee |x'-x| \vee |\xi'-\xi|)^\alpha \tag{2.13}$$

for all $\tau \geq 0$, $(t',x',\xi'), (t,x,\xi) \in [s+\delta^2) \times \mathbb{R}^n \times \mathbb{R}^n$ with $|t'-t| \vee |x'-x| \vee |y'-y| \leq \delta$.

Theorem 2.3 *[Harnack Inequality] There exists a constant $C > 0$ depending only on n, λ and $\|b\|_{L^\infty(BMO)}$ such that given any $v \in L^2(\mathbb{R}^n)$ with $v \geq 0$ and set $u(t,x) = \Gamma_{\tau,t}v(x)$, we have*

$$\sup_{[s,s+R^2]\times B(x_0,R)} u(t,x) \leq C \inf_{[s+3R^2,s+4R^2]\times B(x_0,R)} u(t,y) \tag{2.14}$$

for any $R > 0$, $(x,s) \in \mathbb{R}^n \times [\tau,\infty)$.

The Harnack inequality is also established by G. Seregin and etc. in [29] under a bit additional technical conditions than those stated in the theorem above.

3 Several technique facts

In this and next several sections, we are going to prove the main result, Aronson estimate. In this section, we prove several technique facts which will be needed in the proof of the main result.

The first result we need is a variation of Coifman-Meyer's compensated compactness Theorem [5, 4, 16] which highlights the importance of the Hardy spaces in the study of partial differential equations.

We first recall some facts on BMO functions [30, 14]. A function f is in BMO$(\mathbb{R}^n)$ if

$$\|f\|_{\text{BMO}} = \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|}\int_B |f(x) - f_B|\, dx < \infty \tag{3.1}$$

where $f_B = \frac{1}{|B|}\int_B f(x)\, dx$ and the supremum is taken over all open balls $B \in \mathbb{R}^n$ (in what follows, $B_r(x)$ or $B(x,r)$ denotes the ball centered at x with radius r). If define another norm

$$\|f\|^p_{\text{BMO}_p} = \sup_{B\subset\mathbb{R}^n} \frac{1}{|B|}\int_B |f(x) - f_B|^p\, dx < \infty \tag{3.2}$$

for any $1 \leq p < \infty$, John-Nirenberg inequality [14] (see also for example, Appexdix in D. W. Stroock and S. R. S. Varadhan [33]) implies that $\|\cdot\|_{\text{BMO}_p}$ are equivalent for different p.

The original version of the compensated compactness Theorem, which will be used in our proof of the lower bound of Aronson estimate, can be stated as following

Proposition 3.1 *Let vector fields E, B satisfy $E \in L^p(\mathbb{R}^n)^n$, $B \in L^q(\mathbb{R}^n)^n$ with $\frac{1}{p}+\frac{1}{q} = 1$ ($p \geq 1$, $q \geq 1$) and $\nabla\cdot E = 0$, $\nabla \times B = 0$. Then $E \cdot B \in \mathscr{H}^1$ where $\mathscr{H}^1$ is the Hardy space, and*

$$\|E\cdot B\|_{\mathscr{H}^1} \leq C\|E\|_p\|B\|_q. \tag{3.3}$$

In particular, there is a constant C depending on the dimension n and $p > 1$ such that

$$\|\nabla f \times \nabla g\|_{\mathscr{H}^1} \leq C\,\|\nabla f\|_p\,\|\nabla g\|_q \tag{3.4}$$

for any $f, g \in C_0^\infty(\mathbb{R}^n)$, where $\frac{1}{p}+\frac{1}{q} = 1$. Hence

$$\left|\int_{\mathbb{R}^n} \langle\nabla f(x), b(x)\cdot\nabla g(x)\rangle\, dx\right| \leq C\,\|b\|_{BMO}\,\|\nabla f\|_2\,\|\nabla g\|_2 \tag{3.5}$$

for any $f, g \in H^1(\mathbb{R}^n)$ and for any $b = \left(b^{ij}\right) \in BMO$ which is skew-symmetric, $b^{ij} = -b^{ji}$.

To prove the upper bound of Aronson estimate, we need the following estimate, in the same spirit of compensated compactness.

Proposition 3.2 *There is a universal constant C depending only on the dimension n, such that*

$$\|f\nabla_\xi f\|_{\mathscr{H}^1} \le C|\xi|\|\nabla f\|_2\|f\|_2. \tag{3.6}$$

for any $f \in H^1(\mathbb{R}^n) = W^{1,2}(\mathbb{R}^n)$ *and* $\xi \in \mathbb{R}^n$*, where* $\|\cdot\|_{\mathscr{H}^1}$ *denotes the Hardy norm.*

Proof. Let h be any smooth non-negative function on $\mathbb{R}^n$, with its support in the unit ball $B_1(0)$ such that $\int_{\mathbb{R}^n} h(x)dx = 1$, and for $t>0$, $h_t(x) = t^{-n}h(x/t)$. Notice that $f\nabla_\xi f = \frac{1}{2}\nabla\cdot(f^2\xi)$ in $L^1(\mathbb{R}^n)$, so

$$\begin{aligned}
h_t*(f\nabla_\xi f)(x) &= \frac{1}{2}\int_{B_t(x)} \nabla h_t(x-y)\cdot\xi f^2(y)dy \\
&= \int_{B_t(x)} \frac{1}{t^{n+1}}\nabla h\left(\frac{x-y}{t}\right)\cdot\xi f^2(y)dy \\
&= \int_{B_t(x)} \frac{1}{t^{n+1}}\nabla h\left(\frac{x-y}{t}\right)\cdot\xi f(y)\left[f(y) - \fint_{B_t(x)} f\right]dy \\
&\quad + \int_{B_t(x)} \frac{1}{t^{n+1}}\nabla h\left(\frac{x-y}{t}\right)\cdot\xi f(y)\fint_{B_t(x)} f(y)dy \\
&= I_1 + I_2
\end{aligned}$$

where $\fint_{B_t(x)}$ denotes the average integral over the ball $B_t(x)$, that is, $|B_t(x)|^{-1}\int_{B_t(x)}$. For the first term on the right-hand side, we have

$$|I_1| \le C\left[\fint_{B_t(x)} |\xi f|^\alpha\right]^{\frac{1}{\alpha}}\left[\fint_{B_t(x)}\left|\left(f(y) - \fint_{B_t(x)} f\right)t^{-1}\right|^{\alpha'}\right]^{\frac{1}{\alpha'}} \tag{3.7}$$

where $\alpha \in [1,2)$, $\frac{1}{\alpha}+\frac{1}{\alpha'} = 1$. Choose α,β such that $1 \le \alpha,\beta < 2$ and $\frac{1}{\alpha}+\frac{1}{\beta} = 1+\frac{1}{n}$. Then by the Sobolev-Poincaré inequality, we have

$$\left[\fint_{B_t(x)}\left|\left(f - \fint_{B_t(x)} f\right)t^{-1}\right|^{\alpha'}\right]^{\frac{1}{\alpha'}} \le C\left(\fint_{B_t(x)} |\nabla f|^\beta\right)^{\frac{1}{\beta}}. \tag{3.8}$$

For the second term on the right-hand side, we integrate by part again to obtain

$$|I_2| = \left|\int_{B_t(x)} h\left(\frac{x-y}{t}\right)\frac{1}{t^n}\cdot\mathrm{div}(\xi f(y))\fint_{B_t(x)} f(y)dy\right| \le C|\xi|\fint_{B_t(x)} |\nabla f(y)|dy\fint_{B_t(x)} f(y)dy. \tag{3.9}$$

By using these estimates we thus conclude that

$$\begin{aligned}
\sup_{t>0}|\{h_t*(\xi f\cdot\nabla f)\}(x)| &\le C|\xi|\sup_{t>0}\left(\fint_{B_t(x)} |f|^\alpha\right)^{\frac{1}{\alpha}}\sup_{t>0}\left(\fint_{B_t(x)} |\nabla f|^\beta\right)^{\frac{1}{\beta}} \\
&\quad + C|\xi|\sup_{t>0}\left(\fint_{B_t(x)} |f|\right)\sup_{t>0}\left(\fint_{B_t(x)} |\nabla f|\right)
\end{aligned}$$

$$= \quad C|\xi|[M(|f|^{\alpha})^{\frac{1}{\alpha}}M((|\nabla f|^{\beta})^{\frac{1}{\beta}}+M(|f|)M((|\nabla f|)]$$

where $M(f)$ is the maximal function. Since $1 \le \alpha < 2$, $1 < \beta < 2$, we have $\|M(|f|^{\alpha})^{\frac{1}{\alpha}}\|_2 \le C\|f\|_2$, $\|M(|\nabla f|^{\beta})^{\frac{1}{\beta}}\|_2 \le C\|\nabla f\|_2$ and similarly $\|M(|f|)\|_2 \le C\|f\|_2$, $\|M(|\nabla f|)\|_2 \le C\|\nabla f\|_2$. So $\sup_{t>0}|h_t * (\xi f\cdot\nabla f)| \in L^1$ and

$$\|f\cdot\nabla_\xi f\|_{\mathscr{H}^1} \le C|\xi|\|\nabla f\|_2\|f\|_2. \tag{3.10}$$

■

Given a function $b \in L^\infty([0,\infty),\mathrm{BMO}(\mathbb{R}^n))$, we want to approximate it by a mollified sequence, which is not trivial as it looks. A simple example is a vector field $b(t)$ which depends only on t, not on the space variables. Then it may not be in L^1_{loc} and there is no approximations by mollifying sequences. However, the problem considered here allow us to add a constant to it, i.e. consider $b(t,x)+f(t)$, where $f(t)$ is skew-symmetric so that it will not alter the weak solution formulation of the corresponding parabolic equations. So by subtracting the mean value of b on a unit ball, we may assume that

$$b_{B(0,1)}(t) = \fint_{B(0,1)} b(t,x)\,dx = 0 \qquad \text{for all } t\in[0,\infty). \tag{3.11}$$

Then for any $r>0$

$$\begin{aligned} |b_{B(r)}(t)| &= |b_{B(r)}(t) - b_{B(1)}(t)| = |\fint_{B(1)} b_{B(r)}(t) - b(t,x)\,dx| & (3.12)\\ &\le r^n \fint_{B(r)} |b_{B(r)}(t)-b(t,x)|\,dx \le r^n\|b\|_{L^\infty(\mathrm{BMO}(\mathbb{R}^n))} & (3.13)\end{aligned}$$

where $B(r)=B(0,r)$. By the definition of BMO functions, we have

$$\fint_{B(r)} |b(t,x)-b_{B(r)}(t)|^p\,dx \le C\|b\|^p_{L^\infty(\mathrm{BMO}(\mathbb{R}^n))}, \tag{3.14}$$

which implies that $b\in L^p_{loc}([0,\infty)\times\mathbb{R}^n)$ for any $1\le p<\infty$.

Proposition 3.3 *Take $\Phi\in C_0^\infty(B(1))$ and $\eta\in C_0^\infty([-1,1])$ with $\Phi,\eta\ge 0$ and*

$$\int_{B(1)}\Phi(x)\,dx = \int_{[-1,1]}\eta(t)\,dt = 1.$$

Let $\Phi_\varepsilon(x) = \frac{1}{\varepsilon^n}\Phi(\frac{x}{\varepsilon})$ and $\eta_\varepsilon(t) = \frac{1}{\varepsilon}\eta(\frac{t}{\varepsilon})$. Suppose $b\in L^\infty(BMO(\mathbb{R}^n))$ and satisfies (3.11). *Define*

$$b_\varepsilon(t,x) = \int_{-\varepsilon}^{+\varepsilon}\int_{B(0,\varepsilon)}\Phi_\varepsilon(y)\eta_\varepsilon(s)b(t-s,x-y)\,dyds. \tag{3.15}$$

Then $b_\varepsilon\to b$ locally in L^p for any $1\le p<\infty$, and

$$\|b_\varepsilon\|_{L^\infty(BMO(\mathbb{R}^n))} \le \|b\|_{L^\infty(BMO(\mathbb{R}^n))}. \tag{3.16}$$

Proof. Let x_0, and $r>0$ be fixed. Let $\fint$ denote the average integral over $B(x_0,r)$, that is,

$$\fint \phi(y)dy = \fint_{B(x_0,r)}\phi(y)\,dy.$$

For $x \in B(x_0,r)$ we have

$$\begin{aligned}
&\left| b_\varepsilon(t,x) - \fint b_\varepsilon(t,y)\,dy \right| \\
= &\left| \int_{-\varepsilon}^{+\varepsilon} \int_{B(0,\varepsilon)} \Phi_\varepsilon(y)\eta_\varepsilon(s) b(t-s,x-y)\,dyds \right. \\
&\left. - \fint_{B(x_0,r)} \int_{-\varepsilon}^{+\varepsilon} \int_{B(0,\varepsilon)} \Phi_\varepsilon(z)\eta_\varepsilon(s) b(t-s,y-z)\,dzdsdy \right| \\
= &\left| \int_{-\varepsilon}^{+\varepsilon} \int_{B(0,\varepsilon)} \Phi_\varepsilon(y)\eta_\varepsilon(s) [b(t-s,x-y) - \fint b(t-s,z-y)\,dz] dyds \right| \\
\le &\int_{-\varepsilon}^{+\varepsilon} \int_{B(0,\varepsilon)} \Phi_\varepsilon(y)\eta_\varepsilon(s) \left| b(t-s,x-y) - \fint b(t-s,z-y)\,dz \right| dyds,
\end{aligned}$$

so that

$$\begin{aligned}
&\fint \left| b_\varepsilon(t,x) - \fint b_\varepsilon(t,y)\,dy \right| \,dx \\
\le &\fint \int_{-\varepsilon}^{+\varepsilon} \int_{B(0,\varepsilon)} \Phi_\varepsilon(y)\eta_\varepsilon(s) \left| b(t-s,x-y) - \fint b(t-s,z-y)\,dz \right| dydsdx \\
= &\int_{-\varepsilon}^{+\varepsilon} \int_{B(0,\varepsilon)} \Phi_\varepsilon(y)\eta_\varepsilon(s) \fint \left| b(t-s,x-y) - \fint b(t-s,z-y)\,dz \right| \,dxdyds \\
\le &\int_{-\varepsilon}^{+\varepsilon} \int_{B(0,\varepsilon)} \Phi_\varepsilon(y)\eta_\varepsilon(s) \|b\|_{L^\infty(\mathrm{BMO}(\mathbb{R}^n))}\,dyds \\
= &\|b\|_{L^\infty(\mathrm{BMO}(\mathbb{R}^n))}.
\end{aligned}$$

Hence we have proved $\|b_\varepsilon\|_{L^\infty(\mathrm{BMO}(\mathbb{R}^n))} \le \|b\|_{L^\infty(\mathrm{BMO}(\mathbb{R}^n))}$. ■

The lattice property in the proposition below of the BMO space should be well known, for completeness, a proof is attached.

Proposition 3.4 *Suppose $f,g \in BMO(\mathbb{R}^n)$, then $f \wedge g$ and $f \vee g \in BMO(\mathbb{R}^n)$. Moreover, we have*

$$\|f \wedge g\|_{BMO} \le C\left(\|f\|_{BMO} + \|g\|_{BMO}\right) \tag{3.17}$$

where C only depends on n.

Proof. Here we only prove it for $f \wedge g$ and $f \vee g$ follows similar proof. Observe that for any $a,b,c,d \in \mathbb{R}$, we have

$$|a \wedge b - c \wedge d| \le |a-c| + |b-d|. \tag{3.18}$$

Hence for any ball B,

$$\begin{aligned}
\frac{1}{|B|}\int_B |f \wedge g(x) - (f \wedge g)_B|^2\,dx &\le \frac{1}{|B|}\int_B |f \wedge g(x) - f_B \wedge g_B|^2\,dx \\
&\le \frac{2}{|B|}\int_B |f(x) - f_B|^2\,dx + \frac{2}{|B|}\int_B |g(x) - g_B|^2\,dx \\
&\le C(\|f\|_{\mathrm{BMO}}^2 + \|g\|_{\mathrm{BMO}}^2)
\end{aligned}$$

and the proof is done. ■

4 Proof of Aronson's estimate

The proof follows the main lines as in D. W. Stroock [31] and in particular E. B. Davies [6] from which a clever use of the h-transforms from harmonic analysis is borrowed, while we need to overcome several difficulties since A is non-symmetric and the skew-symmetry part b is singular. These ideas are mainly due to J. Nash [26], J. Moser [22, 23, 24, 25].

Let us begin with the proof of the upper bound.

4.1 Proof of the upper bound

In this part we show the upper bound:

$$\varGamma(x,t;\xi,\tau) \leq \frac{C}{(t-\tau)^{\frac{n}{2}}} \exp\left[-\frac{|x-\xi|^2}{C(t-\tau)}\right] \tag{4.1}$$

for any $t > \tau$ and $x,\xi \in \mathbb{R}^n$, where C depends only on n, λ and $\|b\|_{L^\infty(\mathrm{BMO})}$.

The main idea of E. B. Davies [6] is to consider the h-transform of the fundamental solution $\varGamma(x,t;\xi,\tau)$ and apply Nash and Moser's iteration to the h-transforms of the fundamental solution $\varGamma$. J. Nash's idea is to iterate the L^p-norms of solutions to parabolic equations, and to control the growth of the L^p-norms. The main ingredient in J. Nash's argument is the clever use of the Nash inequality

$$\|u\|_2^{2+\frac{4}{n}} \leq C_n \|\nabla u\|_2^2 \|u\|_1^{\frac{4}{n}}, \qquad \forall u \in L^1(\mathbb{R}^n) \cap H^1(\mathbb{R}^n) \tag{4.2}$$

where $C_n > 0$ is a constant depending only on the dimension n.

The Nash iteration is neatly described as the following (D. Stroock [31], Lemma I.1.14, page 322).

Lemma 4.1 *Given positive numbers c_1, c_2, β and $p \geq 2$. Let w be positive and non-decreasing, continuous on $[0,\infty)$, and u be positive and has continuous derivatives on $[0,\infty)$. Suppose the following differential inequality holds:*

$$u'(t) \leq -\frac{c_1}{p} \frac{t^{(p-2)} u(t)^{1+\beta p}}{w(t)^{\beta p}} + c_2 p u(t), \quad t \geq 0. \tag{4.3}$$

Then there exists a $K(c_1,\beta) > 0$ such that

$$t^{(p-1)/\beta p} u(t) \leq \left(\frac{Kp^2}{\delta}\right)^{\frac{1}{\beta p}} e^{\frac{c_2 \delta t}{p}} w(t), \quad t \geq 0 \tag{4.4}$$

for every $\delta \in (0,1]$.

The above iteration procedure works in a very general setting, and has been explored since the publication of J. Nash' paper [26], and it is still the major ingredient in our proof. It is surprising that they work well even in our setting where the diffusion is very singular.

Fortunately as well, E. B. Davies' idea [6, 7] also works well for our parabolic equations. Following E. B. Davies [6] and D. Stroock [31], given a smooth function ψ on $\mathbb{R}^n$, consider

$$\varGamma^\psi(x,t;\xi,\tau) = e^{-\psi(x)} \varGamma(x,t;\xi,\tau) e^{\psi(\xi)} \tag{4.5}$$

and the linear operator

$$\varGamma^\psi_{\tau,t} f(x) = \int_{\mathbb{R}^n} f(\xi) e^{-\psi(x)} \varGamma(x,t;\xi,\tau) e^{\psi(\xi)} d\xi$$

which is defined for non-negative Borel measurable f , and for f which is smooth with a compact support. It is easy to see that the adjoint operator of $\Gamma^{\psi}_{\tau,t}$ can be identified as the following integral operator

$$\Gamma^{\psi\dagger}_{\tau,t}f(x)=\int_{\mathbb{R}^n}f(\xi)e^{\psi(x)}\Gamma(\xi,t;x,\tau)e^{-\psi(\xi)}d\xi.$$

That is

$$\left\langle\Gamma^{\psi}_{\tau,t}f,g\right\rangle=\left\langle f,\Gamma^{\psi\dagger}_{\tau,t}g\right\rangle$$

for any smooth functions f and g with compact supports.

Lemma 4.2 *Let $T>0,\tau\geq 0$. Let $f\in C_0^\infty(\mathbb{R}^n)$ be non-negative, and $\psi(x)=\alpha\cdot x$ where $\alpha\in\mathbb{R}^n$. Define*

$$f_t(x)=\int\limits_{\mathbb{R}^n}f(\xi)e^{\psi(x)-\psi(\xi)}\Gamma(x,t;\xi,\tau)d\xi=\Gamma^{\psi}_{\tau,t}f(x)$$

for $t>\tau$, and

$$f_t^\dagger(x)=\int\limits_{\mathbb{R}^n}f(\xi)e^{\psi(x)-\psi(\xi)}\Gamma(\xi,T;x,T-t)d\xi$$

for $t\in(0,T]$.

There is a constant $C>0$ depending only on n, such that for any $p\geq 1$,

$$\frac{d}{dt}\|f_t\|_{2p}^{2p}\leq-\lambda\left\|\nabla f_t^p\right\|_2^2+C_B\frac{p^2|\alpha|^2}{\lambda}\|f_t\|_{2p}^{2p}\tag{4.6}$$

$t>\tau$, and

$$\frac{d}{dt}\left\|f_t^\dagger\right\|_{2p}^{2p}\leq-\lambda\left\|\nabla f_t^{\dagger p}\right\|_2^2+C_B\frac{p^2|\alpha|^2}{\lambda}\left\|f_t^\dagger\right\|_{2p}^{2p}\tag{4.7}$$

for all $t\in(0,T]$, where

$$C_B=2C\|b\|^2_{L^\infty(BMO)}+2$$

which depends only on n and the $L_t^\infty BMO_x$ of $b(x,t)$.

Proof. We may assume that $\tau=0$ without lose of generality, so that

$$\begin{aligned}\|f_t\|_{2p}^{2p}&=\int_{\mathbb{R}^n}\left(\int\limits_{\mathbb{R}^n}f(\xi)\Gamma^{\psi}(x,t;\xi,0)d\xi\right)^{2p}dx\\&=\int_{\mathbb{R}^n}\left(\int\limits_{\mathbb{R}^n}f(\xi)e^{-\psi(x)+\psi(\xi)}\Gamma(x,t;\xi,0)d\xi\right)^{2p}dx.\end{aligned}$$

Differentiating $\|f_t\|_{2p}^{2p}$ to obtain

$$\frac{d}{dt}\|f_t\|_{2p}^{2p}=2p\int_{\mathbb{R}^n}f_t(x)^{2p-1}\left(\int\limits_{\mathbb{R}^n}f(\xi)e^{-\psi(x)+\psi(\xi)}\frac{\partial}{\partial t}\Gamma(x,t;\xi,0)d\xi\right)dx$$

and by using the equation (2.5) we have

$$\frac{d}{dt}\|f_t\|_{2p}^{2p}=2p\int_{\mathbb{R}^n}\left[f_t(x)^{2p-1}\int\limits_{\mathbb{R}^n}f(\xi)e^{-\psi(x)+\psi(\xi)}\nabla_x\cdot(A(x,t)\nabla_x\Gamma(x,t;\xi,0))d\xi\right]dx.$$

Similarly, by using the backward equation (2.11) we have

$$\frac{d}{dt}\left\|f_t^\dagger\right\|_{2p}^{2p} = 2p\int_{\mathbb{R}^n}\left[f_t^\dagger(x)^{2p-1}\int\limits_{\mathbb{R}^n} f(\xi)e^{\psi(x)-\psi(\xi)}\nabla_x\cdot\left(A^T(x,t)\nabla_x\Gamma(\xi,T;x,T-t)\right)d\xi\right]dx.$$

By using the Fubini theorem, then performing integration by parts we therefore have

$$\begin{aligned}\frac{1}{2p}\frac{d}{dt}\|f_t\|_{2p}^{2p} &= \int_{\mathbb{R}^n}\left(e^{\psi(\xi)}f(\xi)\int\limits_{\mathbb{R}^n} e^{-\psi(x)}f_t(x)^{2p-1}\nabla_x\cdot(A(x,t)\nabla_x\Gamma(x,t;\xi,0))\,dx\right)d\xi\\
&= \int_{\mathbb{R}^n} f_t(x)^{2p}\langle\nabla\psi,a(x,t)\cdot\nabla\psi\rangle\,dx\\
&\quad -\frac{2p-1}{p^2}\int_{\mathbb{R}^n}\langle\nabla f_t(x)^p,a(x,t)\cdot\nabla f_t(x)^p\rangle\,dx\\
&\quad -\frac{2(p-1)}{p}\int_{\mathbb{R}^n} f_t(x)^p\langle\nabla f_t(x)^p,a(x,t)\cdot\nabla\psi\rangle\,dx\\
&\quad -2\int_{\mathbb{R}^n} f_t(x)^p\langle\nabla f_t(x)^p,b(x,t)\cdot\nabla\psi\rangle\,dx\\
&= I_1-I_2-I_3-I_4.\end{aligned}$$

and similarly we have

$$\begin{aligned}\frac{1}{2p}\frac{d}{dt}\left\|f_t^\dagger\right\|_{2p}^{2p} &= \int_{\mathbb{R}^n} f_t^\dagger(x)^{2p}\langle\nabla\psi,a(x,t)\cdot\nabla\psi\rangle\,dx\\
&\quad -\frac{2(2p-1)}{p}\int_{\mathbb{R}^n}\left\langle\nabla f_t^\dagger(x)^p,a(x,t)\cdot\nabla f_t^\dagger(x)^p\right\rangle dx\\
&\quad +\frac{2(p-1)}{p}\int_{\mathbb{R}^n} f_t^\dagger(x)^p\left\langle\nabla\psi,a(x,t)\cdot\nabla f_t^\dagger(x)^p\right\rangle dx\\
&\quad -2\int_{\mathbb{R}^n} f_t^\dagger(x)^p\left\langle\nabla\psi,b(x,t)\cdot\nabla f_t^\dagger(x)^p\right\rangle dx.\end{aligned}$$

Since $\frac{d}{dt}\|f_t\|_{2p}^{2p}$ and $\frac{d}{dt}\left\|f_t^\dagger\right\|_{2p}^{2p}$ are similar, we only need to prove (4.6).

Each term I_j on the right-hand side of (4.6) can be dominated as the following. The first three terms I_1, I_2 and I_3 can be handled exactly as in B. Davies [6] and D. Stroock [31]. Recall that $\nabla\psi=\alpha$ is a constant vector. Hence

$$I_1\le\frac{|\alpha|^2}{\lambda}\|f_t\|_{2p}^{2p}. \tag{4.8}$$

While for I_2 and I_3, by completing squares we first rewrite the terms of I_3+I_2 as following

$$\begin{aligned}-I_2-I_3 &= -\frac{2p-1}{p^2}\int_{\mathbb{R}^n}\langle\nabla f_t(x)^p,a(x,t)\cdot\nabla f_t(x)^p\rangle\,dx\\
&\quad -2\frac{p-1}{p}\int_{\mathbb{R}^n} f_t(x)^p\langle\nabla f_t(x)^p,a(x,t)\cdot\alpha\rangle\,dx\\
&= -\frac{1}{p}\int_{\mathbb{R}^n}\langle\nabla f_t(x)^p,a(x,t)\cdot\nabla f_t(x)^p\rangle\,dx+(p-1)\int_{\mathbb{R}^n} f_t(x)^{2p}\langle\alpha,a(x,t)\cdot\alpha\rangle\,dx\\
&\quad -\frac{p-1}{p^2}\int_{\mathbb{R}^n}\langle(\nabla f_t(x)^p-pf_t(x)^p\alpha),a(x,t)\cdot(\nabla f_t(x)^p-pf_t(x)^p\alpha)\rangle\,dx.\end{aligned}$$

The last term on the right-hand side is non-positive as $a(x,t)$ is positive definite, so by using inequalities

$$\langle \nabla f_t(x)^p, a(x,t)\cdot \nabla f_t(x)^p\rangle \geq \lambda \left|\nabla f_t(x)^p\right|^2$$

and

$$\langle \alpha, a(x,t)\cdot \alpha\rangle \leq \frac{1}{\lambda}|\alpha|^2$$

we deduce that

$$-I_2 - I_3 \leq -\frac{\lambda}{p}\left\|\nabla f_t^p\right\|_2^2 + \frac{p-1}{\lambda}|\alpha|^2 \|f_t\|_{2p}^{2p}. \tag{4.9}$$

The main innovation in our proof is the handling of the skew-symmetric part I_4 which does not appear in the symmetric case. The idea is to apply the compensated estimate, Proposition 3.2 , to obtain

$$\begin{aligned} |I_4| &= \left|2\int_{\mathbb{R}^n} f_t(x)^p \langle \nabla f_t(x)^p, b(x,t)\cdot\alpha\rangle\, dx\right| \\ &\leq C\|b\|_{L^\infty(\mathrm{BMO})}|\alpha|\left\|f_t^p\right\|_2\left\|\nabla f_t^p\right\|_2 \end{aligned} \tag{4.10}$$

where C is a constant depending only on n. Therefore

$$|I_4| \leq \frac{\lambda}{2p}\left\|\nabla f_t^p\right\|_2^2 + C\|b\|_{L^\infty(\mathrm{BMO})}^2 \frac{p}{\lambda}|\alpha|^2\|f_t\|_{2p}^{2p}. \tag{4.11}$$

Putting these estimates together we thus obtain (4.6). ■

Now we can follow arguments in D. W. Stroock [31] to obtain the upper bound, yet again by using the special feature of our elliptic operator. We contain the major steps only for completeness.

First we can prove the following by exactly the same argument in D. W. Stroock [31].

Lemma 4.3 *There is a constant $C>0$ depending only on n and the $L_t^\infty(BMO)$-norm of the skew-symmetric part of $\left(A^{ij}(x,t)\right)$ such that*

$$\left\|\Gamma_{\tau,t}^{\psi} f\right\|_\infty \leq \frac{C}{(t-\tau)^{n/4}} e^{\frac{C|\alpha|^2(t-\tau)}{\lambda}}\|f\|_2 \tag{4.12}$$

and

$$\left\|\Gamma_{\tau,t}^{\psi\dagger} f\right\|_\infty \leq \frac{C}{(t-\tau)^{n/4}} e^{\frac{C|\alpha|^2(t-\tau)}{\lambda}}\|f\|_2 \tag{4.13}$$

for every $f\in L^2(\mathbb{R}^n)$, $0\leq\tau<t$ and $\alpha\in\mathbb{R}^n$, where $\psi(x)=\alpha\cdot x$.

Proof. We only need to prove (4.12) for the case that $0=\tau<t$. The proof of (4.13) is similar, and uses the inequality (4.7) instead and uses the fact that the constant appears in that inequality is independent of any $T>0$.

To show (4.12), applying Nash's inequality (4.2) to the first term on the right-hand side of (4.6) to deduce that

$$\frac{d}{dt}\|f_t\|_{2p} \leq -\frac{\lambda}{2pC_n}\frac{\|f_t\|_{2p}^{1+4p/n}}{\|f_t\|_p^{4p/n}} + C_B\frac{|\alpha|^2}{\lambda}p\|f_t\|_{2p} \tag{4.14}$$

for every $p>1$. Let $u_p(t)=\|f_t\|_{2p}$ and $w_p(t)=\sup_{0\leq s\leq t} s^{n(p-2)/4p}u_{p/2}(s)$. Then (4.14) may be written as

$$u_p'(t) \leq -\frac{\lambda}{2pC_n}\frac{t^{p-2}u_p(t)^{1+4p/n}}{(w_p(t))^{4p/n}} + C\frac{|\alpha|^2}{\lambda}pu_p(t)$$

so that, according to Lemma 4.1, we have

$$\begin{aligned} w_{2p}(t) &= \sup_{0\le s\le t} s^{n(p-1)/4p} u_p(s) \\ &\le \sup_{0\le s\le t} (\frac{Kp^2}{\delta})^{\frac{n}{4p}} \exp(\frac{C|\alpha|^2\delta s}{p\lambda}) w_p(s) \\ &= (\frac{Kp^2}{\delta})^{\frac{n}{4p}} \exp(\frac{C|\alpha|^2\delta t}{p\lambda}) w_p(t). \end{aligned}$$

According to (4.14), if take $p=1$, we have

$$w_2(t) = \sup_{0\le s\le t} \|f_s\|_2 \le e^{C|\alpha|^2 t/\lambda} \|f\|_2.$$

Now we set $\delta = 1$ and iterate it to get

$$w_{2^m}(t) \le C\exp\left(\frac{C|\alpha|^2 t}{\lambda}\right) w_2(t) \le C\exp\left(\frac{C|\alpha|^2 t}{\lambda}\right) \|f\|_2$$

Letting $m\to\infty$, we therefore obtain that

$$\|f_t\|_\infty \le \frac{C}{t^{n/4}} \exp\left(\frac{C|\alpha|^2 t}{\lambda}\right) \|f\|_2$$

which completes the proof. ■

Proof of the upper bound (4.1). Let us use the same notations as in the proof of the above lemma. By (4.13) and the fact that $\Gamma^{\psi\dagger}_{\tau,t}$ is the adjoint operator of $\Gamma^{\psi}_{\tau,t}$, we have

$$\|f_t\|_2 \le \frac{C}{t^{n/4}} \exp\left(\frac{C|\alpha|^2 t}{\lambda}\right) \|f\|_1.$$

Since $\Gamma^{\psi}_{0,2t} = \Gamma^{\psi}_{t,2t} \circ \Gamma^{\psi}_{0,t}$, we thus deduce that

$$\|f_{2t}\|_\infty \le \frac{C}{t^{n/2}} \exp\left(\frac{C|\alpha|^2 t}{\lambda}\right) \|f\|_1,$$

which is equivalent to

$$\Gamma(x,2t;\xi,0) \le \frac{C}{t^{n/2}} \exp\left[\frac{C|\alpha|^2 t}{\lambda} + \alpha\cdot(\xi - x)\right].$$

Letting $\alpha = \frac{\lambda}{2Ct}(x-\xi)$ and adjusting $2t$ to t and 0 to τ, we therefore derive the upper bound

$$\Gamma(x,t;\xi,\tau) \le \frac{C}{(t-\tau)^{\frac{n}{2}}} \exp\left(-\frac{|x-\xi|^2}{C(t-\tau)}\right)$$

for any $t > \tau \ge 0$. This completes the proof of the upper bound.

4.2 Proof of the lower bound

In this part, we prove the lower bound,

$$\Gamma(x,t;\xi,\tau) \geq \frac{1}{C(t-\tau)^{\frac{n}{2}}} \exp\left[-\frac{C|x-\xi|^2}{t-\tau}\right] \tag{4.15}$$

following the idea due to J. Nash [26], where C depends only on n, λ and $\|b\|_{L^\infty(\mathrm{BMO})}$.

According to Nash's arguments, the lower bound is local in nature, and follows easily from the following

Lemma 4.4 *There is a constant $C_0 > 0$ depending only on the dimension n, $\lambda > 0$ and the $L^\infty(BMO)$ norm of $\left(b^{ij}\right)$, such that*

$$\int_{\mathbb{R}^n} \ln(\Gamma(x,1;\xi,0))\, \mu(d\xi) \geq -C_0, \qquad \forall x \in B(0,2) \tag{4.16}$$

and

$$\Gamma(x,2;\xi,0) \geq e^{-2C_0}, \qquad x,\xi \in B(0,2) \tag{4.17}$$

where μ denotes the standard Gaussian measure in $\mathbb{R}^n$, i.e.

$$\mu(d\xi) = \mu(\xi)\, d\xi \qquad \textit{where } \mu(\xi) = \frac{1}{(2\pi)^{n/2}} e^{-\frac{|\xi|^2}{2}}.$$

Proof. The proof follow the same ideas as in J. Nash, as explained in D. W. Stroock [26]. We have to overcome difficulties arising from the additional non-symmetric part $b(x,t) = \left(b^{ij}(x,t)\right)$. The idea is to consider for any $x \in B(0,2)$ the following function

$$G(t) = \int_{\mathbb{R}^n} \ln(\Gamma(x,1;\xi,1-t))\, \mu(d\xi)$$

where $t \in (0,1]$ and $x \in B(0,2)$. Since $\left(A^{ij}\right)$ is uniformly elliptic with bounded derivatives, $\Gamma(x,t;\xi,\tau)$ is a probability density in x (when others are fixed) and also in ξ (as other variables are fixed). Hence $\int_{\mathbb{R}^n} \Gamma(x,t;\xi,0)\, d\xi = 1$ for every $t \in (0,1]$. According to Jensen's inequality $G(t) \leq 0$ and what we want to show is that $G(1)$ is bounded from below uniformly in $x \in B(0,2)$. To this end we consider the derivative of G. By a simple calculation with integration by parts we obtain

$$\begin{aligned} G'(t) &= \int_{\mathbb{R}^n} \left\langle \xi, a(\xi,1-t)\cdot\nabla_\xi \ln\Gamma(x,t;\xi,1-t)\right\rangle \mu(d\xi) \\ &+ \int_{\mathbb{R}^n} \left\langle \nabla_\xi \ln\Gamma(x,1;\xi,1-t), a(\xi,1-t)\cdot\nabla_\xi \ln\Gamma(x,t;\xi,1-t)\right\rangle \mu(d\xi) \\ &+ \frac{1}{\delta}\int_{\mathbb{R}^n} \left\langle \nabla\mu(\xi)^\delta, b(\xi,1-t)\cdot\nabla_\xi\left(\mu(\xi)^{1-\delta}\ln\Gamma(x,t;\xi,1-t)\right)\right\rangle d\xi \end{aligned} \tag{4.18}$$

for any $\delta \in (0,1)$. We have used the facts that $\nabla \ln\mu(\xi) = -2\xi$, the backward equation for the fundamental solution $\Gamma(x,t;\xi,\tau)$ and the following fact that

$$\langle \nabla\mu, b\cdot\nabla\ln\Gamma\rangle = \frac{1}{\delta}\left\langle \nabla\mu^\delta, b\cdot\nabla\left(\mu^{1-\delta}\ln\Gamma\right)\right\rangle$$

as b is skew-symmetric. Using the Cauchy-Schwartz inequality and the compensated compactness inequality (3.5) we deduce that

$$G'(t) \geq -\frac{1}{\varepsilon} + (1-\varepsilon)\int_{\mathbb{R}^n} \left\langle \nabla_\xi \ln\Gamma(x,1;\xi,1-t), a(\xi,1-t)\cdot\nabla_\xi \ln\Gamma(x,t;\xi,1-t)\right\rangle \mu(d\xi)$$

$$
\begin{aligned}
&-\frac{C}{\delta}\|b\|_{\mathrm{BMO}}\left\|\nabla\mu^{\delta}\right\|_{2}\left\|\nabla_{\xi}\left(\mu^{1-\delta}\ln\Gamma(x,t;\cdot,1-t)\right)\right\|_{2}\\
&\geq -\frac{1}{\varepsilon}+(1-\varepsilon)\lambda\left\|\nabla\ln\Gamma(x,1;\cdot,1-t)\right\|_{L^{2}(\mu)}\\
&-\frac{C}{\delta}\|b\|_{\mathrm{BMO}}\left\|\nabla\mu^{\delta}\right\|_{2}\left\|\nabla\left(\mu^{1-\delta}\ln\Gamma(x,t;\cdot,1-t)\right)\right\|_{2}
\end{aligned}
$$

for any $\varepsilon,\delta\in(0,1)$. Choose $\delta\in(0,\frac{1}{2})$. Then

$$
\left\|\nabla\mu^{\delta}\right\|_{2}=\delta\left\|\mu^{\delta}\nabla\ln\mu\right\|_{2}<\infty.
$$

Moreover, for $\delta\in(0,\frac{1}{2})$,

$$
\sup_{\xi}\left|\mu(\xi)^{\frac{1}{2}-\delta}\nabla\ln\mu(\xi)\right|<\infty
$$

and

$$
\sup_{\xi}\left|\mu(\xi)^{\frac{1}{2}-\delta}\right|<\infty,
$$

so that

$$
\begin{aligned}
\left\|\nabla\left(\mu^{1-\delta}\ln\Gamma(x,t;\cdot,1-t)\right)\right\|_{2}&\leq\left\|(\nabla\mu^{1-\delta})\ln\Gamma(x,t;\cdot,1-t)\right\|_{2}\\
&+\left\|\mu^{1-\delta}\nabla\ln\Gamma(x,t;\cdot,1-t)\right\|_{2}\\
&=(1-\delta)\left\|(\mu^{\frac{1}{2}-\delta}\nabla\ln\mu)\ln\Gamma(x,t;\cdot,1-t)\right\|_{L^{2}(\mu)}\\
&+\left\|\mu^{\frac{1}{2}-\delta}\nabla\ln\Gamma(x,t;\cdot,1-t)\right\|_{L^{2}(\mu)}\\
&\leq C\left\{\left\|\ln\Gamma(x,t;\cdot,1-t)\right\|_{L^{2}(\mu)}+\left\|\nabla\ln\Gamma(x,t;\cdot,1-t)\right\|_{L^{2}(\mu)}\right\}
\end{aligned}
$$

for some constant C depending only on n and $\delta\in(0,\frac{1}{2})$. By substituting this estimate into the inequality for G' we obtain

$$
\begin{aligned}
G'(t)&\geq-\frac{1}{\varepsilon}+(1-\varepsilon)\lambda\left\|\nabla\ln\Gamma(x,1;\cdot,1-t)\right\|_{L^{2}(\mu)}^{2}\\
&-C\|b\|_{\mathrm{BMO}}\left\{\left\|\ln\Gamma(x,t;\cdot,1-t)\right\|_{L^{2}(\mu)}+\left\|\nabla\ln\Gamma(x,t;\cdot,1-t)\right\|_{L^{2}(\mu)}\right\}\\
&\geq-\frac{1}{\varepsilon}-\frac{1}{4\lambda^{2}\varepsilon^{2}}C^{2}\|b\|_{\mathrm{BMO}}^{2}+(1-2\varepsilon)\lambda\left\|\nabla\ln\Gamma(x,1;\cdot,1-t)\right\|_{L^{2}(\mu)}^{2}\\
&-C\|b\|_{\mathrm{BMO}}\left\|\ln\Gamma(x,t;\cdot,1-t)\right\|_{L^{2}(\mu)}
\end{aligned}
$$

for any $\varepsilon\in(0,\frac{1}{2})$. By choosing $\varepsilon=1/3$, we thus have the following differential inequality:

$$
G'(t)\geq-C+\tfrac{\lambda}{3}\left\|\nabla\ln\Gamma(x,1;\cdot,1-t)\right\|_{L^{2}(\mu)}^{2}-C\left\|\ln\Gamma(x,t;\cdot,1-t)\right\|_{L^{2}(\mu)} \tag{4.19}
$$

for all $t\in(0,1)$, for some constant $C>0$ depending only on n and the $L^{\infty}(\mathrm{BMO})$ norm of the skew-symmetric part $b(x,t)$.

The remaining arguments of the proof are more or less the same as D. W. Stroock [31]. Firstly, by the Poincaré inequality for the Gaussian measure, we obtain

$$
J(x,t)\equiv\left\|\ln\Gamma(x,1;\cdot,1-t)-G(t)\right\|_{L^{2}(\mu)}^{2}\leq 2\left\|\nabla\ln\Gamma(x,1;\cdot,1-t)\right\|_{L^{2}(\mu)}^{2}.
$$

On the other hand, since $G(t) < 0$, we have

$$
\begin{aligned}
J(x,t) &= \|\ln \Gamma(x,1;\cdot,1-t) - G(t)\|^2_{L^2(\mu)} \\
&= \int_{\mathbb{R}^n} (\ln \Gamma(x,1;\xi,1-t) - G(t))^2 \mu(d\xi) \\
&\geq \int_{\{\ln \Gamma(x,1;\xi,1-t) \geq -K\}} (\ln \Gamma(x,1;\xi,1-t) - G(t))^2 \mu(d\xi) \\
&\geq \int_{\{\ln \Gamma(x,1;\xi,1-t) \geq -K\}} (\ln \Gamma(x,1;\xi,1-t) + K - G(t) - K)^2 \mu(d\xi) \\
&\geq \frac{1}{2} \int_{\{\ln \Gamma(x,1;\xi,1-t) \geq -K\}} (\ln \Gamma(x,1;\xi,1-t) + K - G(t))^2 \mu(d\xi) - K^2 \\
&\geq \frac{1}{2} \int_{\{\ln \Gamma(x,1;\xi,1-t) \geq -K\}} G(t)^2 \mu(d\xi) - K^2 \\
&= \frac{1}{2} G(t)^2 \mu \{\xi \in \mathbb{R}^n : \ln \Gamma(x,1;\xi,1-t) \geq -K\} - K^2.
\end{aligned}
$$

According to the upper bound

$$
\Gamma(x,1;\xi,1-t) \leq \frac{C}{t^{n/2}} \exp\left[-\frac{|x-\xi|^2}{Ct}\right],
$$

we have

$$
\ln \Gamma(x,1;\xi,1-t) \leq \ln C - \frac{n}{2} \ln t - \frac{|x-\xi|^2}{Ct}
$$

for $x \in B(0,2)$ and $t \in (\frac{1}{2},1)$. Hence

$$
\begin{aligned}
\int_{\{|\xi|>r\}} \Gamma(x,1;\xi,1-t) d\xi &\leq \int_{\{|\xi|>r\}} \frac{C}{t^{n/2}} \exp\left[-\frac{|x-\xi|^2}{Ct}\right] d\xi \\
&= C_1 \mu \left[\left| \sqrt{\frac{C}{2}} t\xi + x \right| > r \right] \\
&\leq C_1 \mu \left[|\xi| > \frac{r-2}{\sqrt{\frac{C}{2} t}} \right] \leq C_1 \mu \left[|\xi| > \frac{r-2}{\sqrt{\frac{C}{2}}} \right]
\end{aligned}
$$

and therefore, there is a positive number R depending on C such that for any $r > R$

$$
\int_{\{|\xi|>r\}} \Gamma(x,1;\xi,1-t)\, d\xi < \frac{1}{4}, \qquad \text{for all } t \in (0,1], x \in B(0,2).
$$

Thus for any $t \in [\frac{1}{2},1]$, there is M such that $\Gamma(x,1;\xi,1-t) \leq M$, so that

$$
\begin{aligned}
\frac{3}{4} \leq \int_{B(0,r)} \Gamma(x,1;\xi,1-t)\, d\xi &\leq |B(0,r)| e^{-K} \\
&+ (2\pi)^{n/2} M e^{r^2/2} \mu \{\xi \in \mathbb{R}^n : \ln \Gamma(x,1;\xi,1-t) \geq -K\}.
\end{aligned}
$$

Choose $K > 0$ such that $|B(0,r)| e^{-K} = \frac{1}{4}$, to obtain

$$
\mu \{\xi \in \mathbb{R}^n : \ln \Gamma(x,1;\xi,1-t) \geq -K\} \geq \frac{1}{2(2\pi)^{n/2} M e^{r^2/2}}
$$

$$\equiv \kappa(r) > 0.$$

By using this estimate we deduce that

$$\begin{aligned}
G'(t) &\geq -C + \frac{\lambda}{6} J(x,t) - C \| \ln \Gamma(x,t;\cdot,1-t) \|_{L^2(\mu)} \\
&\geq -C + \frac{\lambda}{6} J(x,t) - C \left[\sqrt{J(x,t)} + |G(t)| \right] \\
&\geq -C(\varepsilon,\lambda) + \frac{\lambda}{12} J(x,t) - \varepsilon G(t)^2 \\
&\geq -C(\varepsilon,\lambda) + \frac{\lambda}{24} G(t)^2 \mu \{ \xi \in \mathbb{R}^n : \ln \Gamma(x,1;\xi,1-t) \geq -K \} - \frac{\lambda}{12} K^2 \\
&\geq -C(\varepsilon,K,\lambda) + \left(\frac{\lambda}{12} \kappa(r) - \varepsilon \right) G(t)^2
\end{aligned}$$

for $\varepsilon > 0$ such that $\frac{\lambda}{12}\kappa(r) - \varepsilon > 0$. Now we obtain

$$G'(t) \geq -C_1 + C_2 G(t)^2 \tag{4.20}$$

for any $t \in [\frac{1}{2}, 1]$, where $C_1 > 0$, $C_2 \in (0,1]$. The previous inequality (4.20) may be written as

$$G'(t) \geq C_2 \left(G - \sqrt{\frac{C_1}{C_2}} \right) \left(G + \sqrt{\frac{C_1}{C_2}} \right)$$

together with the fact that $G < 0$, it follows that

$$G(1) \geq \min \left\{ -C_1 - 2\sqrt{\frac{C_1}{C_2}}, -\frac{8}{3C_2} \right\} = -C_0. \tag{4.21}$$

The lower bound in (4.16) follows from the Chapman-Kolmogrov equation and Jensen's inequality. In fact

$$\begin{aligned}
\ln \Gamma(x,2;\xi,0) &= \ln \left(\int_{\mathbb{R}^n} \Gamma(x,2;z,1) \Gamma(z,1;\xi,0) dz \right) \\
&= \ln \left(\int_{\mathbb{R}^n} (2\pi)^{n/2} e^{|z|^2/2} \Gamma(x,2;z,1) \Gamma(z,1;\xi,0) \mu(dz) \right) \\
&\geq \ln \left(\int_{\mathbb{R}^n} \Gamma(x,2;z,1) \Gamma(z,1;\xi,0) \mu(dz) \right) \\
&\geq \int_{\mathbb{R}^n} \ln \Gamma(x,2;z,1) \mu(dz) + \int_{\mathbb{R}^n} \ln \Gamma(z,1;\xi,0) \mu(dz) \\
&\geq -2C_0
\end{aligned}$$

which yields (4.17). ■

Proof of the lower bound (4.15). By using scaling invariant properties, i.e. for any $r > 0$ and $z \in \mathbb{R}^n$,

$$\Gamma(rx+z, r^2 t; r\xi + z, 0) = r^{-n} \Gamma^{A_{r,z}}(x,t;\xi,0) \tag{4.22}$$

where $A_{r,z}(x,t) = A(rx+z, r^2 t)$ and Γ^A is the fundamental solution associated with A. The transformation $A \to A^{r,z}$ preserves the elliptic constant λ and more importantly the $L^\infty(\mathrm{BMO})$ norms, so we may apply (4.17) to $\Gamma^{A_{r,z}}$ to deduce that

$$\Gamma(x,2t;\xi,0) \geq \frac{e^{-2A}}{t^{n/2}}, \quad |\xi - x| < 4t^{\frac{1}{2}}. \tag{4.23}$$

Together with the same chain argument as in D. W. Stroock [31] by using the Chapman-Kolmogrov equation, we obtain the lower bound accordingly.

5 Weak solutions for non-symmetric parabolic equations

Let us consider the parabolic equation

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x^i}\left[A^{ij}(x,t)\frac{\partial}{\partial x^j}u(x,t)\right] - \frac{\partial}{\partial t}u(x,t) = 0, \tag{5.1}$$

where $A^{ij} = a^{ij} + b^{ij}$, $\left(a^{ij}\right)$ is symmetric satisfying the uniform elliptic condition that $\lambda \leq (a^{ij}) \leq \lambda^{-1}$ in the matrix sense, $\left(b^{ij}\right)$ is skew-symmetric. We only assume that A^{ij} are Borel measurable in (x,t), and $b^{ij}(x,t)$ belong to the BMO space for every $t \geq 0$, such that the BMO norms $t \to \|b(\cdot,t)\|_{\mathrm{BMO}}$ is bounded, whose supremum norm is denoted by $\|b\|_{L^\infty(\mathrm{BMO})}$, as before.

5.1 Weak solutions

Let us consider Cauchy's initial and Dirichlet boundary problem associated with (5.1). Let $D \subset \mathbb{R}^n$ be an open subset with a smooth boundary. Given $\tau > 0$, $u(x,t)$, which is a locally integrable and Borel measurable function in $(x,t) \in D \times [\tau, \infty)$, is a weak solution to the Dirichlet boundary problem of (5.1) with initial data $u(\cdot,\tau) = f$, if it holds that

$$-\int_\tau^\infty \int_D \nabla\varphi(x,t)\cdot A(x,t)\nabla u(x,t)dxdt + \int_\tau^\infty \int_D u(x,s)\frac{\partial}{\partial s}\varphi(x,s)dsdx + \int_D f(x)\varphi(x,\tau)dx = 0 \tag{5.2}$$

for any smooth function $\varphi(x,s)$ which has a compact support in $D \times [\tau,\infty)$. To ensure that (5.2) is well defined, we need to assume that $u \in L^2\left([\tau,T];H^1(D)\right)$ and $u \in L^\infty\left([\tau,T];L^2(D)\right)$, and the initial data f is locally integrable. Let $\Gamma^D(x,t;\xi,\tau)$ denote the corresponding fundamental solution.

Lemma 5.1 *Suppose in addition that A^{ij} are smooth, so that the fundamental solution $\Gamma(x,t;\xi,\tau)$ exists, is smooth, and satisfies Aronson estimate, and therefore*

$$0 < \Gamma^D(x,t;\xi,\tau) \leq \Gamma(x,t;\xi,\tau) \leq \frac{C}{(t-\tau)^{n/2}}\exp\left(-\frac{|x-\xi|^2}{C(t-\tau)}\right)$$

for all $t > \tau$ and $x,\xi \in D$. If $f \in L^2(D)$, then $u(x,t) = \Gamma^D_{\tau,t}f(x)$ belongs to

$$C([\tau,\infty),L^2(D)) \cap L^\infty([\tau,\infty),L^2(D)) \cap L^2([\tau,\infty),H^1(D)).$$

Moreover, we have the energy inequality

$$||u(\cdot,t)||_2^2 + 2\lambda\int_\tau^t \|\nabla u(\cdot,s)\|_2^2 \leq ||f||_2^2 \tag{5.3}$$

for all $t \geq \tau$, and $u(x,t)$ is also a weak solution to (5.2).

Proof. This is a well known result in the theory of parabolic equations, and its proof is easy. Suppose f is smooth with compact support in D, then $u(x,t)=\Gamma_{\tau,t}^{D}f(x)$ is a classical solution to the parabolic equation (5.1), so that

$$\sum_{i,j=1}^{n}\frac{\partial}{\partial x^{i}}\left(A^{ij}(x,t)\frac{\partial}{\partial x^{j}}u(x,t)\right)-\frac{\partial}{\partial t}u(x,t)=0$$

for all $x\in D$ and $t\geq\tau$. It follows that

$$-\int_{D}\sum_{i,j=1}^{n}A^{ij}(x,t)\frac{\partial}{\partial x^{i}}u(x,t)\frac{\partial}{\partial x^{j}}u(x,t)dx-\frac{1}{2}\frac{\partial}{\partial t}\int_{D}u(x,t)^{2}dx=0$$

for all $t>\tau$, and therefore we have the energy inequality

$$||u(\cdot,t)||_{2}^{2}+2\lambda\int_{\tau}^{t}\|\nabla u(\cdot,s)\|_{2}^{2}ds\leq||f||_{2}^{2}\tag{5.4}$$

for all $t>\tau$. From the energy inequality above, we deduce that for every $f\in L^{2}(D)$, $u(x,t)=\Gamma_{\tau,t}^{D}f(x)$ belongs to $L^{\infty}([\tau,\infty),L^{2}(D))$ and also to $L^{2}([\tau,\infty),H^{1}(D))$, and the energy inequality remains true. Therefore for any $\varphi(x,t)$ which is smooth with compact support in $D\times[\tau,\infty)$, we have

$$\int_{\tau}^{t}\int_{D}u(x,s)\frac{\partial}{\partial s}\varphi(x,s)\,dxds-\int_{\tau}^{t}\int_{D}\nabla\varphi(x,s)\cdot A(x,s)\nabla u(x,s)dxds+\int_{D}f(x)\varphi(x,\tau)dx=0,\tag{5.5}$$

which is true for smooth f with compact support, so that it remains true for $f\in L^{2}(D)$ by the energy inequality above. That is $u(x,t)=\Gamma_{\tau,t}^{D}f(x)$ is a weak solution with initial data $f\in L^{2}(D)$. ■

Next we prove a uniqueness theorem for weak solutions. To this end we need the following fact.

Lemma 5.2 *Let $T>\tau\geq0$. Then*

$$\|u(\cdot,T)\|_{2}^{2}-\|u(\cdot,0)\|_{2}^{2}=2\left\langle\frac{\partial}{\partial t}u,u\right\rangle_{L^{2}\left([\tau,T],H^{-1}(\mathbb{R}^{n})\right),L^{2}\left([\tau,T],H^{1}(\mathbb{R}^{n})\right)}\tag{5.6}$$

for any $u\in L^{2}\left([\tau,T],H^{1}(\mathbb{R}^{n})\right)$, such that $\frac{\partial}{\partial t}u\in L^{2}\left([\tau,T],H^{-1}(\mathbb{R}^{n})\right)$ and $u\in C\left([\tau,T],L^{2}(\mathbb{R}^{n})\right)$, where $\langle\cdot,\cdot\rangle_{W^{\star},W}$ denotes the pairing between a Banach space W and its dual Banach space W^{*}.

Proof. If $u(x,t)=\sum\varphi_{i}(x)\eta_{i}(t)$ where $\varphi_{i}\in H^{1}(\mathbb{R}^{n})$ and η_{i} are smooth with compact support in $[\tau,T]$, then

$$\int_{\mathbb{R}^{n}}u(x,t)^{2}dx=\sum_{i,j}\eta_{i}(t)\eta_{j}(t)\int_{\mathbb{R}^{n}}\varphi_{i}(x)\varphi_{j}(x)dx$$

so that

$$\frac{d}{dt}\int_{\mathbb{R}^{n}}u(x,t)^{2}dx=2\sum_{i,j}\eta_{i}'(t)\eta_{j}(t)\int_{\mathbb{R}^{n}}\varphi_{i}(x)\varphi_{j}(x)dx,$$

and therefore

$$\begin{aligned}\int_{\tau}^{T}\frac{d}{dt}\int_{\mathbb{R}^{n}}u(x,t)^{2}dxdt&=2\int_{\tau}^{T}\sum_{i,j}\eta_{i}'(t)\eta_{j}(t)\int_{\mathbb{R}^{n}}\varphi_{i}(x)\varphi_{j}(x)dxdt\\&=2\left\langle\frac{\partial}{\partial t}u,u\right\rangle_{L^{2}\left([\tau,T],H^{-1}(\mathbb{R}^{n})\right),L^{2}\left([\tau,T],H^{1}(\mathbb{R}^{n})\right)}.\end{aligned}$$

Hence

$$\|u(\cdot,T)\|_2^2-\|u(\cdot,\tau)\|_2^2=2\left\langle\frac{\partial}{\partial t}u,u\right\rangle_{L^2\left([\tau,T],H^{-1}(\mathbb{R}^n)\right),L^2\left([\tau,T],H^1(\mathbb{R}^n)\right)}.$$

The above remains true for any $u\in L^2\left([\tau,T),H^1(\mathbb{R}^n)\right)$, such that $\frac{\partial}{\partial t}u\in L^2\left([\tau,T),H^{-1}(\mathbb{R}^n)\right)$ and $u\in C\left([\tau,T],L^2(\mathbb{R}^n)\right)$, by the density property. ■

We are now in a position to show the following uniqueness theorem for weak solutions.

Theorem 5.3 *Suppose $A=a+b$ satisfies conditions stated at the beginning of the section, i.e. $\lambda\le\left(a^{ij}(x,t)\right)\le\lambda^{-1}$ and $\|b\|_{L^\infty(BMO)}<\infty$. Let $\tau\ge0$. Suppose $u(x,t)\in C([\tau,\infty),L^2(\mathbb{R}^n))\cap L^2([\tau,\infty),H^1(\mathbb{R}^n))$, and satisfies*

$$\int_\tau^\infty\int_{\mathbb{R}^n}u(x,t)\frac{\partial}{\partial t}\varphi(x,t)\,dxdt=\int_\tau^\infty\int_{\mathbb{R}^n}\nabla\varphi(x,t)\cdot A(x,t)\nabla u(x,t)\,dxdt \tag{5.7}$$

for any $\varphi(x,t)$ is smooth with compact support in $\mathbb{R}^n\times(\tau,\infty)$, then

$$\frac{\partial u}{\partial t}\in L^2\left([\tau,\infty),H^{-1}(\mathbb{R}^n)\right). \tag{5.8}$$

Hence the following energy inequality holds:

$$\|u(\cdot,T)\|_2^2+2\lambda\int_\tau^T\int_{\mathbb{R}^n}|\nabla u(x,t)|^2\,dxdt\le\|u(\cdot,\tau)\|_2^2 \tag{5.9}$$

for every $T>\tau$, and the uniqueness of weak solutions holds for the initial problem of (5.1) in space $C([\tau,\infty),L^2(\mathbb{R}^n))\cap L^2([\tau,\infty),H^1(\mathbb{R}^n))$.

Proof. Consider the linear functional

$$F_t(\psi)=\int_{\mathbb{R}^n}(A(x,t)\nabla u(x,t))\cdot\nabla\psi(x)\,dx$$

for $\psi\in H^1(\mathbb{R}^n)$. By using the compensated compactness inequality(3.5) we have

$$F_t(\psi)\le\left(\|a\|_{L^\infty(\mathbb{R}^n\times[0,\infty))}+C\|b\|_{L^\infty(\mathrm{BMO})}\right)\|\nabla u(\cdot,t)\|_{L^2(\mathbb{R}^n)}\|\nabla\psi\|_{L^2(\mathbb{R}^n)} \tag{5.10}$$

for any $\psi\in H^1(\mathbb{R}^n)$. Hence by Riesz' representation theorem, there exists a unique $w(\cdot,t)\in H^1(\mathbb{R}^n)$ for every t such that

$$F_t(\psi)=\int_{\mathbb{R}^n}\left(\nabla w(x,t)\cdot\nabla\psi(x)+w(x,t)\psi(x)\right)\,dx, \tag{5.11}$$

where

$$\|w(\cdot,t)\|_{H^1(\mathbb{R}^n)}\le\left(\|a\|_{L^\infty(\mathbb{R}^n\times[0,\infty))}+C\|b\|_{L^\infty(\mathrm{BMO})}\right)\|\nabla u(\cdot,t)\|_{L^2(\mathbb{R}^n)}$$

which implies that $w\in L^2([\tau,\infty),H^1(\mathbb{R}^n))$.

In terms of $w(x,t)$, (5.7) becomes

$$\int_\tau^T\int_{\mathbb{R}^n}u(x,t)\varphi(x)\eta'(t)\,dxdt=\int_\tau^T\int_{\mathbb{R}^n}(\nabla w(x,t)\cdot\nabla\varphi(x)+w(x,t)\varphi(x))\,\eta(t)\,dxdt \tag{5.12}$$

for any $\eta\in C_0^\infty((\tau,T))$ and $\varphi\in C_0^\infty(\mathbb{R}^n)$, which can be written as

$$\int_\tau^T\langle u(\cdot,t),\varphi\rangle_{L^2}\,\eta'(t)dt=\int_\tau^T\langle w(\cdot,t),\varphi\rangle_{H^1}\,\eta(t)dt$$

and can be extended to any $\varphi \in H^1(\mathbb{R}^n)$. Since

$$\left|\int_\tau^T \langle w(\cdot,t),\varphi\rangle_{H^1}\eta(t)dt\right| \le \int_\tau^T \|w(\cdot,t)\|_{H^1}\|\varphi\|_{H^1}\eta(t)dt$$
$$= \|\varphi\|_{H^1}\int_\tau^T \|w(\cdot,t)\|_{H^1}\eta(t)dt$$
$$\le \|\varphi\|_{H^1}\sqrt{\int_\tau^T \|w(\cdot,t)\|_{H^1}^2 dt}\,\|\eta\|_{L^2([\tau,T))},$$

we obtain

$$\left|\int_\tau^T \langle u(\cdot,t),\varphi\rangle_{L^2}\eta'(t)dt\right| \le \|\varphi\|_{H^1}\sqrt{\int_\tau^T \|w(\cdot,t)\|_{H^1}^2 dt}\,\|\eta\|_{L^2([\tau,T))}$$

which implies that

$$\frac{d}{dt}\langle u(\cdot,t),\varphi\rangle_{L^2} \in L^2([\tau,T])$$

for every $\varphi \in H^1(\mathbb{R}^n)$. Moreover, according to F. Riesz' representation

$$\left\|\frac{d}{dt}\langle u(\cdot,t),\varphi\rangle_{L^2}\right\|_{L^2[\tau,T]} \le \|\varphi\|_{H^1}\sqrt{\int_\tau^\infty \|w(\cdot,t)\|_{H^1}^2 dt}$$

for any $\varphi \in H^1(\mathbb{R}^n)$. Therefore, there is $\frac{\partial}{\partial t}u \in L^2\left([\tau,T],H^{-1}(\mathbb{R}^n)\right)$ such that

$$\int_\tau^T \left\langle \frac{\partial}{\partial t}u(\cdot,t),\varphi\right\rangle_{H^{-1},H^1}\eta(t)dt = -\int_\tau^T \langle u(\cdot,t),\varphi\rangle_{L^2}\eta'(t)dt$$

for every $\varphi \in H^1(\mathbb{R}^n)$ and $\eta \in C_0^\infty(\tau,T)$. The above equality can be written as

$$\left\langle \frac{\partial}{\partial t}u,\varphi\otimes\eta\right\rangle = -\int_\tau^T\int_{\mathbb{R}^n} u(x,t)\frac{\partial}{\partial t}(\varphi(x)\eta(t))\,dxdt$$
$$= -\int_\tau^T\int_{\mathbb{R}^n} \nabla(\varphi(x)\eta(t))\cdot A(x,t)\nabla u(x,t)\,dxdt$$

where and in the remaining part of the proof, for simplicity, we use $\langle\cdot,\cdot\rangle$ to denote the pairing between $L^2\left([\tau,T),H^1(\mathbb{R}^n)\right)$ and its dual space $L^2\left([\tau,T],H^{-1}(\mathbb{R}^n)\right)$. Since

$$\operatorname{span}\left\{\varphi\otimes\eta : \varphi\in H^1(\mathbb{R}^n) \text{ and } \eta\in C_0^\infty(\tau,T)\right\}$$

is dense in $L^2\left([\tau,T],H^1(\mathbb{R}^n)\right)$, we have

$$\left\langle \frac{\partial}{\partial t}u,\psi\right\rangle = -\int_\tau^T\int_{\mathbb{R}^n}\nabla\psi(x,t)\cdot A(x,t)\nabla u(x,t)\,dxdt$$

for any $\psi \in L^2\left([\tau,T],H^1(\mathbb{R}^n)\right)$. In particular,

$$\left\langle \frac{\partial}{\partial t}u,u\right\rangle = -\int_\tau^T\int_{\mathbb{R}^n}\nabla u(x,t)\cdot A(x,t)\nabla u(x,t)\,dxdt$$
$$\le -\lambda\int_\tau^T\int_{\mathbb{R}^n}|\nabla u(x,t)|^2 dxdt.$$

Now, by combining with Lemma 5.2, we deduce that

$$\|u(\cdot,T)\|_2^2 - \|u(\cdot,0)\|_2^2 = 2\left\langle \frac{\partial}{\partial t}u, u \right\rangle \le -2\lambda \int_\tau^T \int_{\mathbb{R}^n} |\nabla u(x,t)|^2 dxdt$$

which in turn yields the energy inequality (5.9). The other conclusions of the theorem follow easily. ■

Remark 5.4 *1) The same results has been proved by H. Osada [28] when A^{ij} are uniformly bounded. In fact the result is classical if A^{ij} are bounded (see for example Theorem 6.2. on page 102, G. M. Lieberman [18]).*

2) $u \in L^2([\tau,\infty), H^1(\mathbb{R}^n))$ and $\frac{\partial u}{\partial t} \in L^2([\tau,\infty), H^{-1}(\mathbb{R}^n))$ actually implies that $u \in C([\tau,\infty), L^2(\mathbb{R}^n))$ with possibly modification on a measure zero subset of $[\tau,\infty)$. Therefore we have proved the uniqueness of weak solution u in space $L^\infty([\tau,\infty), L^2(\mathbb{R}^n)) \cap L^2([\tau,\infty), H^1(\mathbb{R}^n))$.

We are now in a position to state and to prove the following theorem.

Theorem 5.5 *Suppose $(A^{ij}) = (a^{ij}) + (b^{ij})$, where a and b are symmetric and skew-symmetric parts of A respectively, is uniformly elliptic: $\lambda \le a(x,t) \le \lambda^{-1}$ in matrix sense for some constant $\lambda > 0$, and $\|b\|_{L^\infty(BMO)} < \infty$. Then there is a unique positive function $\Gamma(x,t;\xi,\tau)$ defined for $t > \tau \ge 0$ and $x, \xi \in \mathbb{R}^n$, which possesses the following properties.*

1) Γ is a Markov transition density: $\Gamma(x,t;\xi,\tau) > 0$,

$$\int_{\mathbb{R}^n} \Gamma(x,t;\xi,\tau)d\xi = 1 \quad \text{and} \quad \int_{\mathbb{R}^n} \Gamma(x,t;\xi,\tau)dx = 1$$

for any $t > \tau \ge 0$, and

$$\Gamma(x,t;\xi,\tau) = \int_{\mathbb{R}^n} \Gamma(x,t;z,s)\Gamma(z,s;\xi,\tau)dz$$

for any $t > s > \tau \ge 0$.

2) There is a constant $M > 0$ depending only on n, λ and $\|b\|_{L^\infty(BMO)}$ such that

$$\frac{1}{M(t-\tau)^{n/2}} \exp\left(-\frac{M|x-\xi|^2}{t}\right) \le \Gamma(x,t;\xi,\tau) \le \frac{M}{(t-\tau)^{n/2}} \exp\left(-\frac{|x-\xi|^2}{Mt}\right)$$

for all $t > \tau$.

3) For every $f \in L^2(\mathbb{R}^n)$, $u(x,t) = \int_{\mathbb{R}^n} f(\xi)\Gamma(x,t;\xi,\tau)d\xi$ (for any $t \ge \tau$) is the unique weak solution with initial data f, which belongs to

$$C([\tau,\infty), L^2(\mathbb{R}^n)) \cap L^\infty([\tau,\infty), L^2(\mathbb{R}^n)) \cap L^2([\tau,\infty), H^1(\mathbb{R}^n)).$$

Proof. Since $b \in L^\infty(\mathrm{BMO}(\mathbb{R}^n))$, we can choose a $\varepsilon > 0$ such that

$$\|\varepsilon \log(|x|)\|_{\mathrm{BMO}} \le \|b\|_{L^\infty(\mathrm{BMO})}.$$

Define

$$U^{(m)}(x) = (-\varepsilon\log(|x|) + m) \wedge m \vee 0, \qquad L^{(m)}(x) = (\varepsilon\log(|x|) - m) \wedge 0 \vee (-m) \tag{5.13}$$

which are compactly supported BMO functions with

$$\|U^{(m)}\|_{\mathrm{BMO}} = \|L^{(m)}\|_{\mathrm{BMO}} \le C\|b\|_{L^\infty(\mathrm{BMO})},$$

where constant $C > 0$ depends only on the dimension n. Let

$$b^{(m)}(x,t) = b(x,t) \wedge U^{(m)}(x) \vee L^{(m)}(x). \tag{5.14}$$

By Proposition 3.3, we can mollify it to define $b^{(m)}_{\frac{1}{m}}$. Then, there is a C independent of b and m, such that $\|b^{(m)}_{\frac{1}{m}}\|_{L^\infty(\mathrm{BMO})} \le C\|b\|_{L^\infty(\mathrm{BMO})}$. Each $b^{(m)}_{\frac{1}{m}}$ is smooth with compact support and $b^{(m)}_{\frac{1}{m}} \to b$ in $L^p_{loc}([0,\infty)\times\mathbb{R}^n)$ for any $1 \le p < \infty$. For simplicity denote $b^{(m)}_{\frac{1}{m}}$ by b_m. Similarly a_m denotes the mollified one of a for $m = 1,2,\cdots$. $a_m(x,t)$ and $b_m(x,t)$ are smooth, bounded and have bounded derivatives of all orders, and $a_m \to a$ and $b_m \to b$ in $L^p_{loc}([0,\infty)\times\mathbb{R}^n)$ for every $p \in [1,\infty)$.

Now for each $A_m(x,t) = a_m(x,t) + b_m(x,t)$, a_m is uniformly elliptic with elliptic constant 2λ and

$$\|b_m\|_{L^\infty(\mathrm{BMO})} \le C\|b\|_{L^\infty(\mathrm{BMO})}$$

for some constant depending only on the dimension n, thus there is a unique fundamental solution $\Gamma^m(x,t;\xi,\tau)$ which satisfies the Aronson estimate with the same constant. According to Theorem 2.2, $\Gamma^m(x,t;\xi,\tau)$ are Hölder continuous in any compact sub-set of $t > \tau \ge 0$ and $x,\xi \in \mathbb{R}^n$ with the same Hölder exponent and the same Hölder constant for all $m = 1,2,\cdots$. Therefore by the Arzela-Ascoli Theorem, there is a sub-sequence of Γ^m, for simplicity the sub-sequence is still denoted by Γ^m, which converges locally uniformly to some $\Gamma(x,t;\xi,\tau)$ for $t > \tau \ge 0$ and $x,\xi \in \mathbb{R}^n$. Clearly $\Gamma(x,t;\xi,\tau)$ still satisfies 1) and 2).

We now prove 3). By our construction, if $\tau > 0$ and $f \in L^2(\mathbb{R}^n)$,

$$u^m(x,t) = \Gamma^m_{\tau,t}f(x) \to u(x,t) = \Gamma_{\tau,t}f(x)$$

point-wisely. According to Lemma 5.1, u^m (actually $u^m(x,t)$ is Hölder continuous too in $t > \tau$ and x) is a strong solution to the Cauchy problem of the parabolic equation associated with the diffusion matrix A_m, so that the energy inequality holds:

$$||u^m(\cdot,t)||_2^2 + \lambda \int_\tau^t \|\nabla u^m(\cdot,s)\|_2^2 \le ||f||_2^2, \tag{5.15}$$

which implies that $\{u^m\}$ is uniformly bounded in $L^2([\tau,\infty),H^1(\mathbb{R}^n))$. Hence there is a sub-sequence converges weakly, whose limit must be u, and u also satisfies the energy inequality above. In particular

$$u \in C\left([\tau,\infty),L^2(\mathbb{R}^n)\right) \cap L^\infty([\tau,\infty),L^2(\mathbb{R}^n)) \cap L^2([s,\infty),H^1(\mathbb{R}^n)).$$

For each m, we have

$$\int_\tau^\infty \int_{\mathbb{R}^n} u^m(x,t)\frac{\partial}{\partial t}\varphi(x,t)\,dxdt - \int_\tau^\infty \int_{\mathbb{R}^n} \nabla\varphi(x,t)\cdot A_m(x,t)\nabla u^m(\tau,x))\,dxdt = 0$$

for any $\varphi \in C_0^\infty((\tau,\infty)\times\mathbb{R}^n)$. Since $A_m \to A$ in $L^p_{loc}([s,\infty)\times\mathbb{R}^n)$ for any $1 \le p < \infty$ and $u_m \to u$ weakly in $L^2([\tau,\infty),H^1(\mathbb{R}^n))$. By letting $m \to \infty$ in the equality above, we obtain that

$$\int_\tau^\infty \int_{\mathbb{R}^n} u(x,t)\frac{\partial}{\partial t}\varphi(x,t)\,dxdt - \int_\tau^\infty \int_{\mathbb{R}^n} \nabla\varphi(x,t)\cdot A(x,t)\nabla u(x,t))\,dxdt = 0$$

for any $\varphi \in C_0^\infty((\tau,\infty)\times\mathbb{R}^n)$. That is, u is a weak solution to the Cauchy problem of the parabolic equation (5.1) with initial data f. Now according to Lemma 5.3, $\frac{\partial u}{\partial t} \in L^2([\tau,\infty),H^{-1}(\mathbb{R}^n))$ and therefore, by Theorem 5.5,

$$||u(\cdot,t)||_2^2 + \lambda \int_\tau^t \|\nabla u(\cdot,s)\|_2^2 \le ||f||_2^2. \tag{5.16}$$

The uniqueness of the fundamental solution Γ follows from the energy inequality easily. In fact, suppose there is another sub-sequence of Γ^m converges to $\tilde{\Gamma}$. Then $\tilde{u}(x,t) = \tilde{\Gamma_{\tau,t}} f(x)$ satisfies all the results above. Especially they both satisfy the energy inequality. Therefore $w = u - \tilde{u}$ also satisfies the previous integral equation, and by Theorem 5.5, we deduce that

$$\int_{\mathbb{R}^n} w(x,t)^2 \, dx + \lambda \int_\tau^t \int_{\mathbb{R}^n} |\nabla w(x,t)|^2 \, dxdt \leq 0$$

for any $t > \tau$ and we have $w = 0$. This implies $u = \tilde{u}$ and hence $\Gamma = \tilde{\Gamma}$. The proof is complete. ■

References

[1] D. G. Aronson: Uniqueness of positive solutions of second order parabolic equations, *Ann. Polon. Math.*, 16 (1965).

[2] D. G. Aronson: Bounds for the fundamental solution of a parabolic equation, *Bull. Am. Math. Soc*. 73, 890-896 (1967).

[3] D. G. Aronson: Non-negative solutions of linear parabolic equations, *Ann. Sci. Norm. sup. Pisa* (3) 22, 607-94 (1968).

[4] R. Coifman, P.-L. Lions, Y. Meyer and S. Semmes: Compensated compactness and Hardy spaces, *J. Math. Pures et Appl.,* 72, 247-286 (1992).

[5] R. Coifman, G. Weiss: Analyse harmonique non-commutative sur certains sepaces homogènes, *Lecture Notes in Math.*, Springer-Verlag (1971).

[6] E. B. Davies: Explicit Constants for Gaussian Upper Bounds on Heat Kernels, *American J. of Mathematics*, Vol. 109, No. 2 (Apr., 1987), pp. 319-333.

[7] E. B. Davies: *Heat Kernels and Spectral Theory*, Cambridge Tracts in Math. 92, Cambridge University Press (1989).

[8] E. de Giorgi: Sulla differenziabilita e l'analicita delle estremali degli integrali multipli regolari, *Mem. Accad. Sci. Toino. Cl. Sci. Mat. Nat.,* Ser. 3, 3 (1957).

[9] E. B. Fabes and D. W. Stroock: A New Proof of Moser's Parabolic Harnack Inequality Using the old Ideas of Nash, *Arch. for Ratl. Mech. and Anal.* 96, no. 4, 327-338 (1986).

[10] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, INC. (1964).

[11] M. Fukushima: *Dirichlet Forms and Markov Processes*, North-Holland Pub. Amsterdam, Oxford, New York (1980).

[12] A. A. Grigor'yan: The heat equation on noncompact Riemannian manifolds, *Math. USSR Sbornik* Vol. 72, No.1, 47-77 (1992).

[13] S. Hofmann and S. Mayboroda: Hardy and BMO spaces associated to divergence form elliptic operators, arXiv:math/0611804v2 (2007).

[14] F. John, L. Nirenberg: On functions of bounded mean oscillation, *Comm. Pure Appl. Math.* 14, 785-799 (1961).

[15] O. A. Ladyženskaja, V. A. Solonnikov, N. N. Uralćeva: *Linear and Quasilinear Equations of Parabolic Type*, Translations of Math. Monographs 23, AMS Providence RI (1968).

[16] P. G. Lemarié-Rieusset: *The Navier-Stokes Problem in the 21st Century*, CRC Press (2016).

[17] J. Leray: Sur le mouvement d'un liquide visqueux emplissant l'espace, *Acta Math*. 63, 193 - 248 (1934).

[18] G. M. Lieberman: *Second Order Parabolic Differential Equations*, World Scientific (1996).

[19] A. S. Monin, A. M. Yaglom: *Statistical Fluid Mechanics: Mechanics of Turbulence* Volume 1, The MIT Press (1971).

[20] A. S. Monin, A. M. Yaglom: *Statistical Fluid Mechanics: Mechanics of Turbulence* Volume 2, The MIT Press (1975).

[21] C. B. Morrey, Jr.: *Multiple Integrals in the Calculus of Variations,* Springer-Verlag New York Inc. (1966).

[22] J. Moser: A new proof of de Giorgi's theorem cocerning the regularity problem for elliptic differential equations, *Comm. Pure Appl. Math.*, Vol XIII, 457-469 (1960).

[23] J. Moser: A Harnack inequality for parabolic differential equations, *Comm. Pure Appl. Math.*, Vol XVII, 101-134 (1964).

[24] J. Moser: A Harnack inequality for parabolic differential equations, *Comm. Pure Appl. Math*., Vol. 17, 1964, pp. 101-134, and correction in Comm. Pure Appl. Math., Vol. 20, 1967, pp. 231-236.

[25] J. Moser: On a Pointwise Estimate for Parabolic Differential Equations, *Comm. Pure Appl. Math.*, VOL. XXIV, 727-740 (1971).

[26] J. Nash: Continuity of Solutions of Parabolic and Elliptic Equations, *American J. of Mathematics*, Vol. 80, No. 4. (Oct., 1958), pp. 931-954.

[27] J. R. Norris and D. W. Stroock: Estimates on the fundamental solution to heat flows with uniformly elliptic coefficients, *Proc. London. Math. Soc*. (3) 62 (1991), 373-402.

[28] H. Osada: Diffusion processes with generators of generalized divergence form, *J. Math. Kyoto Univ.* (JMKYAZ), 27-4 (1987) 597-619.

[29] G. Seregin, L. Silvestre, V. Šverák, A. Zlatoš: On divergence-free drifts, *J. Differential Equations* 252 (2012) 505–540.

[30] E. M. Stein: *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, New Jersey (1970).

[31] D. W. Stroock: Diffusion semigroups corresponding to uniformly elliptic divergence form operators, *Seminaire de Probabilites*, tome 22 (1988), p. 316 - 347.

[32] D. W. Stroock: *Partial Differential Equations for Probabilists*, Cambridge University Press (2008).

[33] D. W. Stroock and S. R. S. Varadhan: *Multidimensional Diffusion Processes*, Grundlehren Series 233, Springer-Verlag, New York and Heidelberg (1979).

[34] G. I. Taylor: Diffusion by continuous movements, *Proc. London Math. Soc.* 20, 196 - 211 (1921).

[35] G. I. Taylor: Statistical theory of turbulence, *Proc. London Math. Soc.* A 151, 421 - 478 (1935).

[36] W. von Wahl: *The Equations of Navier-Stokes and Abstract Parabolic Equations*, Aspects of Mathematics, Friedr. Vieweg & Sohn (1985).