# Mapping Incidences 

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October 23, 2018


#### Abstract

We show that any finite set $S$ in a characteristic zero integral domain can be mapped to $\mathbb{Z} / p \mathbb{Z}$, for infinitely many primes $p$, preserving all algebraic incidences in $S$. This can be seen as a generalization of the well-known Freiman isomorphism lemma, which asserts that any finite subset of a torsion-free group can be mapped into $\mathbb{Z} / p \mathbb{Z}$, preserving all linear incidences.

As applications, we derive several combinatorial results (such as sum-product estimates) for a finite set in a characteristic zero integral domain. As $\mathbb{C}$ is a characteristic zero integral domain, this allows us to obtain new proofs for some recent results concerning finite sets of complex numbers, without relying on the topology of the plane.


## 1 Introduction

Many problems and results in arithmetic combinatorics deal with algebraic incidences in a finite set $S$. Classical examples are the Szemerédi-Trotter theorem, and sum-product estimates.

A well-studied situation is when $S$ is a subset of $\mathbb{Z} / p \mathbb{Z}$, the finite field with $p$ elements where $p$ is a large prime. In this case, the special structure of the field and powerful techniques such as discrete Fourier analysis provide many tools to attack these problems. These features are not available in other settings and it seems one needs to invent new tricks. For example, when $S$ is a

[^0]subset of the complex numbers, most studies previous to this paper relied on some very clever use of properties of the plane. Thus, it seems desirable to have a tool that reduces a problem from a general setting to the special case of $\mathbb{Z} / p \mathbb{Z}$.

Such a tool exists, if one only cares about the linear relations among the elements of $S$. In this case, the famous Freiman isomorphism lemma (see, for example, [33, Lemma 5.25]) asserts that any finite subset of an arbitrary torsion-free group can be mapped into $\mathbb{Z} / p \mathbb{Z}$, given that $p$ is sufficiently large, preserving all additive (linear) relations in $S$. Thanks to this result, it has now become a common practice in additive combinatorics to translate additive problems in general torsion-free groups to corresponding problems in $\mathbb{Z} / p \mathbb{Z}$.

The goal of this paper is to show that the desired reduction is possible in general. Technically speaking, we prove that any finite set $S$ in a characteristic zero integral domain can be mapped to $\mathbb{Z} / p \mathbb{Z}$, for infinitely many primes $p$, preserving all algebraic incidences in $S$.

Some notable characteristic zero integral domains include the integers, the complex numbers, and the field of rational functions $\mathbb{C}\left(t_{1}, t_{2}, \ldots\right)$ in any number of formal variables $t_{i}$. As applications, we obtain some new results and short proofs of some known results. In particular, it is shown that sum-product estimates and bounds for incidence geometry problems over $\mathbb{Z} / p \mathbb{Z}$ imply the same bounds for the analogous problems over any characteristic zero integral domain (including the real and complex numbers).

Throughout this paper, we assume that all rings are commutative with identity 1 and that all ring homomorphisms take 1 to 1 . Let $D$ be a characteristic zero integral domain (so $D$ is a commutative ring with identity that has no zero divisors). We will identify the subring of $D$ generated by the identity with the integers $\mathbb{Z}$ (since the two are isomorphic). For a subset $S$ of $D$, we will use $\mathbb{Z}[S]$ to denote the smallest subring of $D$ containing $S$.

Theorem 1.1. Let $S$ be a finite subset of a characteristic zero integral domain $D$, and let $L$ be a finite set of non-zero elements in the subring $\mathbb{Z}[S]$ of $D$. There exists an infinite sequence of primes with positive relative density such that for each prime $p$ in the sequence, there is a ring homomorphism $\phi_{p}: \mathbb{Z}[S] \rightarrow \mathbb{Z} / p \mathbb{Z}$ satisfying $0 \notin \phi_{p}(L)$.

By positive relative density, we mean that the sequence has positive density in the sequence of all primes. It is important to note that Theorem 1.1 is not true for all primes. For example, if $S=\{i\} \subset \mathbb{C}$ and $L$ is arbitrary, then the desired map does not exist for $p=-1(\bmod 4)$, since the equation $x^{2}=-1$ is not solvable in $\mathbb{Z} / p \mathbb{Z}$ for these $p$. Note that for the applications of Theorem 1.1 in this paper, we only need that there exist infinitely many primes such that a map $\phi_{p}$ exists, which follows from those primes having positive relative density.

The role of $L$ in Theorem 1.1 is to guarantee that the homomorphism $\phi_{p}$ is injective on certain subsets of $\mathbb{Z}[S]$. Such injectivity is often necessary when applying Theorem 1.1, for example, if one were interested in the cardinality of $S$, one could guarantee that $\phi_{p}$ is injective on $S$ (and thus preserves the cardinality of $S$ ) by setting $L:=\left\{s_{1}-s_{2}: s_{1}, s_{2} \in S\right\}$.

Theorem 1.1 does not give upper bounds on the sizes of the smallest primes $p$ in the infinite sequence it produces. It would be an interesting question to study whether a version of Theorem 1.1 can be proven that includes, for example, an upper bound for at least one prime in the infinite sequence, where the bound would depend on both $S$ and $L$ (see Remark (7.2). Another interesting question is the following: Given a set $A \subset \mathbb{Z} / p \mathbb{Z}$, are there conditions on $A$ and $p$ (say, that $A$ is very small with respect to $p$ ) that allow one to construct a map that preserves algebraic incidences and that sends $A$ into some characteristic zero integral domain (for example, $\mathbb{Z}$ )?

Readers interested in the methods of the current paper may also be interested in the lecture by Serre [22] (posted on the Math ArXiv) titled "How to use finite fields for problems concerning infinite fields," which focuses on problems in algebraic geometry. An excellent discussion of Serre's lecture from a general mathematical viewpoint may be found on Terence Tao's blog [29], and Tao also mentions some relations between Serre's lecture and the current paper.

This paper is organized as follows. In the next few sections, we present a few sample applications of Theorem [1.1, Combining arguments from [2] with Theorem [1.1, we prove a Szemerédi-Trottertype result in Section 2, In Section 3, we use Theorem 1.1 to demonstrate a sum-product estimate for characteristic zero integral domains, based on well-known sum-product estimates in $\mathbb{Z} / p \mathbb{Z}$. Section 4 is focused on combining a product result for $\mathrm{SL}_{2}(\mathbb{Z} / p \mathbb{Z})$ from [16] with Theorem 1.1 to get an analogous product result for $\mathrm{SL}_{2}(D)$, where $D$ is a characteristic zero integral domain. In Section 5, we show that a random matrix taking finitely many values in a characteristic zero integral domain is singular with exponentially small probability. This extends earlier results on integer matrices to the complex setting. Finally, the proof of Theorem 1.1 is given in Section 7.

## 2 A Szemerédi-Trotter-type result for characteristic zero integral domains

In this section, we apply Theorem 1.1 to the problem of bounding the maximum number of incidences between a finite set of lines and a finite set of points. The well-known Szemerédi-Trotter Theorem [28] solves this problem in the case of points and lines in $\mathbb{R} \times \mathbb{R}$. Recently, in [2], an analogous result was proven for $\mathbb{Z} / q \mathbb{Z} \times \mathbb{Z} / q \mathbb{Z}$ where $q$ is a prime.
Theorem 2.1 ((Theorem 6.2 in [2])). Let $q$ be a prime, and let $\mathcal{P}$ and $\mathcal{L}$ be sets of points and lines, respectively, in $\mathbb{Z} / q \mathbb{Z} \times \mathbb{Z} / q \mathbb{Z}$ such that the cardinalities $|\mathcal{P}|,|\mathcal{L}| \leq N \leq q$. Then there exist positive absolute constants $c$ and $\delta$ such that

$$
\begin{equation*}
|\{(p, \ell) \in \mathcal{P} \times \mathcal{L}: p \in \ell\}| \leq c N^{3 / 2-\delta} . \tag{1}
\end{equation*}
$$

Remark 2.2. The original version of Theorem[2.1]proven in [2] relied on the best known sum-product result at the time (also found in [2]), which worked only for subsets of $\mathbb{Z} / q \mathbb{Z}$ with cardinality between $q^{\alpha}$ and $q^{1-\alpha}$ for a constant $\alpha$. In particular, the proof in [2] assumed that Inequality (1) was false and used this assumption to construct a subset $A$ of $\mathbb{Z} / q \mathbb{Z}$ with cardinality $N^{1 / 2-C \delta}$, for some constant $C$, such that $\max \{|A+A|,|A A|\}$ was small, a contradiction of the sum-product estimate proven in [2]. Thus, the version of Theorem 2.1 in [2] required the additional assumption that $N=q^{\alpha}$ for a constant $\alpha$.

To prove Theorem 2.1 as stated above, one can simply replace the sum-product results in [2] by more recent estimates that apply for all subsets of $\mathbb{Z} / q \mathbb{Z}$ (for example, [3, 15, 19]).

In a general ring $R$, we define a line to be the set of solutions $(x, y)$ in $R \times R$ to an equation $y=m x+b$, where $m$ and $b$ are fixed elements of $R$. Using Theorem 1.1, we prove that the same bound as in Theorem 2.1 holds for an arbitrary characteristic zero integral domain:

Theorem 2.3. Let $D$ be a characteristic zero integral domain, and let $\mathcal{P}$ and $\mathcal{L}$ be sets of points and lines (respectively) in $D \times D$ with cardinalities $|\mathcal{P}|,|\mathcal{L}| \leq N$. Then there exist positive absolute constants $c$ and $\delta$ such that

$$
|\{(p, \ell) \in \mathcal{P} \times \mathcal{L}: p \in \ell\}| \leq c N^{3 / 2-\delta} .
$$

The constants $c$ and $\delta$ are the same as those in Theorem [2.1. Any improvement to Theorem 2.1, for example, better constants or giving a good bound when $\mathcal{P}$ and $\mathcal{L}$ have very different cardinalities, would also immediately translate to Theorem 2.3 above. In the case of $\mathbb{R} \times \mathbb{R}$, this theorem is true with $\delta$ being replaced with the optimal constant $1 / 6$ (by the Szemerédi-Trotter Theorem [28]).

Restricting to the case of complex numbers, Solymosi [24, Lemma 1] has proven a Szemerédi-Trotter-type result over $\mathbb{C}$ with $\delta=1 / 6$, under the additional assumption that the set of points form a Cartesian product in $\mathbb{C}^{2}$. Our result has a small $\delta$ but does not require this additional assumption. One would expect that $\delta=1 / 6$ holds without any additional assumptions, and indeed, a tight result appears in a paper on the Math ArXiv by Csaba D. Tóth [35].

We conjecture that one can set $\delta=1 / 6$ in $\mathbb{Z} / p \mathbb{Z}$ given that $N$ is sufficiently small compared to $p$. (This implies $\delta=1 / 6$ for the complex case.)

Proof of Theorem [2.3. Without loss of generality, assume that $|\mathcal{P}|=|\mathcal{L}|=N$, adding "dummy" points and lines if necessary. Say that $\mathcal{P}=\left\{\left(x_{i}, y_{i}\right): i=1, \ldots, N\right\}$, and, uniquely parameterizing a line $y=m x+b$ by the ordered pair $(m, b)$, say that $\mathcal{L}=\left\{\left(m_{i}, b_{i}\right): i=1, \ldots, N\right\}$. Let $S:=\bigcup_{i=1}^{N}\left\{x_{i}, y_{i}, m_{i}, b_{i}\right\}$, set

$$
\begin{aligned}
L_{0}:= & \left\{x_{i}-x_{j}: 1 \leq i<j \leq N\right\} \cup\left\{y_{i}-y_{j}: 1 \leq i<j \leq N\right\} \cup \\
& \cup\left\{m_{i}-m_{j}: 1 \leq i<j \leq N\right\} \cup\left\{b_{i}-b_{j}: 1 \leq i<j \leq N\right\},
\end{aligned}
$$

and let $L:=L_{0} \backslash\{0\}$. By Theorem 1.1, there exists a prime $q>N$ and a ring homomorphism $\phi_{q}: \mathbb{Z}[S] \rightarrow \mathbb{Z} / q \mathbb{Z}$ such that $0 \notin \phi_{q}(L)$. Define a map $\Phi_{q}: \mathbb{Z}[S] \times \mathbb{Z}[S] \rightarrow \mathbb{Z} / q \mathbb{Z} \times \mathbb{Z} / q \mathbb{Z}$ by $\Phi_{q}(a, b)=\left(\phi_{q}(a), \phi_{q}(b)\right)$. Because $0 \notin \phi_{q}(L)$, we know that $\left|\Phi_{q}(\mathcal{P})\right|=\left|\Phi_{q}(\mathcal{L})\right|=N$. Thus, by Theorem 2.1, there exist absolute constants $c$ and $\delta$ such that

$$
\left|\left\{\left(p^{\prime}, \ell^{\prime}\right) \in \Phi_{q}(\mathcal{P}) \times \Phi_{q}(\mathcal{L}): p^{\prime} \in \ell^{\prime}\right\}\right| \leq c N^{3 / 2-\delta} .
$$

Since $\phi_{q}$ is a ring homomorphism, the equation $y=m x+b$ implies that $\phi_{q}(y)=\phi_{q}(m x+b)=$ $\phi_{q}(m) \phi_{q}(x)+\phi_{q}(b)$; and thus,

$$
|\{(p, \ell) \in \mathcal{P} \times \mathcal{L}: p \in \ell\}| \leq\left|\left\{\left(p^{\prime}, \ell^{\prime}\right) \in \Phi_{q}(\mathcal{P}) \times \Phi_{q}(\mathcal{L}): p^{\prime} \in \ell^{\prime}\right\}\right| \leq c N^{3 / 2-\delta}
$$

completing the proof.

## 3 A sum-product result for characteristic zero integral domains

Given a subset $A$ of a ring, we define $A+A:=\left\{a_{1}+a_{2}: a_{1}, a_{2} \in A\right\}$ and $A A:=\left\{a_{1} a_{2}: a_{1}, a_{2} \in A\right\}$. Heuristically, sum-product estimates state that one cannot find a subset $A$ such that both $A+A$ and $A A$ have small cardinality, unless $A$ is close to a subring. The first sum-product result was proven in 1983 by Erdős and Szemerédi [11] for the integers, and there have been numerous improvements and generalizations, see for example [20], [12], 10], and [5]. Proving sum-product estimates in $\mathbb{Z} / p \mathbb{Z}$, where $p$ is a prime, has been the focus of some recent work (see, for example, [2], [1], and [3), with the best known bound due to Katz and Shen [19], slightly improving a result of Garaev [15):
Theorem $3.1(([19]))$. Let $p$ be a prime and let $A$ be a subset of $\mathbb{Z} / p \mathbb{Z}$ such that $|A|<p^{1 / 2}$. Then, there exist absolute constants $C$ and $\alpha$ such that

$$
C|A|^{14 / 13}(\log |A|)^{\alpha} \leq \max \{|A+A|,|A A|\} .
$$

Theorem 3.2 demonstrates the same lower bound on $\max \{|A+A|,|A A|\}$ for any finite subset $A$ of a characteristic zero integral domain.

Theorem 3.2. There are positive absolute constants $C$ and $\alpha$ such that, for every finite subset $A$ of a characteristic zero integral domain,

$$
C|A|^{14 / 13}(\log |A|)^{\alpha} \leq \max \{|A+A|,|A A|\} .
$$

The constants $C$ and $\alpha$ in this result is the same as those in Theorem 3.1.
Theorem 3.2 applies to a very general class of rings; however, our mapping approach requires that the rings be commutative and have characteristic zero. For some results in the noncommutative case, see [5]; and for some results in $\mathbb{Z} / m \mathbb{Z}$ where $m$ is a composite, see [6].

Proof of Theorem 3.2. Because we are interested in a lower bound on $|A+A|$ and $|A A|$, all we need in order to apply Theorem 3.1 is a ring homomorphism $\phi$ from the given characteristic zero integral domain to $\mathbb{Z} / p \mathbb{Z}$ satisfying $|\phi(A)|=|A|$ (since any ring homomorphism automatically satisfies $|\phi(A)+\phi(A)|=|\phi(A+A)| \leq|A+A|$ and $|\phi(A) \phi(A)| \leq|A A|)$. However, Theorem 1.1 also makes it easy to find a ring homomorphism that preserves the cardinalities of $A+A$ and $A A$, as we will show below (such a map would be useful for proving upper bounds on $|A+A|$ and $|A A|$ ).

Let

$$
L_{0}:=\left\{a_{1}-a_{2}: a_{1}, a_{2} \in A\right\} \cup\left\{a_{1}+a_{2}-\left(a_{3}+a_{4}\right): a_{i} \in A\right\} \cup\left\{a_{1} a_{2}-a_{3} a_{4}: a_{i} \in A\right\}
$$

and let $L:=L_{0} \backslash\{0\}$.
By Theorem 1.1, there exists a prime $p>|A|^{2}$ and a ring homomorphism $\phi_{p}: \mathbb{Z}[A] \rightarrow \mathbb{Z} / p \mathbb{Z}$ such that

$$
\begin{align*}
& \left|\phi_{p}(A)\right|=|A|,  \tag{i}\\
& \left|\phi_{p}(A)+\phi_{p}(A)\right|=|A+A|, \text { and }  \tag{ii}\\
& \left|\phi_{p}(A) \phi_{p}(A)\right|=|A A| . \tag{iii}
\end{align*}
$$

All three facts above follow from the definition of a ring homomorphism, along with the definition of $L$ and the fact that $0 \notin \phi_{p}(L)$. We can now apply Theorem 3.1 to get that there exist positive constants $C$ and $\alpha$ such that

$$
C\left|\phi_{p}(A)\right|^{14 / 13}(\log |A|)^{\alpha} \leq \max \left\{\left|\phi_{p}(A)+\phi_{p}(A)\right|,\left|\phi_{p}(A) \phi_{p}(A)\right|\right\} .
$$

Finally, substituting (i), (ii), and (iii) into this inequality gives the desired result.

## 4 A matrix product result for $S L_{2}(D)$

In this section, we will consider finite subsets of the special linear group $\mathrm{SL}_{2}(D)$ of 2 by 2 matrices with determinant 1 and entries in a characteristic zero integral domain $D$. For $A$ a finite subset of $\mathrm{SL}_{2}(D)$, let $\langle A\rangle$ denote the smallest subgroup of $\mathrm{SL}_{2}(D)$ (under inclusion) that contains $A$. We will refer to $\langle A\rangle$ as the group generated by $A$. In general, the goal of this section will be to give conditions on $\langle A\rangle$ so that cardinality of the triple product $A A A:=\left\{a_{1} a_{2} a_{3}: a_{i} \in A\right\}$ is large.

Helfgott proved the following theorem in [16]:

Theorem $4.1(([16]))$. Let $p$ be a prime. Let $A$ be a subset of $\mathrm{SL}_{2}(\mathbb{Z} / p \mathbb{Z})$ not contained in any proper subgroup, and assume that $|A|<p^{3-\epsilon}$ for some fixed $\epsilon>0$. Then

$$
|A A A|>c|A|^{1+\delta}
$$

where $c>0$ and $\delta>0$ depend only on $\epsilon$.
In this section, we will prove the following related result by combining Theorem 4.1 with Theorem 1.1. A group $G$ is metabelian if $G$ has an abelian normal subgroup $N$ such that the quotient group $G / N$ is also abelian.

Theorem 4.2. Let $A$ be a finite subset of $\mathrm{SL}_{2}(D)$, where $D$ is a characteristic zero integral domain, and let $\langle A\rangle$ be the subgroup generated by $A$. If $\langle A\rangle$ has infinite cardinality and $\langle A\rangle$ is not metabelian, then

$$
|A A A|>c|A|^{1+\delta}
$$

where $c>0$ and $\delta>0$ are absolute constants.
One should note that Chang [7] has already proven a very similar product result for $\mathrm{SL}_{2}(\mathbb{C})$, in which "metabelian" is replaced by "virtually abelian". A group $G$ is virtually abelian if $G$ has a finite index subgroup $H$ such that $H$ is abelian.

Theorem $4.3(([7]))$. Let $A$ be a finite subset of $\mathrm{SL}_{2}(\mathbb{C})$, and let $\langle A\rangle$ be the subgroup generated by A. If $\langle A\rangle$ is not virtually abelian (which implies that $\langle A\rangle$ has infinite cardinality), then

$$
|A A A|>c|A|^{1+\delta}
$$

where $c>0$ and $\delta>0$ are absolute constants.
There are many groups that are both metabelian and virtually abelian, for example all abelian groups satisfy both properties. However, neither property implies the other. For example, the group $G:=\prod_{i=1}^{\infty} S_{3}$ (the product of infinitely many copies of the symmetric group on three elements) is metabelian (since $N:=\prod_{i=1}^{\infty}\langle(123)\rangle$ is an abelian, normal subgroup of $G$ such that $G / N$ is also abelian), but $G$ is not virtually abelian. On the other hand, $G:=S_{4} \times \mathbb{Z}$ is virtually abelian (since $H:=\langle(1)\rangle \times \mathbb{Z}$ is a finite-index abelian subgroup of $G$ ), but $G$ is not metabelian (since $S_{4}$ is not metabelian).

One major difference between Theorem 4.2 and Theorem 4.3 is in how the two results are proved. Below, we will prove Theorem 4.2 using Helfgott's Theorem 4.1 as a black box along with some group theory and an easy application of Theorem 1.1. On the other hand, Theorem 4.3 is proven in [7] by adapting Helfgott's methods in [16] from the case of $\mathrm{SL}_{2}(\mathbb{Z} / p \mathbb{Z})$ to $\mathrm{SL}(\mathbb{C})$ and using tools from additive combinatorics.

The constants $\delta>0$ in Theorems 4.2 and 4.3 are not the best possible if one restricts to a subgroup. For example, $\mathrm{SL}_{2}(\mathbb{Z})$ contains a subgroup isomorphic to $F_{2}$, the free group on 2 generators, and the following product result has recently been shown by Razborov [21]:

Theorem $4.4\left(([21))\right.$. Let $A$ be a finite subset of a free group $F_{m}$ (on $m$ generators) with at least two non-commuting elements. Then,

$$
|A A A| \geq \frac{|A|^{2}}{(\log |A|)^{O(1)}}
$$

One should note that neither Theorem 4.2 nor Theorem 4.3 fully characterizes finite subsets of $\mathrm{SL}_{2}(\mathbb{C})$ that have expanding triple product. For example, neither theorem applies when $A$ is contained in an abelian subgroup, but letting

$$
A:=\left\{\left(\begin{array}{cc}
1 & 2^{j} \\
0 & 1
\end{array}\right): 1 \leq j \leq n\right\}
$$

we have that $|A A A| \geq|A A|=\binom{n+1}{2}>n^{2} / 2=|A|^{2} / 2$. One should also note that a sum-product theorem similar to Theorem 3.2] does not hold in general for matrices. As pointed out in [8, Remark 0.2 ], the subset

$$
A:=\left\{\left(\begin{array}{ll}
1 & j \\
0 & 1
\end{array}\right): 1 \leq j \leq n\right\}
$$

has the property that both the sumset and product set are small: $|A+A|=|A A|=2 n-1$. However, it is also shown by Chang [8] that by adding the assumption that the matrices in $A$ are symmetric, one can prove a sum-product result similar to Theorem 3.2,

We now turn our attention to the proof of Theorem 4.2.
Proof of Theorem 4.2. Say that $A$ is a finite subset of $\mathrm{SL}_{2}(D)$, where $D$ is a characteristic zero integral domain. Let $G:=\langle A\rangle$, the subgroup generated by $A$, and assume that $G$ has infinite cardinality and is not metabelian. Let $T$ be the set of all normal subgroups $N$ of $G$ such that $G / N$ is abelian (note that we include $G$ in the set $T$ ), and define

$$
N_{0}:=\bigcap_{N \in T} N .
$$

Then $N_{0}$ is a normal subgroup of $G$ and $G / N_{0}$ is abelian. Since $G$ is not metabelian by assumption, we know that $N_{0}$ is not abelian, and so there exists $B_{1}, B_{2} \in N_{0}$ such that $B_{1} B_{2} \neq B_{2} B_{1}$. Also, let $M_{1}, M_{2}, M_{3}, \ldots, M_{121}$ be 121 distinct elements of $G$ (note $G$ is infinite by assumption). We may now define a set $L_{0}$ as follows:

$$
\begin{aligned}
L_{0}:= & \left\{b_{i}-c_{j}: \begin{array}{l}
i, j \in\{1,2,3,4\} \text { and } b_{i} \text { and } c_{j} \text { are entries in ma- } \\
\text { trices }\left(\begin{array}{ll}
b_{1} & b_{2} \\
b_{3} & b_{4}
\end{array}\right),\left(\begin{array}{ll}
c_{1} & c_{2} \\
c_{3} & c_{4}
\end{array}\right) \in A
\end{array}\right. \\
& \cup\left\{b_{i}-c_{j}:\left(\begin{array}{ll}
i, j \in\{1,2,3,4\} \text { and } b_{i} \text { and } c_{j} \text { are entries in matrices } \\
b_{3} & b_{4}
\end{array}\right) \in M_{k_{1}} \text { and }\left(\begin{array}{ll}
c_{1} & c_{2} \\
c_{3} & c_{4}
\end{array}\right) \in M_{k_{2}} \text { for some } 1 \leq k_{1}, k_{2} \leq 121\right\} \\
& \cup\left\{b_{1}-1, b_{2}, b_{3}, b_{4}-1: \text { where }\left(\begin{array}{ll}
b_{1} & b_{2} \\
b_{3} & b_{4}
\end{array}\right)=B_{1} B_{2} B_{1}^{-1} B_{2}^{-1} \neq\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right)\right\} .
\end{aligned}
$$

Let $L:=L_{0} \backslash\{0\}$, and let $S$ be the set of all entries that appear in matrices in $A$. By Theorem 1.1, there exists $p>|A|$ and $\phi_{p}: \mathbb{Z}[S] \rightarrow \mathbb{Z} / p \mathbb{Z}$ such that $0 \notin \phi_{p}(L)$. Let $\Phi_{p}: \mathrm{SL}_{2}(D) \rightarrow \mathrm{SL}_{2}(\mathbb{Z} / p \mathbb{Z})$ be defined by $\left(\begin{array}{ll}b_{1} & b_{2} \\ b_{3} & b_{4}\end{array}\right) \mapsto\left(\begin{array}{ll}\phi_{p}\left(b_{1}\right) & \phi_{p}\left(b_{2}\right) \\ \phi_{p}\left(b_{3}\right) & \phi_{p}\left(b_{4}\right)\end{array}\right)$. Let $\underline{A}:=\Phi_{p}(A)$ and let $\underline{G}:=\langle\underline{A}\rangle$. Note that by construction $|A|=|\underline{A}|$ and $|A A A| \geq|\underline{A} \underline{A} \underline{A}|$, and also note that $|\underline{G}| \geq 121$.

Assume for the sake of a contradiction that $\underline{G}$ is a proper subgroup of $\mathrm{SL}_{2}(\mathbb{Z} / p \mathbb{Z})$. In [27], Suzuki gives the following classification of the proper subgroups of $\mathrm{SL}_{2}(\mathbb{Z} / p \mathbb{Z})$ :

Theorem 4.5 ((cf. Theorem 6.17 of [27], page 404$)$ ). Let $\underline{G}$ be a proper subgroup of $\mathrm{SL}_{2}(\mathbb{Z} / p \mathbb{Z})$ where $p \geq 5$. Then $\underline{G}$ is isomorphic to one of the following groups (or to a subgroup of one of the following groups):
(i) a cyclic group,
(ii) the group with presentation $\left\langle x, y \mid x^{m}=y^{2}, y^{-1} x y=x^{-1}\right\rangle$, which has order $4 m$,
(iii) a group $H$ of order $p(p-1)$ having a Sylow-p subgroup $Q$ such that $H / Q$ is cyclic and $Q$ is elementary abelian,
(iv) the special linear group $\mathrm{SL}_{2}(\mathbb{Z} / 3 \mathbb{Z})$ on a field of three elements, which has order 24,
(v) $\widehat{S}_{4}$, the representation group of $S_{4}$ (the symmetric group on 4 letters), which has order 48, or
(vi) the special linear group $\mathrm{SL}_{2}(\mathbb{Z} / 5 \mathbb{Z})$ on a field of five elements, which has order 120.

Since $|\underline{G}|>120$, we may eliminate (iv), (v), and (vi) as possibilities. The remaining possibilities (namely, (i), (ii), and (iii)) are all metabelian; and thus, $\underline{G}$ must have a normal subgroup $\underline{N}$ such that $\underline{N}$ is abelian and $\underline{G} / \underline{N}$ is also abelian.

Let $N:=\Phi_{p}^{-1}(\underline{N})$. Then $N$ is a normal subgroup of $G$, and by the third isomorphism theorem $G / N \simeq\left(G / \operatorname{ker}\left(\Phi_{p}\right)\right) /\left(N / \operatorname{ker}\left(\Phi_{p}\right)\right) \simeq \underline{G} / \underline{N}$, which is abelian. Thus, $N_{0}$ is a subgroup of $N$, and so $B_{1}, B_{2} \in N$. We know that $B_{1} B_{2} B_{1}^{-1} B_{2}^{-1} \neq\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right)$, and by the definition of $\Phi_{p}$, we also have that

$$
\Phi_{p}\left(B_{1}\right) \Phi_{p}\left(B_{2}\right) \Phi_{p}\left(B_{1}\right)^{-1} \Phi_{p}\left(B_{2}\right)^{-1}=\Phi_{p}\left(B_{1} B_{2} B_{1}^{-1} B_{2}^{-1}\right) \neq\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right)
$$

But, this contradicts the fact that $\underline{N}$ is abelian. Thus, the assumption that $\underline{G}$ is a proper subgroup of $\mathrm{SL}_{2}(\mathbb{Z} / p \mathbb{Z})$ is false, and we have that $\langle\underline{A}\rangle=\underline{G}=\mathrm{SL}_{2}(\mathbb{Z} / p \mathbb{Z})$.

Finally, by Theorem 4.1, there exist absolute constants $c>0$ and $\delta>0$ such that

$$
|A A A| \geq|\underline{A} \underline{A} \underline{A}| \geq c|\underline{A}|^{1+\delta}=c|A|^{1+\delta}
$$

Another way to show that $\Phi_{p}(A)$ generates all of $\mathrm{SL}_{2}(\mathbb{Z} / p \mathbb{Z})$ would be to assume that $\langle A\rangle$ is not virtually solvable, which implies by Tits Alternative Theorem [34] that $\langle A\rangle$ has a nonabelian free subgroup. Then, following [14, Section 2], it is possible to bound the girth of a certain Cayley graph from below in terms of $p$, eventually showing (via an appeal to Theorem 4.5) that $\left\langle\Phi_{p}(A)\right\rangle=\mathrm{SL}_{2}(\mathbb{Z} / p \mathbb{Z})$.

Also, the proof above uses the following implicit corollary of Theorem4.5 if $\underline{G}$ is a proper subgroup of $\mathrm{SL}_{2}(\mathbb{Z} / p \mathbb{Z})$ and $|\underline{G}|>120$, then $\underline{G}$ in metabelian. A very similar result for $\mathrm{PSL}_{2}(\mathbb{Z} / p \mathbb{Z}) \simeq$ $\mathrm{SL}_{2}(\mathbb{Z} / p \mathbb{Z}) /( \pm I)$ (where $I$ is the identity matrix) appears in [9, Theorem 3.3.4, page 78].

## 5 Random matrices with entries in a characteristic zero integral domain

In [18, 30, it is shown that a random Bernoulli matrix of size $n$ is singular with probability $\exp (-\Omega(n))$. One may ask what happens for random matrices with complex entries. We are going to give a quick proof of the following:

Theorem 5.1. For every positive number $\rho<1$, there is a positive number $\delta<1$ such that the following holds. Let $\xi$ be a random variable with finite support in a characteristic zero integral domain, where $\xi$ takes each value with probability at most $\rho$. Let $M_{n}$ be an $n$ by $n$ random matrix whose entries are iid copies of $\xi$. Then the probability that $M_{n}$ is singular is at most $\delta^{n}$, for all $n$ sufficiently large with respect to $\rho$ and the size of the support of $\xi$.

Remark 5.2. In the case when the characteristic zero integral domain is $\mathbb{C}$, more quantitative bounds are available (see [4, 31]).

Theorem 5.1 follows directly from the following two results.
Theorem 5.3. For every positive number $\rho<1$, there is a positive number $\delta<1$ such that the following holds. Let $n$ be a large positive integer and $p \geq 2^{n^{n}}$ be a prime. Let $\xi$ be a random variable with finite support in $\mathbb{Z} / p \mathbb{Z}$, where $\xi$ takes each value with probability at most $\rho$. Let $M_{n}$ be an $n$ by $n$ random matrix whose entries are iid copies of $\xi$. Then the probability that $M_{n}$ is singular is at most $\delta^{n}$, for all $n$ sufficiently large with respect to $\rho$ and the size of the support of $\xi$.

This theorem was implicitly proved in 30. The bound $2^{n^{n}}$ is not essential, we simply want to guarantee that $p$ is much larger than $n$. The reason that the proof from [30] does not extend directly to the complex case (or characteristic zero integral domains in general) is that in [30] one relied on the identity

$$
\mathbf{I}_{x=0}=\int_{0}^{1} \exp (2 \pi i x t) d t
$$

where $\mathbf{I}$ is the indicator function. This identity holds for $x$ an integer, but it is not true for complex numbers in general. Theorem 1.1 provides a simple way to overcome this obstacle. (For other methods, see [31, 32.)

Lemma 5.4. Let $S$ be a finite subset of a characteristic zero integral domain. There exist arbitrarily large primes $p$ such that there is a ring homomorphism $\phi_{p}: \mathbb{Z}[S] \rightarrow \mathbb{Z} / p \mathbb{Z}$ satisfying the following two properties:
(i) the map $\phi_{p}$ is injective on $S$, and
(ii) for any $n$ by $n$ matrix $\left(s_{i j}\right)$ with entries $s_{i j} \in S$, we have

$$
\operatorname{det}\left(s_{i j}\right)=0 \quad \text { if and only if } \quad \operatorname{det}\left(\phi_{p}\left(s_{i j}\right)\right)=0 .
$$

Proof. Let $L:=\left\{\operatorname{det}\left(s_{i j}\right): s_{i j} \in S\right\} \backslash\{0\}$. Applying Theorem 1.1 gives us a ring homomorphism $\phi_{p}: \mathbb{Z}[S] \rightarrow \mathbb{Z} / p \mathbb{Z}$ (for some arbitrarily large prime $p$ ) such that $0 \notin \phi_{p}(L)$. Since $\phi_{p}$ is a ring homomorphism, $\phi_{p}\left(\operatorname{det}\left(s_{i j}\right)\right)=\operatorname{det}\left(\phi_{p}\left(s_{i j}\right)\right)$ and also $\phi_{p}(0)=0$; thus, we have satisfied condition (ii).

In this particular case, we will show that (i) follows from (ii). If $S$ contains more than one element, we can find $s \neq t \neq 0$ both lying in $S$, and thus

$$
\operatorname{det}\left(\left(\begin{array}{ccccc}
s & t & \cdots & t & t \\
t & s & t & \cdots & t \\
\vdots & t & \ddots & t & \vdots \\
t & \cdots & t & s & t \\
t & t & \cdots & t & t
\end{array}\right)\right)=\operatorname{det}\left(\left(\begin{array}{ccccc}
s-t & 0 & \cdots & 0 & 0 \\
0 & s-t & 0 & \cdots & 0 \\
\vdots & 0 & \ddots & 0 & \vdots \\
0 & \cdots & 0 & s-t & 0 \\
0 & 0 & \cdots & 0 & t
\end{array}\right)\right)=(s-t)^{n-1} t \neq 0 .
$$

Thus, by (ii), $0 \neq\left(\phi_{p}(s)-\phi_{p}(t)\right)^{n-1} \phi_{p}(t)$, and so $\phi_{p}(s) \neq \phi_{p}(t)$ and we see that $\phi_{p}$ is injective on $S$.

The fact that (ii) happens to imply injectivity on $S$ is not important-in fact, for any given finite subset $A \subset \mathbb{Z}[S]$ we can find $\phi_{\widetilde{Q}}$ satisfying (ii) above that is also injective on $A$ by adding $\left\{a_{1}-a_{2}: a_{1} \neq a_{2}\right.$ and $\left.a_{1}, a_{2} \in A\right\}$ to $\widetilde{L}$ in the proof above. For example, we could find $\phi_{\widetilde{Q}}$ that is injective on the set of all determinants of $n$ by $n$ matrices with entries in $S$.

One should note that it is easy to prove results similar to Lemma 5.4 where the determinant is replaced by some polynomial $f\left(x_{1}, x_{2}, \ldots, x_{k}\right)$ with integer coefficients and one wants a map $\phi_{p}$ such that $f$ evaluated at points in $S$ is zero if and only if $f$ evaluated at points in $\phi_{p}(S)$ is zero. This can also easily be extended to the case where $f$ is replaced by a list of polynomials, each of which is evaluated on some subset of $S$.

## 6 The density theorem

The number 7 is a prime in the ring of integers $\mathbb{Z}$; however, if one extends $\mathbb{Z}$ to $\mathbb{Z}[\sqrt{2}]$, the prime 7 splits: $7=(3-\sqrt{2})(3+\sqrt{2})$. This fact has the same mathematical content as the following: the polynomial $x^{2}-2$ is irreducible in $\mathbb{Z}[x]$; however, in $(\mathbb{Z} / 7 \mathbb{Z})[x]$, where the coefficients of the polynomial are viewed as elements of $\mathbb{Z} / 7 \mathbb{Z}$, the polynomial splits: $x^{2}-2=(x-3)(x+3)$. The Frobenius Density Theorem describes how frequently such splitting occurs. In modern formulations, the Frobenius Density Theorem quantifies the proportion of primes that split in a given Galois extension of the rational numbers. We will use the following historical version given in [26, page 32], which is phrased in terms of polynomials splitting modulo $p$. Note that the relative density of a set of primes $S$ is defined to be

$$
\lim _{x \rightarrow \infty} \frac{|\{p \leq x: p \in S\}|}{\mid\{p \leq x: p \text { is prime }\} \mid} .
$$

Theorem 6.1 ((Frobenius Density Theorem)). Let $g(z) \in \mathbb{Z}[z]$ be a polynomial of degree $k$ with $k$ distinct roots in $\mathbb{C}$, and let $G$ be the Galois group of the polynomial $g$, viewed as a subgroup of $S_{k}$ (the symmetric group on $k$ symbols). Let $n_{1}, n_{2}, \ldots, n_{t}$ be positive integers summing to $k$. Then, the relative density of the set of primes $p$ for which $g$ modulo $p$ has a given decomposition type $n_{1}, n_{2}, \ldots, n_{t}$ exists and is equal to $1 /|G|$ times the number of $\sigma \in G$ with cycle pattern $n_{1}, n_{2}, \ldots, n_{t}$.

For example, since the identity element corresponds to the cycle pattern $1,1, \ldots, 1$ and every group has one identity, the relative density of primes $p$ such that $g$ decomposes into $k$ distinct linear factors modulo $p$ is $1 /|G|$.

Theorem 6.1 is the version proven by Frobenius in 1880 and published in 1896 [13]. In [26], Stevenhagen and Lenstra give numerous examples and an illuminating discussion of the original motivation for the Frobenius Density Theorem and how it relates to the stronger Chebotarev Density Theorem.

## 7 Proof of Theorem 1.1

The first step towards proving Theorem 1.1 is proving the following lemma.
Lemma 7.1. Let $S$ be a finite subset of a characteristic zero integral domain $D$, and let $L$ be a finite set of non-zero elements in the subring $\mathbb{Z}[S]$ of $D$. Then there exists a complex number $\theta$ that is algebraic over $\mathbb{Q}$ and a ring homomorphism $\phi: \mathbb{Z}[S] \rightarrow \mathbb{Z}[\theta]$ such that $0 \notin \phi(L)$.

By itself, this lemma allows one to extend sum-product and incidence problem results proven in the complex numbers to any characteristic zero integral domain (in much the same way that Theorem 1.1 allows one to extend such results proven in $\mathbb{Z} / p \mathbb{Z}$ to any characteristic zero integral domain).

Lemma 7.1 is proved using three main steps: applying the primitive element theorem, applying Hilbert's Nullstellensatz to pass to the case of only algebraic numbers, and applying the primitive element again to get to a ring of the form $\mathbb{Z}[\theta]$. Each of these three steps requires negotiating between the rings we are interested in and their fraction fields. Theorem 1.1 is proved by combining Lemma 7.1 with the Frobenius Density Theorem (or the stronger Chebotarev Density Theorem) to pass to a quotient isomorphic to $\mathbb{Z} / p \mathbb{Z}$.
Remark 7.2 (An effective version of Theorem 1.1). It would be interesting to prove a version of Theorem 1.1 that included an upper bound on at least one (or more) of the primes $p$ (in terms of $S$ and $L$ ) for which desired homomorphism $\phi_{p}$ exists. One possible program for proving such a result would be to follow the general outline of the proof of Theorem 1.1 given in this section, combined with effective versions of the primitive element theorem, Hilbert's Nullstellensatz, and the Chebotarev Density Theorem.

Proof of Lemma 7.1. Let $S$ be a finite subset of a characteristic zero integral domain $D$. Recall that we identify the subring of $D$ generated by the identity with $\mathbb{Z}$ and so we use $\mathbb{Z}[S]$ to denote the smallest subring of $D$ containing $S$.

We can write $S=\left\{x_{1}, x_{2}, \ldots, x_{j}, \theta_{1}, \theta_{2}, \ldots, \theta_{k}\right\}$, such that $\left\{x_{1}, x_{2}, \ldots, x_{j}\right\}$ are independent transcendentals over $\mathbb{Q}$ and such that $K$, the fraction field of $\mathbb{Z}[S]$, is algebraic over $\mathbb{Q}\left(x_{1}, x_{2}, \ldots, x_{j}\right)$. Using the primitive element theorem, we can find $\widetilde{\theta}$ in $K$ also algebraic over $\mathbb{Q}\left(x_{1}, x_{2}, \ldots, x_{j}\right)$ such that

$$
\mathbb{Q}\left(x_{1}, x_{2}, \ldots, x_{j}, \theta_{1}, \theta_{2}, \ldots, \theta_{k}\right)=\mathbb{Q}\left(x_{1}, x_{2}, \ldots, x_{j}, \widetilde{\theta}\right)
$$

To get the analogous statement for $\mathbb{Z}$ instead of $\mathbb{Q}$, we write, for each $i$

$$
\theta_{i}=\sum_{k} \frac{f_{i, k}}{g_{i, k}} \widetilde{\theta}^{k},
$$

where $f_{i, k}, g_{i, k} \in \mathbb{Z}\left[x_{1}, x_{2}, \ldots, x_{j}\right]$, and we then define $\theta_{0}$ to be $\widetilde{\theta}$ divided by the product of the $g_{i, k}$. Thus, we can find $\theta_{0}$ in $K$ also algebraic over $\mathbb{Q}\left(x_{1}, x_{2}, \ldots, x_{j}\right)$ such that

$$
\mathbb{Z}[S] \subset \mathbb{Z}\left[x_{1}, x_{2}, \ldots, x_{j}, \theta_{0}\right] \simeq \mathbb{Z}\left[y_{1}, y_{2}, \ldots, y_{j+1}\right] / f_{0}
$$

where the $y_{i}$ are formal variables and $f_{0}$ is an irreducible element in $\mathbb{Z}\left[y_{1}, y_{2}, \ldots, y_{j+1}\right]$ that is non-constant or zero and that gives zero when evaluated at $y_{i}=x_{i}$ for $i=1, \ldots, j$ and $y_{j+1}=\theta_{0}$.

Let $\overline{\mathbb{Q}}$ be the algebraic closure of the rational numbers, let $\mathcal{L}^{\prime}:=\prod_{\ell \in L} \ell$, and let $\mathcal{L} \in \mathbb{Z}\left[y_{1}, \ldots, y_{j+1}\right]$ be the lowest degree representative of the image of $\mathcal{L}^{\prime}$ under the above inclusion and isomorphism. We will use the following corollary to Hilbert's Nullstellensatz:
Proposition 7.3 (c.f. the corollary on page 282 of [23]). If $\mathcal{L}, f_{0} \in \overline{\mathbb{Q}}\left[y_{1}, \ldots, y_{j+1}\right]$ and if on points of $\overline{\mathbb{Q}}^{j+1}$ we have that $\mathcal{L}$ is zero whenever $f_{0}$ is zero, then there exists $m \geq 0$ and $k \in \overline{\mathbb{Q}}\left[y_{1}, \ldots, y_{j+1}\right]$ such that $\mathcal{L}^{m}=k f_{0}$.

Say that $\mathcal{L}^{m}=k f_{0}$ for some $k \in \overline{\mathbb{Q}}\left[y_{1}, \ldots, y_{j+1}\right]$. Since $\mathcal{L}^{m}, f_{0} \in \mathbb{Z}\left[y_{1}, \ldots, y_{j+1}\right]$, we have that $k$ is in $\mathbb{Q}\left(y_{1}, \ldots, y_{j+1}\right)$ (the fraction field of $\left.\mathbb{Z}\left[y_{1}, \ldots, y_{j+1}\right]\right)$. Thus, $k$ is in the ring $\mathbb{Q}\left[y_{1}, \ldots, y_{j+1}\right]$, and so there is a positive integer $c$ such that $c k \in \mathbb{Z}\left[y_{1}, \ldots, y_{j+1}\right]$. We now have $c \mathcal{L}^{m}=(c k) f_{0}$. Since $f_{0}$ is irreducible in $\mathbb{Z}\left[y_{1}, y_{2}, \ldots, y_{j+1}\right]$, we must have that $f_{0}$ divides $\mathcal{L}$ ( $f_{0}$ cannot divide the positive integer $c$ since $f_{0}$ is either non-constant or zero). But this is impossible since by assumption, $\mathcal{L}$ is non-zero in the quotient ring $\mathbb{Z}\left[y_{1}, \ldots, y_{j+1}\right] / f_{0}$. Thus, for every $m \geq 0$ and for every $k \in \overline{\mathbb{Q}}\left[y_{1}, \ldots, y_{j+1}\right]$ we must have that $\mathcal{L}^{m} \neq k f_{0}$. Therefore, by the contrapositive of Proposition [7.3, there exist algebraic numbers $q_{1}, \ldots, q_{j+1} \in \overline{\mathbb{Q}}$ such that $\left.f_{0}\right|_{y_{i}=q_{i}}=0$ while $\left.\mathcal{L}\right|_{y_{i}=q_{i}} \neq 0$. Thus, we have a homomorphism

$$
\psi_{0}: \mathbb{Z}\left[y_{1}, y_{2}, \ldots, y_{j+1}\right] / f_{0} \rightarrow \mathbb{Z}\left[q_{1}, \ldots, q_{j+1}\right]
$$

defined by $y_{i} \mapsto q_{i}$, such that $\psi_{0}(\mathcal{L}) \neq 0$.
Applying the primitive element theorem and clearing denominators as before, we have

$$
\mathbb{Z}\left[q_{1}, \ldots, q_{j+1}\right] \subset \mathbb{Z}\left[\theta_{1}\right]
$$

with $\theta_{1} \in \overline{\mathbb{Q}}$. Combining the inclusions and isomorphisms from the applications of the primitive element theorem with $\psi_{0}$ completes the proof of Lemma 7.1 .

Recall the statement of Theorem 1.1:
Theorem 1.1. Let $S$ be a finite subset of a characteristic zero integral domain $D$, and let $L$ be a finite set of non-zero elements in the subring $\mathbb{Z}[S]$ of $D$. There exists an infinite sequence of primes with positive relative density such that for each prime $p$ in the sequence, there is a ring homomorphism $\phi_{p}: \mathbb{Z}[S] \rightarrow \mathbb{Z} / p \mathbb{Z}$ satisfying $0 \notin \phi_{p}(L)$.

The proof of Theorem 1.1 picks up where the proof of Lemma 7.1 left off.
Proof of Theorem 1.1. By Lemma 7.1, there exists a ring homomorphism

$$
\phi: \mathbb{Z}[S] \rightarrow \mathbb{Z}\left[\theta_{1}\right] \simeq \mathbb{Z}[z] / f_{1},
$$

such that $0 \notin \phi(L)$, where $z$ is a formal variable and $f_{1}$ is an irreducible element in $\mathbb{Z}[z]$ that gives zero when evaluated at $z=\theta_{1}$.

Let $\widehat{L}:=\prod_{\ell \in L} \ell$, let $\widetilde{L}(z) \in \mathbb{Z}[z]$ denote the lowest-degree representative of $\phi(\widehat{L})$ in $\mathbb{Z}[z] / f_{1}$, and let $L_{1}(z)$ denote the product of all distinct irreducible factors of $\widetilde{L}(z)$ in $\mathbb{Z}[z]$. Note that a homomorphism of integral domains will map $\widetilde{L}(z)$ to zero if and only if it maps $L_{1}(z)$ to zero. By
assumption, $\widetilde{L}(z)$ is non-zero, so we must have that $f_{1}(z)$ does not divide $\widetilde{L}(z)$ in $\mathbb{Z}[z]$; and thus $f_{1}(z)$ does not divide $L_{1}(z)$. Therefore, $L_{1}(z)$ has no roots (in $\mathbb{C}$, say) in common with $f_{1}(z)$, since $f_{1}(z)$ is irreducible.

By Theorem6.1(the Frobenius Density Theorem) there exists a sequence of primes ( $p_{1}, p_{2}, p_{3}, \ldots$ ) in $\mathbb{Z}$ (with positive relative density) such that for any prime $p$ in the sequence, the polynomial $f_{1}(z) L_{1}(z)$ factors completely modulo $p$ into a product of $\operatorname{deg}\left(f_{1}(z) L_{1}(z)\right)$ distinct linear factors.

Let $(z-a)$ be a linear factor of $f_{1}(z)$ modulo $p$, where $p$ is any prime in the sequence $\left(p_{1}, p_{2}, p_{3}, \ldots\right)$. Since, modulo $p$, the linear factors of $f_{1}(z)$ are all distinct from those of $L_{1}(z)$, we know that $(z-a)$ does not divide $L_{1}(z)$ modulo $p$. Thus, for infinitely many primes $p$, we may quotient out by $p$ and by $(z-a)$ to get a canonical quotient map

$$
\psi_{1}: \mathbb{Z}[z] / f_{1} \longrightarrow \mathbb{Z}[z] /(p, z-a) \simeq \mathbb{Z} / p \mathbb{Z}
$$

where $\psi_{1}\left(L_{1}(z)\right) \neq 0$. One can think of $\psi_{1}$ as modding out by $p$ and then sending $z$ to the element $a$ in $\mathbb{Z} / p \mathbb{Z}$.

Letting $\phi_{p}:=\psi_{1} \circ \phi$ completes the proof.

## 8 Acknowledgments

We would like to thank Ellen Eischen for providing numerous clarifications and simplifications on an early version of Theorem 1.1 and its proof. Thanks is also due to John Bryk, Wei Ho, Peter Stevenhagen, Terence Tao, and J.B. Tunnell for useful comments. Finally, we would like to thank the anonymous referee for the many exceptionally careful and useful comments on the manuscript, all of which have improved the paper.

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[^0]:    Mathematics Subject Classification: 05B25 (primary), 11T99 (secondary).
    The first author is supported by research grants DMS-0901216 and AFOSAR-FA-9550-09-1-0167. The second and third authors were supported by National Defense Science and Engineering Graduate Fellowships and by National Science Foundation Graduate Research Fellowships.

