EXPLOSION OF SMOOTHNESS FOR CONJUGACIES BETWEEN MULTIMODAL MAPS

JOSÉ F. ALVES, VILTON PINHEIRO, AND ALBERTO A. PINTO

ABSTRACT. Let f and g be smooth multimodal maps with no periodic attractors and no neutral points. If a topological conjugacy h between f and g is C^1 at a point in the nearby expanding set of f, then h is a smooth diffeomorphism in the basin of attraction of a renormalization interval of f. In particular, if $f: I \to I$ and $g: J \to J$ are C^r unimodal maps and h is C^1 at a boundary of I then h is C^r in I.

1. INTRODUCTION

There is a well-known theory in hyperbolic dynamics that studies properties of the dynamics and of the topological conjugacies that lead to additional regularity for the conjugacies. D. Mostow [21] proved that if \mathbb{H}/Γ_X and \mathbb{H}/Γ_Y are two closed hyperbolic Riemann surfaces covered by finitely generated Fuchsian groups Γ_X and Γ_Y of finite analytic type, and $\phi: \overline{\mathbb{H}} \to \overline{\mathbb{H}}$ induces the isomorphism $i(\gamma) = \phi \circ \gamma \circ \phi^{-1}$, then ϕ is a Möbius transformation if, and only if, ϕ is absolutely continuous. M. Shub and D. Sullivan [25] proved that for any two analytic orientation preserving circle expanding endomorphisms f and q of the same degree, the conjugacy is analytic if, and only if, the conjugacy is absolutely continuous. Furthermore, they proved that if f and g have the same set of eigenvalues, then the conjugacy is analytic. R. de la Llave [11] and J.M. Marco and R. Moriyon [19, 20] proved that if Anosov diffeomorphisms have the same set of eigenvalues, then the conjugacy is smooth. For maps with critical points, M. Lyubich [12] proved that C^2 unimodal maps with Fibonnaci combinatorics and the same eigenvalues are C^1 conjugate. W. de Melo and M. Martens [16] proved that if topological conjugate unimodal maps, whose attractors are cycles of intervals, have the same set of eigenvalues, then the conjugacy is smooth. N. Dobbs [22] proved that if a multimodal map f has an absolutely continuous invariant measure, with a positive Lyapunov exponent, and f is absolutely continuous conjugate to another multimodal map, then the conjugacy is C^r in the domain of some induced Markov map of f.

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Here, we study the explosion of smoothness for topological conjugacies, i.e. the conditions under which the smoothness of the conjugacy in a single point extends to an open set. P. Tukia [28] extended the result above of D. Mostow proving that if \mathbb{H}/Γ_X and \mathbb{H}/Γ_Y are two closed hyperbolic Riemann surfaces covered by finitely generated Fuchsian groups Γ_X and Γ_Y of finite analytic type, and $\phi: \overline{\mathbb{H}} \to \overline{\mathbb{H}}$ induces the isomorphism $i(\gamma) = \phi \circ \gamma \circ \phi^{-1}$. then ϕ is a Möbius transformation if, and only if, ϕ is differentiable at one radial limit point with non-zero derivative. Sullivan [26] proved that if a topological conjugacy between analytic orientation preserving circle expanding endomorphisms of the same degree is differentiable at a point with non-zero derivative, then the conjugacy is analytic. Extensions of these results for Markov maps and hyperbolic basic sets on surfaces were developed by E. Faria [3], Y. Jiang [8, 9] and A. Pinto, D. Rand and F. Ferreira [4, 23], among others. For maps with critical points, Y. Jiang [5, 6, 7, 10] proved that quasi-hyperbolic one-dimensional maps are smooth conjugated in an open set with full Lebesgue measure if the conjugacy is differentiable at a point with uniform bound. In this paper, we define the nearby expanding set NE(f) of a multimodal map f and characterize NE(f) in terms of the basins of attraction of renormalization intervals. We prove that if a topological conjugacy between multimodal maps is C^1 at a point in the nearby expanding set NE(f) of f, then the conjugacy is a smooth diffeomorphism in the basin of attraction of a renormalization interval.

2. Explosion of smoothness

Let I be a compact interval and $f: I \to I$ a C^{1+} map. By C^{1+} we mean that f is a differentiable map whose derivative is Hölder. We say that c is a *non-flat turning point* of f, if there exist $\alpha > 1$ and a C^r diffeomorphism ϕ defined in a small neighborhood K of 0 such that

$$f(c+x) = f(c) + \phi(|x|^{\alpha}), \quad \text{for every } x \in K.$$
(2.1)

We say that α is the order of the turning point c and denote it by $\operatorname{ord}_f(c)$. We say that f is a multimodal map if the next three conditions hold: i) $f(\partial I) \subset \partial I$; ii) f has a finite number of turning points points that are all non-flat; and iii) $\# \operatorname{Fix}(f^n) < \infty$ for all $n \in \mathbb{N}$. A unimodal map $f: I \to I$ is a non-flat multimodal map with a unique turning point $c \in I$.

The non-critical backward orbit $\mathcal{O}_{nc}^{-}(p)$ of p is the set of all points q such that there is $n = n(q) \geq 0$ with the property that $f^{n}(q) = p$ and $(f^{n})'(q) \neq 0$. The non-critical alpha limit set $\alpha_{nc}(p)$ of p is the set of all accumulation points of $\mathcal{O}_{nc}^{-}(p)$. Let $\mathcal{O}_{nc}^{-}(\operatorname{PR}(f))$ be the union $\cup_{p \in \operatorname{PR}(f)} \mathcal{O}_{nc}^{-}(p)$ of the non-critical backward orbits $\mathcal{O}_{nc}^{-}(p)$ for every repellor periodic points of $p \in \operatorname{PR}(f)$. Let $\alpha_{nc}(\operatorname{PR}(f))$ be the union $\cup_{p \in \operatorname{PR}(f)} \alpha_{nc}(p)$ of the non-critical alpha backward orbits $\mathcal{O}_{nc}(p)$ for every repellor periodic points of $p \in \operatorname{PR}(f)$.

A set $A \subset J$ is said to be forward invariant if $f(A) \subset A$. The basin $\mathcal{B}(A)$ of a forward invariant set A is the set of all points $x \in A$ such that its omega limit set $\omega(x)$ is contained in A. An invariant compact set $A \subset J$ is called a *(minimal) attractor*, in Milnor's sense [17, 18], if the Lebesgue measure of its basin is positive and there is no forward invariant compact set A' strictly contained in A such that $\mathcal{B}(A')$ has non zero measure. The attractors of a C^r non-flat multimodal map are of one of the following three types: i) a periodic attractor; ii) a minimal set with zero Lebesgue measure; or iii) a cycle of intervals such that the omega limit set of almost every point in the cycle is the whole cycle (see [27]). According to S. van Strien and E. Vargas [27], if $f: I \to I$ is a C^r non-flat multimodal map, then there is a finite set of attractors $A_1, \ldots, A_l \subset I$ such that the union of their basins has full Lebesgue measure in I.

An open interval J(c) containing a critical point c is a renormalization interval of a multimodal (resp. unimodal) map f, if there is $n = n(J(c)) \ge 1$ such that $f^n|_{\overline{J(c)}}$ is also a multimodal (resp. unimodal) map. Hence, the forward orbit of J(c) is a positive invariant set. A multimodal map f is no renormalizable inside a renormalization interval J(c), if there is no renormalization interval strictly contained in J. A multimodal map f is infinitely renormalizable around a critical point c if there is an infinite sequence of renormalization intervals $J_1(c), J_2(c), \ldots$ such that $J_{n+1}(c)$ is strictly contained in $J_n(c)$ and $c = \bigcap_{n \ge 1} J_n(c)$. The basin of attraction $\mathcal{B}(J(c))$ of J(c) is the set of points whose forward orbit intersects J(c).

Definition 1 (Expanding and nearby expanding points). A point $p \in I$ is called *nearby* expanding if there are

- (1) a sequence of points p_n converging to p,
- (2) a sequence of open intervals V_n containing p_n ,
- (3) a sequence of positive integers k_n tending to infinity, and
- $(4) \ \delta = \delta(p) > 0,$

with the following properties:

(1) $f^{k_n}|_{V_n}$ is a diffeomorphism and

(2)
$$f^{k_n}(V_n) = B_{\delta}(f^{k_n}(p_n)).$$

Furthermore, a point $p \in I$ is called *expanding* if $p \in I$ is a nearby expanding point with $p_n = p$ for every $n \in \mathbb{N}$.

The nearby expanding set NE(f) is the set of all nearby expanding points of f and the expanding set E(f) is the set of all expanding points of f.

Lemma 2.1 (Fatness of E(f) and NE(f)). Let f be C^r a multimodal map with $r \ge 3$ and no periodic attractors nor neutral periodic points. Then:

- (1) $E(f) \supset \mathcal{O}_{nc}^{-}(\mathrm{PR}(f))$ and $\mathrm{NE}(f) \supset \alpha_{nc}(\mathrm{PR}(f))$;
- (2) if f is infinitely renormalizable around a critical point c, then there is a renormalization interval J(c) such that E(f) and NE(f) are dense in $\mathcal{B}(J(c))$;
- (3) if f is no renormalizable inside a renormalizable interval J, then E(f) is dense in $\mathcal{B}(J)$ and NE(f) contains $\overline{\mathcal{B}(J)}$.

If $f: I \to I$ is a unimodal map, for every renormalization interval J, $\partial \mathcal{B}(J)$ is uniformly expanding, $\partial I \subset \partial \mathcal{B}(J)$ and $\mathcal{B}(J)$ is an open set with full Lebesgue measure. Hence, by Lemma 2.1, if f is a unimodal map whose attractor is a cycle of intervals then E(f) is dense in I and NE(f) = I. Furthermore, if f is a unimodal map that is infinitely renormalizable then E(f) and NE(f) are dense in I. *Proof.* Let f be infinitely renormalizable around a critical point c. By Lemma A.5, there is a renormalization interval J(c) such that $\mathcal{O}_{nc}^{-}(\operatorname{PR}(f))$ is a dense set in J(c). Since $E(f) \supset \mathcal{O}_{nc}^{-}(\operatorname{PR}(f))$, we obtain that E(f) and $\operatorname{NE}(f)$ are dense in J(c).

Let f be no renormalizable inside a renormalizable interval J. By Lemma A.5, $\alpha_{nc}(\text{PR}(f))$ contains J. Hence, E(f) is dense in J and NE(f) contains J. \Box

Definition 2 (Puncture set P(J)). Let $C_P(I)$ be the set of all critical points c whose noncritical alpha limit sets $\alpha_{nc}(c)$ do not intersect the interior of I. The puncture set P(I) of I is $P(I) = \bigcup_{c \in C_P(I)} \mathcal{O}_{nc}^-(c)$. Let J be a renormalization interval and n the smallest integer such that $F = f^n | J$ is a renormalization of f. Let $C_P(J)$ be the set of all critical points cwhose non-critical alpha limit sets $\alpha_{nc}(c)$ with respect to F | J do not intersect the interior of J. The puncture set P(J) of J is $P(J) = \bigcup_{c \in C_P(J)} \mathcal{O}_{nc}^-(c)$.

Hence, the puncture set P is either empty or a discrete set. Furthermore, we observe that the puncture set is not located in the central part of the dynamics, i.e. (i) if f is infinitely renormalizable there is a renormalization interval J(c) such that $P \cap J(c) = \emptyset$ and (ii) if the Milnor's attractor A of f is a cycle of intervals then $P \cap A = \emptyset$, because $\alpha_{nc}(c)$ is dense in A for every critical point c in the interior of A.

For every connected component $G \in D(J)$, let m = m(G) be the smallest integer such that $f^m(G) \subset J(c)$. If m = 0 the puncture set $G_P \subset G$ of G is $G_P = P(J)$, and if m > 0 the puncture set $G_P \subset G$ of G be the union of all points $x \in G$ such that (i) $(f^m)'(x) = 0$ or (ii) $(f^m)'(x) \in P(J)$. We observe that $G_P \cap G$ is either a discrete set or empty. The *punctured* basin of attraction $\mathcal{B}_P(J(c))$ of J(c) is the union $\cup_{G \in D(J)} G \setminus G_P$. A renormalization domain $J = \bigcup_{c \in CR} J(c)$ of a multimodal map f is the union of renormalization intervals J(c) for a given subset $CR \subset C_f$. Set $\mathcal{B}_P(J) = \bigcup_{c \in CR} \mathcal{B}_P(J(c))$. We observe that $\overline{\mathcal{B}_P(J)} = \overline{\mathcal{B}(J)}$.

Definition 3 (C¹ at a point). We say that a map $h: I \to I'$ is C¹ at a point $p \in I$, if

$$\lim_{\substack{x,y \to p \\ x \neq y}} \frac{h(x) - h(y)}{x - y} = h'(p) \neq 0.$$

We observe that h is C^1 at every point belonging to an interval $K \subset I$ if, and only if, f is a C^1 local diffeomorphism in that interval K.

We say that a topological conjugacy $h: I \to L$ between $f: I \to I$ and $g: I' \to I'$ preserves the order of the critical points, if $\operatorname{ord}_f(c) = \operatorname{ord}_g(h(c))$ for every critical point $c \in C_f$.

Theorem 1 (Explosion of smoothness). Let f and g be C^r multimodal maps with $r \ge 3$ and no periodic attractors nor neutral periodic points. Let h be a topological conjugacy between f and g preserving the order of the critical points. If h is C^1 at a point $p \in NE(f)$, then either

- (1) h is a C^r diffeomorphism in $I \setminus P(I)$; or
- (2) there is a unique maximal renormalization domain J such that h is a C^r diffeomorphism in $J \setminus P(J)$. Furthermore,
 - (a) h is a C^r diffeomorphism in the punctured basin of attraction $\mathcal{B}_P(J)$;

(c) h is not C^1 at any point in $E(f) \cap \partial \mathcal{B}(J)$.

We observe that Theorem 1 still holds if we replace the hypotheses of h being C^1 at a point $p \in E(f)$ by h being C^r in an open set. N. Dobbs [22] proved that if (i) a multimodal map f has an absolutely continuous invariant measure with a positive Lyapunov exponent and (ii) the conjugacy h between f and another multimodal map g is absolutely continuous, then h is C^r in an open set. Hence, Theorem 1 applies to this case.

The proof of Theorem 1 is given at the end of Section 6.

Corollary 2 (Full measure explosion of smoothness for unimodal maps). Let f and g be C^r unimodal maps with $r \ge 3$ and no periodic attractors nor neutral periodic points. Let h be a topological conjugacy between f and g preserving the order of the critical points. If h is C^1 at a point $p \in NE(f)$, then either

- (1) h is a C^r diffeomorphism in the full interval I; or
- (2) there is a unique maximal renormalization interval $J \subseteq I$ such that
 - (a) h is a C^r diffeomorphism in the basin $\mathcal{B}(J)$, and
 - (b) h is not C^1 at any point in $\partial \mathcal{B}(J)$.

We observe that if $f: I \to I$ is a unimodal map, then (i) $\partial \mathcal{B}(J)$ is uniformly expanding, (ii) $\partial I \subset \partial \mathcal{B}(J)$, and (iii) $\mathcal{B}(J)$ is an open set with full Lebesgue measure in I. By Corollary 2, the map h is C^1 at a point $p \in \partial I$ if, and only if, h is a C^r diffeomorphism in I.

3. Zooming pairs

We will prove that, in Theorem 3 and in its two corollaries, the hypothesis h is C^1 at a point p can be weakened to h being (uaa) uniformly asymptotically affine at p. We will define the zooming pairs that we will use to show if h is uaa at a point then h and h^{-1} are C^r in small open sets.

Let $h: I \to I'$ be a homeomorphism. For every (x,y,z) of points $x, y, z \in I$, such that x < y < z, we define the logarithmic ratio distortion $\operatorname{Ird}_h(x, y, z)$ by

$$\operatorname{lrd}_{h}(x, y, z) = \left| \log \frac{|h(z) - h(y)|}{|h(y) - h(x)|} \frac{|y - x|}{|z - y|} \right|$$

Definition 4 (uaa). Let $h : I \to I'$ be a homeomorphism. The map h is uniformly asymptotically affine (uaa) at a point p if, for every $C \ge 1$, there is a continuous function $\epsilon_C : \mathbb{R}^+_0 \to \mathbb{R}^+_0$, with $\epsilon_C(0) = 0$, such that

$$\operatorname{lrd}_h(x, y, z) \le \epsilon_C(|x - p|) , \qquad (3.1)$$

for all x < y < z with $C^{-1} < |z - y|/|y - x| < C$.

Lemma 3.1 (C^1 implies uaa). Let $h: I \to I'$ be a homeomorphism. If h is C^1 at a point $p \in I$, then h is uaa at p.

⁽b) h is not C^r at any open interval contained in $I \setminus \overline{\mathcal{B}(J)}$;

Proof. If h is C^1 at p, then there is a sequence θ_m converging to 0, when m tends to ∞ , such that

$$\left|\log\frac{|h(y) - h(x)|}{|y - x|}h'(p)\right| \le O\left(\frac{1}{m}\right)$$
(3.2)

for all $x, y \in B_{\theta_m}(p)$. Hence, for all $x, y, z \in B_{\theta_m}(p)$, we obtain

$$\left|\log\frac{|h(z) - h(y)|}{|h(y) - h(x)|}\frac{|y - x|}{|z - y|}\right| \le O\left(\frac{1}{m}\right) ,$$
(3.3)

and so, h is uaa at p.

Definition 5 (α -bounded distortion). We say that a C^r multimodal map f has α -bounded distortion with respect to a sequence V_1, V_2, \ldots of intervals and a sequence of integers k_n tending to ∞ , if there is $C \geq 1$ such that

$$\operatorname{Ird}_{f^{k_n}}(x, y, z) \le C |f^{k_n}(z) - f^{k_n}(x)|^{\alpha} , \qquad (3.4)$$

for all $x, y, z \in V_n$, with x < y < z, and all $n \ge 1$.

Definition 6 (Zooming pair (p, V)). Let $f : I \to I$ and $g : I' \to I'$ be C^r maps, with $r \ge 2$, and $h : I \to I'$ a topological conjugacy between f and g. An α -zooming pair (p, V) consists of a point $p \in I$ and an open interval $V \subset I$ such that

- (1) there is a sequence V_1, V_2, \ldots of intervals in I and
- (2) a sequence of integers k_n tending to ∞ ,

with the following properties:

- (1) $\sup_{x \in V_n} |x p| \to 0$ when $n \to \infty$;
- (2) $f^{k_n}|_{V_n}$ and $g^{k_n}|_{h(V_n)}$ are diffeomorphisms onto the intervals V and h(V) respectively;
- (3) f has α -bounded distortion with respect to the sequences V_1, V_2, \ldots and k_1, k_2, \ldots ;
- (4) g has α -bounded distortion with respect to the sequences $h(V_1), h(V_2), \ldots$ and k_1, k_2, \ldots

A central zooming pair (p, V) is a zooming pair (p, V) with the property that $p \in V_n$ for some $n \in \mathbb{N}$.

Lemma 3.2 (Explosion of smoothness from p to V). Let f and g be C^r maps, with $r \ge 3$, topologically conjugated by a homeomorphism h. Assume that (p, V) is an α -zooming pair for some $0 < \alpha < 1$. If h is uaa at p, then h|V is a $C^{1+\alpha}$ diffeomorphism onto its image. Furthermore, if (p, V) is a central zooming pair then $h|V_0$ is a $C^{1+\alpha}$ diffeomorphism onto its image, for some open interval V_0 containing p.

Proof. Given $a, b, c \in V$, with a < b < c, let $a_n, b_n, c_n \in V_n$ be such that $f^{k_n}(a_n) = a$, $f^{k_n}(b_n) = b$ and $f^{k_n}(c_n) = c$. Since f has α -uniformly bounded distortion,

$$\operatorname{Ird}_{f^{k_n}}(a_n, b_n, c_n) \le O(|c-a|^{\alpha}). \tag{3.5}$$

Since g has has uniformly bounded distortion, we get

$$\operatorname{Ird}_{g^{k_n}}(h(a_n), h(b_n), h(c_n)) \le O(|h(c) - h(a)|^{\alpha}) .$$
(3.6)

By the definition of zooming, there is a sequence $\sigma_n \to 0$ such that, for all $x \in V_n$,

$$|x-p| < \sigma_n . \tag{3.7}$$

Since f is (uaa) at p, by (3.1), we have

$$\operatorname{lrd}_h(a_n, b_n, c_n) \leq \epsilon_C(\sigma_n)$$
.

Hence, by (3.7), there is n large enough such that

$$\operatorname{lrd}_h(a_n, b_n, c_n) \le |c - a| . \tag{3.8}$$

Combining (3.5), (3.6) and (3.8), we have

$$\begin{aligned} \operatorname{lrd}_{h}(a, b, c) &\leq \operatorname{lrd}_{g^{k_{n}}}(h(a_{n}), h(b_{n}), h(c_{n})) + \operatorname{lrd}_{h}(a_{n}, b_{n}, c_{n}) + \operatorname{lrd}_{f^{k_{n}}}(a_{n}, b_{n}, c_{n}) \\ &\leq O(|c - a|^{\alpha} + |h(c) - h(a)|^{\alpha}) . \end{aligned}$$

$$(3.9)$$

Therefore, the homeomorphism h is quasi-symmetric in V. Hence, there is $\gamma > 0$, such that h|V is γ -Hölder continuous. Thus, we obtain that (3.9) is bounded by $C_1|c-a|^{\alpha\gamma}$, for some $C_1 > 1$. Hence, by [24], we get that h|V and $h^{-1}|h(V)$ are $C^{1+\alpha\gamma}$ maps. Therefore, $|h(c) - h(a)| \leq O(|c-a|)$ and, so, (3.9) is also bounded by $C_2|c-a|^{\alpha}$, for some $C_2 > 1$. Hence, again by [24], we get that h|V and $h^{-1}|h(V)$ are $C^{1+\alpha}$ maps.

Furthermore, if (p, V) is a central zooming pair then there is an open interval V_0 containing p and an integer n such that $f^n|V_0$ is a C^r diffeomorphism and $f^n(V_0) \subset V$. Hence $h|V_0 = (g^n|h(V_0))^{-1} \circ h \circ f^n$ is a C^r diffeomorphism.

Lemma 3.3 (Building up smoothness from $C^{1+\alpha}$ to C^r). Let f and g be C^r maps, with $r \geq 3$, topologically conjugated by a homeomorphism h. If h|V is a $C^{1+\alpha}$ diffeomorphism in some open set V, then h|W is a C^r diffeomorphism for some open set $W \subset V$.

Proof. By Lemma A.5, there is a reppelor $p \in I$ and integers m and l such that $p \in int(f^m(V))$ and $f^l(p) = p$. Since p is a reppelor there is an open interval $W \subset int(f^n(V))$ with $p \in W$ such that $|f^{l_j}(x)| > \lambda > 1$, for all $x \in W$. Let W_0, W_1, \ldots be a sequence of open intervals contained in W such that (i) $f^l(W_{n+1}) = W_n$, (ii) $W_{n+1} \subset W_n$, and (iii) $|W_n| \to 0$ for every $n \ge 0$. Let $i_n : W_n \to (0, 1)$ be the affine map with the property that $i_n(W_n) = (0, 1)$ and let $f_n = i_0 \circ f^{nl} \circ i_n^{-1}$. By Lemma E13 in [23], there is b > 0 such that $\|\ln df_n\|_{C^{r-1}} \le b$, for every $n \ge 1$. Hence, by Lemma E15 in [23], there is a small $\epsilon > 0$ and a subsequence f_{k_n} converging to a C^r diffeomorphism $f: (0, 1) \to (0, 1)$ in the $C^{r-\epsilon}$ norm.

Let $W'_n = h(W_n)$ and $j_n : W'_n \to (0,1)$ be the affine map with the property that $j_n(W'_n) = (0,1)$, for every $n \ge 1$. Let $g_n = j_0 \circ g^{nl} \circ j_n^{-1}$. By Lemma E13 in [23], there is b > 0 such that $\|\ln dg_n\|_{C^{r-1}} \le b$, for all $n \ge 1$. Hence, by Lemma E15 in [23], there is a small $\epsilon > 0$ and a subsequence m_n of the sequence k_n such that g_{m_n} converges to a C^r diffeomorphism g in the $C^{r-\epsilon}$ norm.

Let $h_n = j_n \circ \overline{h} \circ i_n^{-1}$. Since h is a $C^{1+\alpha}$ diffeomorphism, there is a sequence λ_n tending to 1 such that

$$\frac{|h_n(z) - h_n(y)|}{|h_n(y) - h_n(x)|} \frac{|y - x|}{|z - y|} \le \lambda_n$$

for all $x, y, z \in (0, 1)$. Hence, $\underline{h} = \lim h_n$ is an affine map.

We note that $h|W_0 = j_0^{-1} \circ g_n \circ h_n \circ f_n^{-1} \circ i_0$, for every $n \ge 1$. Hence,

$$h|W_0 = \lim j_0^{-1} \circ g_{m_n} \circ h_{m_n} \circ f_{m_n}^{-1} \circ i_0 = j_0^{-1} \circ \underline{g} \circ \underline{h} \circ \underline{f}^{-1} \circ i_0.$$

Since, g, \underline{h} and f are C^r diffeomorphisms, we obtain that $h|W_0$ is a C^r diffeomorphism. \Box

4. Nearby expanding set

We will prove that for every nearby expanding point $p \in NE(f)$ there is an open set V such that (p, V) is a 1-zooming pair.

Given any $K \subset \mathbb{R}$ and r > 0, set $B_r(K) = \bigcup_{p \in K} B_r(p)$, where $B_r(p) = (p - r, p + r)$.

Recall that the *Schwarzian derivative of* f in the complement of the critical points is defined by

$$\mathcal{S}f := \frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'}\right)^2.$$

Lemma 4.1 (Nearby expanding point originates a zooming pair). Let f and g be C^3 multimodal maps topologically conjugated by h, with no periodic attractors and no neutral periodic points. For every $x \in NE(f)$ there is an interval V such that (x, V) is a 1-zooming pair. Furthermore, for every $x \in E(f)$ there is an interval V such that (x, V) is a central 1-zooming pair.

Proof. By [27], there is $\gamma > 0$ such that, for every point $x \in I$, with

$$f^n(x) \in \bigcup_{c \in C(f)} B_{\gamma}(c) \text{ and } g^n(h(x)) \in \bigcup_{c \in C(f)} h(B_{\gamma}(c)),$$

we have $Sf^{n+1}(x) < 0$ and $Sg^{n+1}(h(x)) < 0$.

By Lemma A.4, one find $\gamma_0 < \gamma_1 < \gamma_2 < \gamma_3 < \gamma_4 < \gamma$ and nice sets J_0, J_1, J_2 such that

$$B_{\gamma_0}(\mathcal{C}_f) \subset J_0 \subset B_{\gamma_1}(\mathcal{C}_f) \subset B_{\gamma_2}(\mathcal{C}_f) \subset J_1 \subset B_{\gamma_3}(\mathcal{C}_f) \subset B_{\gamma_4}(\mathcal{C}_f) \subset J_2 \subset B_{\gamma}(\mathcal{C}_f)$$

Let $J_i = \bigcup_{c \in \mathcal{C}_f} J_i(c), c \in J_i(c) = (a_i(c), b_i(c))$ for every $c \in \mathcal{C}_f$ and i = 0, 1, 2.

Given $x \in NE(f)$, for some small $\delta > 0$, take a sequence of points $x_j \to x$ and intervals $W_j^0 \ni x_j$ such that $f^{m_j}|_{\overline{W_j^0}}$ is a diffeomorphism and $f^{m_j}(W_j^0) = B_{2\delta}(f^{m_j}(x_j))$ for $m_j \to \infty$. Let $W_j \subset W_j^0$ be the interval such that $f^{m_j}(W_j) = B_{\delta}(x_j)$ and let L_j^0, R_j^0 be the connected components of $W_j^0 \setminus W_j$.

For every $j \ge 1$, define n_j as follows: If $f^i(x_j) \notin J_1$, for every $0 \le i < m_j$, take $n_j = -1$; otherwise, take $n_j < m_j$ as the biggest integer such that $f^i(x_j) \in J_1$.

Our goal is to obtain a sequence $j_i \to +\infty$ and intervals $V_{j_i} \subset W_{j_i}^0$ containing x_{j_i} with the following properties: $\inf_i |f^{m_{j_i}}(V_{j_i})| > 0$ and the ratio distortion of $f^{m_{j_i}}|_{V_{j_i}}$ uniformly bounded. If $n_j = -1$ take $V_j = W_j$. In this case, $|f^{m_j}(V_j)| = 2\delta$ and the bounded of the ratio distortion follows from Theorem A.1, because $J_1 \supset B_{\gamma_2}(\mathcal{C}_f)$. Thus, we assume from now on that $n_j \neq -1$.

If $\liminf_j m_j - n_j < \infty$, let V_j be the maximal interval such that $x_j \in V_j \subset W_j$ and $f^{n_j}(V_j) \subset J_2$. Taking a subsequence, we assume that there is K > 0 such that $m_j - n_j \leq K$ for every j.

Since $Df^{m_j} \neq 0$ in W_j and $f^{n_i}(W_i) \cap J_1 \neq \emptyset$ and by maximality of V_j , if $W_j \neq V_j$ then

$$f^{n_j}(V_j) \supset (c_j - \gamma_4, c_j - \gamma_3) \text{ or } f^{n_j}(V_j) \supset (c_j + \gamma_3, c_j + \gamma_4)$$

for some $c_j \in C_f$. In particular, $|f^{n_j}(V_j)| \geq \gamma_4 - \gamma_3$. Thus, there is $\varepsilon > 0$ such that, for every j, $|f^{m_j}(V_j)| > \varepsilon > 0$, $|f^{m_j}(V_j)| = |f^{m_j}(W_j)| = 2\delta$ or $f^{m_j}(V_j)$ is a finite iteration of an interval with length greater than $\gamma_4 - \gamma_3$. Furthermore, since

$$|f^{m_j}(L_j)|/|f^{m_j}(W_j)| = |f^{m_j}(R_j)|/|f^{m_j}(W_j)| = 1/2 \text{ for every } j,$$
(4.1)

we get that

$$\frac{|f^{n_j+1}(L_j)|}{|f^{n_j+1}(V_j)|} \ge \frac{|f^{n_j+1}(L_j)|}{|f^{n_j+1}(W_j)|}$$

and

$$\frac{|f^{n_j+1}(R_j)|}{|f^{n_j+1}(V_j)|} \ge \frac{|f^{n_j+1}(R_j)|}{|f^{n_j+1}(W_j)|}$$

are bounded away from zero. Since $Sf^{n_j+1}(z) < 0$, for every

$$z \in f^{-n_j}(B_{\gamma}(\mathcal{C}_f)) \supset f^{-n_j}(B_{\gamma_5}(\mathcal{C}_f)) \supset f^{-n_j}(J_2) \supset V_j,$$

the ratio distortion of $f^{n_j+1}|_{V_j}$ is uniformly bounded $(V_j \subset W_j)$. Thus, the ratio distortion of $f^{m_j}|_{V_j}$ is also uniformly bounded and $|f^{m_j}(V_j)| > \varepsilon > 0$ for every j.

Let us consider the case $\liminf_j m_j - n_j = \infty$. Taking a subsequence, if necessary, we assume that $\lim_j m_j - n_j = \infty$.

Claim 1. $f^{n_j}(W_i^0) \subset J_2$ for every $j \in \mathbb{N}$.

Proof of the claim. Let V_i^0 be the maximal interval such that

$$x_j \in V_j^0 \subset W_j^0$$
 and $f^{n_j}(V_j^0) \subset J_2$.

We will show that $W_j^0 = V_j^0$.

By the maximality of V_j^{0} , if $W_j^0 \neq V_j^0$ then there is $p_{2,j} \in \partial J_2 \cap \partial (f^{n_j}(V_j^0))$. On the other hand, since $f^{n_j}(x_j) \in J_1$, there is $p_{1,j} \in \partial J_1$ such that

$$f^{n_j}(V_j^0) \supset (p_{1,j}, p_{2,j}) \text{ or } f^{n_j}(V_j^0) \supset (p_{2,j}, p_{1,j})$$

If $p_{1,j} < p_{2,j}$ take $T_j = (p_{1,j}, p_{2,j})$; otherwise, take $T_j = (p_{2,j}, p_{1,j})$. Since J_1 and J_2 are nice sets with $J_1 \subset J_2$, it follows that $f^k(\partial T_j) \cap J_1 = \emptyset$ for every $k \ge 0$. Hence, if $\ell_j \ge 0$ is the smaller integer such that $f^{\ell_j}(T_j) \cap J_1 \ne \emptyset$, then $f^{\ell_j}(T_j) \cap J_1(c_j) \ne \emptyset$ for some $c_j \in \mathcal{C}_f$. Furthermore, $f^{\ell_j}(T_j) \supset J_1(c_j)$. However, since $Df^{m_j} \ne 0$ on W_j^0 , we get $\ell_j \ge m_j - n_j$. Thus, it follows from Theorem A.1 that

$$4\delta = |f^{m_j}(W_j^0)| \ge |f^{m_j}(V_j^0)| \ge |f^{m_j - n_j}(T_j)| \ge C\lambda^{m_j - n_j}|T_j| \ge C\lambda^{m_j - n_j}(\gamma_4 - \gamma_3) \to \infty \text{ (for a subsequence).}$$

Hence, we get a contradiction.

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By Theorem A.1, if $f^i(W_j^0) \cap J_0 = \emptyset$, for every $n_j < i < m_j$, then $f^{m_j - (n_j + 1)}$ has uniformly bounded distortion on $f^{n_j + 1}(W_j^0)$ not dependent upon j. In particular,

$$|f^{n_j+1}(L_j)|/|f^{n_j+1}(W_j)|$$
 and $|f^{n_j+1}(R_j)|/|f^{n_j+1}(W_j)|$

are bounded away from zero. Since $f^{n_j}(W_j^0) \subset B_{\gamma}(\mathcal{C}_f)$ and $Sf^{n_j+1}(z) < 0$, for every $z \in W_j^0$, the ratio distortion of $f^{n_j+1}|_{W_j}$ is uniformly bounded. Thus, taking $V_j = W_j$, the ratio distortion of $f^{m_j}|_{V_j}$ is uniformly bounded and $|f^{m_j}(V_j)| = 2\delta$ for every j.

From now on, we will assume not only that $m_j - n_j \to \infty$, but also that $f^i(W_j^0) \cap J_0 \neq \emptyset$ for some $n_j < i < m_j$.

Let k_j be the smaller integer $\ell > n_j$ such that $f^{\ell}(W_j^0) \cap J_0 \neq \emptyset$, i.e.

$$k_j = \min\{\ell > n_j; f^{\ell}(W_j^0) \cap J_0 \neq \emptyset\}.$$

Claim 2. There is K > 0 such that $m_j - k_j \leq K$, for every $j \in \mathbb{N}$.

Proof of the claim. Since $f^{\ell}(x_j) \notin J_1$, for all $n_j < j < m_j$, there is a connected component T_j of $J_1 \setminus \overline{J_0}$ such that $T_j \subset f^{k_j}(W_j^0)$. Since J_0 and J_1 are nice sets with $J_0 \subset J_1$, it follows that

$$f^i(\partial T_j) \cap J_0 = \emptyset$$

for all $i \ge 0$. So, if $\ell_j \ge 0$ is the smaller integer such that

$$f^{\ell_j}(T_j) \cap J_0 \neq \emptyset,$$

i.e. $f^{\ell_j}(T_j) \cap J_0(c_j) \neq \emptyset$ for some $c_j \in \mathcal{C}_f$. Thus, $f^{\ell_j}(T_j) \supset J_0(c_j)$. Since $f^{m_j}|_{W_j^0}$ in a diffeomorphism, we get $\ell_j \geq m_j - k_j$. Thus, from Theorem A.1, it follows that

$$4\delta = |f^{m_j}(W_j^0)| \ge |f^{m_j - n_j}(T_j)| \ge C\lambda^{m_j - k_j} |T_j| \ge C\lambda^{m_j - n_j} (\gamma_2 - \gamma_1),$$

for every $j \in \mathbb{N}$. Since $\lambda > 1$, we necessarily have $m_j - k_j$ bounded.

Using Theorem A.1, we conclude that $f^{k_j-(n_j+1)}$ has uniformly bounded distortion on $f^{n_j+1}(W_j^0)$ (not dependent upon j). Since $0 \le m_j - k_j \le K$ and $f^{m_j}|_{\overline{W_j^0}}$ is a diffeomorphism, we obtain that $f^{m_j-(n_j+1)}$ has uniformly bounded distortion on $f^{n_j+1}(W_j^0)$ (also not dependent upon j). Thus,

$$|f^{n_j+1}(L_j)|/|f^{n_j+1}(W_j)|$$
 and $|f^{n_j+1}(R_j)|/|f^{n_j+1}(W_j)|$

are bounded away from zero. Since $Sf^{n_j+1}(z) < 0$ the ratio distortion of $f^{n_j+1}|_{W_j}$ is uniformly bounded for all $z \in W_j^0$. Again, taking $V_j = W_j$, the ratio distortion of $f^{m_j}|_{V_j}$ is uniformly bounded and $|f^{m_j}(V_j)| = 2\delta$ for all j.

Thus, replacing j by a subsequence, we get intervals $V_j \subset W_j^0$ containing x_j with the following properties: $\inf_j |f^{m_j}(V_j)| > 0$, the ratio distortion of $f^{m_j}|_{V_j}$ is uniformly bounded and the ratio distortion of $g^{m_j}|_{h(V_j)}$ is also uniformly bounded.

By compactness, taking a subsequence, there is an open interval V and a sequence of intervals $x_j \in V'_j \subset V_j$, $j \ge 1$, such that $f^{s_j}(V'_j) = V$, for all j. Thus, (x, V) is a 1-zooming pair. Similarly, if $x \in E(f)$ there is an interval V such that (x, V) is a central 1-zooming pair.

Lemma 4.2 (Explosion of smoothness at expanding points). Let f and g be C^3 multimodal maps topologically conjugated by h, with no periodic attractors and no neutral periodic points. Let the conjugacy h be C^1 at a point x. If $x \in NE(f)$, then there is an open interval V such that h|V is C^r .

Proof. By Lemma 4.1, if $x \in NE(f)$ there is an interval V such that (x, V) is a 1-zooming pair. Since h is C^1 at x, then by Lemma 3.1 we have that h is uaa at x. Thus, it follows from Lemma 3.2 that h|V is a $C^{1+\alpha}$ diffeomorphism. Hence, by Lemma 3.3, h|W is a C^r diffeomorphism for some $W \subset V$.

5. Smooth conjugacy and renormalization intervals

In this section we assume that f and g are C^r multimodal maps, with $r \geq 3$ and no periodic attractors nor neutral periodic points. Furthermore, we assume that h is a topological conjugacy between f and g preserving the order of the critical points. We define

$$s = \min_{\{c \in C_f\}} \operatorname{ord}_f(c).$$

Definition 7 (Smooth conjugacy domain). For $s \leq t \leq r$, the t-smooth conjugacy interval V is an open set V such that h|V is a C^t diffeomorphism. The set $C_f^t \subseteq C_f$ consists of all critical points c such that there is a t-smooth conjugacy open interval V containing $c \in V$. For every $c \in C_f^s$, the s-smooth conjugacy maximal interval $J^s(c)$ of c is the maximal open interval $J^s(c)$ containing c such that h is C^s in $J^s(c)$. The s-smooth conjugacy domain J^s is

$$J^s = \bigcup_{c \in C^s_{\mathfrak{s}}} J^s(c).$$

We say that a critical point $c \in C_f$ is *s*-recurrent, if there is $n = n(c, s) \ge 1$ such that $J^s(c) \cap f^n J^s(c) \neq \emptyset$. Let $CR^s \subset C_f$ be the set of all *s*-recurrent critical points. Let $J_R^s = \bigcup_{c \in CR^s} J^s(c)$.

Lemma 5.1 (Spreading smooth conjugacy intervals). Let h be a topological conjugacy between f and g and let $s \le t \le r$. Then

- (1) if V is a t-smooth conjugacy interval then int(f(V)) is a t-smooth conjugacy interval;
- (2) if V is a t-smooth conjugacy interval then the connected components of $f^{-1}(V) \setminus (f^{-1}(V) \cap C_f)$ are t-smooth conjugacy intervals; and
- (3) if V is an s-smooth conjugacy interval then $f^{-1}(V)$ is an s-smooth conjugacy interval.

Furthermore,

- (1) if $c \in C_f$ and, for some small open interval V containing c and some n, $f^n(V) \subset J^s$ then $c \in C_f^s$; and
- (2) if $c \in C_f$ and, for some small open interval $V \subset J^r$ and some $n, c \in int(f^n(V))$ then $c \in C_f^r$.

Proof. Since f is a multimodal map, the interior of f(V) is an open interval and for every $x \in f(V)$ there is an open interval W such that $x \in f(W)$ and f|W is a C^r diffeomorphism. Hence, $h|f(W) = g \circ h \circ (f|W)^{-1}$ is a C^r diffeomorphism.

For every $x \in f^{-1}(V) \setminus (f^{-1}(V) \cap C_f)$, there is an open interval W such that $x \in W$ and f|W is a C^r diffeomorphism. Hence, $h|W = (g|_{h(W)})^{-1} \circ h \circ f$ is a C^r diffeomorphism.

Let $c \in f^{-1}(V) \cap C_f$ and c' = h(c). Let b = f(c) and b' = g(h(c)). Recall that $f(c+x) = f(c) + \phi(|x|^{\alpha})$ and $g(c'+x) = g(c') + \psi(|x|^{\alpha})$. Hence, there is a small open interval W containing c, such that for every $c + x \in W$

$$(g|_{h(W)})^{-1}(y) = c' + (\psi^{-1}(y - g(c'))^{1/\alpha},$$

if $x \ge 0$ and

$$(g|_{h(W)})^{-1}(y - g(c')) = c' - (\psi^{-1}(y - g(c')))^{1/\alpha},$$

if $x \leq 0$. Hence, if $x \geq 0$

$$h|W = (g|_{h(W)})^{-1} \circ h \circ f = c' + [\psi^{-1}(-g(c') + h(f(c) + \phi(|x|^{\alpha})))]^{1/\alpha}$$

and if $x \leq 0$

$$h|W = (g|_{h(W)})^{-1} \circ h \circ f = c' - [\psi^{-1}(-g(c') + h(f(c) + \phi(|x|^{\alpha})))]^{1/\alpha}.$$

The map $\psi^{-1}(-g(c') + h(f(c) + y))$ is a C^r diffeomorphism. Hence, by Taylor's theorem, there is a constant c and C^r diffeomorphism θ such that $\psi^{-1}(-g(c') + h(f(c) + y)) = y(c + y\theta(y))$. Therefore,

$$h|W = x(c + |x|^{\alpha}\theta(|x|^{\alpha}))^{1/\alpha}.$$

Hence, h is a C^s diffeomorphism.

Denoting by p and q the points that form the boundary of an interval V, the set V is dynamically symmetric if either f(p) = f(q) or f(p) and f(q) form the boundary of f(V).

Lemma 5.2 (Nice J^s). If h is a C^s diffeomorphism in an open set V then the s-conjugacy maximal domain J^s is a non-empty and there is $n \ge 0$ such that $int(f^n(V)) \subset J^s$. Furthermore,

- (1) if $c \in C_f^s$ the set $J^s(c)$ is dynamically symmetric;
- (2) for all $c_1, c_2 \in C_f^s$ the sets $J^s(c_1)$ and $J^s(c_2)$ are either disjoint or equal; and
- (3) the s-conjugacy maximal domain J^s is a nice set.

Proof. Let us assume that h is a C^r diffeomorphism in an open set V. It follows from Lemma A.2 that there is an $n \in \mathbb{N}$ and $c \in C_f$ such that $f^n|_V$ is a diffeomorphism C^r and $c \in \operatorname{int}(f^n(V))$. Hence, by Lemma 5.1 (i), h is a C^r diffeomorphism in $f^n(V)$. Hence, $J^s(c) \supset f^n(V)$ is a non-empty closed interval and $c \in C_f^r$.

Let us denote J^s by J. Let us denote by p and q the boundary points of J(c). Let us prove that the interval J(c) is dynamically symmetric, i.e. either f(p) = f(q), or f(p)and f(q) form the boundary of f(J(c)). Let us suppose, by contradiction, that there is $z \in \text{int } J(c)$ that is not a critical point such that f(z) = f(q) (or, similarly, f(z) = f(q)). Let V_z and V_q be small neighborhoods of z and q, respectively, such that $f|_{V_z}$ is a C^r diffeomorphism and $f(V_q) \subset f(V_z)$. Hence, by Lemma 5.1 (i), h is a C^s diffeomorphism in

 $f(V_q) \subset f(V_z)$ and, again by Lemma 5.1, h is a C^r diffeomorphism in V_q . Hence, h has a C^r diffeomorphic extension to a neighborhood of q which is absurd.

By construction, if $J(c_1) \cap J(c_2) \neq \emptyset$, for some $c_1, c_2 \in C_f$, then $J(c_1) = J(c_2)$

Let us prove that the set J is nice. Let us suppose, by contradiction, that there is a point $p \in \partial J(c)$ and $n \ge 0$ such that $f^n(p) \in J$ and $f^m(p) \notin J$, for all 0 < m < n. Hence, there is a small neighborhood V of p such that $f^n(V) \subset J$. By Lemma 5.1 (i) and (iii), h is a C^s diffeomorphism in V which is absurd. The proof of case (ii) is similar. \Box

Given a nice set J, let I(J) be the set of all points $x \in I$ whose forward orbit intersects J. Let D(J) be the set of all connected components G of I(J), i.e.

$$I(J) = \bigcup_{G \in D(J)} G.$$

The open intervals $G \in D(J)$ are called the gaps of I(J). We note that the boundary $\partial I(J)$ of I(J) is totally disconnected.

Lemma 5.3 (The basin of attraction of J^s). Let $\emptyset \neq J^s \subset int(I)$ For every $G \in D(J^s)$ with $G \cap J^s = \emptyset$, there is $n = n(G) \ge 1$ such that

- (1) $f^n | G$ is a diffeomorphism;
- (2) there is $c \in C_f^s$ such that $f^n(G) = J^s(c)$;
- (3) $f^{j}(G) \cap J^{s} = \emptyset$, for every $0 \leq j < n$.

Proof. For every $x \in I(J) \setminus J$, let n(x) > 1 be such that $f^n(x) \in J$ and $f^j(x) \notin J$ for every $0 \leq j < n$. Let $E = \{x, \ldots, f^{n-1}(x)\}$. By Lemma 5.2, $E \cap C_f = \emptyset$ and so there is a small open set V such that $f^n | V$ is a C^r diffeomorphism and $f^n(V) \subset J$. Let us prove by contradiction that there is a small open interval $W \subset V$ containing x such that n(y) = n(x) for every $y \in W$. If there is not a small open interval $W \subset V$ containing x such that that n(y) = n(x), for every $y \in W$, then there is a sequence of points $x_n \in V$ converging to x with $n(x_n) = j < n(x)$. Hence, $f^j(x) \in \partial J$. Since J is nice $f^{n-j}(f^j(x)) \cap J = \emptyset$ which is a contradiction. Let V = (x, a) be the maximal open interval containing x such that n(y) = n(x) for every $y \in V$. Let us prove, by contradiction, that $f^n(a) \in \partial J$. By the above argument, If $f^n(a) \in J$ then there is an open interval W_a such that n(y) = n(a) for every $y \in W_a$ which is absurd by maximality of V. Hence, for every $x \in I(J) \setminus J$, there is a maximal open interval G such that n(y) = n(x), for every $y \in G$, and $f^n(G) \subset \partial J$. Hence, $f^n | G$ is a C^r diffeomorphism and $f^n(G) = J(c)$ for some c.

Lemma 5.4 $(J_R^s \text{ is a renormalization domain})$. Let $\emptyset \neq J^s \subset \text{int}(I)$. For every $c \in C_f^s$, there is n(c) and $c'(c) \in C_f^s$ with the following properties:

- (1) $f^{n(c)}(J^s(c)) \subset \overline{(J^s(c'(c)))};$
- (2) $\partial f^{n(c)}(J^{s}(c)) \subset \partial J^{s}(c'(c));$
- (3) $f^i(J^s(c)) \cap J^s = \emptyset$, for every $1 \le i < n(c)$;
- (4) J_R^s is a renormalization domain;
- (5) $\mathcal{B}(J_R^s) \subseteq I(J^s)$ and $\overline{\mathcal{B}(J_R^s)} = \overline{I(J^s)};$

Proof. By Lemma 5.3, for every gap $G \in D(J)$ there are $n = n(G) \ge 1$ and $c(G) \in C_f \cap J$ such that $f^n(G) = J(c(G)), f^n|G$ is a C^r diffeomorphism and $f^n(G) \cap J = \emptyset$ for every $o \le i < n$.

For every $c \in C_f$, either (A) int $f(J(c)) \cap \partial I(J) = \emptyset$; or (B) int $f(J) \cap \partial I(J) \neq \emptyset$

Case (A). Since int $f(J(c)) \cap \partial I(J) = \emptyset$, there is an open interval K, that is either a) an interval J(c') or b) a gap G, such that $f(J(c)) \subset \overline{K}$. In case a), this lemma follows from noting that J is nice, and so $\partial f(J(c)) \subset \partial J(c')$. In case b), there is $n = n(G) \ge 1$ such that $f^{n+1}J(c) \subset J(c(G))$ and $f^{i+1}J(c) \cap J = \emptyset$ for every $0 \le i < n$. Furthermore, since J is nice, $f^{n+1}(\partial J(c)) \subset \partial J(c(G))$ that proves this lemma in case b).

Case (B). Let us suppose that there is a point $x \in \partial I(J) \cap \inf f(J(c))$. Let V be a small neighborhood contained in J(c) such that f|V is a C^r diffeomorphism and x is contained in the interior of f(V). Since $\partial I(J)$ is a totally disconnected set, there are gaps G_y and G'_y with a boundary point $y \in f(V)$. Let $z \in V$ be such that f(z) = y and take a smaller neighborhood $V_0 \subset V$ of z such that $f(V_0 \setminus \{z\}) \subset G_y \cup G'_y$. By Lemma 5.1, if there is

$$w \in f^{n(G_y)+1}(V_0 \setminus \{z\}) \cap \partial J(c(G_y)),$$

then there is an open interval $W \subset f^{n(G_y)+1}(V_0 \setminus \{z\})$ containing w such that h|W is a C^s diffeomorphism. Since $w \in \partial J(c(G_y))$, we obtain a contradiction. Hence, for some $0 \leq i < n(G_y)$, there is a critical point $c_y \in C_f$ such that $c_y = f^i(y)$. Therefore, $J(c(G_y)) =$ $J(c(G'_y))$ and $n(G_y) = n(G'_y)$. Since the set of critical points is finite, $\partial I(J) \cap \inf f(J(c_0))$ is also finite and for every $w \in \partial I(J) \cap \inf f(J(c_0))$, there are gaps G_w and G'_w with $w \in \partial G_w \cap \partial G'_w$ such that

$$J(c(G_w)) = J(c(G'_w)) = J(c(G_y))$$
 and $n(G_w) = n(G'_w) = n(G_y).$

Furthermore, since J is nice, $f^{n(G_y)}(\partial J(c)) \subset \partial J(c(G_y))$, that proves this lemma in case (B).

Hence, Lemma 5.4 (i) and (ii) hold. Therefore, J_R^s is a renormalization domain. Lemma 5.4 (i) and (ii) also imply for every gap $G \subset I(J^s)$ there is a gap $G' \subset \mathcal{B}(J_R^s)$ such that $G \setminus G'$ is either (i) empty or (ii) it is a finite set of points $S_G = G \setminus G'$ with the following properties: for every $x \in S_G$ there is i = i(x) and j = j(x) such that (i) $0 \leq i < j$, (ii) $f^i(x) \in C_f^s$, (iii) $f^i(x) \notin J_R^s$, and (iv) $f^j(x) \in \partial J_R^s$. Hence, Lemma 5.4 (iv) holds.

Theorem 3 (Explosion of smoothness). Let f and g be C^r multimodal maps with $r \ge 3$ and no periodic attractors and no neutral periodic points. Let h be a topological conjugacy between f and g preserving the order of the critical points. If h is C^1 at a point $p \in NE(f)$, then either

- (1) h is a C^s diffeomorphism in the full interval I or in its interior int(I); or
- (2) there is a unique maximal renormalization domain $J \subseteq I$ such that h is a C^s diffeomorphism in J. Furthermore,
 - (a) h is a C^s diffeomorphism in the basin of attraction $\mathcal{B}(J)$;
 - (b) h is not C^s at any open interval contained in $I \setminus \overline{\mathcal{B}(J)}$;
 - (c) h is not C^1 at any point in $E(f) \cap \partial \mathcal{B}(J)$.

Proof. By Lemma 4.2, there is an open interval W such that h|W is C^s and so the ssmooth conjugacy maximal domain $J^s \neq \emptyset$. If h is not a C^s diffeomorphism in I or int(I), then, by Lemma 5.4, there is a renormalization domain J^s_R such that (i) $h|\mathcal{B}(J^s_R)$ is a C^s diffeomorphism and (ii) there is no open interval $V \subset I \setminus \overline{\mathcal{B}(J^s_R)} = I \setminus \overline{I(J^s)}$ such that h|is a C^s diffeomorphism. Let us prove, by contradiction, that h is not C^1 at any point in $E(f) \cap \partial \mathcal{B}(J)$. By Lemma 4.2, if h is C^1 at some point $x \in E(f) \cap \partial \mathcal{B}(J)$ then there is an open interval W containing x such that h|W is C^s which is a contradiction.

Theorem 4 below gives a criterium for non-smoothness of the conjugacy when the conjugacy does not preserve the order of the critical points. The non-critical forward orbit $\mathcal{O}_{nc}^+(p)$ of p is the set of all points q such that there is $n = n(q) \ge 0$ with the property that $f^n(p) = q$ and $(f^n)'(p) \ne 0$. The non-critical omega limit set $\omega_{nc}(p)$ of p is the set of all accumulation points of $\mathcal{O}_{nc}^+(p)$.

Theorem 4 (Implosion of non smoothness). Let f and g be C^r multimodal maps with $r \geq 3$ and no periodic attractors and no neutral periodic points. Let h be a topological conjugacy, between f and g, not preserving the order of the critical points c_f and $c_g = h(c_f)$. The conjugacy h is not C^1 simultaneously at (i) a point belonging to $E(f) \cap \alpha_{nc}(c_f)$ and a point belonging to $E(f) \cap \omega_{nc}(c_f)$.

If f is a Collet-Eckmann map with negative Schwarzian derivative, then $E(f) \cap \omega_{nc}(c_f) \neq \emptyset$ and $\alpha_{nc}(c_f)$ contains the Milnor's attractor cycle.

Proof. Let us prove, by contradiction, that h is not C^1 at any point belonging to $E(f) \cap \alpha(c_f)$. If h is C^1 at a point $x \in E(f) \cap \alpha_{nc}(c_f)$ then, by Lemma 4.2, there is an open interval V_1 containing x such that $h|V_1$ is C^r . Since $x \in \alpha_{nc}(c_f)$, there is an integer n such that $c \in int(f^n(V_1))$. Hence, by Lemma 5.1, h is a C^r diffeomorphism in an open set V_c containing c.

If h is C^1 at a point $x \in E(f) \cap \omega_{nc}(c_f)$ then, by Lemma 4.2, there is an open interval W_1 containing x such that $h|W_1$ is C^r . Since $x \in \omega_{nc}(c_f)$, there is an open set $W_{f(c)}$ containing f(c) and an integer n such that $f^n(W_{f(c)}) \subset W_1$ and $f^n|W_{f(c)}$ is a C^r diffeomorphism. Hence, by Lemma 5.1, $h|W_{f(c)}$ is a C^r diffeomorphism.

Since h does not preserve the order of the critical points c_f and $c_g = h(c_f)$, h can not be C^1 at c_f and $f(c_f)$ simultaneously which is an absurd.

6. C^r smoothness of the conjugacy

In this section, we prove Theorem 1.

Lemma 6.1 $(K(c') \subseteq J_R^s \text{ is a renormalization interval})$. Let h be a C^r diffeomorphism in an open set V_1 . There is a maximal renormalization interval $K(c') \subseteq J_R^s$ and a puncture set $P(c') \subset K(c')$ such that

- (1) h is a C^r diffeomorphism in $K(c') \setminus P(c')$, and
- (2) $int(V_1 \cap \mathcal{B}(K(c'))) \neq \emptyset$.

Furthermore, $\partial K(c') \subset E(f)$ and h is not C^1 at the boundary $\partial K(c')$ points.

Proof. Using Lemma A.2, there is a sequence of open sets V_1, V_2, V_3, \ldots such that (i) $V_{i+1} \cap C_f \neq \emptyset$; (ii) $f^{n_i}(V_i) \supset V_{i+1}$, and (iii) $|V_i| \to 0$. Since C_f is finite, (i) there is $c' \in C_f \cap J$ and (ii) a subsequence $V_{n_1}, V_{n_2}, V_{n_3}, \ldots$ such that $f^{m_i}(V_{n_i}) \supset V_{n_{i+1}}$, where $m_i = \sum_{j=n_i}^{n_{i+1}-1} n_j$, and (iii) $c' \in V_{n_i}$ for every $i \ge 1$. By Lemma 5.1, $h \mid \operatorname{int}(f^{m_i}(V_{n_i}))$ is a C^r diffeomorphism and so $h \mid V_{n_{i+1}}$ is also a C^r diffeomorphism. By Lemma 5.4, there is a non-empty maximal renormalization interval $J = J^s(c') \subseteq J_R^s$ containing V_{n_i} for all i. Let l be the smallest integer such that $F = f^l \mid J$ is a renormalization of f restricted to J.

Let C (possibly empty) be the set of all critical point $c \in C_F$ of F|J such that there is no open interval $V_c \subset J$ with the property that $c \in V_c$ and $h|V_c$ is a C^r diffeomorphism. For every $c \in C$, let $\alpha_{nc}(c)$ be the non-critical alpha limit set of c with respect to F|J. Set $\alpha_{nc}(C) = \bigcup_{c \in C} \alpha_{nc}(c)$.

Let us prove that the open connected component H of $J \setminus \alpha_{nc}(C)$ containing c' is a renormalization interval for F. Let us start proving, by contradiction, that H is nonempty. If $H = \emptyset$, there are (i) $c_1 \in C$, (ii) an open interval $U \in V_{n_1}$ and (iii) an integer lsuch that $c_1 \in \operatorname{int} F^l(U)$. By Lemma 5.1, $c_1 \in C_F^r$ which is absurd. Take i_0 large enough such that, for every $i \geq i_0$, $c' \in V_{n_i} \subset H$ and $c' \in V_{n_{i+1}} \subset H$. Since $f^{m_i}(V_{n_i}) \supset V_{n_{i+1}}$, there is l_i such that (i) $F^{l_i}(V_{n_i}) = f^{m_i}(V_{n_i})$ and (ii) $F^{l_i}(V_{n_i}) \cap H \neq \emptyset$. Since $\alpha_{nc}(C)$ is forward invariant, $\partial F^{l_i}(H) \subset \alpha_{nc}(C)$. Let us prove, by contradiction, that (i) $\partial F^{l_i}(H) \subset \partial H$ and (ii) $F^{l_i}(H) \subset \overline{H}$. If $F^{l_i}(H) \notin \overline{H}$ then there is $x \in \partial H$ such that $x \in \operatorname{int}(F^{l_i}(H))$. Hence, by Lemma 5.1, h is C^r in an open set containing x which is a contradiction. Hence, $F^{l_i}(H) \subset \overline{H}$ and, by forward invariance of $\alpha_{nc}(C)$, $\partial F^{l_i}(H) \subset \partial H$. Thus, H is a renormalization interval for F. Take k the smallest integer such that $F_1 = F^k | H$ is a renormalization of F restricted to H.

For every open interval $H_1 \subset H$, let C_{H_1} be the set of all critical point $c \in H_1$ of $F_1|H$ such that there is no open interval $V_c \subset H$ with the property that $c \in V_c$ and $h|V_c$ is a C^r diffeomorphism. For every $c \in C_{H_1}$, let $\mathcal{O}_{nc}^-(c)$ be the non-critical backward orbit of c with respect to $F_1|H$. Set $\mathcal{O}_{nc}^-(C_{H_1}) = \bigcup_{c \in C_{H_1}} \mathcal{O}_{nc}^-(c)$. Since the accumulation set of $\mathcal{O}_{nc}^-(C_H)$ is contained in $\alpha_{nc}(C)$, the set $\mathcal{O}_{nc}^-(C_{H_1})$ is a discrete set of H, for every open interval $H_1 \subset H$.

Now, let $H_1 \subset H$ be the maximal open set such that $h|H_1 \setminus \mathcal{O}_{nc}^-(C_{H_1})$ is C^r . Either (i) $H_1 = H$, or (ii) $H_1 \neq H$ is non-empty.

Case (i). The interval K(c') = H is the maximal interval of renormalization containing c' and $P(c') = \mathcal{O}_{nc}^{-}(C_H)$ is the punctured set of K(c') with the property that $h|K(c') \setminus P(c')$ is C^r . Furthermore, $int(V_1 \cap \mathcal{B}(K(c'))) \neq \emptyset$.

Case (ii). There is *i* large enough such that $V_{n_i} \subset H_1$ and $F_1^l(V_{n_i}) \cap H_1 \neq \emptyset$.

Let us prove by contradiction that $\partial H_1 \cap \mathcal{O}_{nc}^-(C_H) = \emptyset$. If $x \in \partial H_1 \cap \mathcal{O}_{nc}^-(C_H)$ take the smallest m such that $F_1^m(x) \in C_H$. Let a and b be close enough to x such that (i) either (a, x) or (x, b) is contained in H_1 , (ii) $F_1^{m+1}(a) = F_1^{m+1}(b)$, (iii) $F_1^m|(a, b)$, $F_1^{m+1}|(a, x)$ and $F_1^{m+1}|(x, b)$ are diffeomorphisms. Hence, $(a, b) \subset H_1$ that is a contradiction.

Let us prove by contradiction that if $x \in \partial H_1$ then x is not contained in the preorbit of a critical point. Take the smallest m such that $F_1^m(x) = c$ is a critical point. Since $c \notin \mathcal{O}_{nc}^-(C_H)$, there is a small open set W containing c such that h|W is a C^r diffeomorphism. Furthermore, there is a small enough open set V such that (i) V contains x, (ii) $F_1^m|V$ is a diffeomorphism and (iii) $F_1^m(V) \subset W$. Thus, by Lemma 5.1, h|V is also a C^r diffeomorphism that is a contradiction.

Let us prove by contradiction that $F_1(\partial H_1) \cap H_1 = \emptyset$. If $x \in \partial H_1$ and $F_1(x) \in H_1$ then there are small enough open sets V and W such that (i) V contains x, (ii) $F_1|V$ is a diffeomorphism because x is not a critical point of F_1 , (iii) $F_1(V) = W$, (iv) $W \subset H_1$ and so (v) h|W is a C^r diffeomorphism. Hence h|V is also a C^r diffeomorphism that is a contradiction.

There is *i* large enough such that $V_{n_i} \,\subset \, H_1$ and $c' \in F_1^k(V_{n_i})$, for some *k*, and so $F_1^k(H_1) \cap H_1 \neq \emptyset$. Hence, to prove that H_1 is a renormalization maximal interval it is enough to prove, by contradiction, that $F_1(\partial H_1) \subset \partial H_1$. If $x \in \partial H_1$ and $F_1(x) \notin \partial H_1$ and so $F_1(x) \notin \overline{H_1}$, then (i) there is $y \in H_1$ such that $F_1(y) = x$ and (ii) open intervals *V* and *W* with the following properties: (i) *V* contains *x*, (ii) $W \subset H_1$ contains *y*, (iii) $F_1|W$ is a diffeomorphism, (iv) $F_1^m(W) = V$. Since h|W is a C^r diffeomorphism, by Lemma 5.1, we get that h|V is also a C^r diffeomorphism that is a contradiction. Therefore, $K(c') = H_1$ is a renormalization interval containing c' and $P(c') = \mathcal{O}_{nc}^-(C_{H_1})$ is the punctured set of K(c') such that $h|K(c') \setminus P(c')$ is C^r . Furthermore, $int(V_1 \cap \mathcal{B}(K(c'))) \neq \emptyset$.

Proof of Theorem 1. By Lemma 4.2, there is an open interval V_1 such that $h|V_1$ is C^r . If h is not a C^r diffeomorphism in $I \setminus P$, then, by Lemma 6.1, there is a maximal renormalization interval K(c') and a punctured set $P(c') \subset K(c')$ such that h is a C^r diffeomorphism in $K(c') \setminus P(c')$. By Lemma 5.1, h is a C^r diffeomorphism in the punctured basin of attraction $\mathcal{B}_P(J(c'))$.

Let C_f^r be the union of all critical points $c \in C_f$ such that $K(c) \neq \emptyset$ is a maximal renormalization interval and $P(c) \subset K(c)$ is a punctured subset such that h is a C^r diffeomorphism in $K(c) \setminus P(c)$. Let $J = \bigcup_{c \in C_f^r} K(c)$ be the maximal renormalization domain and $P = \bigcup_{c \in C_f^r} P(c)$ the punctured set of J. By Lemma 5.1, h is a C^r diffeomorphism in the punctured basin of attraction $\mathcal{B}_P(J) = \bigcup_{c \in C_f^r} \mathcal{B}_P(J(c'))$.

Let us prove, by contradiction, h is not a C^r diffeomorphism at any open interval $V \subset I \setminus \overline{\mathcal{B}}(J)$. If h is a C^r diffeomorphism at V then, by Lemma 6.1, there is $c \in C_f^r$ such that $int(V \cap \mathcal{B}(K(c))) \neq \emptyset$ which is a contradiction.

Let us prove, by contradiction, that h is not C^1 at any point in $E(f) \cap \partial \mathcal{B}(J)$. By Lemma 4.2, if h is C^1 at some point $x \in E(f) \cap \partial \mathcal{B}(J)$ then there is an open interval Wcontaining x such that h|W is C^s which is a contradiction.

Appendix A. Properties of multimodal maps

A periodic point p with period $n \in \mathbb{N}$ is called a *periodic attractor* if there is an open set V with $p \in \partial V$ such that $\lim_{j \to +\infty} f^{jn}(V) = p$. A periodic point p with period $n \in \mathbb{N}$ is called *neutral* if $|Df^n(p)| = 1$. A periodic point p with period $n \in \mathbb{N}$ is *weak repelling* if p is neutral and there is an open set V with $p \in V$ such that $f^n|V$ is a diffeomorphism and $\lim_{j \to +\infty} (f^n|V)^{-n}(x) = p$ for all $x \in V$. A periodic point p with period $n \in \mathbb{N}$ is a *repellor* if $|Df^n(p)| > 1$. Let us denote by PR(f) the set of all repellor periodic points of f.

Theorem A.1 (Mañé). Let $f: I \to I$ be a C^2 map without weak repelling periodic points and such that $\# \operatorname{Fix}(f^n) < \infty$ for all $n \in \mathbb{N}$. For every $\gamma > 0$, there are C > 0 and $\lambda > 1$ with the following properties:

(1) if $J \subset I$ is an interval whose $\omega(J)$ does not intersect any periodic attractor, and

(2) if $n \in \mathbb{N}$ is such that, for every $0 \leq j \leq n$, $f^j(J) \cap B_{\gamma}(C_f) = \emptyset$,

then

$$\operatorname{Ird}_{f^n}(x, y, z) \leq C |f^n(z) - f^n(x)|$$
 and $|f^n(J)| \geq C \lambda^n |J|$,

for every $x, y, z \in J$ with x < y < z.

Proof. It follows from Mañe's Theorem [13] and the fact that the logarithm of a C^2 map is locally Lipschitz outside the critical set.

Lemma A.2 (Forward capture of a critical point). Let $f: I \to I$ be a C^2 map and $\# \operatorname{Fix}(f^n) < \infty$ for every $n \in \mathbb{N}$. For each interval $J \subset I$, whose $\omega(J)$ does not intersect a periodic attractor, there is $n \in \mathbb{N}$ such that the interior of $f^n(J)$ contains a critical point.

Proof. Let us suppose, by contradiction, that $f^n | \operatorname{int}(J)$ is a diffeomorphism onto its image for every $n \in \mathbb{N}$. Since $\omega(J)$ does not intersect a periodic attractor and a C^2 map does not admit a wandering interval (see [1, 15]), there is k > l > 0 such that $f^k(J) \cap f^l(J) \neq \emptyset$. The closure D of the set $\bigcup_{n\geq 0} f^{n(k-l)}(f^l(J))$ is a forward invariant interval for $f^{(k-l)}$. Thus, $g = f^{2(k-\ell)}|_D$ is monotone map of D into itself. Thus, $\omega_g(x) \subset \operatorname{Fix}(g)$ for every $x \in D$. Since $\#\operatorname{Fix}(g) < \infty$, we get that there is an attracting fixed point $p \in D$ for g. Hence, $\mathcal{O}_f^+(p)$ is an attracting periodic orbit for f intersecting $\omega_f(J)$, contradicting our hypothesis.

Lemma A.3 (Domain shrinking for iterated local diffeomorphisms). Let $f: I \to I$ be a C^2 map and $\# \operatorname{Fix}(f^n) < \infty$ for every $n \in \mathbb{N}$. If $J_1, J_2, \ldots \in I$ is a sequence of open intervals such that

- (1) $\bigcup_{n\geq 1} \omega(J_n)$ does not intersect a periodic attractor and
- (2) $f^{m_n} J_n$ are diffeomorphisms, with m_n tending to ∞ ,

then $|J_n| \to 0$ when n tends to infinity.

Proof. Let us suppose, by contradiction, that there is $\delta > 0$ such that $|J_n| > \delta$, for every $n \ge 1$. Since I is compact, there is an interval L and an infinite subsequence J_{m_1}, J_{m_2}, \ldots of intervals such that $L \subset J_{m_n}$ for every $n \ge 1$. Hence, $f^{\ell}|L$ is a diffeomorphism, for every $\ell \ge 1$, which, by Lemma A.2, is a contradiction.

Following M. Martens [14], a union $J = \bigcup_i J_i$ of pairwise disjoint open intervals J_1, J_2, \ldots is a *nice set*, if the forward orbit of the boundaries $\bigcup_{i=1}^l \partial J_i$ of J do not intersect J.

Lemma A.4 (Nice infinitesimal neighborhoods of critical points). Let $f : I \to I$ be a multimodal map without periodic attractors. For every small $\varepsilon > 0$, there is a nice set $J = \bigcup_{c \in \mathcal{C}_f} (p_c, q_c) \mathcal{N}$ such that $c \in (p_c, q_c) \subset B_{\varepsilon}(c)$ for all $c \in \mathcal{C}_f$.

We note that, if $\{J_k\}$ is the set of connected components of a nice set J then

$$I' = \bigcup_{J_k \cap \mathcal{C}_f \neq \emptyset} J_k$$

is also a nice set. Let \mathcal{N} be the collection of all nice set $J = \bigcup_k (p_k, q_k)$ such that $\mathcal{C}_f \subset J$ and $(p_k, q_k) \cap \mathcal{C}_f \neq \emptyset$ for all k. We note that, if $U, V \in \mathcal{N}$ then $U \cap V \in \mathcal{N}$.

Proof. First, let us show that there is a nice set J such that $C_f \subset J$. Consider the compact positive invariant set

$$\Lambda = \{ x \in I ; f^j(x) \notin B_{\varepsilon}(\mathcal{C}_f) , \forall j \ge 0 \}.$$

For every $c \in C_f$, there is a connected component $J_{c,\Lambda} \supset B_{\varepsilon}(c)$ of $I \setminus \Lambda$. Let $J = \bigcup_{c \in C_f} J_{c,\Lambda}$. Since $\partial J = \bigcup_{c \in C_f} \partial J_{c,\Lambda} \subset \Lambda$, we get $f^j(\partial J) \subset \Lambda$ for every $j \ge 0$. Hence, $f^j(\partial J) \cap J = \emptyset$ for every $j \ge 0$, i.e. J is a nice set and contains C_f . Thus, \mathcal{N} is not an empty collection.

If $c \in \mathcal{C}_f$, either $V \supset B_{\varepsilon}(c)$, for all $V \in \mathcal{N}$, or there exists $V(c) = \bigcup_{\tilde{c} \in \mathcal{C}_f} V_{\tilde{c}}(c) \in \mathcal{N}$ such that $V_c(c) \subset B_{\varepsilon}(c)$ and $\tilde{c} \in V_{\tilde{c}}(c)$ for all $\tilde{c} \in \mathcal{C}_f$.

Let $\mathcal{C}_f^{\varepsilon}$ be the set of $c \in \mathcal{C}_f$ such that $V \supset B_{\varepsilon}(c)$ for all $V \in \mathcal{N}$. For every $c \in \mathcal{C}_f^{\varepsilon}$, let $H(c) = \operatorname{int} \bigcap_{J \in \mathcal{N}} J_c$, where J_c is the connected component of J containing c. Hence, H(c) is a nice interval and

$$H(c) \subset W \text{ for all } W \in \mathcal{N}.$$
 (A.1)

Claim 3. If $c_0 \in C_f$ is non-wandering then $c_0 \notin C_f^{\varepsilon}$ for all $\varepsilon > 0$.

Proof of the claim. Let $\varepsilon > 0$ and $c_0 \in C_f$ be a non-wandering point. Hence, take the smallest $n \geq 1$ such that $f^n(H(c_0)) \cap H(c_0) \neq \emptyset$. Either (i) $f^n(H(c_0)) \not\subset \overline{H(c_0)}$ or (ii) $f^n(H(c_0)) \subset \overline{H(c_0)}$.

Case (i). Take $q \in H(c_0)$ such that $f^n(q) \in f^n(H(c_0)) \cap \overline{H(c_0)}$ and there is a small interval V_q containing q such that $f^n|V_q$ is a diffeomorphism. For every $c \in \mathcal{C}_f$, let U_c be the connected component of $\operatorname{int}(I) \setminus \{q, \dots, f^{n-1}(q)\}$ containing c. We get that $U = \bigcup_{c \in \mathcal{C}_f} U_c$ belongs to \mathcal{N} and $H(c_0) \not\subset U_{c_0}$, because $q \in H(c_0)$ but $q \notin U_{c_0}$, contradicting (A.1).

Case (ii). Since $f^n(H(c_0)) \subset \overline{H(c_0)}$, $g = f^n|_{\overline{H(c_0)}}$ is a multimodal map and $f^n(\partial H(c_0)) \subset \partial H(c_0)$. Since there is no periodic attractor for g, there is a periodic point $q \in H(c_0)$ for the map g. For every $c \in \mathcal{C}_f$, let U_c be the connected component of $\operatorname{int}(I) \setminus \{q, \dots, f^{m-1}(q)\}$ containing c, where m is the period of q with respect to f. We get that $U = \bigcup_{c \in \mathcal{C}_f} U_c$ belongs to \mathcal{N} and $H(c_0) \not\subset U_{c_0}$, because $q \in H(c_0)$ but $q \notin U_{c_0}$, contradicting (A.1).

Now, we consider the case of the wandering critical points. Let $\varepsilon > 0$ and c_0 be a wandering critical point. From Lemma A.2, there is $n \ge 1$ and a non-wandering $\tilde{c} \in C_f$ such that $\tilde{c} \in f^n(H(c_0))$. By the claim above, $\tilde{c} \notin C_f^{\varepsilon}$. Thus, there is $V = \bigcup_{c \in C_f} V_c \in \mathcal{N}$ such that $\partial V_{\tilde{c}} \cap f^n(H(c_0)) \neq \emptyset$. Let $q \in H(c_0)$ be such that $f^n(q) \in \partial V_{\tilde{c}}$ and there is a small interval V_q containing q such that $f^n|V_q$ is a diffeomorphism. For every $c \in C_f$ consider U_c the connected component of $V_c \setminus \{q, \cdots, f^t(q)\}$ containing c. Thus $U = \bigcup_{c \in C_f} U_c \in \mathcal{N}$ and $H(c_0) \notin U_{c_0}$, contradicting (A.1).

Lemma A.5 (Fatness of repellors). Let f be C^r a multimodal map with $r \ge 3$ and no periodic attractors and no neutral points.

- (1) If f is infinitely renormalizable around a critical point c, then there is a renormalization interval J(c) such that $\mathcal{O}_{nc}^{-}(\operatorname{PR}(f))$ is dense in $\mathcal{B}(J(c))$.
- (2) If f is no renormalizable inside a renormalizable interval J, then $\alpha_{nc}(\operatorname{PR}(f))$ contains $\overline{\mathcal{B}(J)}$.

Proof. Let us prove (1). Since f is infinitely renormalizable around c, there is an infinite sequence of intervals J_1, J_2, \ldots such that J_{n+1} is strictly contained in J_n and there is a sequence m_1, m_2, \ldots such that $f^{m_n}|J_n$ is a multimodal map and $c \in f^{m_n}(J_n)$. By taking J_1 sufficiently small, we assume that for every critical point $c' \in J_1$ with $c' \neq c$, there is a sequence l_1, l_2, \ldots such that $m_n l_n < m_{n+1}$ and $c' \in f^{m_n l_n}(J_{n+1})$. Let p_n be a periodic point contained in the boundary ∂J_n of J_n . Hence p_n is a repellor and the set $S = \bigcup_{n>1} \alpha_{nc}(p_n)$ contains $c \in \partial S$. Let us prove that S is dense in the smallest interval set that contains S. By contradiction, suppose that S is not a dense set. Hence, there is an open interval Ksuch that $K \subset J_1 \setminus S$ and $\partial K \subset S$. By forward invariance of S under $f^{m_1}, f^{m_1k}(K) \subset J_1 \setminus S$ and $\partial f^{m_1k}(K) \subset S$ for every k. By Lemma A.2, there is k_1 such that $f^{m_1k_1}(K)$ contains some critical point $c' \in J_1$. Hence, there is n large enough and l_n such that $f^{m_n l_n}(J_{n+1}) \subset$ $f^{m_1k_1}(K)$. Hence, there is k_2 such that $f^{m_{n+1}}(J_{n+1}) \subset f^{m_1k_2}(K)$. Since $c \in f^{m_{n+1}}(J_{n+1})$, we get $c \in f^{m_1k_2}(K)$. Noting that p_n converges to c, we obtain that $f^{m_1k_2}(K)$ contain some p_n , for n large, which contradicts that $f^{m_1k_2}(K) \subset J_1 \setminus S$. Hence, S is dense in the smallest interval set that contains S. Since $c \in \partial S$ is a turning point, S is dense in a small neighborhood of c. Hence, there is a renormalization interval J(c), small enough, containing c that is contained in the closure of S.

Let us prove (2). Since J is a renormalization interval, there is m such that $f^m|J$ is a multimodal map. Let $p \in J$ be a periodic repellor with period k of the map $f^m|J$. Since $\alpha_{nc}(p)$ is a closed set, it is enough to prove that $\alpha_{nc}(p)$ is dense in J. By contradiction, suppose that $\alpha_{nc}(p)$ is not a dense set. Hence, there is an open interval K such that $K \subset J \setminus \alpha_{nc}(p)$ and $\partial K \subset \alpha_{nc}(p)$. By forward invariance of $\alpha_{nc}(p)$ under f^m , $f^{mk}(K) \subset J_1 \setminus \alpha_{nc}(p)$ and $\partial f^{mk}(K) \subset \alpha_{nc}(p)$ for every k. By Lemma A.2, there is a sequence k_1, k_2, \ldots such that $K_n = f^{mk_n}(K)$ contains some critical point $c_n \in J$. Since, the set of critical points in J is finite, there is a critical point $c \in J$ and $k_{l_1} < k_{l_2}$ such that K_{l_1} and K_{l_2} contain the critical point $c \in J$. Hence, $K_{l_1} \cap K_{l_2} \neq \emptyset$. Since

$$\partial K_{l_1} \subset \alpha_{nc}(p)$$
, $\partial K_{l_1} \subset \alpha_{nc}(p)$, $K_{l_1} \cap \alpha_{nc}(p) = \emptyset$ and $K_{l_1} \cap \alpha_{nc}(p) = \emptyset$.

we obtain that $K_{l_1} = K_{l_2}$. In particular, $f^{m(k_l-k_{l_1})}|K_{l_1}$ is a multimodal and K_{l_1} is strictly contained in J which contradicts that f is no renormalizable inside of the renormalizable interval J. Hence, $\alpha_{nc}(p)$ contains the closure of J. Hence, by definition of alpha limit, $\alpha_{nc}(p)$ contains $\overline{\mathcal{B}}(J)$.

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José F. Alves, Centro de Matemática da Universidade do Porto, Rua do Campo Alegre 687, 4169-007 Porto, Portugal

E-mail address: jfalves@fc.up.pt *URL*: http://www.fc.up.pt/cmup/jfalves

VILTON PINHEIRO, DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE FEDERAL DA BAHIA, AV. ADEMAR DE BARROS S/N, 40170-110 SALVADOR, BRAZIL. *E-mail address*: viltonj@ufba.br

URL: http://www.pgmat.ufba.br

ALBERTO A. PINTO, LIAAD-INESC PORTO LA E DEPARTAMENTO DE MATEMÁTICA, FACULDADE DE CIÊNCIAS DO PORTO, RUA DO CAMPO ALEGRE 687, 4169-007 PORTO, PORTUGAL *E-mail address*: aapinto@fc.up.pt

URL: http://www.ccog.up.pt/index.php/alberto-adrego-pinto