HYPERCYCLIC OPERATORS AND ROTATED ORBITS WITH POLYNOMIAL PHASES

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ABSTRACT. An important result of León-Saavedra and Müller says that the rotations of hypercyclic operators remain hypercyclic. We provide extensions of this result for orbits of operators which are rotated by unimodular complex numbers with polynomial phases. On the other hand, we show that this fails for unimodular complex numbers whose phases grow to infinity too quickly, say at a geometric rate. A further consequence of our work is a notable strengthening of a result due to Shkarin which concerns variants of León-Saavedra and Müller's result in a non-linear setting.

1. INTRODUCTION

Linear dynamical systems have surprising properties. By a linear dynamical system, we mean a couple (X, T) where X is a (complex) topological vector space and $T \in \mathfrak{L}(X)$ is a bounded linear operator on X. We are interested in transitive linear dynamical systems; namely we ask that T is hypercyclic: there exists $x \in X$, called a hypercyclic vector for T, such that its T-orbit $\{T^n x; \ n \geq 0\}$ is dense in X. We shall denote by $HC(T)$ the set of hypercyclic vectors for T. A comprehensive treatment on linear dynamical systems is carried out in the recent books [3], [8].

Linear dynamical systems have rigid properties which are not shared by general dynamical systems. For instance, Ansari has shown in [1] that, for any hypercyclic operator T and for any $q \geq 1$, T^q is hypercyclic and $HC(T^q) = HC(T)$. Some years later, León and Müller were interested in the rotations of a hypercyclic vector. They proved in [12] that $\{T^n x; \ n \geq 1\}$ is dense in X iff $\{\lambda T^n x; \ n \geq 1, \ \lambda \in \mathbb{T}\}$ is dense in X, where $\mathbb{T}$ is the unit circle. In particular, for any $\theta \in \mathbb{R}$, $HC(T) = HC(e^{i\theta} T)$. For further results along this line of research we refer to [2], [5], [12], [13], [14], [16].

To understand the influence of "rotations" in linear dynamics it would be desirable to fully answer the following question.

Question. Let $T \in \mathfrak{L}(X)$ be a hypercylic operator and let $x \in X$. For which sequences $(\lambda_n) \subset \mathbb{T}^{\mathbb{N}}$ do we have $\overline{\{T^n x; \ n \geq 1\}} = X$ if and only if $\overline{\{\lambda_n T^n x; \ n \geq 1\}} = X$?

Let us first show that an affirmative answer to the above question already imposes some restrictions to the sequence (λ_n). Suppose $T \in \mathfrak{L}(X)$ is a hypercyclic operator and consider $x \in HC(T)$. Take any non-zero functional $x^* \in X^*$ and define $\lambda_n = |x^*(T^n x)|/x^*(T^n x)$ if $x^*(T^n x) \neq 0$ and $\lambda_n = 1$ if $x^*(T^n x) = 0$. Observe that $\lambda_n \in \mathbb{T}$ for every $n \in \mathbb{N}$. We claim that the set $\{\lambda_n T^n x; \ n \geq 1\}$ is not dense in X. To see this, we argue by contradiction and so assume that $\{\lambda_n T^n x; \ n \geq 1\}$ is dense in X. Then, the set $\{x^*(\lambda_n T^n x); \ n \geq 1\}$ is dense in $\mathbb{C}$. However, the last is a contradiction since, by the construction of λ_n, $x^*(\lambda_n T^n x) \geq 0$ for every $n \in \mathbb{N}$.

We may also obtain a counterexample if we require moreover that (λ_n) belongs to a finite subset of $\mathbb{T}$. Indeed, let $T \in \mathfrak{L}(X)$ be a hypercyclic operator on a Banach space X and take $x \in HC(T)$. Then define $\lambda_n = 1$ if $\|T^n x - x\| \geq \|x\|/2$ and $\lambda_n = -1$ if $\|T^n x - x\| < \|x\|/2$. It readily follows that $\|\lambda_n T^n x - x\| \geq \|x\|/2$ for every $n \geq 1$.

On the contrary, the León-Saavedra and Müller result shows that the answer is true provided $(\lambda_n) = (e^{in\theta})$, $\theta \in \mathbb{R}$. Our main result says that the rotations of the orbit by unimodular complex numbers with linear phases can be replaced by rotations consisting of unimodular complex numbers with polynomial phases.

Theorem 1.1. *Let X be a complex topological vector space, let $T \in \mathfrak{L}(X)$ and let $x \in X$. The following are equivalent.*

    (i) *x is hypercyclic for T.*
    (ii) *$\overline{\{e^{iP(n)}T^n x; \ n \geq 1\}} = X$ for any polynomial $P \in \mathbb{R}[t]$.*
    (iii) *$\overline{\{e^{iP(n)}T^n x; \ n \geq 1\}}^o \neq \emptyset$ for some polynomial $P \in \mathbb{R}[t]$.*

The equivalence of (ii) and (iii) in this theorem is in fact an extension of Bourdon-Feldman's theorem, which says that $\{T^n x; \ n \geq 0\}$ is dense if and only if it is somewhere dense, see [4]. Actually, this result holds even for the projective orbit i.e., if the set $\{\lambda T^n x; \ \lambda \in \mathbb{C}, n \geq 0\}$ is somewhere dense then it is everywhere dense. In this case T is called supercyclic and x is called a supercyclic vector for T. The supercyclic version of Bourdon-Feldman's theorem will be used in the proof of Theorem 1.1. That (i) implies (ii) will depend on an extension of a result of Shkarin which has its own interest.

The problem of rotations of hypercyclic vectors can also be studied for stronger forms of hypercyclicity. We recall that the lower density of a set of natural numbers A is defined by

$$\underline{\mathrm{dens}}(A) := \liminf_{N \to \infty} \frac{\mathrm{card}(A \cap [1, N])}{N}.$$

An operator $T \in \mathfrak{L}(X)$ is called frequently hypercyclic provided there exists $x \in X$ such that, for any $U \subset X$ open and nonempty, the set $\{n \in \mathbb{N}; \ T^n x \in U\}$ has positive lower density. The vector x is then called frequently hypercyclic for T and the set of T-frequently hypercyclic vectors will be denoted by $FHC(T)$.

It has be shown in [3] that frequent hypercyclicity is invariant under rotation: for any $\lambda \in \mathbb{T}$, $FHC(T) = FHC(\lambda T)$. Here is the polynomial version of this property, to be proved in Section 3 of the paper.

Theorem 1.2. *Let X be a complex topological vector space, let $T \in \mathfrak{L}(X)$ and let $x \in X$. The following are equivalent.*

    (i) *$x \in FHC(T)$.*
    (ii) *For any $P \in \mathbb{R}[t]$, for any $U \subset X$ open and nonempty, $\{n \in \mathbb{N}; \ e^{iP(n)}T^n x \in U\}$ has positive lower density.*
    (iii) *There exists $P \in \mathbb{R}[t]$ such that, for any $U \subset X$ open and nonempty, $\{n \in \mathbb{N}; \ e^{iP(n)}T^n x \in U\}$ has positive lower density.*

In Section 4, we investigate other choices of phases. We show that $\{e^{if(n)}T^n x; \ n \geq 1\}$ does not need to be dense for all $x \in HC(T)$ provided f goes sufficiently fast to infinity, for instance if f has exponential growth. On the contrary, we extend Theorem 1.1 to sequences which do not grow too quickly to infinity, for instance for sequences like $f(n) = n^a \log^b(n+1)$. The polynomial case (Theorem 1.1) plays a crucial role in this extension.

Finally, in Section 5, we study the link between the problem of rotations of hypercyclic vectors and the theory of uniformly distributed sequences. In particular, we point out that the uniform distribution of $(f(n))$ is not sufficient to ensure that $\{e^{if(n)}T^n x;\ n \geq 1\}$ is dense for any $x \in HC(T)$. Nevertheless, uniform distribution will be a useful tool to obtain generic statements.

2. Proof of the main result and an extension of a theorem of Shkarin

2.1. **The strategy.** As mentioned in the introduction, the equivalence between (i) and (ii) in Theorem 1.1 is already known when P is a polynomial of degree 1. This was first done when $P(n) = 2\pi n\theta$ with $\theta = \frac{p}{q} \in \mathbb{Q}$, $q \geq 1$. Indeed, in that case, $e^{2\pi iqn\theta}T^{qn}x = T^{qn}x$ and Ansari has shown in [1] that $HC(T) = HC(T^q)$ for any $q \geq 1$. If θ does not belong to $\mathbb{Q}$, this result goes back to the paper [12] by León-Saavedra and Müller: they showed that for any $\theta \in \mathbb{R}$, $HC(e^{i\theta}T) = HC(T)$, namely that given $x \in HC(T)$ and any $y \in X$, there exists a sequence (n_k) such that $e^{2\pi i n_k \theta}T^{n_k}x \to y$. This was later improved by Shkarin in [16] who proved the following result (see also [14] for a similar abstract result).

Theorem 2.1. *(Shkarin) Let T be a hypercyclic continuous linear operator on a topological vector space X and let g be a generator of a compact topological group G. Then $\{(T^n x, g^n);\ n \geq 1\}$ is dense in $X \times G$ for any $x \in HC(T)$.*

In particular, one may apply this result with $G = \mathbb{T}$ and $g = e^{2\pi i\theta}$, which generates $\mathbb{T}$ provided $\theta \notin \mathbb{Q}$. This implies the León-Saavedra and Müller theorem since for any $y \in X$ we may pick a sequence (n_k) such that $T^{n_k}x \to y$ and $e^{2\pi i n_k \theta} \to 1$.

This observation in the starting point of our investigations. We shall prove by induction a polynomial variant of Shkarin's result. Roughly speaking, it will say that, for any polynomial $P \in \mathbb{Z}[t]$ and for any $(x, y) \in HC(T) \times X$ there exists a sequence of positive integers (n_k) such that $T^{n_k}x \to y$ and $g^{P(n_k)} \to 1_G$. At a first glance this seems weaker than Theorem 2.1, since we do not get the density of $\{(T^n x, g^{P(n)});\ n \geq 1\}$. And of course we can not do better because we do not require that g is a generator of G. This allows us to handle the case of a rational phase in the same process. Moreover, this variant is exactly what is needed to deduce Theorem 1.1. If we further assume that $\{g^{P(n)};\ n \geq 1\}$ is dense in G, in which case g is necessarily a generator of G, then we will be able to show the density of $\{(T^n x, g^{P(n)});\ n \geq 1\}$. Thus, we will obtain the full extension of Shkarin's result in our setting.

At this stage we need to introduce notations. Let G be an abelian compact topological group and let $p \geq 1$. For $\mathbf{g} = (g_1, \ldots, g_p) \in G^p$, $\mathbf{d} = (d_1, \ldots, d_p) \in \mathbb{N}^p$, and $u, v \in X$, we set

$$F_{u,v}^{\mathbf{g},\mathbf{d}} = \left\{ h \in G;\ (v, h, 1_G, \ldots, 1_G) \in N_u^{\mathbf{g},\mathbf{d}} \right\}$$

where

$$N_u^{\mathbf{g},\mathbf{d}} = \overline{\left\{ (T^n u, g_1^{n^{d_1}}, \ldots, g_1^n, g_2^{n^{d_2}}, \ldots, g_2^n, \ldots, g_p^{n^{d_p}}, \ldots, g_p^n);\ n \geq 1 \right\}}.$$

Our variants of Shkarin theorem read as follows.

Theorem 2.2. *Let T be a hypercyclic continuous linear operator on a topological vector space X, let $p \geq 1$ and let $g_1, \ldots, g_p$ be elements of a compact topological group G. Let also $\mathbf{d} = (d_1, \ldots, d_p) \in \mathbb{N}^p$. Then for any $u \in HC(T)$ and any $v \in X$, $1_G \in F_{u,v}^{\mathbf{g},\mathbf{d}}$.*

Theorem 2.3. *Let T be a hypercyclic continuous linear operator on a topological vector space X. Let $P \in \mathbb{Z}[t]$ and let $g \in G$ be such that $\{g^{P(n)}; n \geq 1\}$ is dense in G. Then $\{(T^n x, g^{P(n)}); n \geq 1\}$ is dense in $X \times G$ for any $x \in HC(T)$.*

The proof of (iii) implies (i) in Theorem 1.1 will need the following variant of the Bourdon-Feldman theorem.

Theorem 2.4. *Let $T \in \mathcal{L}(X)$ and let $x \in X$. If $\overline{\{\lambda T^n x;\ \lambda \in \mathbb{T}, n \geq 0\}}^o \neq \emptyset$ then $\overline{\{\lambda T^n x;\ \lambda \in \mathbb{T}, n \geq 0\}} = X$.*

Before to prove Theorems 2.2, 2.3 and 2.4 let us show how to deduce Theorem 1.1 from them. For simplicity, throughout this section, we will assume that X and G are metrisable. The same proofs work when X or G are not metrisable, replacing everywhere sequences by nets. To show that (i) implies (ii), let $x \in HC(T)$, let $G = \mathbb{T}$ and let $P(n) = \theta_p n^p + \cdots + \theta_1 n$ (we may always assume that it has no constant term). Setting $g_k = e^{i\theta_k}$ and $d_k = k$ for $k = 1, \ldots, p$, we may apply Theorem 2.2. In particular, given any $y \in X$, one may find a sequence $(n_k)_k$ such that

$$T^{n_k} x \to y,\ e^{i n_k^p \theta_p} \to 1, \ldots, e^{i n_k \theta_1} \to 1.$$

This clearly implies that $e^{iP(n_k)} T^{n_k} x \to y$. It remains to prove that (iii) implies (i). (iii) yields $\overline{\{\lambda T^n x;\ \lambda \in \mathbb{T}, n \geq 0\}}^o \neq \emptyset$ so, Theorem 2.4 gives $\overline{\{\lambda T^n x;\ \lambda \in \mathbb{T}, n \geq 0\}} = X$ and we conclude by the León-Saavedra and Müller theorem.

2.2. **Preparatory lemmas.** Let us turn to the proof of Theorem 2.2. We fix an operator T acting on a topological vector space X and a compact group G. We will need the following elementary lemma.

Lemma 2.5. *Let $g, h \in G$, $d \geq 1$, $m \geq 1$ and let (n_k) be a sequence of integers such that*

$$g^{n_k} \to 1_G,\ g^{n_k^2} \to 1_G, \ldots, g^{n_k^{d-1}} \to 1_G,\ g^{n_k^d} \to h.$$

Then

$$g^{(n_k+m)^d} \to h g^{m^d}.$$

Proof. Write

$$
\begin{aligned}
g^{(n_k+m)^d} &= \prod_{j=0}^{d} g^{n_k^j \binom{d}{j} m^{d-j}} \\
&= g^{n_k^d} \left(\prod_{j=1}^{d-1} \left(g^{n_k^j} \right)^{\binom{d}{j} m^{d-j}} \right) g^{m^d} \\
&\to h g^{m^d}.
\end{aligned}
$$

$\square$

The sets $F_{u,v}^{\mathbf{g},\mathbf{d}}$ share some properties which are summarized below.

Lemma 2.6. *Let $u, v, w \in X$, $p \geq 1$, $\mathbf{g} \in G^p$, $\mathbf{d} \in \mathbb{N}^p$. The following hold.*
  (i) *$F_{u,v}^{\mathbf{g},\mathbf{d}}$ is closed; $F_{u,v}^{\mathbf{g},\mathbf{d}} \subset F_{Tu,Tv}^{\mathbf{g},\mathbf{d}}$.*
  (ii) *$F_{u,v}^{\mathbf{g},\mathbf{d}} F_{v,w}^{\mathbf{g},\mathbf{d}} \subset F_{u,w}^{\mathbf{g},\mathbf{d}}$.*
  (iii) *Let $(v_k) \subset X$ and $(h_k) \subset G$ be such that $v_k \to v$, $h_k \to h$ and $h_k \in F_{u,v_k}^{\mathbf{g},\mathbf{d}}$. Then $h \in F_{u,v}^{\mathbf{g},\mathbf{d}}$.*

Proof. (i) is trivial. (ii) follows from Lemma 2.5. Indeed, for $\mathcal{O}$ an open neighbourhood of 1_G in G and $h \in G$, let us denote

$$\mathcal{O}_h = (h.\mathcal{O}) \times \mathcal{O} \times \cdots \times \mathcal{O} \subset G^{d_1 + \cdots + d_p}.$$

Let $h_1 \in F_{u,v}^{\mathbf{g},\mathbf{d}}$, $h_2 \in F_{v,w}^{\mathbf{g},\mathbf{d}}$ and let $W \times \mathcal{O}_{h_1 h_2}$ be an open neighborhood of the point $(w, h_1 h_2, 1_G, \ldots, 1_G)$ in $X \times G^{d_1 + \cdots + d_p}$. Since $h_2 \in F_{v,w}^{\mathbf{g},\mathbf{d}}$, there exists $m \in \mathbb{N}$ such that

$$T^m v \in W \text{ and } (g_1^{m^{d_1}}, \ldots, g_p^m) \in \mathcal{O}_{h_2}.$$

Let V be an open neighbourhood of v such that $T^m V \subset W$. Since $h_1 \in F_{u,v}^{\mathbf{g},\mathbf{d}}$, an application of Lemma 2.5 gives the existence of an integer n satisfying

- $T^n u \in V \implies T^{n+m} u \in W$;
- $g_1^{(n+m)^{d_1}} \in h_1 h_2 \mathcal{O}$;
- $g_l^{(n+m)^k} \in \mathcal{O}$ provided $l = 1$ and $k \le d_1 - 1$ or $l \ge 2$ and $k \le d_l$.

This shows that $h_1 h_2 \in F_{u,w}^{\mathbf{g},\mathbf{d}}$.

The proof of (iii) goes along the same lines. Let $V \times \mathcal{O}_h$ be an open neighbourhood of $(v, h, 1_G, \ldots, 1_G)$ in $X \times G^{d_1 + \cdots + d_p}$. There exists $k \ge 1$ such that $(v_k, h_k, 1_G, \ldots, 1_G) \in V \times \mathcal{O}_h$ and since $h_k \in F_{u,v_k}^{\mathbf{g},\mathbf{d}}$, there exists $n \ge 1$ such that $(T^n u, g_1^{n^{d_1}}, \ldots, g_p) \in V \times \mathcal{O}_h$. Thus, $h \in F_{u,v}^{\mathbf{g},\mathbf{d}}$. $\qquad\square$

For a proof of the following lemma see, for instance, [9].

Lemma 2.7. *A closed subsemigroup of a compact topological group is a subgroup.*

2.3. **Proof of Theorem 2.2.** We are now ready for the proof of Theorem 2.2. We proceed by induction on $d_1 + \cdots + d_p$. We first assume that $d_1 + \cdots + d_p = 1$ and let $g \in G$, $u \in HC(T)$, $v \in X$. Define $G_0 = \overline{\{g^n; \ n \ge 0\}}$. G_0 is an abelian compact topological group and g is a generator of G_0. By applying Shkarin's result, $\{(T^n u, g^n); \ n \ge 1\}$ is dense in $X \times G_0$. In particular, $1_G \in F_{u,v}^{\mathbf{g},\mathbf{d}}$.

Suppose now that $d_1 + \cdots + d_p \ge 2$ and let $u \in HC(T)$, $v \in X$. We set $\mathbf{d}' = (d_1 - 1, \ldots, d_p)$. We consider any $x, y \in HC(T)$. By the induction hypothesis, $1_G \in F_{x,y}^{\mathbf{g},\mathbf{d}'}$. This yields, by compactness of G, that $F_{x,y}^{\mathbf{g},\mathbf{d}}$ is nonempty. In particular, Lemma 2.6 tells us that $F_{x,x}^{\mathbf{g},\mathbf{d}}$ is a closed subsemigroup of G, hence a closed subgroup of G. Moreover, if we use again Lemma 2.6, we observe that

$$\begin{cases} F_{x,x}^{\mathbf{g},\mathbf{d}} F_{x,y}^{\mathbf{g},\mathbf{d}} & \subset & F_{x,y}^{\mathbf{g},\mathbf{d}} \\ F_{x,y}^{\mathbf{g},\mathbf{d}} F_{y,x}^{\mathbf{g},\mathbf{d}} & \subset & F_{x,x}^{\mathbf{g},\mathbf{d}}. \end{cases}$$

Since $F_{x,y}^{\mathbf{g},\mathbf{d}}$ and $F_{y,x}^{\mathbf{g},\mathbf{d}}$ are both nonempty, $F_{x,y}^{\mathbf{g},\mathbf{d}}$ contains a coset of $F_{x,x}^{\mathbf{g},\mathbf{d}}$ and is contained in a coset of $F_{x,x}^{\mathbf{g},\mathbf{d}}$. Thus it is a coset of $F_{x,x}^{\mathbf{g},\mathbf{d}}$ (at this point, it is important to notice that we need that G is abelian).

We apply this for $x = u$ and $y = Tu$: there exists $g \in G$ such that $F_{u,Tu}^{\mathbf{g},\mathbf{d}} = g F_{u,u}^{\mathbf{g},\mathbf{d}}$. Now, using again (i) and (ii) of Lemma 2.6, we get

$$\begin{cases} F_{Tu,T^2u}^{\mathbf{g},\mathbf{d}} \supset F_{u,Tu}^{\mathbf{g},\mathbf{d}} \supset g F_{u,u}^{\mathbf{g},\mathbf{d}} \\ F_{u,T^2u}^{\mathbf{g},\mathbf{d}} \supset F_{u,Tu}^{\mathbf{g},\mathbf{d}} F_{Tu,T^2u}^{\mathbf{g},\mathbf{d}} \supset g^2 F_{u,u}^{\mathbf{g},\mathbf{d}}. \end{cases}$$

Since $F^{\mathbf{g},\mathbf{d}}_{u,T^2u}$ is a coset of $F^{\mathbf{g},\mathbf{d}}_{u,u}$, this in turn yields $F^{\mathbf{g},\mathbf{d}}_{u,T^2u} = g^2 F^{\mathbf{g},\mathbf{d}}_{u,u}$. Repeating the process, we obtain that, for any $n \geq 1$, $F^{\mathbf{g},\mathbf{d}}_{u,T^nu} = g^n F^{\mathbf{g},\mathbf{d}}_{u,u}$. Now, we use again the induction hypothesis, but for $d_1 + \cdots + d_p = 1$. This gives a sequence (n_k) such that $g^{n_k} \to 1_G$ and $T^{n_k}u \to v$. By the last part of Lemma 2.6, $1_G \in F^{\mathbf{g},\mathbf{d}}_{u,v}$, which achieves the proof of Theorem 2.2.

2.4. **Proof of Theorem 2.3.** Fix $x \in HC(T)$ and let $y \in X$, $m \in \mathbb{N}$. Let also d be the degree of the polynomial P, $d \geq 1$. Then $T^m x \in HC(T)$ and by Theorem 2.2 there exists a sequence of positive integers (n_k) such that

$$T^{n_k}(T^m x) \to y, \; g^{n_k} \to 1_G, \; g^{n_k^2} \to 1_G, \ldots, \; g^{n_k^d} \to 1_G.$$

From the last we deduce that

$$T^{n_k+m}x \to y, \; g^{(n_k+m)^j} \to g^{m^j} \; \text{ for every } \; j = 0, 1, \ldots, d$$

and this in turn implies $T^{n_k+m}x \to y$, $g^{P(n_k+m)} \to g^{P(m)}$. Thus,

$$(y, g^{P(m)}) \in \overline{\{(T^n x, g^{P(n)}); \; n \geq 1\}} \; \text{ for every pair } (y,m) \in X \times \mathbb{N}.$$

Since $\{g^{P(m)}; m \geq 1\}$ is dense in G the conclusion follows.

2.5. **An extension of the Bourdon-Feldman result.** The next series of lemmas will be used in the proof of Theorem 2.4. This kind of approach has appeared in [6] and borrows ideas from [15].

Lemma 2.8. *Let x,y be vectors in X. If*

$$\overline{\{\lambda T^n x; \; \lambda \in \mathbb{T}, n \geq 0\}}^o \cap \overline{\{\lambda T^n y; \; \lambda \in \mathbb{T}, n \geq 0\}}^o \neq \emptyset$$

then

$$\overline{\{\lambda T^n x; \; \lambda \in \mathbb{T}, n \geq 0\}}^o = \overline{\{\lambda T^n y; \; \lambda \in \mathbb{T}, n \geq 0\}}^o.$$

Proof. There exist $\alpha \in \mathbb{T}$ and a positive integer k such that

$$\alpha T^k x \in \overline{\{\lambda T^n x; \; \lambda \in \mathbb{T}, n \geq 0\}}^o \cap \overline{\{\lambda T^n y; \; \lambda \in \mathbb{T}, n \geq 0\}}^o.$$

From the last we deduce that $\overline{\{\lambda T^n x; \; \lambda \in \mathbb{T}, n \geq k\}} \subset \overline{\{\lambda T^n y; \; \lambda \in \mathbb{T}, n \geq 0\}}$ and since $\overline{\{\lambda T^n x; \; \lambda \in \mathbb{T}, n \leq k\}}^o = \emptyset$ the inclusion

$$\overline{\{\lambda T^n x; \; \lambda \in \mathbb{T}, n \geq 0\}}^o \subset \overline{\{\lambda T^n y; \; \lambda \in \mathbb{T}, n \geq 0\}}^o$$

follows. Interchanging the roles of x and y in the previous argument we conclude the reverse inclusion and we are done. $\qquad\square$

Lemma 2.9. *Let $x \in X$. For every non-zero complex number μ, $\overline{\{\lambda T^n x; \; \lambda \in \mathbb{T}, n \geq 0\}}^o = \overline{\{\lambda T^n(\mu x); \; \lambda \in \mathbb{T}, n \geq 0\}}^o.$*

Proof. We first assume that $\overline{\{\lambda T^n(x); \lambda \in \mathbb{T}, n \geq 0\}}^o \neq \emptyset$ and let U be a nonempty open subset of X such that $U \subset \overline{\{\lambda T^n x; \; \lambda \in \mathbb{T}, n \geq 0\}}^o$. There exist a complex number ρ and a positive integer m such that

(2.1) $$\rho U \cap U \neq \emptyset$$

and

(2.2) $$\rho^m = \mu.$$

The inclusion $\rho^l U \subset \overline{\{\lambda T^n(\rho^l x); \lambda \in \mathbb{T}, n \geq 0\}}^o$ trivially holds for every non-negative integer l. Because of (2.1) we get

$$\rho^l U \cap \rho^{l+1} U \neq \emptyset \text{ for every } l = 0, 1, 2, \ldots.$$

Hence, for every $l = 0, 1, \ldots, m - 1$

$$\overline{\{\lambda T^n(\rho^l x); \lambda \in \mathbb{T}, n \geq 0\}}^o \cap \overline{\{\lambda T^n(\rho^{l+1} x); \lambda \in \mathbb{T}, n \geq 0\}}^o \neq \emptyset$$

and by (2.2) and Lemma 2.8 we conclude that

$$\overline{\{\lambda T^n x; \ \lambda \in \mathbb{T}, n \geq 0\}}^o = \overline{\{\lambda T^n(\mu x); \ \lambda \in \mathbb{T}, n \geq 0\}}^o.$$

If we now assume that $\overline{\{\lambda T^n(x); \lambda \in \mathbb{T}, n \geq 0\}}^o = \emptyset$, then we shall have

$$\overline{\{\lambda T^n(\mu x); \lambda \in \mathbb{T}, n \geq 0\}}^o = \emptyset$$

for any $\mu \neq 0$, otherwise

$$\overline{\{\lambda T^n(x); \lambda \in \mathbb{T}, n \geq 0\}}^o = \overline{\left\{\lambda T^n\left(\frac{1}{\mu}\mu x\right); \lambda \in \mathbb{T}, n \geq 0\right\}}^o$$

would be nonempty. $\qquad\square$

Proof of Theorem 2.4. Since $\overline{\{\lambda T^n x; \ \lambda \in \mathbb{T}, n \geq 0\}}^o \neq \emptyset$, Bourdon-Feldman's theorem implies that T is supercyclic, in fact x is a supercyclic vector for T. By the density of supercyclic vectors there exists a supercyclic vector z for T such that $z \in \overline{\{\lambda T^n x; \lambda \in \mathbb{T}, n \geq 0\}}^o$. Applying Lemma 2.9 we get $\mu z \in \overline{\{\lambda T^n x; \lambda \in \mathbb{T}, n \geq 0\}}^o$ for every non-zero complex number μ. Since z is supercyclic for T and the set $\{\lambda T^n x; \ \lambda \in \mathbb{T}, n \geq 0\}$ is T-invariant, the result follows. $\qquad\square$

2.6. **Consequences and remarks.** Theorem 2.1 is a particular case of a more general result which can be found in [16]. Let us recall that a continuous map $T : X \to X$, where X is a topological space, is universal provided there exists $x \in X$, called universal vector for T, such that $\{T^n x; \ n \geq 1\}$ is dense in X. We denote by $\mathcal{U}(T)$ the set of universal vectors for T.

Theorem 2.1 can be extended to nonlinear dynamical systems whose set of universal vectors satisfies connectedness assumptions. Precisely, Shkarin has proved the following result:

> Let X be a topological space, let $T : X \to X$ be a continuous map and let g be a generator of a compact topological group G. Assume also that there is a nonempty subset Y of $\mathcal{U}(T)$ such that $T(Y) \subset Y$ and Y is path connected, locally path connected and simply connected. Then the set $\{(T^n x, g^n); \ n \geq 1\}$ is dense in $X \times G$ for any $x \in Y$.

Starting from this result and with exactly the same proof, we can get the following statement.

> Let X be a topological space, let $T : X \to X$ be a continuous map, let $g_1, \ldots, g_p$ be elements of a compact topological group G and let $\mathbf{d} = (d_1, \ldots, d_p) \in \mathbb{N}^p$. Assume also that there is a nonempty subset Y of $\mathcal{U}(T)$ such that $T(Y) \subset Y$ and Y is path connected, locally path connected and simply connected. Then $X \times \{(1_G, \ldots, 1_G)\} \subset N_u^{\mathbf{g},\mathbf{d}}$ for any $u \in Y$.

We shall need later a variant of Theorem 2.3, where we allow the use of several polynomials.

Corollary 2.10. *Let $P_1, \ldots, P_r$ be real polynomials and let E be the closure of the set $\{e^{iP_1(n)}, \ldots, e^{iP_r(n)}\}; \ n \geq 1\}$. Let $T \in \mathfrak{L}(X)$ be hypercyclic and let $x \in HC(T)$. Then $\{(T^n x, e^{iP_1(n)}, \ldots, e^{iP_r(n)}); \ n \geq 1\}$ is dense in $X \times E$.*

Proof. Let $d = \max(\deg(P_1), \ldots, \deg(P_r))$ and let us write $P_p(x) = \sum_{j=0}^{d} \theta_{j,p} x^j$ for $p = 1, \ldots, r$. Let $y \in X$, $m \geq 1$ and let (n_k) be a sequence of integers such that

$$T^{n_k}(T^m x) \to y, \ e^{i n_k^l \theta_{j,p}} \to 1 \text{ for } 1 \leq j, l \leq d, \ 1 \leq p \leq r.$$

Then, as in the proof of Theorem 2.3, $T^{n_k + m} \to y$ and $e^{iP_p(n_k + m)} \to e^{iP_p(m)}$ for any p in $\{1, \ldots, r\}$. ∎

This corollary is particularly interesting when the closure of $\{(e^{iP_1(n)}, \ldots, e^{iP_r(n)}); \ n \geq 1\}$ is equal to $\mathbb{T}^r$. This a well-known phenomenon in the theory of uniformly distributed sequences.

Definition 2.11. We say that the real polynomials $P_1, \ldots, P_r$ are $\pi\mathbb{Q}$-independent provided for any $h \in \mathbb{Z}^r$, $h \neq 0$, the polynomial $h_1 P_1 + \cdots + h_r P_r$ does not belong to $\pi\mathbb{Q}[t]$.

A fundamental theorem in the theory of uniformly distributed sequences, see [10], says that if $P_1, \ldots, P_r$ is a $\pi\mathbb{Q}$-independent family of real polynomials, then the sequence $(P_1(n), \ldots, P_r(n))$ is uniformly distributed modulo 1. Hence, $\{(e^{iP_1(n)}, \ldots, e^{iP_r(n)}); \ n \geq 1\}$ is dense in $\mathbb{T}^r$.

Corollary 2.12. *Let $P_1, \ldots, P_r$ be real polynomials which are $\pi\mathbb{Q}$-independent. Let $T \in \mathfrak{L}(X)$ be hypercyclic and let $x \in HC(T)$. Then $\{(T^n x, e^{iP_1(n)}, \ldots, e^{iP_r(n)}); \ n \geq 1\}$ is dense in $X \times \mathbb{T}^r$.*

3. Rotations of frequently hypercyclic vectors

This section is devoted to the proof of Theorem 1.2. We first need an elementary lemma on sets with positive lower density. Its proof can be found e.g. in [3, Lemma 6.29].

Lemma 3.1. *Let $A \subset \mathbb{N}$ have positive lower density. Let also $I_1, \ldots, I_q \subset \mathbb{N}$ with $\bigcup_{j=1}^{q} I_j = \mathbb{N}$, and let $n_1, \ldots, n_q \in \mathbb{N}$. Then $B := \bigcup_{j=1}^{q} (n_j + A \cap I_j)$ has positive lower density.*

As recalled before, if $P_1, \ldots, P_r$ is a $\pi\mathbb{Q}$-independent family of real polynomials, then $\{(e^{iP_1(n)}, \ldots, e^{iP_r(n)}); \ n \geq 1\}$ is dense in $\mathbb{T}^r$. We need a variant of this result when there exist relations between the polynomials.

Proposition 3.2. *Let $P_1, \ldots, P_r$ be real polynomials without constant term. Assume that there exist $p \in \{1, \ldots, r\}$, integers m and $(a_{j,k})_{\substack{p+1 \leq j \leq r \\ 1 \leq k \leq p}}$, polynomials $(R_j)_{p+1 \leq j \leq r}$ in $\pi\mathbb{Z}[t]$ so that*

    (i) *$P_1, \ldots, P_p$ are $\pi\mathbb{Q}$-independent;*
    (ii) *For any j in $\{p+1, \ldots, r\}$,*

$$mP_j = a_{j,1} P_1 + \cdots + a_{j,p} P_p + R_j.$$

Then the closure of $\{(e^{iP_1(n)}, \ldots, e^{iP_r(n)}); \ n \geq 0\}$ is equal to

$$\left\{ (e^{i\theta_1}, \ldots, e^{i\theta_r}); \ \forall j \geq p+1, \ m\theta_j = a_{j,1}\theta_1 + \cdots + a_{j,p}\theta_p \right\}.$$

Proof. Let $(\theta_1, \ldots, \theta_r) \in \mathbb{R}^r$ be such that $m\theta_j = a_{j,1}\theta_1 + \cdots + a_{j,p}\theta_p$ for $j \geq p+1$. We define Q_j and T_j by

$$Q_j(x) = \begin{cases} \frac{1}{m}P_j(2mx) & \text{if } j \leq p \\ P_j(2mx) & \text{if } j \geq p+1 \end{cases}$$

$$T_j(x) = \frac{1}{m}R_j(2mx), \; j \geq p+1.$$

We may observe that $T_j(n) \in 2\pi\mathbb{Z}$ for any integer n since R_j has no constant term and belongs to $\pi\mathbb{Z}[t]$. It is also easy to check that the family $(Q_1, \ldots, Q_p)$ remains $\pi\mathbb{Q}$-independent. Hence we can find a sequence of integers (n_k) such that $e^{iQ_j(n_k)}$ goes to $e^{i\theta_j/m}$ for any j in $\{1, \ldots, p\}$. Now, for $j \geq p+1$,

$$Q_j = a_{j,1}Q_1 + \cdots + a_{j,p}Q_p + T_j$$

so that $e^{iQ_j(n_k)}$ goes to $e^{i(a_{j,1}\theta_1 + \cdots + a_{j,p}\theta_p)/m} = e^{i\theta_j}$. Finally, we have shown that for any $j \leq r$, $e^{iP_j(2mn_k)}$ converges to $e^{i\theta_j}$, which implies the proposition. □

We shall use this proposition under the form of the following corollary.

Corollary 3.3. *Let $P_1, \ldots, P_r$ be real polynomials without constant term. Then the closure of $\{(e^{iP_1(n)}, \ldots, e^{iP_r(n)}); \; n \geq 1\}$ is invariant under complex conjugation.*

Proof. We extract from $(P_1, \ldots, P_r)$ a maximal family $(P_j)_{j \in J}$ which is $\pi\mathbb{Q}$-independent. Without loss of generality, we may assume that $J = \{1, \ldots, p\}$. This means that the assumptions of the previous proposition are satisfied. Hence, the result of the proposition describes the closure of $\{(e^{iP_1(n)}, \ldots, e^{iP_r(n)}); \; n \geq 1\}$. And this closure is clearly invariant under complex conjugation. □

We turn to the proof of Theorem 1.2. We shall in fact prove a variant of it which looks significantly stronger, since we control simultaneously several rotated orbits.

Theorem 3.4. *Let $P_1, \ldots, P_r$ be real polynomials without constant terms. Let also $T \in \mathcal{L}(X)$ be frequently hypercyclic and let $x \in FHC(T)$. Then, for any nonempty open set $U \subset X$,*

$$\left\{ n \in \mathbb{N}; \; \forall l \in \{1, \ldots, r\}, \; e^{iP_l(n)}T^n x \in U \right\}$$

has positive lower density.

Proof. We denote by d the maximum of the degree of $P_1, \ldots, P_r$ and we argue by induction on d. The case $d = 0$ is trivial since the polynomials have to be equal to zero. So, let us assume that the theorem has been proved until rank $d-1$ and let us prove it for $d \geq 1$. Let $V \subset X$ and let $\varepsilon > 0$ be such that V is open and nonempty and $D(1, \varepsilon)V \subset U$, where $D(a, \varepsilon)$ means the disk $|z - 1| < \varepsilon$. Let us set

$$E = \overline{\left\{ e^{iP_1(k)}, \ldots, e^{iP_r(k)}); \; k \geq 0 \right\}} \subset \mathbb{T}^r.$$

By the compactness of E, there exist integers $m_1, \ldots, m_q$ such that, for any $k \geq 0$, one may find $j \in \{1, \ldots, q\}$ so that $|e^{iP_l(k)} - e^{iP_l(m_j)}| < \varepsilon$ for any $l = 1, \ldots, r$. We then set, for $j = 1, \ldots, q$,

$$I_j = \left\{ k \geq 0; \; \forall l \in \{1, \ldots, r\}, \; |e^{iP_l(k)} - e^{iP_l(m_j)}| < \varepsilon \right\}.$$

Therefore, $\bigcup_{j=1}^{q} I_j = \mathbb{N}$.

By Corollary 3.3, E is invariant under complex conjugation. In particular, for any $j = 1, \ldots, q$, $(e^{-iP_1(m_j)}, \ldots, e^{-iP_r(m_j)})$ belongs to E. We now apply Corollary 2.10. For any $j = 1, \ldots, q$, we may find an integer n_j such that, for any $l \in \{1, \ldots, r\}$,

$$e^{iP_l(m_j)} e^{iP_l(n_j)} T^{n_j} x \in V.$$

Since T is continuous, there exists an open neighbourhood W of x such that

$$\forall j \in \{1, \ldots, q\}, \ \forall l \in \{1, \ldots, r\}, \ e^{iP_l(m_j)} e^{iP_l(n_j)} T^{n_j} W \subset V.$$

Now, there exist a sequence of polynomials $(Q_{j,l})_{\substack{1 \leq j \leq q \\ 1 \leq l \leq r}}$ with degree at most $d-1$ and without constant term such that, for any j in $1, \ldots, q$, for any l in $1, \ldots, r$, for any $k \geq 0$,

$$P_l(n_j + k) = P_l(n_j) + P_l(k) + Q_{l,j}(k).$$

We set

$$A = \big\{ k \geq 0; \ \forall (l, j) \in \{1, \ldots, r\} \times \{1, \ldots, q\}, \ e^{iQ_{l,j}(k)} T^k x \in W \big\}.$$

By the induction hypothesis, A has positive lower density. By Lemma 3.1, this remains true for

$$B = \bigcup_{j=1}^{q} (n_j + A \cap I_j).$$

Now, pick any $n \in B$. There exist j in $\{1, \ldots, q\}$ and $k \in A \cap I_j$ such that $n = n_j + k$. This leads to

$$e^{iP_l(n_j+k)} T^{n_j+k}(x) = \underbrace{e^{iP_l(k)} e^{-iP_l(m_j)}}_{\in D(1,\varepsilon)} \underbrace{e^{iP_l(m_j)} e^{iP_l(n_j)} T^{n_j} \underbrace{\big(e^{iQ_{j,l}(k)} T^k x \big)}_{\in W}}_{\in V} .$$

$$\underbrace{\phantom{e^{iP_l(k)} e^{-iP_l(m_j)} e^{iP_l(m_j)} e^{iP_l(n_j)} T^{n_j} e^{iQ_{j,l}(k)} T^k x}}_{\in U}$$

Thus, $B \subset \big\{ n \in \mathbb{N}; \ \forall l \in \{1, \ldots, r\}, \ e^{iP_l(n)} T^n x \in U \big\}.$ $\qquad\square$

Proof of Theorem 1.2. That (i) implies (ii) is a direct consequence of the previous theorem: when we have a single polynomial, we may allow a constant term since $e^{i\theta} x \in FHC(T)$ iff $x \in FHC(T)$.

It remains to prove (iii) implies (i). The proof follows the same lines; let $U, V \subset X$ be open and nonempty and let $\varepsilon > 0$ with $D(1, \varepsilon)V \subset U$. Let $\lambda_1, \ldots, \lambda_q \in \mathbb{T}$ be such that $\mathbb{T}$ is contained in $\bigcup_{j=1}^{q} D(\lambda_j^{-1}, \varepsilon)$. For $j = 1, \ldots, q$, let us set

$$I_j = \big\{ k \geq 0; \ e^{iP(k)} \in D(\lambda_j^{-1}, \varepsilon) \big\}.$$

Moreover, for any j in $\{1, \ldots q\}$, one may find an integer n_j such that $T^{n_j} x \in \lambda_j V$ since the assumption and Theorem 1.1 imply $x \in HC(T)$. Let now $W \subset X$ open and nonempty be such that $T^{n_j}(W) \subset \lambda_j V$ for any $j \in \{1, \ldots, q\}$. We finally set

$$A = \big\{ k \geq 0; \ e^{iP(k)} T^k x \in W \big\}$$

$$B = \bigcup_{j=1}^{q} (n_j + A \cap I_j).$$

A, thus B, have positive lower density. Moreover, if $n = n_j + k$ belongs to B, then

$$T^{n_j+k}(x) = \underbrace{\underbrace{e^{-iP(k)}}_{\in D(\lambda_j^{-1}, \varepsilon)} T^{n_j} \underbrace{(e^{iP(k)}T^k x)}_{\in W}}_{\in \lambda_j V}.$$
$$\underbrace{\phantom{e^{-iP(k)} T^{n_j} (e^{iP(k)}T^k x)}}_{\in U}$$

$\square$

Remark 3.5. Even if the result of Theorem 3.4 looks stronger than condition (ii) of Theorem 1.2, it is the natural statement which comes from our proof. Indeed, if you follow the proof for a single polynomial, then you have to apply the induction hypothesis for several polynomials!

4. OTHER SEQUENCES

In this section, we study if Theorem 1.1 remains true when we replace the sequence $(P(n))$ by other classical sequences. We first show that this is not the case if the sequence grows to infinity too quickly. The counterexample is very easy. It is just a backward shift on $\ell^2(\mathbb{Z}_+)$. Wy denote by B the unweighted backward shift. As usual $(e_n)_{n\geq 0}$ denotes the standard basis on $\ell^2(\mathbb{Z}_+)$.

Proposition 4.1. *Let $T = 2B$ acting on $\ell^2(\mathbb{Z}_+)$ and let $(f(n))$ be a sequence of positive integers with $f(n+1) > af(n)$, $n \in \mathbb{N}$, for some $a > 1$. Then for every $x \in HC(T)$ there exists $\theta \in \mathbb{R}$ such that the set $\left\{ e^{2\pi i f(n)\theta}T^n x; \ n \geq 1 \right\}$ is not dense in $\ell^2(\mathbb{Z}_+)$.*

Proof. Let $x \in HC(T)$. Choose $N \geq 1$ such that $\sum_{j\geq N+1} a^{-j} \leq 1/4$ and define the set

$$A = \left\{ n \in \mathbb{N}; \ \exists \lambda \in \mathbb{T}, \ \|\lambda T^n x - e_N\| < 1/2 \right\}.$$

Observe that if n belongs to A, then $n+k$ does not belong to A for any $k \leq N$. Indeed, one can write $T^n x = \mu e_N + y$ with $\|y\| \leq 1/2$ and $|\mu| = 1$, so that $T^{n+k}x = \mu 2^k e_{N-k} + T^k y$. This yields, for any $\lambda \in \mathbb{T}$,

$$\begin{aligned}
\|\lambda T^{n+k}x - e_N\| &= \|\lambda \mu 2^k e_{N-k} - e_N + T^k y\| \\
&\geq 2^k - 2^{k-1} \geq 1.
\end{aligned}$$

We define a sequence (α_n) as follows:

- $\alpha_n = 0$ provided $n \notin A$;
- if $n \in A$, we set $\theta_{n-1} = \sum_{j=0}^{n-1} \frac{\alpha_j}{f(j)}$. We then choose $\alpha_n \in \{0, 1/2\}$ such that

$$\Re e \left(e^{2\pi i(f(n)\theta_{n-1}+\alpha_n)} e_N^*(T^n x) \right) \leq 0.$$

We finally define $\theta = \sum_{n\geq 0} \frac{\alpha_n}{f(n)}$ and we claim that $\{e^{2\pi i f(n)\theta}T^n x; \ n \geq 1\}$ is not dense in $\ell^2(\mathbb{Z}_+)$. Precisely, let us show that e_N does not belong to the closure of $\{e^{2\pi i f(n)\theta}T^n x; \ n \geq 1\}$. Indeed, when n does not belong to A, we are sure that $\|e^{2\pi i f(n)\theta}T^n x - e_N\| \geq 1/2$. Otherwise, when n belongs to A, we can write

$$\|e^{2\pi i f(n)\theta}T^n x - e_N\| \geq |e_N^*(e^{2\pi i f(n)\theta}T^n x) - 1|.$$

Now,

$$\begin{aligned}
e_N^*(e^{2\pi i f(n)\theta}T^n x) &= e^{2\pi i(f(n)\theta_{n-1}+\alpha_n)} e_N^*(T^n x) e^{2\pi i f(n)\left(\sum_{j\geq n+1} \frac{\alpha_j}{f(j)}\right)} \\
&= e^{2\pi i(f(n)\theta_{n-1}+\alpha_n)} e_N^*(T^n x) e^{2\pi i f(n)\left(\sum_{j\geq n+N+1} \frac{\alpha_j}{f(j)}\right)}.
\end{aligned}$$

We set $z = e^{2\pi i(f(n)\theta_{n-1}+\alpha_n)}e_N^*(T^nx)$ and $\beta = 2\pi f(n)\left(\sum_{j \geq n+N+1}\frac{\alpha_j}{f(j)}\right)$. By the construction of α_n, $\Re e(z) \leq 0$. Moreover, it is easy to check that $\beta \in [0, \pi/4]$:

$$0 \leq 2\pi f(n)\left(\sum_{j \geq n+N+1}\frac{\alpha_j}{f(j)}\right) \leq \pi f(n)\sum_{j \geq N+n+1}\frac{1}{a^{j-n}f(n)} \leq \frac{\pi}{4}.$$

Thus, $e_N^*(e^{2\pi if(n)\theta}T^nx)$ does not belong to the cone $\{\rho e^{i\gamma}; \rho > 0, \ |\gamma| \leq \pi/4\}$. In particular, there exists $\delta > 0$ such that

$$\|e^{2\pi if(n)\theta}T^nx - e_N\| \geq \delta.$$

$\square$

Remark 4.2. In the previous proposition one cannot conclude the stronger assertion that there exists $\theta \in \mathbb{R}$ such that for every $x \in HC(2B)$ the set $\{e^{2\pi if(n)\theta}(2B)^nx; \ n \geq 1\}$ is not dense in $\ell^2(\mathbb{Z}_+)$. The reason for this is very simple and comes from the fact that $2B$ is hypercyclic in a very strong sense, namely it satisfies the hypercyclicity criterion. For a comprehensive discussion on the hypercyclicity criterion and its several equivalent forms we refer to [3], [8]. To explain now briefly, take any operator $S \in \mathcal{L}(X)$ satisfying the hypercyclicity criterion and let (λ_n) be any sequence of unimodular complex numbers. It is then immediate that the sequence of operators (λ_nS^n) also satisfies the hypercyclicity criterion; hence, the set of $y \in X$ such that $\{\lambda_nS^ny; n \geq 1\}$ is dense in X is G_δ and dense in X. By an appeal of Baire's category theorem one can find $x \in HC(S)$ such that $\overline{\{\lambda_nS^nx; \ n \geq 0\}} = X$. Applying the last for $S := 2B \in \mathcal{L}(\ell^2(\mathbb{Z}_+))$ and $\lambda_n := e^{2\pi if(n)\theta}$, $n \geq 1$ we see that for every $\theta \in \mathbb{R}$ there exists $x \in HC(2B)$ such that the set $\{e^{2\pi if(n)\theta}(2B)^nx; \ n \geq 1\}$ is dense in $\ell^2(\mathbb{Z}_+)$.

On the contrary, we have an analog to Theorem 1.1 for sequences which grow slowly to infinity. The growth condition which comes into play here is based on the increases of the function.

Theorem 4.3. *Let X be a Banach space, let $T \in \mathcal{L}(X)$, let $x \in X$ and let $(f(n))$ be a sequence of real numbers satisfying the following condition: there exist an integer $d \geq 0$, sequences $(g_l(n))_n$ for $0 \leq l \leq d$ and $(\varepsilon_k(n))_n$ for any $k \geq 1$, such that, for any $n, k \geq 1$,*

$$f(n+k) - f(n) = \sum_{l=0}^{d} g_l(n)k^l + \varepsilon_k(n)$$

and, for a fixed $k \geq 1$, $|\varepsilon_k(n)| \xrightarrow{n \to +\infty} 0$. Then the following are equivalent.

  (i) $x \in HC(T)$;
  (ii) $\{e^{if(n)}T^nx; \ n \geq 1\}$ *is dense in X.*

Proof. We just need to prove that $(i) \implies (ii)$. We are going to apply Theorem 1.1 (observe that a polynomial P satisfies the assumptions of Theorem 4.3 with $d = \deg(P)$ and $\varepsilon_k(n) = 0$!) and a compactness argument.

Lemma 4.4. *Let $d \geq 0$, $x \in HC(T)$, $y \in X$ and $\varepsilon > 0$. There exists $K \geq 1$ such that, for any $P \in \mathbb{R}_d[t]$, for any $\mu \in \mathbb{T}$, there exists $k \leq K$ such that*

$$(4.1) \qquad\qquad \|e^{iP(k)}T^kx - \mu y\| < \varepsilon.$$

Proof of Lemma 4.4. Let us observe that if (P, μ) satisfies (4.1) for a fixed k, this inequality is also satisfied with the same k for any (Q, λ) in a neighbourhood of (P, μ). Moreover, by Theorem 1.1, given any $(P, \mu) \in \mathbb{R}_d[t] \times \mathbb{T}$, we know that we may always find an integer k such that (4.1) is true. Define now

$$\mathbb{R}_d^\pi[t] = \left\{ P = \sum_{j=0}^d \theta_j t^j; \ \theta_j \in [0, 2\pi] \right\}.$$

$\mathbb{R}_d^\pi[t]$ is a compact subset of $\mathbb{R}_d[t]$. Moreover, for any $P \in \mathbb{R}_d[t]$, there exists some $Q \in \mathbb{R}_d^\pi[t]$ such that $e^{iQ(k)} = e^{iP(k)}$ for any $k \in \mathbb{Z}$. Hence, Lemma 4.4 follows from the compactness of $\mathbb{R}_d^\pi[t] \times \mathbb{T}$. $\square$

We come back to the proof of Theorem 4.3. We fix $\varepsilon > 0$ and $y \in X$. We apply Lemma 4.4 to the 4-tuple (d, x, y, ε). We then set $\delta = \varepsilon / \|T\|^K$. Since $x \in HC(T)$, there exist $n \geq 1$ as large as necessary and $\alpha_n \in \mathbb{T}$ such that

$$e^{if(n)} T^n x = \alpha_n x + z, \ \|z\| < \delta.$$

Then, there exists $k \leq K$ such that, setting $P_n(k) = \sum_{l=0}^d g_l(n) k^l$,

$$e^{iP_n(k)} T^k x = \alpha_n^{-1} y + z', \ \|z'\| < \varepsilon.$$

Thus,

$$
\begin{aligned}
e^{if(n+k)} T^{n+k} x &= e^{i\left(f(n+k) - f(n)\right)} T^k \left(e^{if(n)} T^n x\right) \\
&= e^{i\varepsilon_k(n)} e^{iP_n(k)} T^k \left(\alpha_n x + z\right) \\
&= e^{i\varepsilon_k(n)} \left(y + \alpha_n z' + e^{iP_n(k)} T^k z\right).
\end{aligned}
$$

Since $\sup_{0 \leq k \leq K} |\varepsilon_k(n)|$ goes to 0 as n goes to infinity, we get

$$\left\| e^{if(n+k)} T^{n+k} x - y \right\| < 3\varepsilon,$$

provided n has been chosen large enough. $\square$

This theorem covers the cases of many sequences which do not grow too quickly to infinity, like $f(n) = n^a \log^b(n+1)$, $(a, b) \in \mathbb{R}^2$, or finite linear combinations of such functions. Interestingly, we may also observe that Theorem 2.2 does not extend to this level of generality.

Example 4.5. Let $T = 2B$ acting on $\ell^2(\mathbb{Z}_+)$. There exists $x \in HC(T)$ such that $(e_0, 1)$ does not belong to $\overline{\{(T^n x, e^{i \log(n)}); \ n \geq 1\}}$.

Proof. Observe that

$$\Re e(e^{i \log(n)}) \geq 0 \iff n \in \bigcup_{k \geq 0} [\exp(2k\pi - \pi/2), \exp(2k\pi + \pi/2)] \iff n \in \bigcup_{k \geq 0} (a_k, b_k)$$

where (a_k) and (b_k) are sequences of integers satisfying $a_{k+1} - b_k \to +\infty$. Let (x_k) be a dense subset of $\ell^2(\mathbb{Z}_+)$ such that for any $k \geq 1$, $\|x_k\| \leq k$ and x_k has a finite support contained in $[0, a_{k+1} - b_k - 1]$. We set

$$x = \sum_{j \geq 1} \frac{1}{2^{b_j 2}} S^{b_j 2} x_j,$$

where S is the forward shift. Provided $n \in \bigcup_k (a_k, b_k)$, we know that $\|T^n x - e_0\| \geq 1$ since $\langle T^n x, e_0 \rangle = 0$. In particular, $(e_0, 1)$ does not belong to $\overline{\{(T^n x, e^{i \log(n)}); \ n \geq 1\}}$. Nevertheless, x is a hypercyclic vector for T. Indeed,

$$\|T^{b_{k^2}} x - x_k\| \leq \sum_{j \geq k+1} \left\| \frac{1}{2^{b_{j^2} - b_{k^2}}} S^{b_{j^2} - b_{k^2}} x_j \right\| \leq \sum_{j \geq k+1} \frac{j}{2^{b_{j^2} - b_{k^2}}} \leq \sum_{j \geq k+1} \frac{j}{2^j} \xrightarrow{k \to +\infty} 0.$$

$\square$

As a consequence of Theorem 4.3, given $T \in \mathfrak{L}(X)$ and a sequence of real numbers $(f(n))$ such that $f(n+1) - f(n) \to 0$, then $x \in HC(T)$ if and only if $\{e^{2\pi i f(n)} T^n x; \ n \geq 1\}$ is dense in X. Rephrasing this, we can say that for every sequence $(\lambda_n) \subset \mathbb{T}$ with $\lambda_{n+1}/\lambda_n \to 1$, $x \in HC(T)$ if and only if $\{\lambda_n T^n x; \ n \geq 1\}$ is dense in X. One now may wonder whether the assumption "$(\lambda_n) \subset \mathbb{T}$" can be dropped. However, this is not the case. Indeed, León-Saavedra proved in [11] that there exists a hypercyclic operator $T \in \mathfrak{L}(X)$ such that for every $x \in X$ the set $\{\lambda_n T^n x; \ n \geq 1\}$ is not dense in X where $\lambda_n = 1/n$, $n = 1, 2, \ldots$ and of course $\lambda_{n+1}/\lambda_n \to 1$. In general, if we keep the assumption $\lambda_{n+1}/\lambda_n \to 1$ but we move away form the unit circle $\mathbb{T}$, that is $\lambda_n \in \mathbb{C}$, then none of the implications $(ii) \implies (i)$, $(iii) \implies (ii)$ in Theorem 1.1 hold. This follows by Propositions 2.4 and 2.5 from [7]. On the other hand, under the additional assumption that $T \in \mathfrak{L}(X)$ is hypercyclic it is known [6] that for (λ_n) a sequence of non-zero complex numbers with $\lambda_{n+1}/\lambda_n \to 1$ and $x \in X$, if the set $\{\lambda_n T^n x; \ n \geq 1\}$ is somewhere dense then it is everywhere dense.

5. Uniformly distributed sequences and generic statements

In Section 4 we showed that Theorem 1.1 is no longer true for sequences of unimodular complex numbers whose phases grow to infinity at a geometric rate, see Proposition 4.1. In particular, denoting by B the unweighted backward shift on $\ell^2(\mathbb{Z}_+)$ then, for every $x \in HC(2B)$ there exists $\theta \in \mathbb{R}$ such that the set $\left\{e^{2\pi i 2^n \theta}(2B)^n x; \ n \geq 1\right\}$ is not dense in $\ell^2(\mathbb{Z}_+)$. In this section we shall establish results, both in measure and category, going to the opposite direction. For instance, a consequence of our result, Proposition 5.2, is that for every $x \in HC(2B)$ the set of θ's in $\mathbb{R}$ such that $\left\{e^{2\pi i 2^n \theta}(2B)^n x; \ n \geq 1\right\}$ is dense in $\ell^2(\mathbb{Z}_+)$ is residual and of full Lebesgue measure in $\mathbb{R}$. This is in sharp contrast with Proposition 4.1. Such kind of behavior comes as a natural consequence from the general metric theorem of Koksma, see Theorem 4.3 in Chapter 1 of [10]. Koksma's theorem generalizes the beautiful result of Weyl [10]: *if (n_k) is a distinct sequence of integers then the sequence $(n_k \theta)$ is uniformly distributed for almost every $\theta \in \mathbb{R}$* . Here, we shall need the following corollary of Koksma's theorem, see Corollary 4.3 in Chapter 1 of [10], which we state as a theorem.

Theorem 5.1. *(Koksma) Let $(f(n))$ be a sequence of real numbers such that for some $\delta > 0$, $|f(n) - f(m)| > \delta$ for $n \neq m$. Then the sequence $(f(n)\theta)$ is uniformly distributed for almost every $\theta \in \mathbb{R}$.*

Proposition 5.2. *Let X be a Banach space and let $T \in \mathfrak{L}(X)$ be a hypercyclic operator. Let also $(f(n))$ be a sequence of real numbers such that for some $\delta > 0$, $|f(n) - f(m)| > \delta$ for $n \neq m$. Then for every $x \in HC(T)$ there exists a residual subset A of $\mathbb{R}$ with full measure such that for every $\theta \in A$ the set $\{(T^n x, e^{2\pi i f(n)\theta}); \ n \geq 1\}$ is dense in $X \times \mathbb{T}$.*

Proof. Take $x \in HC(T)$. Let $\{x_j; \ j \in \mathbb{N}\}$, $\{t_l; l \in \mathbb{N}\}$ be dense subsets of X and $\mathbb{T}$ respectively and define $A := \bigcap_{j,l,s \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} A_{j,l,s,n}$, where

$$A_{j,l,s,n} := \{\theta \in \mathbb{R}; \ |e^{2\pi i f(n)\theta} - t_l| < 1/s \ \text{and} \ \|T^n x - x_j\| < 1/s\},$$

for $j, l, s, n \in \mathbb{N}$. Clearly, for every $\theta \in A$ the set $\{(T^n x, e^{2\pi i f(n)\theta}); \ n \geq 1\}$ is dense in $X \times \mathbb{T}$. Let us first prove that A is residual in $\mathbb{R}$. It is easy to see that $A_{j,l,s,n}$ is open in $\mathbb{R}$ and by Baire's category theorem it remains to show that for every $j, l, s \in \mathbb{N}$ the set $\bigcup_{n \in \mathbb{N}} A_{j,l,s,n}$ is dense in $\mathbb{R}$. To this end, fix $j, l, s \in \mathbb{N}$ and let $\alpha \in \mathbb{R}$, $\epsilon > 0$. Consider the following set of positive integers $\{n_1 < n_2 < \cdots\} := \{n \in \mathbb{N}; \ \|T^n x - x_j\| < 1/s\}$. By Theorem 5.1 the sequence $(f(n_k)\theta)$ is uniformly distributed for almost every $\theta \in \mathbb{R}$. In particular, there exists $\theta \in \mathbb{R}$ with $|\theta - \alpha| < \epsilon$ such that the set $\{e^{2\pi i f(n_k)\theta}; \ k \in \mathbb{N}\}$ is dense in $\mathbb{T}$. Thus, $|\alpha - \theta| < \epsilon$, $|e^{2\pi i f(n_m)\theta} - t_l| < 1/s$ and $\|T^{n_m} x - x_j\| < 1/s$ for some positive integer m. This shows the density result. In the above, it is implicit that for $j, l, s \in \mathbb{N}$ the set $\bigcup_{n \in \mathbb{N}} A_{j,l,s,n}$ has full measure. Hence, A has full measure. $\square$

In view of the above, one may be tempted to ask the following question.

Question. Let X be a Banach space, let $T \in \mathfrak{L}(X)$ be hypercyclic and let $x \in HC(T)$. Consider a sequence of real numbers $(f(n))$ such that the sequence $(f(n)\theta)$ is uniformly distributed for some $\theta \in \mathbb{R}$. Is it true that $\{e^{2\pi i f(n)} T^n x; \ n \geq 1\}$ is dense in X?

Although in many cases the above question admits an affirmative reply, for instance when $f(n)$ is a polynomial in n, the answer in general is no! To see this, fix a hypercyclic vector $x \in \ell^2(\mathbb{Z}_+)$ for $2B \in \mathfrak{L}(\ell^2(\mathbb{Z}_+))$, where B is the unweighted backward shift. By Proposition 4.1 there exists $\theta' \in \mathbb{R}$ such that the set $\{e^{2\pi i 2^n \theta'}(2B)^n x; n \geq 1\}$ is not dense in $\ell^2(\mathbb{Z}_+)$. Define now $f(n) = 2^n \theta'$. Now, on the one hand, by Theorem 5.1 there exists $\theta \in \mathbb{R}$ such that the sequence $(f(n)\theta)$ is uniformly distributed and, on the other hand, $\{e^{2\pi i f(n)}(2B)^n x; \ n \geq 1\}$ is not dense in $\ell^2(\mathbb{Z}_+)$. Regarding the previous question it is important to note that there are sequences of real numbers $(f(n))$ such that for no $\theta \in \mathbb{R}$ the sequence $(f(n)\theta)$ is uniformly distributed and yet $\overline{\{e^{2\pi i f(n)} T^n x; \ n \geq 1\}} = X$ for every $x \in HC(T)$. This is the case for the sequence $f(n) = \log(n)$, $n = 1, 2, \ldots$. Indeed, this follows by Theorem 4.3 and the fact that for every $\theta \in \mathbb{R}$ the sequence $(\log(n)\theta)$ is not uniformly distributed, see Examples 2.4, 2.5 in Chapter 1 of [10]. It is clear now that the right question along this line is the following.

Question. Let X be a Banach space, let $T \in \mathfrak{L}(X)$ be hypercyclic and let $x \in HC(T)$. Consider a sequence of real numbers $(f(n))$ which is uniformly distributed. Is it true that the set $\{e^{2\pi i f(n)} T^n x; \ n \geq 1\}$ is dense in X?

In turn, the answer to this question is negative.

Proposition 5.3. *Let $T = 2B$ acting on $X = \ell^2(\mathbb{Z}_+)$. There exists a uniformly distributed sequence of real numbers $(f(n))$ and a vector $x \in HC(T)$ such that $\{e^{2\pi i f(n)} T^n x; \ n \geq 1\}$ is not dense in X.*

Proof. We start from any uniformly distributed sequence $(g(n))$. The idea of the proof is to change slightly the sequence $(g(n))$ to a sequence $(f(n))$ which remains uniformly distributed and such that $e^{2\pi i f(n)}$ can be arbitrarily chosen for n belonging to some subset $A \subset \mathbb{N}$ containing arbitrarily large intervals. We define simultaneously $x \in HC(T)$ such that

$$\begin{cases} e_0^*(T^n x) &= 0 \quad \text{provided } n \notin A \\ \Re e\big(e_0^*(e^{2\pi i f(n)} T^n x)\big) &\geq 0 \quad \text{provided } n \in A. \end{cases}$$

Hence, $\{e^{2\pi if(n)}T^n x;\ n \geq 1\}$ will not be dense in X.

We now proceed with the details. Let (n_k) be an increasing sequence of integers such that $n_1 = 1$, $n_{k+1} > n_k + (k+1)$ for any $k \geq 1$ and $k^2/n_k \to 0$. Let (x_k) be a dense sequence in $\ell^2(\mathbb{Z}_+)$ such that, for any $k \geq 1$, $\|x_k\| \leq k$ and x_k has finite support contained in $[0, k]$. We set

$$x = \sum_{j \geq 1} \frac{1}{2^{n_j}} S^{n_j} x_j$$

and we observe that $x \in HC(T)$. Indeed,

$$\|T^{n_k} x - x_k\| = \left\| \sum_{j > k} \frac{1}{2^{n_j - n_k}} S^{n_j - n_k} x_j \right\| \leq \sum_{j > k} \frac{j}{2^j} \xrightarrow{k \to +\infty} 0.$$

We then define $(f(n))$ by

- $f(n) = g(n)$ provided $n \notin \bigcup_{k \geq 1} [n_k, n_k + k]$;
- $f(n)$ is any positive real number such that $\Re e\left(e_0^*(e^{2\pi if(n)}T^n x)\right) \geq 0$ provided $n \in \bigcup_{k \geq 1} [n_k,\ n_k + k]$.

As already observed, $\{e^{2\pi if(n)}T^n x;\ n \geq 1\}$ cannot be dense in $\ell^2(\mathbb{Z}_+)$. Thus it remains to show that $(f(n))$ is uniformly distributed. Let I be a subarc of $\mathbb{T}$ and let $n \geq 1$. Let $k \geq 1$ be such that $n_k \leq n < n_{k+1}$. Then, since $f(j)$ and $g(j)$ may only differ for $j \leq n$ if j belongs to $\bigcup_{l=1}^{k} [n_l, n_l + l]$ which has cardinality less than $(k+1)^2$,

$$-\frac{(k+1)^2}{n} + \frac{1}{n}\text{card}\left(\left\{1 \leq j \leq n;\ e^{2\pi ig(j)} \in I\right\}\right) \leq \frac{1}{n}\text{card}\left(\left\{1 \leq j \leq n;\ e^{2\pi if(j)} \in I\right\}\right)$$

and

$$\frac{1}{n}\text{card}\left(\left\{1 \leq j \leq n;\ e^{2\pi if(j)} \in I\right\}\right) \leq \frac{1}{n}\text{card}\left(\left\{1 \leq j \leq n;\ e^{2\pi ig(j)} \in I\right\}\right) + \frac{(k+1)^2}{n}.$$

Since $k^2/n \leq k^2/n_k$ goes to zero, $(f(n))$ remains uniformly distributed. $\square$

We conclude the paper with a generic result which is related to the question we asked in the introduction. The space $\mathbb{T}^{\mathbb{N}}$ is endowed with the metric $d(\Lambda, M) = \sum_{n=1}^{+\infty} \frac{|\lambda_n - \mu_n|}{2^n}$ for $\Lambda = (\lambda_n) \in \mathbb{T}^{\mathbb{N}}$, $M = (\mu_n) \in \mathbb{T}^{\mathbb{N}}$, and becomes a complete metric space.

Proposition 5.4. *Let X be a Banach space and let $T \in \mathfrak{L}(X)$ be a hypercyclic operator. Then for every $x \in HC(T)$ there exists a residual subset B of $\mathbb{T}^{\mathbb{N}}$ such that for every $(\lambda_1, \lambda_2, \ldots) \in B$ the set $\{\lambda_n T^n x;\ n \geq 1\}$ is dense in X.*

Proof. Fix $x \in HC(T)$. Let $\{x_j;\ j \in \mathbb{N}\}$ be a dense set in X and define the set $B_{j,s,n} := \{\Lambda = (\lambda_m) \in \mathbb{T}^{\mathbb{N}};\ \|\lambda_n T^n x - x_j\| < 1/s\}$ for $j, s, n \in \mathbb{N}$. It is straightforward to check that $B_{j,s,n}$ is open in $\mathbb{T}^{\mathbb{N}}$ for every $j, s, n \in \mathbb{N}$. We then define $B := \bigcap_{j,s \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} B_{j,s,n}$. Observe that if $\Lambda = (\lambda_n) \in B$ then the set $\{\lambda_n T^n x;\ n \geq 1\}$ is dense in X. Hence, in view of Baire's category theorem it suffices to show that for every $j, s \in \mathbb{N}$ the set $\bigcup_{n \in \mathbb{N}} B_{j,s,n}$ is dense in $\mathbb{T}^{\mathbb{N}}$. Fix $j, s \in \mathbb{N}$ and let $M = (\mu_n) \in \mathbb{T}^{\mathbb{N}}$, $\epsilon > 0$. There exists a positive integer N such that $\sum_{n=N+1}^{+\infty} \frac{2}{2^n} < \epsilon$. Define the vector $\Lambda := (\mu_1, \ldots, \mu_N, 1, 1, 1, \ldots) \in \mathbb{T}^{\mathbb{N}}$. From the above we get $d(\Lambda, M) < \epsilon$ and $\|\lambda_n T^n x - x_j\| = \|T^n x - x_j\| < 1/s$ for some $n \in \mathbb{N}$ with $n > N$. Hence, the set $\bigcup_{n \in \mathbb{N}} B_{j,s,n}$ is dense in $\mathbb{T}^{\mathbb{N}}$. $\square$

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