# Regular circle actions on 2-connected 7-manifolds 

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#### Abstract

We determine the homeomorphism (resp. diffeomorphism) types of those 2 -connected 7 -manifolds (resp. smooth 2 -connected 7 -manifolds) that admit regular circle actions (resp. smooth regular circle actions).


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## 1 Introduction

In this paper all manifolds under consideration are closed, oriented and topological, unless otherwise stated. Moreover, all homeomorphisms and diffeomorphisms are to be orientation preserving. Given a positive integer $n$ let $S^{n}$ (resp. $D^{n+1}$ ) be the diffeomorphism type of the unit $n$-sphere (resp. the unit $(n+1)$-disk) in $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$.

Definition 1.1. A circle action $S^{1} \times M \rightarrow M$ on a manifold $M$ is called regular if this action is free and the orbit space $N:=M / S^{1}$ (with quotient topology) is a manifold.

Similarly, a smooth circle action $S^{1} \times M \rightarrow M$ on a smooth manifold $M$ is called regular if this action is free (see [26, p.38, Proposition 5.2]).

For a given manifold $M$ one can ask
Problem 1.2. Does $M$ admit a regular circle action?

Solutions to Problem 1.2 can have direct implications in contact topology. For example, the Boothby-Wang theorem implies that the existence

[^0]of a smooth regular circle action on a smooth manifold $M$ is a necessary condition to the existence of a regular contact form on $M$ (see [8, p.341]).

Problem 1.2 has been solved for all 1-connected 5 -manifolds by Duan and Liang [6]. In particular, it was shown that all 1-connected 4-manifolds with second Betti number $r$ can be realized as the orbit spaces of some regular circle actions on the single 5 -manifold $\#_{r-1} S^{2} \times S^{3}$, the connected sums of $r-1$ copies of the product $S^{2} \times S^{3}$. In this paper we study Problem 1.2 for the 2 -connected 7 -manifolds.

Our main result is stated in terms of a family $M_{l, k}^{c}, c \in\{0,1\}, l, k \in \mathbb{Z}$ of 2 -connected 7 -manifolds. The manifolds $M_{l, k}^{0}$ are the total spaces of the $S^{3}$-bundles $\pi_{M}: M_{l, k}^{0} \rightarrow S^{4}$ with characteristic map $\left[f_{l, k}\right] \in \pi_{3}\left(S O_{4}\right)$ defined by

$$
f_{l, k}(u) v=u^{l+k} v u^{-l}, v \in \mathbb{R}^{4}, u \in S^{3}
$$

where the space $\mathbb{R}^{4}$ and the sphere $S^{3}$ are identified with the algebra of quaternions and the space of unit quaternions respectively, and where quaternion multiplication is understood on the right hand side of the formula. Complete classification on the manifolds $M_{l, k}^{0}$ has been obtained by Milnor [19], Crowley and Escher [3].

The manifold $M_{l, k}^{1}$ is an analogue of the manifold $M_{l, k}^{0}$ in the non-smooth category when $k \equiv 0 \bmod 2$. Write $\mathcal{S}^{T O P}\left(M_{l, k}^{0}\right)$ for the set of all equivalence classes $[M, h]$ of the pairs $(M, h)$ with $h: M \rightarrow M_{l, k}^{0}$ a homotopy equivalence of 7 -manifolds. Two such pairs $(M, h)$ and $\left(M^{\prime}, h^{\prime}\right)$ are called equivalent if there is a homeomorphism $f: M \rightarrow M^{\prime}$ such that $h$ is homotopic to $h^{\prime} \circ f$ (see [18, Chapter 2]). Adapting the arguments of [3, Section 5] from the PL case to the TOP case we have the composition of isomorphisms

$$
\mathcal{S}^{T O P}\left(M_{l, k}^{0}\right) \xrightarrow{\eta}\left[M_{l, k}^{0}, G / T O P\right] \xrightarrow{d} H^{4}\left(M_{l, k}^{0} ; \pi_{4}(G / T O P)\right) \cong H^{4}\left(M_{l, k}^{0}\right)
$$

where the space $G$ (resp. TOP) is the direct limit of the set of self homotopy equivalences of $S^{n-1}$ (resp. the topological monoid of origin-preserving homeomorphisms of $\mathbb{R}^{n}$ ), $\eta$ is the one to one correspondence in the surgery exact sequence (see [18, p.40-44]) and where the isomorphism $d$ is induced by the primary obstruction to null-homotopy. Write $\left[M_{l, k}^{1}, h_{M}\right]$ for the element $(d \circ \eta)^{-1}\left(\pi_{M}^{*}(\iota)\right) \in \mathcal{S}^{T O P}\left(M_{l, k}^{0}\right)$ where $\iota \in H^{4}\left(S^{4}\right)$ is the generator as in [3] and $H^{4}\left(M_{l, k}^{0}\right) \cong \mathbb{Z}_{k}$ is generated by $\pi_{M}^{*}(\iota)$. Clearly, the 7 -manifold $M_{l, k}^{1}$ is 2 -connected and unique up to homeomorphism.

The group $\Gamma_{7}$ of exotic 7 -spheres is cyclic of order 28 with generator $M_{1,1}^{0}$ [7, Section 6]. Let $\Sigma_{r}:=r M_{1,1}^{0} \in \Gamma_{7}, r \in \mathbb{Z}$. Our main result is stated below, where $\mathbb{N}$ is the set of all nonnegative integers.

Theorem 1.3. All homeomorphism classes of the 2 -connected 7 -manifolds that admit regular circle actions are represented by

$$
\#{ }_{2 r} S^{3} \times S^{4} \# M_{6 m,(1+c) k}^{c}, c \in\{0,1\}, r \in \mathbb{N} \text { and } m, k \in \mathbb{Z}
$$

All diffeomorphism classes of the smooth 2 -connected 7 -manifolds that admit smooth regular circle actions are represented by

$$
\#{ }_{2 r} S^{3} \times S^{4} \# M_{6(a+1) m,(a+1) k}^{0} \# \Sigma_{(1-a) m}, a \in\{0,1\}, r \in \mathbb{N}, m, k \in \mathbb{Z}
$$

In the course to establish Theorem 1.3 we obtain also a classification of the 6 -manifolds that can appear as the orbit spaces of some regular circle actions on 2 -connected 7 -manifolds, see Lemmas 2.1 and 2.2 in Section 2. In addition, Theorem 1.3 has some direct consequences which are discussed in Section 4.

## 2 The homeomorphism types of the orbit spaces

In this section we determine the homeomorphism types of those 6-manifolds which can appear as the orbit spaces of some regular circle actions on 2connected 7 -manifolds.

A 2-connected 7 -manifold $M$ with a regular circle action defines a principal $S^{1}$-bundle $M \rightarrow N$ with base space $N=M / S^{1}[9]$. Fixing an orientation on $S^{1}$ once and for all, and let $N$ be furnished with the orientation compactible with that on $M$. From the homotopy exact sequence

$$
0 \rightarrow \pi_{2}(M) \rightarrow \pi_{2}(N) \rightarrow \pi_{1}\left(S^{1}\right) \rightarrow \pi_{1}(M) \rightarrow \pi_{1}(N) \rightarrow 0
$$

of the fibration one finds that

$$
\pi_{1}(N)=0 ; \pi_{2}(N) \cong \pi_{1}\left(S^{1}\right) \cong \mathbb{Z}
$$

Consequently $N$ is a 1 -connected 6 -manifold with $H_{2}(N) \cong \mathbb{Z}$.
Conversely, for a 1 -connected 6 -manifold $N$ with $H_{2}(N) \cong \mathbb{Z}$ let $t \in$ $H^{2}(N) \cong \mathbb{Z}$ be a generator and let

$$
S^{1} \hookrightarrow N_{t} \rightarrow N
$$

be the oriented circle bundle over $N$ with Euler class $t$. From the homotopy exact sequence of this fibration we find that $N_{t}$ is 2 -connected with the canonical orientation as the total space of the circle bundle. Summarizing we get

Lemma 2.1. Let $S^{1} \times M \rightarrow M$ be a regular circle action on a 2 -connected 7 -manifold $M$ with orbit space $N$. Then $N$ is a 1 -connected 6 -manifold with $H_{2}(N) \cong \mathbb{Z}$.

Conversely, every 1-connected 6-manifold $N$ with $H_{2}(N) \cong \mathbb{Z}$ can be realized as the orbit space of some regular circle action on a 2 -connected 7-manifold.

In view of Lemma 2.1 the classification of those 1-connected 6 -manifolds $N$ with $H_{2}(N) \cong \mathbb{Z}$ amounts to a crucial step toward a solution to Problem 1.2. In terms of the known invariants for 1 -connected 6 -manifolds due to Jupp [14] and Wall [27] we can enumerate all these manifolds in the next result.

Denote by $\Theta$ the set of equivalence classes $[N, t]$ of the pairs $(N, t)$ with $N$ a 1-connected 6-manifold whose integral cohomology satisfies

$$
H^{r}(N)=\left\{\begin{array}{l}
\mathbb{Z} \text { if } r=0,2,4,6 \\
0 \text { otherwise }
\end{array}\right.
$$

and with $t \in H^{2}(N)$ a fixed generator. Two elements $\left(N_{1}, t_{1}\right),\left(N_{2}, t_{2}\right)$ are called equivalent if there is a homeomorphism $f: N_{1} \rightarrow N_{2}$ such that $f^{*} t_{2}=t_{1}$. For each $(N, t)$ fix a generator $x \in H^{4}(N)$ such that the value $\langle t \cup x,[N]\rangle$ of the product $t \cup x$ on the fundamental class [ $N$ ] is equal to 1 . Consider the functions

$$
k, p: \Theta \rightarrow \mathbb{Z} ; \varepsilon: \Theta \rightarrow\{0,1\} ; \delta: \Theta \rightarrow\{0,1\}
$$

determined by the following properties
i) $t^{2}=k([N, t]) x$;
ii) the second Stiefel-Whitney class $w_{2}(N)$ and the first Pontrjagin class $p_{1}(N)$ of $N$ are given by $\varepsilon([N, t]) t \bmod 2$ and $p([N, t]) x$, respectively;
iii) the class $\Delta(N) \equiv \delta([N, t]) x \bmod 2 \in H^{4}\left(N ; \mathbb{Z}_{2}\right)$ is the KirbySiebenmann invariant of $N$,
where the Kirby-Siebenmann invariant $\Delta(V)$ of a manifold $V$ is the obstruction to lift the classifying map $V \rightarrow B T O P$ for the stable tangent bundle of $V$ to $B P L$, and where $B T O P$ and $B P L$ are the classifying spaces for the stable $T O P$ bundles and $P L$ bundles, respectively [15].

Lemma 2.2. For each 1 -connected 6 -manifold $M$ with $H_{2}(M) \cong \mathbb{Z}$ there exists an $r \in \mathbb{N}$ and an element $[N, t] \in \Theta$ such that $M \cong \#_{r} S^{3} \times S^{3} \# N$.

Moreover, the system $\{k, p, \varepsilon, \delta\}$ is a set of complete invariants for elements $[N, t] \in \Theta$ that is subject to the following constraints:

> i)If $k([N, t]) \equiv 1 \bmod 2$, then $\varepsilon([N, t])=0$ and $p([N, t])=24 m+4 k([N, t])+24 \delta([N, t])$ for some $m \in \mathbb{Z}$
ii)If $k([N, t]) \equiv 0 \bmod 2$, then for some $m \in \mathbb{Z}$

$$
p([N, t])=\left\{\begin{array}{cl}
24 m+4 k([N, t])+24 \delta([N, t]) & \text { if } \varepsilon([N, t])=0 \\
48 m+k([N, t])+24 \delta([N, t]) & \text { if } \varepsilon([N, t])=1
\end{array}\right.
$$

In addition, the manifold $N$ is smoothable if and only if $\delta([N, t])=0$.
Proof. This is a direct consequence of [14, Theorem 0; Theorem 1]. In particular, the expressions of the function $p$ are deduced from the following relation on $H^{6}(N)$ which holds for all $d \in \mathbb{Z}$ :

$$
(2 d t+\varepsilon([N, t]) t)^{3} \equiv(p([N, t]) x+24 \delta([N, t]) x)(2 d t+\varepsilon([N, t]) t) \bmod 48 .
$$

## 3 Circle bundles over $[N, t] \in \Theta$

Lemma 2.2 singles out the family $\Theta$ of 1 -connected 6 -manifolds which plays a key role in presenting the orbit spaces of regular circle actions on $2-$ connected 7 -manifolds. In this section we determine the homeomorphism and diffeomorphism type of the total space $N_{t}$ of the circle bundle over $[N, t] \in \Theta$, i.e. the oriented circle bundle over $N$ with Euler class $t$. For this purpose we shall recall in Section 3.1 that the definition of the known invariant system for 2 -connected 7 -manifolds. The main results in this section are Lemmas 3.3 and 3.5, which identify the homeomorphism and diffeomorphism types of the manifolds $N_{t}$ with certain $M_{l, k}^{c}$.

### 3.1 Invariants for 2-connected 7-manifolds

Recall from Eells, Kuiper [7], Kreck, Stolz [16] and Wilkens [29] that associated to each 2 -connected 7 -manifold $M$ there is a system $\left\{H, \frac{p_{1}}{2}, b, \Delta, \mu, s_{1}\right\}$ of invariants characterized by the following properties:
i) $H$ is the forth integral cohomology group $H^{4}(M)$ [29];
ii) $\frac{p_{1}}{2}(M) \in H$ is the first spin characteristic class [25] introduced by Wilkens [29] in smooth category and extended to topological category by Kreck-Stolz [16, Lemma 6.5];
iii) $b: \tau(H) \otimes \tau(H) \rightarrow \mathbb{Q} / \mathbb{Z}$ is the linking form on the torsion part $\tau(H)$ of the group $H$ [29];
iv) $\Delta(M) \in H^{4}\left(M ; \mathbb{Z}_{2}\right)$ is the Kirby-Siebenmann invariant of $M$ [16].

Furthermore, if the manifold $M$ is smooth and bounds a smooth 8-manifold $W$ with the induced map $j^{*}: H^{4}(W, M ; \mathbb{Q}) \rightarrow H^{4}(W ; \mathbb{Q})$ an isomorphism, then
v) the invariant $\mu \in \mathbb{Q} / \mathbb{Z}$ is firstly defined in [7] for a spin $W$ and extended in [16] for a general $W$, whose value is given by the formula
$\mu(M) \equiv-\frac{1}{2^{5} \cdot 7} \sigma(W)+\frac{1}{2^{7} \cdot 7} p_{1}^{2}(W)-\frac{1}{2^{6} \cdot 3} z^{2} \cdot p_{1}(W)+\frac{1}{2^{7} \cdot 3} z^{4} \bmod \mathbb{Z}$,
where $z \in H^{2}(W)$ satisfies $w_{2}(W)=z \bmod 2, \sigma(W)$ is the signature of the intersection form on $H^{4}(W, M ; \mathbb{Q})$, and where $p_{1}^{2}(W), z^{2} \cdot p_{1}(W)$ and $z^{4}$ are the characteristic numbers

$$
\begin{aligned}
& \left\langle p_{1}(W) \cup j^{*-1} p_{1}(W),[W, M]\right\rangle,\left\langle z^{2} \cup j^{*-1} p_{1}(W),[W, M]\right\rangle, \\
& \left\langle z^{2} \cup j^{*-1} z^{2},[W, M]\right\rangle,
\end{aligned}
$$

respectively. Finally, if $M$ is topological and bounds a topological 8-manifold $W$ with the induced map $j^{*}: H^{4}(W, M ; \mathbb{Q}) \rightarrow H^{4}(W ; \mathbb{Q})$ an isomorphism, then
vi) the topological invariant $s_{1} \in \mathbb{Q} / \mathbb{Z}$ is defined in [16] whose value is given by

$$
s_{1}(M) \equiv-\frac{1}{2^{3}} \sigma(W)+\frac{1}{2^{5}} p_{1}^{2}(W)-\frac{7}{2^{4} \cdot 3} z^{2} \cdot p_{1}(W)+\frac{7}{2^{5 \cdot 3}} z^{4} \bmod \mathbb{Z} .
$$

Example 3.1. Let $N_{t}$ be the total space of the circle bundle over $[N, t] \in \Theta$. Then the system $\left\{H, \frac{p_{1}}{2}, b, \Delta, \mu, s_{1}\right\}$ of invariants for the manifold $N_{t}$ can be expressed in terms of the invariants for $\Theta$ introduced in Lemma 2.2 as follows. For simplicity we write $p, k, \varepsilon$ and $\delta$ in place of $p([N, t]), k([N, t])$, $\varepsilon([N, t])$ and $\delta([N, t])$, respectively.
i) $H^{4}\left(N_{t}\right) \cong \mathbb{Z}_{k}$ with generator $\pi^{*}(x)$, where $\pi: N_{t} \rightarrow N$ is the bundle projection and
$\mathbb{Z}_{k}=\left\{\begin{array}{cc}\mathbb{Z} & \text { if } k=0 \\ \mathbb{Z} / k \mathbb{Z} & \text { if } k \neq 0\end{array} ;\right.$
ii) $\Delta\left(N_{t}\right) \equiv \frac{1+(-1)^{k}}{2} \cdot \delta \pi^{*}(x) \bmod 2$;
iii) $\frac{p_{1}}{2}\left(N_{t}\right) \equiv \frac{p+\varepsilon k}{2} \pi^{*}(x) \bmod k$;
iv) $b\left(\pi^{*}(x), \pi^{*}(x)\right) \equiv \frac{1}{k} \bmod \mathbb{Z}$;
v) $\mu\left(N_{t}\right) \equiv-\frac{|k|}{2^{5} \cdot 7 k}+\frac{(p+k)^{2}}{2^{7} \cdot 7 k}+\frac{(\varepsilon-1)(2 p+k)}{2^{7} \cdot 3} \bmod \mathbb{Z}$;
vi) $s_{1}\left(N_{t}\right)=-\frac{|k|}{2^{3} k}+\frac{(p+k)^{2}}{2^{5} k}+\frac{7(\varepsilon-1)(2 p+k)}{2^{5} \cdot 3} \bmod \mathbb{Z}$.

Firstly, from the section $H^{2}(N) \xrightarrow{\cup t} H^{4}(N) \xrightarrow{\pi^{*}} H^{4}\left(N_{t}\right) \rightarrow 0$ in the Gysin sequence of the fibration $N_{t} \xrightarrow{\pi} N$ and from the relation $t^{2}=k x$ on $H^{4}(N)$ we find that $H^{4}\left(N_{t}\right) \cong \mathbb{Z}_{k}$ with generator $\pi^{*}(x)$. This shows i).

Next, let $f: N \rightarrow B T O P$ be the classifying map for the stable tangent bundle of $N$. In view of the decomposition $T N_{t} \cong \pi^{*} T N \oplus \varepsilon^{1}\left(\varepsilon^{1}\right.$ denotes the trivial line bundle) for the tangent bundle of $N_{t}$ the classifying map for the stable tangent bundle of $N_{t}$ is given by the composition $f \circ \pi: N_{t} \rightarrow$ $N \rightarrow B T O P$. It follows that the Kirby-Siebenmann invariant $\Delta\left(N_{t}\right)$ of the manifold $N_{t}$ is $\pi^{*} \Delta(N) \equiv \delta \pi^{*}(x) \bmod 2$. This shows ii).

To calculate the remaining invariants $\frac{p_{1}}{2}, b, \mu, s_{1}$ of the manifold $N_{t}$ we make use of the associated disk bundle $W_{t} \xrightarrow{\pi_{0}} N$ of the oriented 2 -plane bundle $\xi_{t}$ over $N$ with Euler class $t$. If $\varepsilon=1$, It follows from the decomposition $T W_{t} \cong \pi_{0}^{*} T N \oplus \pi_{0}^{*} \xi_{t}$ that $w_{2}\left(W_{t}\right)=0$ and $\frac{p_{1}}{2}\left(W_{t}\right)=\frac{p+k}{2} \pi_{0}^{*} x$. From the relation $\partial W_{t}=N_{t}$ we get

$$
\frac{p_{1}}{2}\left(N_{t}\right)=\frac{p+k}{2} \pi^{*} x .
$$

If $\varepsilon=0$, from the decomposition $T N_{t} \cong \pi^{*} T N \oplus \varepsilon^{1}$ we get

$$
\frac{p_{1}}{2}\left(N_{t}\right)=\frac{p}{2} \pi^{*}(x) .
$$

This shows iii).
To compute the linking form $b$ of $N_{t}$ we can assume that $k \neq 0$. Consider the commutative ladder of exact sequences

$$
\begin{array}{cccccccc}
0 & \rightarrow & H^{4}\left(W_{t}, N_{t}\right) & \xrightarrow{j^{*}} & H^{4}\left(W_{t}\right) & \xrightarrow{i^{*}} & H^{4}\left(N_{t}\right) & \rightarrow
\end{array} 0
$$

with $\phi$ the Thom isomorphism. Since $\pi^{*}(x)=i^{*} \pi_{0}^{*}(x)$ and $y:=\phi(t)$ is a generator of $H^{4}\left(W_{t}, N_{t}\right)$ with $j^{*}(y)=\pi_{0}^{*}\left(t^{2}\right)=k \pi_{0}^{*}(x)$ we get

$$
b\left(\pi^{*}(x), \pi^{*}(x)\right) \equiv \frac{1}{k}<y \cup \pi_{0}^{*} x,\left[W_{t}, N_{t}\right]>\equiv \frac{1}{k} \bmod \mathbb{Z} .
$$

This shows iv).
Since the induced map $j^{*}: H^{4}\left(W_{t}, N_{t} ; \mathbb{Q}\right) \rightarrow H^{4}\left(W_{t} ; \mathbb{Q}\right)$ is clearly an isomorphism when $k \neq 0$, the invariants $\mu$ and $s_{1}$ are defined for $N_{t}$. Moreover, from the Lefschetz duality and the relation $j^{*}(y)=k \pi_{0}^{*}(x)$ we get $\sigma\left(W_{t}\right)=\frac{|k|}{k}$. From the decomposition $T W_{t} \cong \pi_{0}^{*}\left(T N \oplus \xi_{t}\right)$ we get, in addition to

$$
p_{1}\left(W_{t}\right)=\pi_{0}^{*}\left(p_{1}(N)+t^{2}\right)=(p+k) \pi_{0}^{*}(x),
$$

that

$$
w_{2}\left(W_{t}\right) \equiv(\varepsilon+1) \pi_{0}^{*}(t) \bmod 2
$$

Therefore we can take $z=(1-\varepsilon) \pi_{0}^{*}(t)$ in the formulae for $\mu$ and $s_{1}$, and as a result

$$
z^{2}=(1-\varepsilon)^{2} \pi_{0}^{*}\left(t^{2}\right)=k(1-\varepsilon)^{2} \pi_{0}^{*}(x) .
$$

As the group $H^{4}\left(W_{t}, N_{t}\right) \cong \mathbb{Z}$ is generated by $y=\phi(t)$ with the relation $j^{*}(y)=k \pi_{0}^{*}(x)$, the isomorphism

$$
H^{4}\left(W_{t}, N_{t}\right) \otimes H^{4}\left(W_{t}\right) \xrightarrow{\cup} H^{8}\left(W_{t}, N_{t}\right)
$$

by the Lefschetz duality, together with the formulae for $p_{1}\left(W_{t}\right)$ and $z^{2}$ above, implies the relations below

$$
z^{2} p_{1}\left(W_{t}\right)=(1-\varepsilon)^{2}(p+k) ; p_{1}^{2}\left(W_{t}\right)=\frac{1}{k}(p+k)^{2} ; z^{4}=k(1-\varepsilon)^{4} .
$$

Substituting these values in the formulae for $\mu$ and $s_{1}$ yields v) and vi) respectively. This completes the computation of the invariant system for the manifolds $N_{t}$.

Example 3.2. The invariant system $\left\{H, \Delta, \frac{p_{1}}{2}, b, s_{1}, \mu\right\}$ of the manifolds $M_{l, k}^{c}$ has been computed by Crowley and Escher [3] for the case of $c=0$. We extend their calculation as to include the exceptional case of $c=1$.
i) $H^{4}\left(M_{l, k}^{c}\right) \cong \mathbb{Z}_{k}$ with generator $\kappa=\left\{\begin{array}{cl}\pi_{M}^{*}(\iota) & \text { if } c=0 \\ \left(\pi_{M} \circ h_{M}\right)^{*}(\iota) & \text { if } c=1\end{array}\right.$;
ii) $b(\kappa, \kappa) \equiv \frac{1}{k} \bmod \mathbb{Z}$;
iii) $\Delta\left(M_{l, k}^{c}\right) \equiv \frac{1+(-1)^{k}}{2} \cdot c \kappa \bmod 2$;
iv) $\frac{p_{1}}{2}\left(M_{l, k}^{c}\right) \equiv(2 l+12 c) \kappa \bmod k$;
v) $s_{1}\left(M_{l, k}^{c}\right) \equiv \frac{(2 l+k+12 c)^{2}-|k|}{8 k} \bmod \mathbb{Z}$;
vi) $\mu\left(M_{l, k}^{0}\right) \equiv \frac{(k+2 l)^{2}-|k|}{28 \cdot 8 k} \bmod \mathbb{Z}$.

Firstly, since $h_{M}: M_{l, k}^{1} \rightarrow M_{l, k}^{0}$ is a homotopy equivalence we get i) and ii) from the relations $H^{4}\left(M_{l, k}^{0}\right) \cong \mathbb{Z}_{k}$ (with generator $\pi_{M}^{*}(\iota)$ ) and $b\left(\pi_{M}^{*}(\iota), \pi_{M}^{*}(\iota)\right) \equiv \frac{1}{k} \bmod \mathbb{Z}$ when $c=0$.

Next, since the map $\mathcal{S}^{T O P}\left(M_{l, k}^{0}\right) \xrightarrow{\Delta} H^{4}\left(M_{l, k}^{0} ; \mathbb{Z}_{2}\right)$ of taking Kirby- Siebenmann class is a surjective homomorphism [24, Theorem 15.1], and since $\left[M_{l, k}^{1}, h_{M}\right]$ is a generator of the cyclic group $\mathcal{S}^{T O P}\left(M_{l, k}^{0}\right) \cong \mathbb{Z}_{k}$ we have

$$
\Delta\left(M_{l, k}^{1}\right)=\Delta\left(\left[M_{l, k}^{1}, h_{M}\right]\right)=\frac{1+(-1)^{k}}{2} \kappa \bmod 2 .
$$

This shows iii).
To calculate the remaining invariants $\frac{p_{1}}{2}, \mu, s_{1}$ of the manifold $M_{l, k}^{c}$ we construct an 8 -manifold $W_{l, k}^{c}$ with boundary $\partial W_{l, k}^{c} \cong M_{l, k}^{c}$ as follows. Let $\pi_{W}: W_{l, k}^{0} \rightarrow S^{4}$ be the associated disk bundle of the sphere bundle $\pi_{M}$ : $M_{l, k}^{0} \rightarrow S^{4}$. Then $M_{l, k}^{0}=\partial W_{l, k}^{0}$. Write $\mathcal{S}^{T O P}\left(W_{l, k}^{0}\right)$ for the set of equivalence classes $[W, h]$ of the pairs $(W, h)$ with $h:(W, \partial W) \rightarrow\left(W_{l, k}^{0}, M_{l, k}^{0}\right)$ a homotopy equivalence between 8 -manifolds with boundary. Consider the following commutative diagram analoguing to the one [3, (7)] due to Crowley and Escher in the PL-category(see also Section 1)

$$
\begin{aligned}
& \mathcal{S}^{T O P}\left(W_{l, k}^{0}\right) \xrightarrow[\cong]{\cong}\left[W_{l, k}^{0}, G / T O P\right] \xrightarrow{\xrightarrow{\longrightarrow}} \quad H^{4}\left(W_{l, k}^{0}\right) \cong \mathbb{Z} \\
& i^{*} \downarrow \quad i^{*} \downarrow \quad i^{*} \downarrow \\
& \mathcal{S}^{T O P}\left(M_{l, k}^{0}\right) \xrightarrow[\cong]{\cong}\left[M_{l, k}^{0}, G / T O P\right] \xrightarrow[\cong]{\xrightarrow{\geqq}} H^{4}\left(M_{l, k}^{0}\right) \cong \mathbb{Z}_{k}
\end{aligned}
$$

where $i^{*}: \mathcal{S}^{T O P}\left(W_{l, k}^{0}\right) \rightarrow \mathcal{S}^{T O P}\left(M_{l, k}^{0}\right)$ sends each $[W, h]$ to the restriction $\left[\partial W,\left.h\right|_{\partial W}\right]$. Writing $\left[W_{l, k}^{1}, h_{W}\right]$ for the element $(d \circ \eta)^{-1}\left(\pi_{W}^{*}(\iota)\right) \in$ $\mathcal{S}^{T O P}\left(W_{l, k}^{0}\right)$ we get $M_{l, k}^{1} \cong \partial W_{l, k}^{1}$ from the diagram above.

To find the formula of $\frac{p_{1}}{2}\left(\mathcal{M}_{l, k}^{c}\right)$ we compute the first Pontrjagin class $p_{1}\left(W_{l, k}^{c}\right)$ of $W_{l, k}^{c}$. Let $\alpha$ denote the generator of $H^{4}\left(W_{l, k}^{c}\right) \cong \mathbb{Z}$ satisfies

$$
\alpha=\left\{\begin{array}{cl}
\pi_{W}^{*}(\iota) & \text { if } c=0 \\
\left(\pi_{W} \circ h_{W}\right)^{*}(\iota) & \text { if } c=1
\end{array}\right.
$$

and associate an integer $p\left(W_{l, k}^{c}\right)$ to $W_{l, k}^{c}$ such that $p_{1}\left(W_{l, k}^{c}\right)=p\left(W_{l, k}^{c}\right) \alpha$. Let $\bar{i}: G / T O P \rightarrow B T O P$ be the natural inclusion and let $f_{c}: W_{l, k}^{c} \rightarrow B T O P$ be the classifying map for the stable tangent bundle of $W_{l, k}^{c}$. It follows from the isomorphism $\mathcal{S}^{T O P}\left(W_{l, k}^{0}\right) \xrightarrow{\eta}\left[W_{l, k}^{0}, G / T O P\right]$ and the proof of [18, Theorem 2.23] that

$$
\bar{i}_{*} \eta\left(\left[W_{l, k}^{1}, h_{W}\right]\right)=h_{W}^{*-1}\left[f_{1}\right]-\left[f_{0}\right]
$$

and hence

$$
p\left(W_{l, k}^{1}\right) \pi_{W}^{*}(\iota)=h_{W}^{*-1} p_{1}\left(W_{l, k}^{1}\right)=p_{1}\left(W_{l, k}^{0}\right)+f^{*} \bar{i}^{*} p_{1}
$$

where $f=d^{-1}\left(\pi_{W}^{*}(\iota)\right)=\eta\left(\left[W_{l, k}^{1}, h_{W}\right]\right)$ is the generator of $\left[W_{l, k}^{0}, G / T O P\right]$ and $p_{1} \in H^{4}(B T O P)$ is the first Pontrjagin class [14]. It is shown in [24, Lemma 13.3,Proposition 13.4] that a generator $g$ of $\left[S^{4}, G / T O P\right]$ corresponds to a topological bundle $\xi$ with classifying map $\bar{i} \circ g$ and Pontrjagin class $p_{1}(\xi)=g^{*} \bar{i}^{*} p_{1}= \pm 24 \iota$. With an appropriate choice of $\pm d$ : $\left[W_{l, k}^{0}, G / T O P\right] \rightarrow H^{4}\left(W_{l, k}^{0}\right)$ applying $\pi_{W}^{*}$ to this equation we get

$$
f^{*} \bar{i}^{*} p_{1}=24 \pi_{W}^{*}(\iota) .
$$

This, together with the fact $p\left(W_{l, k}^{0}\right)=2(k+2 l)$ [19] and the formula for $p\left(W_{l, k}^{1}\right)$ above, implies that $p\left(W_{l, k}^{c}\right)=2 k+4 l+24 c$. Consequently from $M_{l, k}^{c} \cong \partial W_{l, k}^{c}$ we get iv).

Finally, we compute $s_{1}\left(M_{l, k}^{c}\right)$. The exact sequence

$$
H^{4}\left(W_{l, k}^{c}, M_{l, k}^{c}\right) \xrightarrow{j^{*}} H^{4}\left(W_{l, k}^{c}\right) \rightarrow H^{4}\left(M_{l, k}^{c}\right) \rightarrow 0
$$

together with the isomorphisms $H^{4}\left(M_{l, k}^{c}\right) \cong \mathbb{Z}_{k}$ and $H^{4}\left(W_{l, k}^{c}, M_{l, k}^{c}\right) \cong \mathbb{Z}$ by the Lefschetz duality, implies that we can take a generator $\beta$ of $H^{4}\left(W_{l, k}^{c}, M_{l, k}^{c}\right)$ such that $j^{*}(\beta)=k \alpha$. Since $j^{*}: H^{4}\left(W_{l, k}^{c}, M_{l, k}^{c} ; \mathbb{Q}\right) \rightarrow H^{4}\left(W_{l, k}^{c} ; \mathbb{Q}\right)$ is an isomorphism for $k \neq 0$ the invariant $s_{1}$ is defined for $M_{l, k}^{c}$. It follows from the Lefschetz duality and the relation $j^{*}(\beta)=k \alpha$ that $\sigma\left(W_{l, k}^{c}\right)=\frac{|k|}{k}$. On the other hand, the formula for $p\left(W_{l, k}^{c}\right)$, together with the relation $j^{*}(\beta)=k \alpha$ and the isomorphism

$$
H^{4}\left(W_{l, k}^{c}, M_{l, k}^{c}\right) \otimes H^{4}\left(W_{l, k}^{c}\right) \xrightarrow{\cup} H^{8}\left(W_{l, k}^{c}, M_{l, k}^{c}\right)
$$

by the Lefschetz duality, implies that

$$
p_{1}^{2}\left(W_{l, k}^{c}\right)=\frac{4}{k}(k+2 l+12 c)^{2} .
$$

In addition, the relation $w_{2}\left(W_{l, k}^{1}\right)=w_{2}\left(W_{l, k}^{0}\right)=0$ indicates that we can take $z=0$ in the formula of $s_{1}$. Substituting the values of $\sigma\left(W_{l, k}^{c}\right), z, p_{1}^{2}\left(W_{l, k}^{c}\right)$ in the formula for $s_{1}$ shows v ).

Similarly, we refer vi) to Crowley and Escher [3]. This completes the computation of the invariant system for the manifolds $M_{l, k}^{c}$.

### 3.2 Circle bundles over $[N, t] \in \Theta$

In this section we will prove Lemmas 3.3 and 3.5 which identify the homeomorphism and diffeomorphism types of the manifolds $N_{t}$ with certain $M_{l, k}^{c}$.

Lemma 3.3. Let $N_{t}$ be the total space of the circle bundle over $[N, t] \in \Theta$. Then there is a homeomorphism $N_{t} \cong M_{l, k}^{c}$ where

$$
\begin{aligned}
& (k, c)=\left(k([N, t]), \frac{1+(-1)^{k([N, t])}}{2} \cdot \delta([N, t])\right) ; \\
& l=\frac{p([N, t])+(3 \varepsilon([N, t])-4) \cdot k([N, t])-\left(1+(-1)^{k([N, t])}\right) \cdot 12 \delta([N, t])}{4} .
\end{aligned}
$$

Proof. We divide the proof into two cases depending on $\Delta\left(N_{t}\right) \equiv 0$ or 1 $\bmod 2$.

Case 1. $\Delta\left(N_{t}\right) \equiv 0 \bmod 2$ (i.e. the manifold $N_{t}$ is smoothable, see [18, p.33], [12] and [24, Theorem 5.4]): Module by a $\mathbb{Z}_{2}$ ambiguity Wilkens [29] showed that the system $\left\{H, \frac{p_{1}}{2}, b\right\}$ of invariants classifies $N_{t}$ and $M_{l, k}^{0}$ up to homeomorphism. Moreover, Crowley and Escher [3] proved that this ambiguity can be realized by some $M_{l, k}^{0}$ whose homeomorphism types can be distinguished by the invariant $s_{1}$ and hence the system $\left\{H, \frac{p_{1}}{2}, b, s_{1}\right\}$ classifies the manifolds $N_{t}$ and $M_{l, k}^{0}$. Therefore the proof is completed by comparing these invariants for $N_{t}$ and $M_{l, k}^{0}$ obtained in Example 3.1 and 3.2.

Case 2. $\Delta\left(N_{t}\right) \equiv 1 \bmod 2$ : We only need to show that $N_{t}$ is homeomorphic to $M_{l, k}^{1}$ with $[N, t] \in \Theta$ and

$$
(k, l)=\left(k([N, t]), \frac{p([N, t])+(3 \varepsilon([N, t])-4) \cdot k([N, t])-24}{4}\right) .
$$

It suffices to construct a homotopy equivalence $q: N_{t} \rightarrow M_{l, k}^{0}$ with

$$
\left[N_{t}, q\right]=\left[M_{l, k}^{1}, h_{M}\right] \in \mathcal{S}^{T O P}\left(M_{l, k}^{0}\right)
$$

According to Lemma 2.2 there exists a manifold $N^{\prime}$ with $\left[N^{\prime}, t^{\prime}\right] \in \Theta$ whose invariant system $\left(k\left(\left[N^{\prime}, t^{\prime}\right]\right), p\left(\left[N^{\prime}, t^{\prime}\right]\right), \varepsilon\left(\left[N^{\prime}, t^{\prime}\right]\right), \delta\left(\left[N^{\prime}, t^{\prime}\right]\right)\right.$ is

$$
(k([N, t]), p([N, t])-24, \varepsilon([N, t]), 0) .
$$

Consider the map $\eta: \mathcal{S}^{T O P}\left(N^{\prime}\right) \rightarrow\left[N^{\prime}, G / T O P\right]$ in the surgery exact sequence of $N^{\prime}$. By the argument at the end of the proof of [14, Theorem 1] we find a homotopy equivalence $h_{N}: N \rightarrow N^{\prime}$ such that
i) the class $\eta\left(\left[N, h_{N}\right]\right)$ is trivial on the 2 skeleton of $N^{\prime}$;
ii) the primary obstruction to finding a null-homotopy of $\eta\left(\left[N, h_{N}\right]\right)$
is the generator $x^{\prime} \in H^{4}\left(N^{\prime} ; \pi_{4}(G / T O P)\right)$ with $\left\langle x^{\prime} \cup t^{\prime},\left[N^{\prime}\right]\right\rangle=1$.
Pulling $h_{N}$ back by the bundle projection $\pi^{\prime}: N_{t}^{\prime} \rightarrow N^{\prime}$ induces a homotopy equivalence $h_{t}: N_{t} \rightarrow N_{t}^{\prime}$. On the other hand, by the result of Case 1 we get a homeomorphism $u: M_{l, k}^{0} \rightarrow N_{t}^{\prime}$ such that $u^{*}\left(\pi^{*}\left(x^{\prime}\right)\right)=\pi_{M}^{*}(\iota)$. So it remains to show that $\left[N_{t}, u^{-1} \circ h_{t}\right]=\left[M_{l, k}^{1}, h_{M}\right]$.

Let $\left[N^{\prime}, G / T O P\right]_{2}$ denote the subset of $\left[N^{\prime}, G / T O P\right]$ whose elements are trivial on the 2 skeleton of $N^{\prime}$ and consider the following two commutative diagrams:

$$
\begin{aligned}
& \mathcal{S}^{T O P}\left(N^{\prime}\right) \quad \xrightarrow{\pi^{\prime *}} \quad \mathcal{S}^{T O P}\left(N_{t}^{\prime}\right) \\
& \downarrow \eta \quad \cong \downarrow \eta \\
& {\left[N^{\prime}, G / T O P\right] \xrightarrow{\pi^{\prime *}}\left[N_{t}^{\prime}, G / T O P\right]} \\
& \mathcal{S}^{T O P}\left(N_{t}^{\prime}\right) \xrightarrow{\stackrel{u^{*}}{\geqq}} \mathcal{S}^{T O P}\left(M_{l, k}^{0}\right) \\
& \cong \downarrow \eta \quad \cong \downarrow \eta \\
& {\left[N^{\prime}, G / T O P\right]_{2} \xrightarrow{\pi^{\prime *}}\left[N_{t}^{\prime}, G / T O P\right] \xrightarrow{\stackrel{u^{*}}{\cong}}\left[M_{l, k}^{0}, G / T O P\right]} \\
& \downarrow d \quad \cong \downarrow d \quad \cong \downarrow d \\
& H^{4}\left(N^{\prime}\right) \quad \xrightarrow{\pi^{\prime *}} \quad H^{4}\left(N_{t}^{\prime}\right) \quad \xrightarrow{\xrightarrow{u^{*}}} \quad H^{4}\left(M_{l, k}^{0}\right)
\end{aligned}
$$

where
i) $\mathcal{S}^{T O P}\left(N^{\prime}\right) \xrightarrow{\pi^{\prime *}} \mathcal{S}^{T O P}\left(N_{t}^{\prime}\right)$ maps $\left[N^{\prime \prime}, h^{\prime \prime}\right]$ to $\left[N_{t}^{\prime \prime}, h_{t}^{\prime \prime}\right]$ with $h_{t}^{\prime \prime}$
a pull-back of $h^{\prime \prime}$ by the bundle projection $\pi^{\prime}: N_{t}^{\prime} \rightarrow N^{\prime}$;
ii) $\mathcal{S}^{T O P}\left(N_{t}^{\prime}\right) \xrightarrow{u^{*}} \mathcal{S}^{T O P}\left(M_{l, k}^{0}\right) \operatorname{maps}\left[M^{\prime}, g^{\prime}\right]$ to $\left[M^{\prime}, u^{-1} \circ g^{\prime}\right]$;
iii) the maps $d$ send a homotopy class to its primary obstruction to null-homotopy.

The diagrams above, together with the relations

$$
\pi^{\prime *}\left[N, h_{N}\right]=\left[N_{t}, h_{t}\right], u^{*}\left(\pi^{\prime *}\left(x^{\prime}\right)\right)=\pi_{M}^{*}(\iota) \text { and } d\left(\eta\left[N, h_{N}\right]\right)=x^{\prime}
$$

imply that $u^{*}\left[N_{t}, h_{t}\right]=\left[M_{l, k}^{1}, h_{M}\right]$, i.e. $\left[N_{t}, u^{-1} \circ h_{t}\right]=\left[M_{l, k}^{1}, h_{M}\right]$. This completes the proof of Case 2 .

The following lemma plays a key role in the proof of Lemma 3.5 and its proof will be postponed to the end of this section.

Lemma 3.4. Let $N_{t}$ be the total space of the circle bundle over $[N, t] \in \Theta$ with $\delta([N, t])=0$ and $k([N, t])=0$. Then there exists an 8 -manifold $W$ homotopy equivalent to $S^{4}$ whose boundary satisfies

$$
\partial W \cong\left\{\begin{array}{cl}
N_{t} & \text { if } \varepsilon([N, t])=1 \\
N_{t} \# \Sigma_{\frac{p([N, t])}{24}} & \text { if } \varepsilon([N, t])=0
\end{array}\right.
$$

Lemma 3.5. Let $N_{t}$ be the total space of the circle bundle over $[N, t] \in \Theta$ with $\delta([N, t])=0$. Then one has a diffeomorphism $N_{t} \cong M_{l, k}^{0} \# \Sigma_{r}$ where $N_{t}$ has the smooth structure as the total space of the circle bundle and where

$$
(k, l, r)=\left(k([N, t]), \frac{p([N, t])+(3 \varepsilon([N, t])-4) \cdot k([N, t])}{4}, \frac{(1-\varepsilon([N, t])) \cdot(p([N, t])-4 k([N, t]))}{24}\right) .
$$

Proof. In the case of $k([N, t]) \neq 0$ it is shown in 3 that the system $\left\{H, \frac{p_{1}}{2}, b, \mu\right\}$ classifies $N_{t}$ and $M_{l, k}^{0}$ up to diffeomorphism. Hence the proof is done by comparing those invariants for $N_{t}$ and $M_{l, k}^{0}$ obtained in Examples 3.1 and 3.2.

Assume next that $k([N, t])=0$ and let $W$ be the 8 -manifold given by Lemma 3.4. We can take a closed tubular neighborhood $E$ of an embedding $h: S^{4} \hookrightarrow$ Interior $W$ which is also a homotopy equivalence [20, Lemma 6]. As $H_{i}(W \backslash$ Interior $E, \partial E) \cong H_{i}(W, E)=0$ for all $i$ by the excision theorem, $W \backslash$ Interior $E$ is an $\mathrm{h}-$ cobordism between $\partial W$ and $\partial E$. Hence we get the diffeomorphisms $\partial W \cong \partial E \cong M_{l, k}^{0}$ for some $l, k \in \mathbb{Z}$ by the hcobordism theorem [21, Theorem 9.1] and the fact that $E$ is the total space of the normal disk bundle of the embedding $h$. Comparing the invariants $\left\{H, \frac{p_{1}}{2}\right\}$ of $\partial W$ and $M_{l, k}^{0}$ given in Examples 3.1 and 3.2 we find that $\partial W$ is diffeomorphic to $M_{\frac{p([N, t])}{4}, 0}^{0}$ [3]. This shows

$$
N_{t} \cong\left\{\begin{array}{cl}
M_{\frac{p([N, t])}{4}, 0}^{0} \# \Sigma_{\frac{p([N, t])}{24}} & \text { if } \varepsilon([N, t])=0 \\
M_{\frac{p([N, t])}{4}, 0}^{0} & \text { if } \varepsilon([N, t])=1
\end{array}\right.
$$

which completes the proof.
Proof of Lemma 3.4. The construction of $W$ and the corresponding calculations will be divided into two cases depending on $\varepsilon([N, t])=0$ or 1. Let $\pi_{0}: W_{t} \rightarrow N$ be the associated disk bundle of the circle bundle $\pi: N_{t} \rightarrow N$.

Case 1. $\varepsilon([N, t])=1$ : Take an embedding $f: S^{2} \hookrightarrow$ Interior $W_{t}$ that represents a generator of the group $H_{2}\left(W_{t}\right) \cong \mathbb{Z}$. Since $w_{2}\left(W_{t}\right)=0$ (see Example 3.1), the $f$ extends to an embedding $\bar{f}: S^{2} \times D^{6} \hookrightarrow$ Interior $W_{t}$. The 8 -manifold $W$ is obtained from $W_{t}$ by surgery along $\bar{f}$. On one hand, it is clear that $\partial W \cong \partial W_{t} \cong N_{t}$. On the other hand, from the homotopy equivalences

$$
X \simeq W_{t} \cup_{f} D^{3} \simeq W \cup D^{6}
$$

with $X$ the trace of the surgery [1, P.83-84] we find that $W$ is 3-connected with $H_{4}(W) \cong H_{4}(X) \cong \mathbb{Z}$. Moreover, since $N_{t}$ is 2 -connected and $W$ is 3 -connected we can conclude that $H_{i}(W) \cong H^{8-i}\left(W, N_{t}\right)=0$ for $i \geq 5$ by the Lefschetz duality and the cohomology exact sequence. Hence from the Whitehead theorem we get $W \simeq S^{4}$.

Case 2. $\varepsilon([N, t])=0$ : The desired manifold $W$ is constructed as follows. Represent the generator $x \cap[N] \in H_{2}(N)$ by an embedding $\bar{g}: S^{2} \times D^{4} \hookrightarrow N$ $\left(w_{2}(N)=0\right)$. Let $\bar{h}$ be the pull-back of $\bar{g}$ by the projection $\pi$ as shown in the diagram

$$
\begin{array}{ccc}
S^{3} \times D^{4} & \stackrel{\bar{h}}{\hookrightarrow} & N_{t} \\
\downarrow & & \pi \downarrow,  \tag{3.1}\\
S^{2} \times D^{4} & \stackrel{\bar{g}}{\hookrightarrow} & N
\end{array}
$$

and let $\widetilde{W}:=N_{t} \times[0,1] \cup_{(\bar{h}, 1)} D^{4} \times D^{4}$. Then

$$
\begin{equation*}
\partial \widetilde{W} \cong N_{t} \sqcup\left(-\Sigma_{r}\right) \text { for some } r \in \mathbb{Z} \tag{3.2}
\end{equation*}
$$

since in (3.1) the map $\bar{h}$ induces an isomorphism $\pi_{3}\left(S^{3} \times D^{4}\right) \rightarrow \pi_{3}\left(N_{t}\right)$ [28, Lemma 1]. The manifold $W$ is obtained from $\widetilde{W}$ by removing the tubular neighborhood of a smooth $\operatorname{arc} \alpha:[0,1] \rightarrow \widetilde{W}$ with $\alpha(0) \in N_{t}, \alpha(1) \in \partial W^{\prime}$ and $\alpha(0,1) \subset$ Interior $\widetilde{W}$.

It remains to show that
i) $W \simeq S^{4}$; ii) $r=\frac{p([N, t])}{24}($ in $(3.2))$.

The property i) follows from the facts that the trace $\widetilde{W}$ of the surgery along $\bar{h}$ has the homotpy type $\Sigma_{r} \cup D^{4}$ and the homeomorphism type of $W$ is obtained from $\widetilde{W}$ to collapse the component $\Sigma_{r}$ of $\partial \widetilde{W}$ to a point.

For the property ii) we only need to show $\mu\left(\Sigma_{r}\right) \equiv \frac{p(N)}{24 \cdot 28} \bmod \mathbb{Z}$. By the collar neighborhood theorem, the homotopy sphere $\Sigma_{r}$ in (3.2) bounds an 8 -manifold $W^{\prime}:=W_{t} \cup_{\bar{h}} D^{4} \times D^{4}$. For the convenience of calculation, we make use of an alternative decomposition $W^{\prime}=W_{t} \cup_{\overline{i_{0}}} \mathbb{C} P^{2} \times D^{4}$ where $\overline{i_{0}}$ is the pull-back of $\bar{g}$ by the projection $\pi_{0}$ as in the diagram

$$
\begin{array}{ccc}
V \times D^{4} & \stackrel{\overline{i_{0}}}{\hookrightarrow} & W_{t} \\
\downarrow & & \pi_{0} \downarrow . \\
S^{2} \times D^{4} & \stackrel{\bar{g}}{\hookrightarrow} & N
\end{array}
$$

From the isomorphism

$$
H_{4}\left(\mathbb{C} P^{2}\right) \oplus H_{4}\left(W_{t}\right) \stackrel{i_{1 *} \oplus i_{2 *}}{\cong} H_{4}\left(W^{\prime}\right)
$$

by the Mayer-Vietoris sequence with $\alpha \in H_{4}\left(W_{t}\right) \cong \mathbb{Z}$ the generator satisfies $\left\langle\pi_{0}^{*} x, \alpha\right\rangle=1$ (see Section 2) and $i_{1}: \mathbb{C} P^{2} \rightarrow W^{\prime}, i_{2}: W_{t} \rightarrow W^{\prime}$ the inclusions, we can see below that the intersection matrix of $W^{\prime}$ with respect to the basis $x_{1}, x_{2} \in H^{4}\left(W^{\prime}, \partial W^{\prime}\right)$ is

$$
\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right)
$$

where $i_{1 *}\left[\mathbb{C} P^{2}\right]=x_{1} \cap\left[W^{\prime}, \partial W^{\prime}\right], i_{2 *} \alpha=x_{2} \cap\left[W^{\prime}, \partial W^{\prime}\right]$. First observe that

$$
\left\langle x_{1} \cup x_{1},\left[W^{\prime}, \partial W^{\prime}\right]\right\rangle=0
$$

as the normal bundle of $i_{1}$ is trivial. Next since the self-intersection number of $i_{2 *} \alpha$ is the same as that of $\alpha$ and the homomorphism $j^{*}: H^{4}\left(W_{t}, N_{t}\right) \rightarrow$ $H^{4}\left(W_{t}\right)$ is trivial (see Example 3.1), this implies

$$
\left\langle x_{2} \cup x_{2},\left[W^{\prime}, \partial W^{\prime}\right]\right\rangle=\left\langle j^{*} D_{W_{t}} \alpha \cup D_{W_{t}} \alpha,\left[W_{t}, N_{t}\right]\right\rangle=0[1, \mathrm{p} .115]
$$

where $D_{W_{t}} \alpha$ denotes the Lefschetz duality of $\alpha$. Finally we have

$$
\left\langle x_{1} \cup x_{2},\left[W^{\prime}, \partial W^{\prime}\right]\right\rangle=\left\langle j^{\prime *} x_{1}, i_{2 *} \alpha\right\rangle=\left\langle i_{2}^{*} j^{\prime *} x_{1}, \alpha\right\rangle=\left\langle\pi_{0}^{*} x, \alpha\right\rangle=1
$$

with $j^{\prime}: W^{\prime} \rightarrow\left(W^{\prime}, \partial W^{\prime}\right)$ the inclusion and where $\pi_{0}^{*} x=i_{2}^{*} j^{*} x_{1}$ follows from the relations $\pi_{0}^{-1} f\left[S^{2}\right]=i_{2}^{-1} i_{1}\left[\mathbb{C} P^{2}\right], x=D_{N} f_{*}\left[S^{2}\right]$ and $x_{1}=$ $D_{W^{\prime}} i_{1 *}\left[\mathbb{C} P^{2}\right]$. Thus the signature $\sigma\left(W^{\prime}\right)$ is 0 .

We can take $z$ to be a generator of $H^{2}\left(W^{\prime}\right) \cong \mathbb{Z}$ since $i_{2}^{*} w_{2}\left(W^{\prime}\right)=$ $w_{2}\left(W_{t}\right) \neq 0$ by Example 3.1 and the isomorphism $i_{2}^{*} T W^{\prime} \cong T W_{t}$. To get the values of $z^{2}, p_{1}\left(W^{\prime}\right)$, it is necessary to compute the images of $z^{2}, p_{1}\left(W^{\prime}\right)$ under the isomorphism

$$
i_{1}^{*} \oplus i_{2}^{*}: H^{4}\left(W^{\prime}\right) \rightarrow H^{4}\left(\mathbb{C} P^{2}\right) \oplus H^{4}\left(W_{t}\right)
$$

whose matrix with respect to the basis $\left\{j^{\prime *} x_{1}, j^{\prime *} x_{2}\right\}$ and $\left\{\left[\mathbb{C} P^{2}\right]^{*}, \pi_{0}^{*} x\right\}$ is the same as the intersection matrix of $W^{\prime}$ with respect to the basis $x_{1}, x_{2}$, where $\left[\mathbb{C} P^{2}\right]^{*} \in H^{4}\left(\mathbb{C} P^{2}\right)$ satisfies $\left\langle\left[\mathbb{C} P^{2}\right]^{*},\left[\mathbb{C} P^{2}\right]\right\rangle=1$. Since $i_{1}^{*} z \in$ $H^{2}\left(\mathbb{C} P^{2}\right), i_{2}^{*} z \in H^{2}\left(W_{t}\right)$ are generators, it follows that

$$
i_{1}^{*} \oplus i_{2}^{*}\left(z^{2}\right)=\left(i_{1}^{*} z^{2}, \pi_{0}^{*} t^{2}\right)=\left(\left[\mathbb{C} P^{2}\right]^{*}, 0\right)
$$

Moreover, according to the isomorphisms $i_{2}^{*} T W^{\prime} \cong T W_{t}$ and $i_{1}^{*} T W^{\prime} \cong$ $T \mathbb{C} P^{2} \oplus \varepsilon^{4}$, the relations $\left\langle p_{1}\left(\mathbb{C} P^{2}\right),\left[\mathbb{C} P^{2}\right]\right\rangle=3$ and $p_{1}\left(W_{t}\right)=p([N, t]) \pi_{0}^{*} x$ imply that

$$
i_{1}^{*} \oplus i_{2}^{*} p_{1}\left(W^{\prime}\right)=\left(3\left[\mathbb{C} P^{2}\right]^{*}, p([N, t]) \pi_{0}^{*} x\right)
$$

Therefore we can see that

$$
z^{2}=j^{\prime *} x_{2} ; p_{1}\left(W^{\prime}\right)=p([N, t]) j^{\prime *} x_{1}+3 j^{\prime *} x_{2} .
$$

Again from these relations and the intersection form of $W^{\prime}$ we get

$$
p_{1}^{2}\left(W^{\prime}\right)=6 p([N, t]) ; z^{2} p_{1}\left(W^{\prime}\right)=p([N, t]) ; z^{4}=0
$$

Consequently, substituting these values in the formula of $\mu$, this implies

$$
\mu\left(M^{\prime}\right)=\frac{p(N)}{24 \cdot 28} \bmod \mathbb{Z} .
$$

Remark 3.6. In a communication concerning this work Diarmuid Crowley pointed out that according to a result of Wilkens [30, Theorem 1 (ii)] the decomposition $N_{t} \cong M_{l, k}^{0} \# \Sigma_{r}$ in Lemma 3.4 can be simplified as $N_{t} \cong M_{l, k}^{0}$ when $k([N, t])=0$, which will play a role in the proof of Corollary 4.5 in the coming section.

In the recent paper 44 (see also [2] [5]) Crowley and Nordstrom generalised the classical Eells-Kuiper invariant $\mu$. Their new invariant can be applied to give a simple proof of the diffeomorphism $N_{t} \cong M_{l, k}^{0}$ when $k([N, t])=0$.

## 4 Proof of Theorem 1.3 and applications

We establish Theorem 1.3 and present some applications.
Proof of Theorem 1.3. Let $M$ be a $2-$ connected $7-$ manifold with a regular circle action. By Lemmas 2.1 and $2.2 M$ is the total space of the oriented circle bundle over $N \#_{r} S^{3} \times S^{3}$ with Euler class $\bar{t} \in H^{2}\left(N \#_{r} S^{3} \times S^{3}\right) \cong \mathbb{Z}$ a generator, where $[N, t] \in \Theta, r \in \mathbb{N}$. Identify $\bar{t}$ with the generator $t \in$ $H^{2}(N) \cong \mathbb{Z}$ under the isomorphism $H^{2}(N) \rightarrow H^{2}\left(N \#_{r} S^{3} \times S^{3}\right)$ induced by the map $N \#_{r} S^{3} \times S^{3} \rightarrow N$ collapsing $\#_{r} S^{3} \times S^{3}$ to a point. By Lemmas 3.3 and 3.4 it suffices to show that $M \cong N_{t} \#_{2 r} S^{3} \times S^{4}$.

Consider the decomposition

$$
N \#_{r} S^{3} \times S^{3}=\left(N \backslash \stackrel{\circ}{D}_{1}\right) \cup_{f}\left(\#_{r} S^{3} \times S^{3} \backslash \stackrel{\circ}{D}_{2}\right)
$$

with $D_{i} \cong D^{6}$ and $f: \partial D_{2} \rightarrow \partial D_{1}$ a diffeomorphism. Since the restriction of the bundle $N_{t} \rightarrow N$ on $D_{1}$ is trivial and $N_{t} \cong N_{t} \# S^{7}$ one has the corresponding decomposition

$$
M \cong\left(N_{t} \backslash \stackrel{\circ}{D}_{1} \times S^{1}\right) \cup_{f \times i d}\left(\left(\#_{r} S^{3} \times S^{3} \backslash \stackrel{\circ}{D}_{2}\right) \times S^{1}\right) \cong N_{t} \# M_{0}
$$

where $i d$ is the identity on $S^{1}$, and where

$$
M_{0}=\left(S^{7} \backslash \stackrel{\circ}{D}_{1} \times S^{1}\right) \cup_{f \times i d}\left(\left(\#_{r} S^{3} \times S^{3} \backslash \stackrel{\circ}{D}_{2}\right) \times S^{1}\right)
$$

Since $M_{0}$ can be easily identified with the total space of the oriented circle bundle over $C P^{3} \#_{r} S^{3} \times S^{3}$ with Euler class a proper generator of $H^{2}\left(C P^{3} \#_{r} S^{3} \times S^{3}\right) \cong \mathbb{Z}$, a calculation similar to that in Example 3.1 shows that the invariant system $\left\{H, \frac{p_{1}}{2}, b, \mu\right\}$ for $M_{0}$ and $\#{ }_{2 r} S^{3} \times S^{4}$ coincides. Consequently $M_{0}$ is diffeomorphic to $\#_{2 r} S^{3} \times S^{4}$. This shows that $M \cong N_{t} \#{ }_{2 r} S^{3} \times S^{4}$ which completes the proof.

A classical topic is to decide which homotopy spheres admit smooth regular circle actions ( [13] [17] [22] [23]). Combining Theorem 1.3 with Example 3.2 we recover the classical computation of Montgomery and Yang [22].

Corollary 4.1. Among the 28 homotopy 7 -spheres $\Sigma_{r}, 0 \leq r \leq 27$ the following ones admit smooth regular circle actions

$$
\Sigma_{r}, r=0,4,6,8,10,14,18,20,22,24 .
$$

In term of our notation the unit tangent bundle of the sphere $S^{4}$ is $M_{-1,2}^{0}$. The additive property of the Eells-Kuiper invariant $\mu$ shows that $M_{-1,2}^{0} \# \Sigma_{r}$ with $0 \leq r \leq 27$ represent all the diffeomorphism types of the smooth manifolds homeomorphic to $M_{-1,2}^{0}$. One can deduce from Theorem 1.3 and Example 3.2 that

Corollary 4.2. All the smooth manifolds homeomorphic to the unit tangent bundle of the sphere $S^{4}$ and admitting smooth regular circle actions are

$$
M_{-1,2}^{0} \# \Sigma_{r}, r=0,2,6,7,8,12,14,15,16,19,20,23,26 .
$$

In [11] Grove, Verdiani and Ziller constructed on the manifold $M_{-1,2}^{0} \# \Sigma_{27}$ a metric with positive sectional curvature (see Goette [10, p.34-35]). According to Corollary 4.2 this manifold does not admit any smooth regular circle action.

Definition 4.3 Two regular (resp. smooth regular) circle actions

$$
S^{1} \times M_{i} \rightarrow M_{i}, i=1,2,
$$

on two manifolds (resp. smooth manifolds) $M_{i}$ are called equivalent if there is a equivariant homeomorphism (diffeomorphism) $f: M_{1} \rightarrow M_{2}$. Let $\rho_{T}(M)$ (resp. $\rho_{S}(M)$ ) be the number of all equivalence classes of regular (resp. smooth regular) circle actions on a given manifold (resp. smooth manifold) $M$.

Since the number $\rho_{T}(M)$ can be seen as the number of those elements $[N, t] \in \Theta$ satisfying $N_{t} \cong M$, we get from Lemmas 2.2 and 3.3 that

Corollary 4.4. For the family

$$
M=M_{6 m,(1+c) k}^{c} \#_{2 r} S^{3} \times S^{4}, c \in\{0,1\}, r \in \mathbb{N}, m, k \in \mathbb{Z}
$$

of manifolds that represent all homeomorphism classes of the 2-connected 7 -manifolds with regular circle actions (see Theorem 1.3) we have

$$
\rho_{T}(M)=\left\{\begin{array}{cc}
1 & \text { if } k=0 \text { and } m \equiv 1 \bmod 2, \\
2 & \text { if } k=0 \text { and } m \equiv 0 \bmod 2, \\
\infty & \text { if } k \neq 0
\end{array}\right.
$$

Similarly, in the smooth category we get from Lemmas 2.2 and 3.5, together with Remark 3.6, that

Corollary 4.5. For the family

$$
M=M_{6(1+a) m,(1+a) k}^{0} \# \Sigma_{(1-a) m} \# 2 r S^{3} \times S^{4}, a \in\{0,1\}, r \in \mathbb{N}, m, k \in \mathbb{Z}
$$

of manifolds that represent all diffeomorphism classes of the smooth $2-$ connected 7 -manifolds with smooth regular circle actions (see Theorem 1.3) we have

$$
\sigma(M)=\left\{\begin{array}{cc}
1 & \text { if } k=0, a=0 \text { and } m \equiv 1 \bmod 2, \\
2 & \text { if } k=0 \text { and }(1+a) m \equiv 0 \bmod 2, \\
\infty & \text { if } k \neq 0
\end{array}\right.
$$

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