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Approximation of holomorphic maps from Runge domains to affine algebraic varieties

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Abstract

We present a geometric proof of the theorem saying that holomorphic maps from Runge domains to affine algebraic varieties admit approximation by Nash maps. Next we generalize this theorem.

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1 Introduction

After the seminal papers of Artin [2], [3] the fundamental problem of algebraic approximation of holomorphic maps satisfying polynomial equations has been studied by several mathematicians (see e.g. [7], [9], [10], [14], [19], [21], [31]). The following result, which can be viewed as a global version of Artin's approximation theorem, is due to L. Lempert (see [21], p. 335).

Theorem 1.1 *Let V, W be complex affine algebraic varieties, let $K \subset W$ be a holomorphically convex compact set and let $f : K \to V$ be a holomorphic map. Then f can be uniformly approximated by a sequence $f_\nu : K \to V$ of Nash maps.*

(For the definition of Nash maps see Section 3.2; In the case of V nonsingular Theorem 1.1 had been proved before in [9]. If $W = \mathbf{C}$ and V is arbitrary then it follows from [10].) Artin's approximation theorem is local and its proof uses Weierstrass Preparation. The original proof of Lempert's approximation theorem [21], pp 338-339, relies on the general Néron desingularization, a deep and difficult result of commutative algebra for which the reader is referred to [1], [26], [27], [28], [29], [30].

Theorem 1.1 is expressed in terms of analytic geometry and has had numerous applications in the theory of several complex variables (see [5], [11], [12], [13], [21], [23], [25], [31]). It is natural to ask whether one can replace Néron desingularization by simpler geometric methods. The main purpose of this paper is to present a new geometric proof of Theorem 1.1 based on classical arguments of singularity theory and complex analysis (see Section 4). In the last section, Section 5, we show how our method allows us to generalize Lempert's result.

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Several variants of Artin's approximation theorem turned out to be very useful in singularity theory and complex geometry. It is difficult to give here a full account of this research. Instead we refer the reader to two recent papers [17], [24] and references therein.

Our main results will be preceded by an outline of the proof of Theorem 1.1 where we present the main ideas and explain why the proof is organized in the way it is (see Section 2). The preliminary material is gathered in Section 3.

2 Outline of the proof of Theorem 1.1

First the problem is reduced (by means of standard methods of multidimensional complex analysis) to the case where $W = \mathbf{C}^n$ for some integer n, and K is a compact polydisc (see Section 4). Then it is sufficient to prove the following

Theorem 2.1 *Let $f : U \to V$ be a holomorphic map, where $U \subseteq \mathbf{C}^n$ is an open polydisc and $V \subseteq \mathbf{C}^q$ is an algebraic variety. Then for every open $U_0 \Subset U$ there is a sequence $f_\nu : U_0 \to V$ of Nash maps converging uniformly to $f|_{U_0}$.*

Theorem 2.1 will be obtained by reducing to the case when $\dim f^{-1}(\mathrm{Sing}(V)) < n-1$ and by applying Lemma 4.1.

Throughout the proof the domain on which the relevant functions are defined will be shrunk for several times. For this reason in Section 4 we work with a fixed compact set K, and the functions are defined in some neighborhood of K (which can be changed). In this outline, for simplicity of notation we assume that U has all the properties which actually are obtained after shrinking this domain.

Let $f^{-1}(\mathrm{Sing}(V))_{(n-1)}$ denote the union of all $(n-1)$–dimensional irreducible components of $f^{-1}(\mathrm{Sing}(V))$ and let $\overline{f(U)}^z$ denote the Zariski closure of $f(U)$ (i.e. the smallest algebraic set containing $f(U)$). We shall explain the idea of the proof of Theorem 2.1 additionally assuming that $\overline{f(U)}^z = V$, and that $f^{-1}(\mathrm{Sing}(V))_{(n-1)}$ has a finite number of irreducible components. Making these assumptions we do not lose generality. (This is because first V can be replaced by $\overline{f(U)}^z$. Then one can shrink U to obtain the finiteness condition.) Our aim is to construct a holomorphic map $F_1 : U \to V_1$ such that: $\overline{F_1(U)}^z = V_1$ and $F_1^{-1}(\mathrm{Sing}(V_1))_{(n-1)}$ has fewer irreducible components than $f^{-1}(\mathrm{Sing}(V))_{(n-1)}$, and if F_1 can be approximated by Nash maps into V_1 then f can be approximated by Nash maps into V. When this is accomplished we can replace f by F_1 and repeat the whole process. Such repetitions lead us to the case solved by Lemma 4.1, application of which finishes the proof.

Let us start with preliminary remarks. First $V \subset \mathbf{C}^k \times \mathbf{C}^{q-k}$ can be assumed to be an irreducible normal analytic space with proper projection onto $\mathbf{C}^k$, where $k = \dim V$. A reduction to the case where this assumption is satisfied is standard (for details see the proof of Theorem 2.1 in Section 4). Then the set $\Sigma_V \subset \mathbf{C}^k$ (defined in Section 3.4) is either empty or purely $(k-1)$-dimensional. (This is because otherwise V would not be locally irreducible contradicting normality.)

Consequently, there is a reduced $N \in \mathbf{C}[w_1, \ldots, w_k]$ such that $N^{-1}(0) = \Sigma_V$. Let $G = \pi \circ f$, where $\pi : \mathbf{C}^k \times \mathbf{C}^{q-k} \to \mathbf{C}^k$ denotes the natural projection. Since $\overline{f(U)}^z = V$, we have $G(U) \nsubseteq \Sigma_V$. Therefore $(N \circ G)^{-1}(0)$ is either purely $(n-1)$–dimensional or empty. We can assume, shrinking U if needed, that $(N \circ G)^{-1}(0)$ has a finite number of irreducible components.

Now our main tools are Propositions 4.2, 4.3 proved in Sections 4.3, 4.4, respectively, and Corollary 3.8. First, Proposition 4.2 enables us to reduce the problem to the case when

$$(\flat) \qquad G^{-1}(\mathrm{Sing}(\Sigma_V))_{(n-1)} \subseteq f^{-1}(\mathrm{Sing}(V))_{(n-1)}.$$

To be more precise, Proposition 4.2 provides us with a suitable linear change of the coordinates in $\mathbf{C}^q$ after which $(\flat)$ holds.

Next we will construct a holomorphic map F_* into an algebraic variety V_* with $F_*^{-1}(\mathrm{Sing}(V_*)) \subseteq G^{-1}(\mathrm{Sing}(\Sigma_V))$ and such that if there is a sequence $F_{*,\nu}$ of Nash maps into V_* approximating F_*, then there is a sequence G_ν of Nash maps into $\mathbf{C}^k$ approximating G such that $\{(N \circ G_\nu)^{-1}(0)\}$ converges to $(N \circ G)^{-1}(0)$ in the sense of chains. (We say that a sequence $\{B_\nu\}$ of purely s–dimensional analytic sets converges to a purely s–dimensional analytic set B in the sense of chains if $\{Z_\nu\}$ converges to Z, where Z_ν and Z are chains obtained by assigning multiplicity 1 to all irreducible components of B_ν and B, respectively. For the definition of the convergence of chains see Section 3.3.)

Observe that, by Corollary 3.8, the existence of G_ν as above implies that there is a sequence f_ν of Nash maps into V approximating f. Moreover, by $(\flat)$, the number of the irreducible components of $F_*^{-1}(\mathrm{Sing}(V_*))_{(n-1)}$ does not exceed the number of the irreducible components of $f^{-1}(\mathrm{Sing}(V))_{(n-1)}$. Since the former number can be equal to the latter one, in general we cannot define $V_1 = V_*$, $F_1 = F_*$. However, V_* will have a very special description whose modification will allow us to construct V_1, F_1 with all the required properties.

Let us describe how to obtain V_*, F_* and V_1, F_1. (Details are presented in the proof of Proposition 4.3.) Let $A_1, \ldots, A_p$ denote the (pairwise distinct) irreducible components of $(N \circ G)^{-1}(0)$. Since U is an open polydisc, we have $N \circ G = u_1^{\alpha_1} \cdot \ldots \cdot u_p^{\alpha_p} \bar{R}$, where $\bar{R} \in \mathcal{O}(U)$ is a nowhere vanishing function, $u_1, \ldots, u_p$ are minimal defining functions for $A_1, \ldots, A_p$, and $\alpha_1, \ldots, \alpha_p$ are positive integers. (Recall that $u \in \mathcal{O}(U)$ is called a minimal defining function for A if $A = u^{-1}(0)$ and for every open subset $D \subseteq U$ and $v \in \mathcal{O}(D)$ with $A \cap D \subseteq v^{-1}(0)$, there is $g \in \mathcal{O}(D)$ such that $v = g \cdot u|_D$. It is well known that the existence of minimal defining functions is a consequence of universal solvability of the second Cousin problem on U which, if U is a domain of holomorphy, is equivalent to $H^2(U, \mathbf{Z}) = 0$, cf. [18].)

Now define $F_* = (G, u_1, \ldots, u_p, \bar{R})$,

$$V_* = \{(w_1, \ldots, w_k, u_1, \ldots, u_p, \bar{R}) \in \mathbf{C}^{k+p+1} : N(w_1, \ldots, w_k) = u_1^{\alpha_1} \cdot \ldots \cdot u_p^{\alpha_p} \bar{R}\},$$

and suppose that there are sequences $G_\nu, u_{1,\nu}, \ldots, u_{p,\nu}, \bar{R}_\nu$ of Nash maps converging locally uniformly to $G, u_1, \ldots, u_p, \bar{R}$ such that $N \circ G_\nu = u_{1,\nu}^{\alpha_1} \cdot \ldots \cdot u_{p,\nu}^{\alpha_p} \bar{R}_\nu$. Since $u_1, \ldots, u_p$ are minimal defining functions, $\{(N \circ G_\nu)^{-1}(0)\}$ converges to

$(N \circ G)^{-1}(0)$ in the sense of chains. Since N is reduced, $F_*^{-1}(\mathrm{Sing}(V_*)) \subseteq G^{-1}(\mathrm{Sing}(N^{-1}(0)))$. The functions $u_1, \ldots, u_p, \bar{R}$ will be chosen in such a way that $\overline{F_*(U)}^z = V_*$.

Let us turn to V_1, F_1. If $F_*^{-1}(\mathrm{Sing}(V_*))_{(n-1)} = \emptyset$, then set $V_1 = V_*, F_1 = F_*$. Otherwise one can assume that $A_1 \subseteq F_*^{-1}(\mathrm{Sing}(V_*))_{(n-1)}$, and then we will construct V_1, F_1 with

$$F_1^{-1}(\mathrm{Sing}(V_1))_{(n-1)} \subseteq \overline{G^{-1}(\mathrm{Sing}(N^{-1}(0)))_{(n-1)} \setminus A_1}.$$

(For any $B \subseteq \mathbf{C}^q$, $\overline{B}$ denotes the closure in the Euclidean topology.) The construction will be carried out in α_1 steps. More precisely, one step will be repeated for α_1 times, each time with different input data. In each step we modify the lefthand side of the equation $N(w_1, \ldots, w_k) = u_1^{\alpha_1} \cdot \ldots \cdot u_p^{\alpha_p} \bar{R}$ and add to the system a collection of extra equations of the form $q_j = v_j u_1$, where q_j are suitably chosen polynomials and v_j are new variables. This operation, which allows us to decrease the power of u_1 by 1, can be viewed as some sort of blowing-up. After α_1 repetitions we obtain a system of polynomial equations $N_{\alpha_1}(w_1, \ldots, w_k, v_1, \ldots, v_{t_{\alpha_1}}) = u_2^{\beta_2} \cdot \ldots \cdot u_p^{\beta_p} R_{\alpha_1}$, $q_j = v_j u_1$, for $j = 1, \ldots, t_{\alpha_1}$, which defines some variety containing V_1 as an irreducible component.

Together with the equations we will introduce new functions corresponding to the variables v_j, R_{α_1} (also denoted by v_j, R_{α_1}) which will become components of the map F_1.

3 Preliminaries

3.1 Runge domains and polynomial polyhedra

A domain of holomorphy $\Omega \subset \mathbf{C}^n$ is called a Runge domain if every function $f \in \mathcal{O}(\Omega)$ can be uniformly approximated on every compact subset of Ω by polynomials in n complex variables.

We say that P is a polynomial polyhedron in $\mathbf{C}^n$ if there exist polynomials in n complex variables $q_1, \ldots, q_s$ and real constants $c_1, \ldots, c_s$ such that

$$P = \{x \in \mathbf{C}^n : |q_1(x)| \leq c_1, \ldots, |q_s(x)| \leq c_s\}.$$

The following theorem is a straightforward consequence of Theorem 2.7.3 and Lemma 2.7.4 from [18].

Theorem 3.1 *Let $\Omega \subset \mathbf{C}^n$ be a Runge domain. Then for every $\Omega_0 \Subset \Omega$ there exists a compact polynomial polyhedron $P \subseteq \Omega$ such that $\Omega_0 \subseteq P$.*

The following fact from [18] (p. 55) is well known.

Theorem 3.2 *Let f be a holomorphic function in a neighborhood of a compact polynomial polyhedron $K \subset \mathbf{C}^n$. Then f can be uniformly approximated on K by polynomials in n complex variables.*

3.2 Nash maps and sets

Let Ω be an open subset of $\mathbf{C}^n$ and let f be a holomorphic function on Ω. We say that f is a Nash function at $x_0 \in \Omega$ if there exist an open neighborhood U of x_0 and a polynomial $P : \mathbf{C}^n \times \mathbf{C} \to \mathbf{C}$, $P \neq 0$, such that $P(x, f(x)) = 0$ for $x \in U$. A holomorphic function defined on Ω is said to be a Nash function if it is a Nash function at every point of Ω. A holomorphic mapping defined on Ω with values in $\mathbf{C}^N$ is said to be a Nash mapping if each of its components is a Nash function.

A subset Y of an open set $\Omega \subset \mathbf{C}^n$ is said to be a Nash subset of Ω if and only if for every $y_0 \in \Omega$ there exists a neighborhood U of y_0 in Ω and there exist Nash functions $f_1, \dots, f_s$ on U such that

$$Y \cap U = \{x \in U : f_1(x) = \ldots = f_s(x) = 0\}.$$

The following proposition explains the relation between Nash and algebraic sets (cf. [32]).

Proposition 3.3 *Let X be an irreducible Nash subset of an open set $\Omega \subset \mathbf{C}^n$. Then there exists an algebraic subset Y of $\mathbf{C}^n$ such that X is an analytic irreducible component of $Y \cap \Omega$. Conversely, every analytic irreducible component of $Y \cap \Omega$ is an irreducible Nash subset of Ω.*

3.3 Convergence of closed sets and holomorphic chains

Let U be an open subset in $\mathbf{C}^m$. By a holomorphic chain in U we mean a formal sum $A = \sum_{j \in J} \alpha_j C_j$, where $\alpha_j \neq 0$ for $j \in J$ are integers and $\{C_j\}_{j \in J}$ is a locally finite family of pairwise distinct irreducible analytic subsets of U (see [8], [33], cf. also [4]). The set $\bigcup_{j \in J} C_j$ is called the support of A and is denoted by $|A|$ whereas the sets C_j are called the components of A with multiplicities α_j. The chain A is called positive if $\alpha_j > 0$ for all $j \in J$. If all the components of A have the same dimension n then A will be called an $n-$chain.

Below we introduce the convergence of holomorphic chains in U. To do this we first need the notion of the local uniform convergence of closed sets. Let Y, Y_ν be closed subsets of U for $\nu \in \mathbf{N}$. We say that $\{Y_\nu\}$ converges to Y locally uniformly if:

(1l) for every $a \in Y$ there exists a sequence $\{a_\nu\}$ such that $a_\nu \in Y_\nu$ and $a_\nu \to a$ in the standard topology of $\mathbf{C}^m$,
(2l) for every compact subset K of U such that $K \cap Y = \emptyset$ it holds $K \cap Y_\nu = \emptyset$ for almost all ν.

Then we write $Y_\nu \to Y$. For details concerning the topology of local uniform convergence see [34].

We say that a sequence $\{Z_\nu\}$ of positive n-chains converges to a positive n-chain Z if:

(1c) $|Z_\nu| \to |Z|$,
(2c) for each regular point a of $|Z|$ and each submanifold T of U of dimension

$m-n$ transversal to $|Z|$ at a such that the closure $\overline{T}$ (in U) is compact and $|Z| \cap \overline{T} = \{a\}$, we have $deg(Z_\nu \cdot T) = deg(Z \cdot T)$ for almost all ν.

Then we write $Z_\nu \rightarrowtail Z$. (By $Z \cdot T$ we denote the intersection product of Z and T (cf. [33]). (1c) and the choice of a, T in (2c) imply that the chains $Z_\nu \cdot T$ and $Z \cdot T$ for sufficiently large ν have finite supports and the degrees are well defined. Recall that for a chain $A = \sum_{j=1}^{d} \alpha_j \{a_j\}$, $deg(A) = \sum_{j=1}^{d} \alpha_j$.)

When we say that a sequence $\{X_\nu\}$ of purely n–dimensional analytic sets converges to a purely n-dimensional analytic set X *in the sense of chains*, we mean that the sequence $\{Z_\nu\}$ of n–chains converges to the n–chain Z, where Z_ν, Z are obtained by assigning the multiplicity 1 to all irreducible components of X_ν, X respectively.

3.4 Analytic sets with proper projection

Let $\pi : \mathbf{C}^m \times \mathbf{C}^s \to \mathbf{C}^m$ be the natural projection, let Ω be a domain in $\mathbf{C}^m$, and let Y be a purely m–dimensional analytic subset of $\Omega \times \mathbf{C}^s$ such that $\pi|_Y : Y \to \Omega$ is a proper map. By Σ_Y we denote the set of all points $a \in \Omega$ such that the fiber of $\pi|_Y$ over a does not have the maximal cardinality. Recall that Σ_Y (called the discriminant of $\pi|_Y$) is an analytic subsets of Ω (cf. [8]).

For algebraic sets, we need a slightly more general notion. Let $\mathcal{L}(\mathbf{C}^N, \mathbf{C}^m)$ denote the vector space of all linear maps from $\mathbf{C}^N$ to $\mathbf{C}^m$. Let $V \subset \mathbf{C}^N$ be algebraic of pure dimension m and let $A \in \mathcal{L}(\mathbf{C}^N, \mathbf{C}^m)$ such that $A|_V : V \to \mathbf{C}^m$ is a proper map. By $\Sigma_A \subset \mathbf{C}^m$ we denote the set of points $a \in \mathbf{C}^m$ such that the fiber of $A|_V$ over a does not have the maximal cardinality. Recall that Σ_A (called the discriminant of $A|_V$) is algebraic. Set $s_A := (A|_V)^{-1}(\mathrm{Sing}(\Sigma_A))$. When A is the natural projection from $\mathbf{C}^N = \mathbf{C}^m \times \mathbf{C}^s$ to $\mathbf{C}^m$, we often write Σ_V instead of Σ_A.

3.5 Holomorphic maps into algebraic varieties

For any subset B of $\mathbf{C}^q$ let $\overline{B}^z$ denote the Zariski closure of B i.e. the intersection of all algebraic subvarieties of $\mathbf{C}^q$ containing B. For any algebraic subvariety V of $\mathbf{C}^q$ by $I(V)$ we denote the ideal of all polynomials $p \in \mathbf{C}[y_1, \ldots, y_q]$ such that $V \subseteq p^{-1}(0)$. For any $g_1, \ldots, g_s \in \mathbf{C}[y_1, \ldots, y_q]$ by $I(g_1, \ldots, g_s)$ we denote the ideal generated by $g_1, \ldots, g_s$.

Lemma 3.4 *Let Ω and B be a domain in $\mathbf{C}^n$ and an irreducible analytic subset of Ω, respectively, and let $g = (g_1, g_2) \in \mathcal{O}(\Omega, \mathbf{C}^q \times \mathbf{C}^r)$. Let $\delta, h_1, \ldots, h_{t_1} \in \mathbf{C}[y_1, \ldots, y_q]$ be such that*

$$\overline{g_1(B)}^z \setminus \delta^{-1}(0) = \{y \in \mathbf{C}^q \setminus \delta^{-1}(0) : h_j(y) = 0 \text{ for } j = 1, \ldots, t_1\},$$

where $t_1 = q - \dim \overline{g_1(B)}^z$, and $\delta|_{\overline{g_1(B)}^z} \neq 0$ and for every $a \in \overline{g_1(B)}^z \setminus \delta^{-1}(0)$, the map $(h_1, \ldots, h_{t_1}) : \mathbf{C}^q \to \mathbf{C}^{t_1}$ is a submersion in some neighborhood of a in $\mathbf{C}^q$, and $\delta I(\overline{g_1(B)}^z) \subseteq I(h_1, \ldots, h_{t_1})$. Then there are $t_2 - t_1$ polynomials

$h_{t_1+1}, \ldots, h_{t_2} \in \mathbf{C}[y_1, \ldots, y_q, y_{q+1}, \ldots, y_{q+r}]$, *where* $t_2 = q + r - \dim\overline{g(B)}^z$, *and there is* $\hat{\delta} \in \mathbf{C}[y_1, \ldots, y_q, y_{q+1}, \ldots, y_{q+r}]$ *such that*

$$\overline{g(B)}^z \setminus \hat{\delta}^{-1}(0) = \{y \in \mathbf{C}^q \times \mathbf{C}^r \setminus \hat{\delta}^{-1}(0) : h_j(y) = 0 \text{ for } j = 1, \ldots, t_2\},$$

and $\hat{\delta}|_{\overline{g(B)}^z} \neq 0$, *and for every* $b \in \overline{g(B)}^z \setminus \hat{\delta}^{-1}(0)$, *the map* $(h_1, \ldots, h_{t_2}) : \mathbf{C}^{q+r} \to \mathbf{C}^{t_2}$ *is a submersion in some neighborhood of* b *in* $\mathbf{C}^{q+r}$, *and* $\hat{\delta} I(\overline{g(B)}^z) \subseteq I(h_1, \ldots, h_{t_2})$.

Proof of Lemma 3.4. Let us denote $C_1 = \overline{g_1(B)}^z$, $C_2 = \overline{g(B)}^z$. Since B is irreducible, C_2 is irreducible as well. Then there are $\hat{\delta}_1, \hat{h}_1, \ldots, \hat{h}_{t_2} \in \mathbf{C}[y_1, \ldots, y_{q+r}]$, such that

$$C_2 \setminus \hat{\delta}_1^{-1}(0) = \{y \in \mathbf{C}^q \times \mathbf{C}^r \setminus \hat{\delta}_1^{-1}(0) : \hat{h}_j(y) = 0 \text{ for } j = 1, \ldots, t_2\},$$

and $\hat{\delta}_1|_{C_2} \neq 0$, and for every $b \in C_2 \setminus \hat{\delta}_1^{-1}(0)$, the map $(\hat{h}_1, \ldots, \hat{h}_{t_2}) : \mathbf{C}^{q+r} \to \mathbf{C}^{t_2}$ is a submersion in some neighborhood of b in $\mathbf{C}^{q+r}$, and for every $G \in \mathbf{C}[y_1, \ldots, y_{q+r}]$ with $C_2 \subseteq G^{-1}(0)$ there are $\hat{r}_1, \ldots, \hat{r}_{t_2} \in \mathbf{C}[y_1, \ldots, y_{q+r}]$ such that $\hat{\delta}_1 \cdot G = \sum_{j=1}^{t_2} \hat{r}_j \hat{h}_j$. (See [22], pp. 402-405.)

Let us show that there are $h_{t_1+1}, \ldots, h_{t_2} \in \{\hat{h}_1, \ldots, \hat{h}_{t_2}\}$ with the required properties. Observe that $C_2 \subseteq C_1 \times \mathbf{C}^r$, which implies that $\hat{\delta}_1 \cdot h_i = \sum_{j=1}^{t_2} b_{j,i} \hat{h}_j$, for $i = 1, \ldots, t_1$, where $b_{j,i} \in \mathbf{C}[y_1, \ldots, y_{q+r}]$. Next, $(\hat{\delta}_1 \circ g) \cdot (\delta \circ g_1)|_B \neq 0$. Indeed, otherwise either $\hat{\delta}_1|_{C_2} = 0$ or $\delta|_{C_1} = 0$. Consequently, there is $x_0 \in B$ such that $(\hat{\delta}_1 h_1, \ldots, \hat{\delta}_1 h_{t_1})$ is a submersion in a neighborhood of $g(x_0)$, and therefore there are $j_1, \ldots, j_{t_1}$ such that the determinant $d(y_1, \ldots, y_{q+r})$ of the matrix $[b_{j_k,i}(y_1, \ldots, y_{q+r})]_{k=1,\ldots,t_1; i=1,\ldots,t_1}$ satisfies $d(g(x_0)) \neq 0$. This implies that $d|_{C_2} \neq 0$ and there are $c_{k,i}, d_{l,i} \in \mathbf{C}[y_1, \ldots, y_{q+r}]$, for $k, i = 1, \ldots, t_1$ and $l \in J = \{1, \ldots, t_2\} \setminus \{j_1, \ldots, j_{t_1}\}$ such that $d \cdot \hat{h}_{j_i} = \sum_{k=1}^{t_1} c_{k,i} h_k + \sum_{l \in J} d_{l,i} \hat{h}_l$.

Now it is clear that the assertion of the lemma is satisfied with $\hat{\delta} = d \cdot \hat{\delta}_1 \cdot \delta$ and $(h_1, \ldots, h_{t_1}, h_{t_1+1}, \ldots, h_{t_2})$, where $h_k = \hat{h}_{j_k}$, for $k = t_1 + 1, \ldots, t_2$ where $\{j_{t_1+1}, \ldots, j_{t_2}\} = J$. ■

Remark 3.5 Let $a_1, \ldots, a_p \in \mathbf{C}^{q+r} \setminus C_2$. Then

(a) In the first paragraph of the proof of Lemma 3.4, $\hat{h}_1, \ldots, \hat{h}_{t_2}$, $\hat{\delta}_1$ can be chosen so that $\hat{h}_i(a_j) \neq 0 \neq \hat{\delta}_1(a_j)$ for $i = 1, \ldots, t_2$ and $j = 1, \ldots, p$.
(b) If $h_i(a_j) \neq 0 \neq \delta(a_j)$ for $i = 1, \ldots, t_1$ and $j = 1, \ldots, p$ then $h_{t_1+1}, \ldots, h_{t_2}$ and the $b_{j,i}$'s can be chosen so that $h_i(a_j) \neq 0 \neq \hat{\delta}(a_j)$ for $i = 1, \ldots, t_2$ and $j = 1, \ldots, p$.

Proof of Remark 3.5. As for (a), it is sufficient to prove the assertion with $C_2, a_1, \ldots, a_p$ replaced by $\Phi(C_2), \Phi(a_1), \ldots, \Phi(a_p)$, where Φ is any linear isomorphism. Therefore we may assume, applying a linear change of the variables in $\mathbf{C}^{q+r}$ if needed, that $\pi|_{C_2} : C_2 \to \mathbf{C}^{\dim(C_2)+1}$ is a proper map, where π denotes the projection onto the first $\dim(C_2) + 1$ coordinates of $\mathbf{C}^{q+r}$. Moreover, $\pi(a_j) \notin \pi(C_2)$ for every $j = 1, \ldots, p$ and the fiber of $\pi|_{C_2}$ over a consists of one element for every $a \in \mathrm{Reg}(\pi(C_2))$. We may also assume that

$\rho|_{\pi(C_2)} : \pi(C_2) \to \mathbf{C}^{\dim(C_2)}$ is a proper map, where ρ denotes the projection onto the first $\dim(C_2)$ coordinates of $\mathbf{C}^{\dim(C_2)+1}$ and that the fibers of $\rho|_{\pi(C_2)}$ have maximal cardinality over $\rho(\pi(a_j))$ for every $j = 1, \ldots, p$.

Let $P \in (\mathbf{C}[y_1, \ldots, y_{\dim(C_2)}])[y_{\dim(C_2)+1}]$ be the irreducible monic polynomial with $P^{-1}(0) = \pi(C_2)$. Then it is well known (cf. [22], pp. 402-405) that one can take $\hat{\delta}_1, \hat{h}_1, \ldots, \hat{h}_{t_2}$ such that $\hat{h}_1 = P$ and $\hat{\delta}_1$ is (a power of) the discriminant of P. Of course, $\hat{h}_1, \hat{\delta}_1$ satisfy the requirements. Finally, if needed, we can replace $\hat{h}_j$, for $j \geq 2$, by $\hat{h}_j + \epsilon_j \hat{h}_1$ where $\epsilon_j \in \mathbf{C}$ ($|\epsilon_j|$ small) to obtain (a).

Let us turn to (b) (assuming that we have (a)). In view of (a), the fact that h_j, for $j = t_1 + 1, \ldots, t_2$, are chosen among $\hat{h}_1, \ldots, \hat{h}_{t_2}$, and that $\hat{\delta} = d \cdot \hat{\delta}_1 \cdot \delta$, it is sufficient to observe that the $b_{j,i}$'s can be chosen in such a way that the determinant $d(a_j) \neq 0$ for $j = 1, \ldots, p$. This however is obvious because for every $j \notin J$, $b_{j,i}$ can be replaced by $b_{j,i} + \epsilon_{j,i} \hat{h}_l$ for any $l \in J$ and $\epsilon_{j,i} \in \mathbf{C}$ with $|\epsilon_{j,i}|$ small.■

Lemma 3.6 *Let K be a compact polydisc in $\mathbf{C}^n$ and let $G \in \mathcal{O}(K, \mathbf{C}^k)$ such that $\overline{G(K)}^z = \mathbf{C}^k$. Let $0 \neq N \in \mathbf{C}[y_1, \ldots, y_k]$ and $u_1, \ldots, u_p, R \in \mathcal{O}(K)$ satisfy $N \circ G = u_1^{\alpha_1} \cdot \ldots \cdot u_p^{\alpha_p} R$ for some positive integers $\alpha_1, \ldots, \alpha_p$. Then there are nowhere vanishing functions $v_1, \ldots, v_p \in \mathcal{O}(K)$ and there is $S \in \mathcal{O}(K)$ such that $N \circ G = (u_1 v_1)^{\alpha_1} \cdot \ldots \cdot (u_p v_p)^{\alpha_p} S$ and $\overline{(G, u_1 v_1, \ldots, u_p v_p)(K)}^z = \mathbf{C}^{k+p}$.*

Proof. It is sufficient to show that there are nowhere vanishing $v_1, \ldots, v_p \in \mathcal{O}(K)$ such that $\overline{(G, u_1 v_1, \ldots, u_p v_p)(K)}^z = \mathbf{C}^{k+p}$, by which the other assertions follow immediately. In other words, it is sufficient to show that if $g_1, \ldots, g_t \in \mathcal{O}(K)$ are algebraically independent over $\mathbf{C}$, then for any $u \in \mathcal{O}(K)$, $u \neq 0$, there is a nowhere vanishing $v \in \mathcal{O}(K)$ such that $g_1, \ldots, g_t, u \cdot v$ are also algebraically independent.

We have two cases: either $g_1, \ldots, g_t, u$ are algebraically independent (and there is nothing to prove) or not. In the latter case it is sufficient to show that there is a nowhere vanishing $v \in \mathcal{O}(K)$ such that $g_1, \ldots, g_t, v$ are algebraically independent over $\mathbf{C}$ because then $g_1, \ldots, g_t, u \cdot v$ are also such (cf. [20] for basic facts on algebraic extensions). Define a family of one-variable functions: $\chi_1(x) = \exp(x)$ and $\chi_j(x) = \chi_{j-1}(\exp(x))$ for $j > 1$. Then $\{\chi_j\}_{j=1}^k$ are algebraically independent for any k. Hence, if $g_1, \ldots, g_t$ are algebraically independent, there is j such that $g_1, \ldots, g_t, \chi_j$ are also independent (where χ_j is treated as an n-variable function; cf. [20] for basic facts on transcendental extensions).■

3.6 A discriminant criterion for the existence of algebraic approximations

Let us recall a result from [6] which is one of the main tools in the present paper. Let $U \subset \mathbf{C}^n$ be a domain and let $\pi : U \times \mathbf{C}^k \to U$ denote the natural projection. Let $X \subset U \times \mathbf{C}^k$ be an analytic subset of pure dimension n with proper projection onto U. Recall that $s(\pi|_X)$ denotes the cardinality of the generic fiber in X over U. For any analytic set C, $C_{(n-1)}$ denotes the union of all $(n-1)$–dimensional irreducible components of C. For the definition of Σ_X see Section 3.4.

Theorem 3.7 *Let $\{X_\nu\}$ be a sequence of purely n-dimensional analytic subsets of $U \times \mathbf{C}^k$ with proper projection onto U converging locally uniformly to X such that $s(\pi|_X) = s(\pi|_{X_\nu})$ for $\nu \in \mathbf{N}$. Assume that $\{(\Sigma_{X_\nu})_{(n-1)}\}$ converges to $(\Sigma_X)_{(n-1)}$ in the sense of holomorphic chains. Then for every analytic subset Y of $U \times \mathbf{C}^k$ of pure dimension n such that $Y \subseteq X$ and for every open relatively compact subset $\tilde{U}$ of U there exists a sequence $\{Y_\nu\}$ of purely n-dimensional analytic subsets of $\tilde{U} \times \mathbf{C}^k$ converging to $Y \cap (\tilde{U} \times \mathbf{C}^k)$ in the sense of holomorphic chains such that $Y_\nu \subseteq X_\nu$ for every $\nu \in \mathbf{N}$.*

The assumption that $\{(\Sigma_{X_\nu})_{(n-1)}\}$ converges to $(\Sigma_X)_{(n-1)}$ in the sense of chains is essential. It is not difficult to observe that otherwise X_ν can be (and usually are) irreducible even when X is not. Then, in particular, X can contain the graph Y of some map holomorphic on $\tilde{U}$ whereas X_ν do not contain any such graphs. To prove Theorem 2.1, we will use Theorem 3.7 in the case when Y is the graph of a holomorphic map to be approximated by Nash maps.

Let V be a purely m–dimensional algebraic variety in $\mathbf{C}^m \times \mathbf{C}^k$ with proper projection onto $\mathbf{C}^m$. Assume that $\Sigma_V = N^{-1}(0)$, where N is a polynomial in m variables. Let $f : U \to V$ be a holomorphic map such that $\tilde{f}(U) \nsubseteq N^{-1}(0)$ where $\tilde{f}$ is the map consisting of the first m components of f. Theorem 3.7 implies the following

Corollary 3.8 *Let $\tilde{f}_\nu \in \mathcal{O}(U, \mathbf{C}^m)$ be a sequence of Nash maps converging to $\tilde{f}$ locally uniformly such that $\{(N \circ \tilde{f}_\nu)^{-1}(0)\}$ converges to $(N \circ \tilde{f})^{-1}(0)$ in the sense of chains. Then for every analytic subset Y of $U \times \mathbf{C}^k$ of pure dimension n such that $Y \subseteq (\tilde{f} \times \mathrm{id}_{\mathbf{C}^k})^{-1}(V)$ and for every open relatively compact subset $\tilde{U}$ of U there is a sequence $\{Y_\nu\}$ of purely n–dimensional Nash subsets of $\tilde{U} \times \mathbf{C}^k$ converging to $Y \cap (\tilde{U} \times \mathbf{C}^k)$ in the sense of holomorphic chains such that $Y_\nu \subseteq (\tilde{f}_\nu|_{\tilde{U}} \times \mathrm{id}_{\mathbf{C}^k})^{-1}(V)$ for every $\nu \in \mathbf{N}$. In particular, there is a sequence $f_\nu \in \mathcal{O}(\tilde{U}, V)$ of Nash maps converging to $f|_{\tilde{U}}$ uniformly.*

Proof. Only the last sentence requires explanation. Let $\hat{f}$ be the map consisting of the last k components of f. Fix open $\tilde{U} \Subset \hat{U} \Subset U$. Since $\mathrm{graph}(\hat{f}) \subseteq (\tilde{f} \times \mathrm{id}_{\mathbf{C}^k})^{-1}(V)$, there are purely n–dimensional Nash sets $Y_\nu \subseteq (\tilde{f}_\nu|_{\hat{U}} \times \mathrm{id}_{\mathbf{C}^k})^{-1}(V)$ such that $\{Y_\nu\}$ converges to $\mathrm{graph}(\hat{f}) \cap (\hat{U} \times \mathbf{C}^k)$ in the sense of chains.

The fibers of $(\tilde{f}_\nu|_{\hat{U}} \times \mathrm{id}_{\mathbf{C}^k})^{-1}(V) \subseteq \hat{U} \times \mathbf{C}^k$ over $\hat{U}$ are uniformly bounded by some bound independent of ν (which follows by the fact that V has proper projection onto $\mathbf{C}^m$ and $\hat{U} \Subset U$). Hence the fibers of Y_ν over $\hat{U}$ are also such. This implies, in view of the fact that $\{Y_\nu\}$ converges in the sense of chains to the graph of a map, that the fibers of $Y_\nu \cap (\tilde{U} \times \mathbf{C}^k)$ are singletons for almost all ν. In other words, $Y_\nu \cap (\tilde{U} \times \mathbf{C}^k)$ is the graph of some (Nash) map $\hat{f}_\nu : \tilde{U} \to \mathbf{C}^k$ for almost all ν. Now we can define $f_\nu = (\tilde{f}_\nu|_{\tilde{U}}, \hat{f}_\nu)$.■

4 Proof of Theorem 1.1

4.1 Reduction to the case when K is a polydisc

A proof of Theorem 1.1 can be reduced (by means of standard techniques of multidimensional complex analysis) to the case where $W = \mathbf{C}^n$, K is a compact polynomial polyhedron, and $f : D \to V$, where D is an open neighborhood of K in $\mathbf{C}^n$ (see [21], p. 339). Let us assume that this reduction has been done. As we will show below, a modification of the method presented in [21] (p. 339) allows us to assume that K is a compact polydisc.

First, for some m, there are a compact polydisc E in $\mathbf{C}^{n+m}$, polynomials $P_1, \ldots, P_m$ and a mapping F of a neighborhood U of E into $\mathbf{C}^q$ such that in some neighborhood $\tilde{D}$ of K in D we have $f(z) = F(z, P_1(z), \ldots, P_m(z))$ (cf. [18], p. 55). Moreover, if $Z \subset \mathbf{C}^n \times \mathbf{C}^m$ denotes the graph of the map $z \mapsto (P_1(z), \ldots, P_m(z))$, then $\tilde{K} = (K \times \mathbf{C}^m) \cap Z \subseteq E$ is a polynomially convex compact subset of Z. We can take U to be an open polydisc. It is clear that in order to approximate f it is sufficient to approximate the map $F|_{\tilde{K}} : \tilde{K} \to V$ by Nash maps into V.

Let $Q_1, \ldots, Q_r$ be polynomials in q complex variables such that $V = \{Q_1 = \ldots = Q_r = 0\}$. Let $\tilde{Z}$ denote the union of all analytic irreducible components of $Z \cap U$ which have a non-empty intersection with $\tilde{D} \times \mathbf{C}^m$. Observe that, for $i = 1, \ldots, r$, $Q_i \circ F$ vanishes identically on $\tilde{Z}$. Pick $\alpha \in \mathcal{O}(U)$ non-vanishing identically on any analytic irreducible component of $\tilde{Z}$ but vanishing identically on the other irreducible components of $Z \cap U$ (cf. [18], p. 192). Then $\alpha(y) \cdot Q_i(F(y)) = 0$ for every $y \in Z \cap U$ and $i = 1, \ldots, r$. Therefore, in view of Z being algebraic, there are polynomials $R_1, \ldots, R_t$ in $n + m$ complex variables, vanishing identically on $Z \cap U$, and such that

$$(\natural) \qquad \alpha(y) \cdot Q_i(F(y)) = \textstyle\sum_{j=1}^{t} b_{i,j}(y) R_j(y), \text{ for } y \in U \text{ and } i = 1, \ldots, r$$

with certain holomorphic functions $b_{i,j}$.

In the space $\mathbf{C}^{1+q+n+m+rt}$ with coordinates $x_1, w_1, \ldots, w_q, u_1, \ldots, u_{n+m}, v_{1,1}, v_{1,2}, \ldots, v_{r,t}$ consider the variety T defined by the equations

$$x_1 \cdot Q_i(w) = \sum_{j=1}^{t} v_{i,j} R_j(u), \text{ for } i = 1, \ldots, r.$$

By $(\natural)$, the image of the map

$$g : U \ni y \mapsto (\alpha(y), F(y), y, b_{i,j}(y)) \in \mathbf{C}^{1+q+n+m+rt}$$

is contained in T.

Now suppose that there is an open polydisc U' with $E \subset U' \subset U$ such that g can be approximated on U' by a Nash map $g'(y) = (\alpha'(y), F'(y), y'(y), b'_{i,j}(y))$ whose image is contained in T. If this approximation is close enough, then $y'(y)$ has the inverse $\tilde{y}$ on E close to the identity. Replacing $g'(y)$ by $g'(\tilde{y}(y))$, we can assume that $y'(y) = y$. Consequently, $\alpha'(y) \cdot Q_i(F'(y)) = 0$ for every $y \in \tilde{Z} \cap U'$ and every i. But $\alpha'(y)$ does not vanish identically on any irreducible component of $\tilde{Z} \cap U'$ if the approximation is close enough. Therefore $Q_i(F'(y)) = 0$ for every $y \in \tilde{Z} \cap U'$ and every i, which implies that $F'(\tilde{K}) \subset V$.

Thus in order to obtain the required Nash approximation $F'|_{\tilde{K}}$ of $F|_{\tilde{K}}$ it suffices to approximate $g : E \to T$, where E is a compact polydisc. Now, we see that to prove Theorem 1.1, it is sufficient to prove Theorem 2.1.

4.2 Proof of Theorem 2.1

First we will focus on the case when $\dim f^{-1}(\mathrm{Sing}(V)) \leq n-2$.

Lemma 4.1 *Let $f : U \to V$ be a holomorphic map, where $U \subseteq \mathbf{C}^n$ is a Runge domain and $V \subseteq \mathbf{C}^q$ is an algebraic variety. Assume that $\dim f^{-1}(\mathrm{Sing(V)}) < n-1$. Then for every open $U_0 \Subset U$ there is a sequence $f_\nu : U_0 \to V$ of Nash maps converging uniformly to $f|_{U_0}$.*

Proof. For an elementary proof of the lemma for $n = 1$ the reader is referred to [10]. Let us assume that $n \geq 2$. For any $\mathbf{C}$-linear subspace L of $\mathbf{C}^n$ let $L^\perp$ denote the orthogonal complement of L in $\mathbf{C}^n$. Fix an open set $U_0 \Subset U$.

Since $\dim f^{-1}(\mathrm{Sing}(V)) < n-1$, there are $\epsilon > 0$, $(n-2)$–dimensional linear subspaces $L_1, \ldots, L_t \subset \mathbf{C}^n$ and open bounded balls $B_j \subset L_j$ and $B'_j \subset L_j^\perp$, for $j = 1, \ldots, t$, such that $P_j := B_j + B'_j \Subset U$ and $\overline{U}_0 \subseteq \bigcup_{j=1}^t P_j$, and $(\overline{B_j} + \overline{B'_{j,\epsilon}}) \cap f^{-1}(\mathrm{Sing}(V)) = \emptyset$, for $j = 1, \ldots, t$, where $B'_{j,\epsilon} = \{x \in B'_j : \mathrm{dist}(x, \partial B'_j) < \epsilon\}$.

Observe that for every $i \neq j$ such that $P_i \cap P_j \neq \emptyset$ there are open balls $P_{i,j} \subseteq B_i$, $P_{j,i} \subseteq B_j$ and open connected sets $l_{i,j} \subseteq B'_i$, $l_{j,i} \subseteq B'_j$ such that $\overline{P_{i,j} + l_{i,j}} \cap f^{-1}(\mathrm{Sing}(V)) = \emptyset$, $\overline{P_{j,i} + l_{j,i}} \cap f^{-1}(\mathrm{Sing}(V)) = \emptyset$, $l_{i,j} \cap B'_{i,\epsilon} \neq \emptyset$, $l_{j,i} \cap B'_{j,\epsilon} \neq \emptyset$, and $(P_{i,j} + l_{i,j}) \cap (P_{j,i} + l_{j,i}) \neq \emptyset$. Indeed, pick $z \in (P_i \cap P_j) \setminus f^{-1}(\mathrm{Sing}(V))$. We have $z = u_i + v_i = u_j + v_j$ for some $u_i \in L_i$, $v_i \in L_i^\perp$, $u_j \in L_j$, $v_j \in L_j^\perp$. Next pick $v'_i \in B'_{i,\epsilon}$, $v'_j \in B'_{j,\epsilon}$. Since $u_i + v'_i, u_j + v'_j \notin f^{-1}(\mathrm{Sing}(V))$, there are a path $k_{i,j} \subset B'_i$ connecting v_i, v'_i and a path $k_{j,i} \subset B'_j$ connecting v_j, v'_j such that $(u_i + k_{i,j}) \cap f^{-1}(\mathrm{Sing}(V)) = \emptyset = (u_j + k_{j,i}) \cap f^{-1}(\mathrm{Sing}(V))$. Now it suffices to take $P_{i,j}, P_{j,i}$ to be small balls centered at u_i, u_j in B_i, B_j, respectively, and $l_{i,j}$, $l_{j,i}$ to be small neighborhoods of $k_{i,j}$, $k_{j,i}$ in B'_i, B'_j, respectively.

Define $E = \bigcup_{j=1}^t (B_j + B'_{j,\epsilon}) \cup \bigcup_{i,j=1}^t (P_{i,j} + l_{i,j})$, assuming that $P_{i,j} + l_{i,j} = \emptyset$ if $P_i \cap P_j = \emptyset$. By the facts that U is a Runge domain, $E \Subset U$ and $f(\overline{E}) \cap \mathrm{Sing}(V) = \emptyset$, there is a sequence $f_\nu : E \to V$ of Nash maps approximating $f|_E$ uniformly (cf. [9] p. 334; the idea of the proof is as follows: since $\overline{f(E)} \cap \mathrm{Sing}(V) = \emptyset$, there are an open neighborhood N of $\overline{f(E)}$ in $\mathbf{C}^q$ together with a Nash retraction τ of N onto some open neighborhood of $\overline{f(E)}$ in V. Now $f|_E$ can be approximated by polynomial maps into $\mathbf{C}^q$ whose restrictions to E have images in N. These restrictions can be composed with τ yielding the required Nash approximations of $f|_E$).

Observe that if f_ν has a holomorphic extension to $\bigcup_{j=1}^t P_j$, then the proof will be completed. Indeed, by the maximum principle, if such f_ν approximates f on E then it also approximates f on $\bigcup_{j=1}^t P_j$. Moreover, if f_ν is a holomorphic map on $\bigcup_{j=1}^t P_j$ and a Nash map on E then it is a Nash map on $\bigcup_{j=1}^t P_j$.

For every j put $E_j = (B_j + B'_{j,\epsilon}) \cup \bigcup_{k=1}^{t}(P_{j,k} + l_{j,k})$ (again assuming $P_{j,k} + l_{j,k} = \emptyset$ if $P_j \cap P_k = \emptyset$). The Hartogs extension theorem implies that for every j, $f_\nu|_{E_j}$ has an extension $f_{j,\nu} : P_j \to V$ such that $f_{j,\nu}|_{\{z\}+B'_j}$ is a holomorphic map for every $z \in B_j$. But then, since $f_{j,\nu}|_{B_j+B'_{j,\epsilon}}$ is a holomorphic map, the Cauchy integral formula implies that $f_{j,\nu}$ is a continuous separately holomorphic map. Hence it is a holomorphic map.

It remains to show that for every i, j, $f_{i,\nu}|_{P_i\cap P_j} = f_{j,\nu}|_{P_i\cap P_j}$. Fix $i \neq j$ such that $P_i \cap P_j \neq \emptyset$. Since $C = (P_{i,j} + l_{i,j}) \cap (P_{j,i} + l_{j,i}) \subseteq E_i \cap E_j \subseteq P_i \cap P_j$ we have $f_{j,\nu}|_C = f_\nu|_C = f_{i,\nu}|_C$. But $C \neq \emptyset$ and $P_i \cap P_j$ is connected so $f_{i,\nu}|_{P_i\cap P_j} = f_{j,\nu}|_{P_i\cap P_j}$ and the proof is complete.$\blacksquare$

Notation. Let K be a connected compact subset of $\mathbf{C}^n$ such that $\mathrm{int}K \neq \emptyset$ and let $g : K \to C \subseteq \mathbf{C}^q$. By $g_D : D \to C$ we denote a holomorphic map such that $g_D|_K = g$, where D is an open connected neighborhood of K. If g_D exists, then g is called holomorphic. If g_D is a Nash map then g is called a Nash map. The collection of all holomorphic maps from K to C will be denoted by $\mathcal{O}(K, C)$. For $C = \mathbf{C}$ we write $\mathcal{O}(K)$. A sequence $g_\nu \in \mathcal{O}(K, C)$, for $\nu \in \mathbf{N}$, is said to converge to $g \in \mathcal{O}(K, C)$ uniformly if there is an open $D' \supset K$ for which there are $g_{D'}, g_{\nu,D'}$, $\nu \in \mathbf{N}$, such that $g_{\nu,D'}$ converges to $g_{D'}$ uniformly. Let $h \in \mathcal{O}(D, \mathbf{C}^q)$ for some open $D \subset \mathbf{C}^n$. Let $Y \subset \mathbf{C}^q$ be an analytic set. Then by $h^{-1}(Y)_{(n-1)}$ we denote the union of all $(n-1)$–dimensional irreducible components of $h^{-1}(Y)$.

Proof of Theorem 2.1. Fix an open $U_0 \Subset U$ (which clearly can be assumed to be connected) and a compact polydisc K with $U_0 \subset K \subset U$. One can assume that $\overline{f(K)}^z = V$ (because otherwise V can be replaced by $\overline{f(K)}^z$), and that $(f|_D)^{-1}(\mathrm{Sing}(V))_{(n-1)} \neq \emptyset$ for every open neighborhood D of K (as otherwise Lemma 4.1 finishes the proof).

Put $F_0 = f|_K, V_0 = V$. We iterate the following process starting from F_0. Suppose we have $F_i \in \mathcal{O}(K, V_i)$ such that $\overline{F_i(K)}^z = V_i$, $F_{i,D}^{-1}(\mathrm{Sing}(V_i))_{(n-1)} \neq \emptyset$, for every open neighborhood D of K, where $V_i \subset \mathbf{C}^{q_i}$. We will show that there is $F_{i+1} \in \mathcal{O}(K, V_{i+1})$, where $V_{i+1} \subset \mathbf{C}^{q_{i+1}}$ is an algebraic variety, such that:

(x) if there is a sequence $F_{i+1,\nu} \in \mathcal{O}(K, V_{i+1})$ of Nash maps converging uniformly to F_{i+1}, then there is a sequence $F_{i,\nu} \in \mathcal{O}(K, V_i)$ of Nash maps converging uniformly to F_i,
(y) $\overline{F_{i+1}(K)}^z = V_{i+1}$ and there are an open $D \supset K$ and an irreducible component T of $(F_{i,D})^{-1}(\mathrm{Sing}(V_i))_{(n-1)}$ with $T \cap K \neq \emptyset$ such that

$$(F_{i+1,D})^{-1}(\mathrm{Sing}(V_{i+1}))_{(n-1)} \subseteq \overline{(F_{i,D})^{-1}(\mathrm{Sing}(V_i))_{(n-1)} \setminus T}.$$

Let us show that once F_{i+1} is constructed, the proof will be completed. Set $C_{i,D'} := (F_{i,D'})^{-1}(\mathrm{Sing}(V_i))_{(n-1)}$. Let $I_i \subset \mathcal{O}(K)$ be the ideal of all $\alpha \in \mathcal{O}(K)$ such that $\alpha_{D'}|_{C_{i,D'}} = 0$ for some open $D' \supset K$. It is well known that for every analytic hypersurface H in an open polydisc D' there is $g \in \mathcal{O}(D')$ such that $H = g^{-1}(0)$ (cf. [18]). This fact and (y) imply that $I_i \subsetneq I_{i+1}$. Therefore if $C_{i,D'} \neq \emptyset$ for every i and every open $D' \supset K$, then there is an infinite ascending sequence of ideals in $\mathcal{O}(K)$. But $\mathcal{O}(K)$ is noetherian (cf. [15]) so there must be i_0 and an open $D' \supset K$ such that $C_{i_0,D'} = \emptyset$ (i.e. $\dim F_{i_0,D'}^{-1}(\mathrm{Sing}(V_{i_0})) < n-1$). Now it is clear that Lemma 4.1 allows us to complete the proof if, given F_i, we can construct F_{i+1} satisfying (x) and (y).

Put $k_i = \dim(V_i)$. Let us show how to construct F_{i+1}. First observe that V_i is irreducible (because K is a polydisc and $V_i = \overline{F_i(K)}^z$). We can also assume that V_i is a normal analytic space. Indeed, if V_i is not normal then we can replace V_i, F_i by $\tilde{V}_i, \tilde{F}_i$, where $\pi : \tilde{V}_i \to V_i$ is the normalization of V_i, whereas $\tilde{F}_i : K \to \tilde{V}_i$ is a holomorphic map such that $\pi \circ \tilde{F}_i = F_i$. (The existence of $\tilde{F}_i$ is an immediate consequence of the fact that $\pi|_{\tilde{V}_i \setminus \pi^{-1}(\mathrm{Sing}(V_i))} : \tilde{V}_i \setminus \pi^{-1}(\mathrm{Sing}(V_i)) \to V_i \setminus \mathrm{Sing}(V_i)$ is a biholomorphism (see [22], pp 343-346).)

After this preparation let us construct F_{i+1}. Our main tools are Propositions 4.2 and 4.3 (whose proofs are postponed to Sections 4.3, 4.4, respectively) and Corollary 3.8. For definitions of Σ_A, s_A and $\mathcal{L}(\mathbf{C}^N, \mathbf{C}^m)$ see Section 3.4.

Proposition 4.2 *Let V be an algebraic subset of $\mathbf{C}^N$ of pure dimension m, and let $U \subset \mathbf{C}^n$ be an open polydisc. Let $f : U \to V$ be a holomorphic map such that $\overline{f(U)}^z = V$. Then for every open $U_0 \Subset U$ there is $A \in \mathcal{L}(\mathbf{C}^N, \mathbf{C}^m)$ such that $A|_V : V \to \mathbf{C}^m$ is a proper map and* $\dim(f|_{U_0})^{-1}(s_A \setminus \mathrm{Sing}(V)) \leq n-2$.

By Proposition 4.2, there is a linear $A : \mathbf{C}^{q_i} \to \mathbf{C}^{k_i}$ such that $A|_{V_i} : V_i \to \mathbf{C}^{k_i}$ is proper and

(a) $(A|_{V_i} \circ F_{i,D})^{-1}(\mathrm{Sing}(\Sigma_A))_{(n-1)} \subseteq F_{i,D}^{-1}(\mathrm{Sing}(V_i))_{(n-1)}$, for every sufficiently small open $D \supset K$.

Let $\Phi : \mathbf{C}^{q_i} \to \mathbf{C}^{q_i}$ be a linear automorphism such that $A = \pi \circ \Phi$, where $\pi : \mathbf{C}^{k_i} \times \mathbf{C}^{q_i - k_i} \to \mathbf{C}^{k_i}$ is the natural projection. Then $\Sigma_A = \Sigma_{\Phi(V_i)}$. Since $\Phi(V_i)$ is a normal space, $\Sigma_{\Phi(V_i)}$ is purely $(k_i - 1)$–dimensional or empty. Therefore there is $N \in \mathbf{C}[w_1, \ldots, w_{k_i}]$ such that $N^{-1}(0) = \Sigma_{\Phi(V_i)}$. Set $G = \pi \circ \Phi \circ F_i$. Then (a) can be rewritten as follows

(0) $G_D^{-1}(\mathrm{Sing}(N^{-1}(0)))_{(n-1)} \subseteq F_{i,D}^{-1}(\mathrm{Sing}(V_i))_{(n-1)}$, for every sufficiently small open $D \supset K$.

On the other hand, the facts that $\dim(\Phi(V_i)) = k_i$, $\Phi(V_i)$ has proper projection onto $\mathbf{C}^{k_i}$ and $\overline{\Phi(F_i(K))}^z = \Phi(V_i)$ imply that $\overline{G(K)}^z = \mathbf{C}^{k_i}$. Hence, G, N satisfy the hypotheses of the following

Proposition 4.3 *Let $E \subset \mathbf{C}^n$ be a compact polydisc with* $\mathrm{int}(E) \neq \emptyset$. *Let $G \in \mathcal{O}(E, \mathbf{C}^k)$, $N \in \mathbf{C}[w_1, \ldots, w_k]$ satisfy $\overline{G(E)}^z = \mathbf{C}^k$, $N \neq 0$. Then there are an algebraic subset $\tilde{V}$ of some $\mathbf{C}^q$ and $\tilde{f} \in \mathcal{O}(E, \tilde{V})$ with $\overline{\tilde{f}(E)}^z = \tilde{V}$ such that:*

(1) *either $\tilde{f}_D^{-1}(\mathrm{Sing}(\tilde{V}))_{(n-1)} = \emptyset$ for some open $D \supset E$ or there are an open $D \supset E$ and an irreducible component T of $G_D^{-1}(\mathrm{Sing}(N^{-1}(0)))_{(n-1)}$ with $T \cap E \neq \emptyset$ such that $\tilde{f}_D^{-1}(\mathrm{Sing}(\tilde{V}))_{(n-1)} \subseteq \overline{G_D^{-1}(\mathrm{Sing}(N^{-1}(0)))_{(n-1)} \setminus T}$,*
(2) *if there is a sequence $\tilde{f}_\nu \in \mathcal{O}(E, \tilde{V})$ of Nash maps converging uniformly to $\tilde{f}$, then there are a sequence $G_\nu \in \mathcal{O}(E, \mathbf{C}^k)$ of Nash maps converging uniformly to*

G and an open $D' \supset E$ such that $\{(N \circ G_{\nu,D'})^{-1}(0)\}$ converges to $(N \circ G_{D'})^{-1}(0)$ in the sense of chains.

By Proposition 4.3, there are an algebraic subset V_{i+1} of some $\mathbf{C}^{q_{i+1}}$ and $F_{i+1} \in \mathcal{O}(K, V_{i+1})$ with $\overline{F_{i+1}(K)}^z = V_{i+1}$ such that:

(3) either $F^{-1}_{i+1,D}(\mathrm{Sing}(V_{i+1}))_{(n-1)} = \emptyset$ for some open $D \supset K$ or there are an open $D \supset K$ and an irreducible component T of $G^{-1}_D(\mathrm{Sing}(N^{-1}(0)))_{(n-1)}$ with $T \cap K \neq \emptyset$ such that $F^{-1}_{i+1,D}(\mathrm{Sing}(V_{i+1}))_{(n-1)} \subseteq \overline{G^{-1}_D(\mathrm{Sing}(N^{-1}(0)))_{(n-1)} \setminus T}$,

(4) if there is a sequence $F_{i+1,\nu} \in \mathcal{O}(K, V_{i+1})$ of Nash maps converging uniformly to F_{i+1}, then there are a sequence $G_\nu \in \mathcal{O}(K, \mathbf{C}^{k_i})$ of Nash maps converging uniformly to G and an open $D' \supset K$ such that $\{(N \circ G_{\nu,D'})^{-1}(0)\}$ converges to $(N \circ G_{D'})^{-1}(0)$ in the sense of chains.

Now, by (4) and Corollary 3.8, if there is a sequence $F_{i+1,\nu} \in \mathcal{O}(K, V_{i+1})$ of Nash maps converging uniformly to F_{i+1}, then there is a sequence $\bar{F}_\nu \in \mathcal{O}(K, \Phi(V_i))$ of Nash maps converging uniformly to $\Phi \circ F_i$, which clearly implies that (x) is satisfied. As for (y), it is an immediate consequence of (0) and (3). Thus the proof is complete.■

4.3 Proof of Proposition 4.2

We follow the notation introduced in subsection 3.4. Throughout the proof we fix a nonempty Zariski open subset T of $\mathcal{L}(\mathbf{C}^N, \mathbf{C}^m)$ such that

$$\pi : V \times T \to \mathbf{C}^m \times T, \quad \pi(x, A) = (A(x), A)$$

is proper and $(\Sigma_\pi) \cap (\mathbf{C}^m \times \{A\}) = \Sigma_A \times \{A\}$, for all $A \in T$, where Σ_π denotes the discriminant of π. Then by Bertini Theorem (see for instance [16] Corollary 10.9 and Remark 10.9.2, pp. 274-275) replacing T by a smaller nonempty Zariski open subset of $\mathcal{L}(\mathbf{C}^N, \mathbf{C}^m)$, if necessary, we have

$$\mathrm{Sing}(\Sigma_\pi) \cap (\mathbf{C}^m \times \{A\}) = \mathrm{Sing}(\Sigma_A) \times \{A\}, \quad \text{for all } A \in T.$$

Therefore, if we denote $s_\pi = \pi^{-1}(\mathrm{Sing}(\Sigma_\pi))$, then

(a) $$s_\pi \cap (V \times \{A\}) = s_A \times \{A\}, \quad \text{for all } A \in T.$$

Since $\dim\mathrm{Sing}(\Sigma_A) \leq m-2$ and $A|_V : V \to \mathbf{C}^m$ is proper for $A \in T$, we have

(b) $$\dim(s_A) \leq m-2, \text{ for all } A \in T.$$

For a line L in $\mathcal{L}(\mathbf{C}^N, \mathbf{C}^m)$ we put

$$s_L := \overline{\bigcup_{A \in L \cap T} s_A}^z .$$

We claim that $\dim s_L \leq m-1$. Indeed, $\dim(s_\pi \cap (V \times (L \cap T)) \leq m-1$ by (a) and (b). The image in V of the standard projection $V \times T \to V$ of $s_\pi \cap (V \times (L \cap T))$

is algebraically constructible (cf. e.g. [22], p. 395) and s_L is its Zariski closure. This shows the claim (cf. e.g. [22], pp. 393-394). Finally, for each $x \in \mathrm{Reg}(V)$ there is an $A \in T$ such that $A(x) \notin \Sigma_A$, and hence the set of such A is Zariski open dense, so $\bigcap_{A\in T} s_A \subset \mathrm{Sing}(V)$. Since the Zariski topology is noetherian

(c) there are $k \in \mathbf{N}$ and $A_1, \dots, A_k \in T$ such that $s_{A_1} \cap \ldots \cap s_{A_k} \subset \mathrm{Sing}(V)$.

Now fix an open polydisc $U_0 \Subset U$ and set $Y_A := (f|_{U_0})^{-1}(s_A \setminus \mathrm{Sing}(V))$. For $i \geq 1$ consider the following statement

(c_i) there are $A_1, \dots, A_i \in T$ such that $\dim(\bigcap_{j=1}^{i} Y_{A_j}) \leq n-2$.

By (c), (c_k) holds. We will prove that (c_i) $\Rightarrow$ (c_{i-1}) for $i \geq 2$, thus showing (c_1) and hence Proposition 4.2.

Thus suppose that there are $A_1, \dots, A_i \in T$ such that $\dim(\bigcap_{j=1}^{i} Y_{A_j}) \leq n-2$. Let L be the line in $\mathcal{L}(\mathbf{C}^N, \mathbf{C}^m)$ containing A_{i-1} and A_i and let

$$Y_L := (f|_{U_0})^{-1}(s_L \setminus \mathrm{Sing}(V)).$$

Since $\overline{f(U)}^z = V$ and $\dim s_L \leq m-1$ we obtain

(d) $$\dim(Y_L) \leq \dim(f|_{U_0})^{-1}(s_L) \leq n-1.$$

Let $\mathcal{Z}$ denote the finite family of all $(n-1)$–dimensional analytic irreducible components of Y_L. For each $Z \in \mathcal{Z}$ the set $\mathcal{A}_Z$ of such $A \in L \cap T$ that

(e) $$Z \subset \textstyle\bigcap_{j=1}^{i-2} Y_{A_j} \cap Y_A$$

is Zariski closed, hence either finite or equal $L \cap T$. Indeed, by definitions of Y_A and Y_L, $\mathcal{A}_Z$ equals the set of such $A \in L \cap T$ that $\overline{f(Z)}^z \subset \bigcap_{j=1}^{i-2} s_{A_j} \cap s_A$. But, by (a), $\bigcap_{j=1}^{i-2} s_{A_j} \cap s_A$ depends algebraically on A.

If $\mathcal{A}_Z$ is finite for every $Z \in \mathcal{Z}$ then there is $A \in L \cap T$ such that (e) fails for every $Z \in \mathcal{Z}$ and then $\dim \bigcap_{j=1}^{i-2} Y_{A_j} \cap Y_A \leq n-2$ that completes the proof. Thus suppose that there is $Z \in \mathcal{Z}$ for which (e) holds for every $A \in L \cap T$. Then

$$Z \subset \bigcap_{j=1}^{i-2} Y_{A_j} \cap Y_{A_{i-1}} \cap Y_{A_i}$$

that contradicts the assumption $\dim(\bigcap_{j=1}^{i} Y_{A_j}) \leq n-2$. ■

4.4 Proof of Proposition 4.3

Remark 4.4 The letters v_i, u_i, R_i, $\hat{w}_i$, used below denote either (tuples of) variables or (tuples of) functions in x. It will be clear from the context whether a given letter denotes a variable or a function. When we write that a tuple of functions satisfies some equation, we mean that the equation holds true if every variable is replaced by the function denoted by the same letter.

If $(N\circ G)^{-1}(0)=\emptyset$ then define $\tilde V\subset \mathbf{C}^{k+1}$ by the equation $N(w_1,\dots,w_k)=R_0$ and take $\tilde f(x)=(G(x),R_0(x))=(G(x),N(G(x)))$. Clearly, $\tilde V,\tilde f$ satisfy the requirements.

Let us assume that $(N\circ G)^{-1}(0)\neq\emptyset$. Clearly it is sufficient to prove the proposition in the case where N is reduced which we also assume.

There are an open polydisc $D\supset E$ and a holomorphic extension G_D of G. Let $A_1,\dots,A_p$ be all irreducible components of $(N\circ G_D)^{-1}(0)$ intersecting E and let $u_1,\dots,u_p\in\mathcal{O}(D)$ be minimal defining functions for $A_1,\dots,A_p$ respectively. (Recall that $u\in\mathcal{O}(D)$ is called a minimal defining function for A if $A=u^{-1}(0)$ and for every open subset $U\subseteq D$ and $v\in\mathcal{O}(U)$ with $A\cap U\subseteq v^{-1}(0)$, there is $g\in\mathcal{O}(U)$ such that $v=g\cdot u|_U$. It is well known that the existence of minimal defining functions is a consequence of universal solvability of the second Cousin problem on D which, if D is a domain of holomorphy, is equivalent to $H^2(D,\mathbf{Z})=0$, cf. [18].) Then there are $R_0\in\mathcal{O}(D)$ and positive integers $k_1,\dots,k_p$ such that $N(G_D(x))=u_1(x)^{k_1}\cdot\ldots\cdot u_p(x)^{k_p}R_0(x)$ and $A_l\nsubseteq R_0^{-1}(0)$, for $l=1,\dots,p$. By Lemma 3.6, $u_1,\dots,u_p,R_0$ can be chosen in such a way that $\overline{(G,u_1,\dots,u_p)(E)}^z=\mathbf{C}^{k+p}$.

Set $Z_s=G_D^{-1}(\mathrm{Sing}(N^{-1}(0)))_{(n-1)}$, $Z_r=\overline{G_D^{-1}(N^{-1}(0))\setminus Z_s}$. If $A_j\nsubseteq Z_s$ for every j, then, after shrinking D if needed, we obtain $Z_s=\emptyset$. Then define $\tilde V\subset\mathbf{C}^{k+p+1}$ by the equation $N(w_1,\dots,w_k)=u_1^{k_1}\cdot\ldots\cdot u_p^{k_p}R_0$ and take $\tilde f(x)=(G_D(x),u_1(x),\dots,u_p(x),R_0(x))$. Observe that $\tilde V,\tilde f$ satisfy (2) of Proposition 4.3 because $u_1,\dots,u_p$ are minimal defining functions and $R_0^{-1}(0)\cap E=\emptyset$. As for (1), we will show that $\tilde f^{-1}(\mathrm{Sing}(\tilde V))_{(n-1)}=\emptyset$. Suppose that $\tilde f(C)\subset\mathrm{Sing}(\tilde V)$ for some $(n-1)$–dimensional analytic $C\subset D$. Then $N\circ G_D|_C=0=\frac{\partial N}{\partial w_j}\circ G_D|_C$, for $j=1,\dots,k$. But N is reduced so $G_D(C)\subset\mathrm{Sing}(N^{-1}(0))$. This contradicts the fact that $Z_s=\emptyset$.

If $A_j\subseteq Z_s$ for some j, then we may assume, renumbering the components, that $j=1$. Put $\hat w_1=(w_1,\dots,w_k)$, $\hat w_1(x)=G_D(x)$. Now the construction, the aim of which is to remove A_1 from Z_s, consists of k_1 steps.

Step 1. Define $C_1=\overline{\hat w_1(A_1)}^z$. Then $C_1\subsetneq\mathbf{C}^k$ is irreducible because $C_1\subseteq N^{-1}(0)$ and A_1 is irreducible. By Lemma 3.4 and [22], pp. 402-405, there are $\delta_1,q_1,\dots,q_{t_1}\in\mathbf{C}[\hat w_1]$, where $t_1=k-\dim C_1$, such that

$$C_1\setminus\delta_1^{-1}(0)=\{\hat w_1\in\mathbf{C}^k\setminus\delta_1^{-1}(0):q_1(\hat w_1)=\ldots=q_{t_1}(\hat w_1)=0\},$$

and $\delta_1|_{C_1}\neq0$, and for every $a\in C_1\setminus\delta_1^{-1}(0)$ the map $(q_1,\dots,q_{t_1}):\mathbf{C}^k\to\mathbf{C}^{t_1}$ is a submersion in some neighborhood of a in $\mathbf{C}^k$. Moreover, $\delta_1 I(C_1)\subseteq I(q_1,\dots,q_{t_1})$. Every irreducible component Z of $\overline{\bigcup_{j=1}^p(u_j^{-1}(0))\setminus Z_s}$ satisfies $\hat w_1(Z)\nsubseteq C_1$ because $C_1\subseteq\mathrm{Sing}(N^{-1}(0))$ and $\hat w_1(Z)\nsubseteq\mathrm{Sing}(N^{-1}(0))$. Therefore, in view of Remark 3.5, we may assume that every such component Z satisfies $\hat w_1(Z)\nsubseteq\delta_1^{-1}(0)$.

The inclusion $C_1\subseteq N^{-1}(0)$ implies that $\delta_1N=\sum_{j=1}^{t_1}q_jr_{1,j}$, where $r_{1,j}\in\mathbf{C}[\hat w_1]$ and the fact that u_1 is a minimal defining function implies that there is $v_j\in\mathcal{O}(D)$ such that $q_j(\hat w_1(x))=v_j(x)u_1(x)$ for $j=1,\dots,t_1$.

Let $\hat{v}_1$ denote the tuple $(v_1, \dots, v_{t_1})$ of t_1 variables. Define $N_1 \in \mathbf{C}[\hat{w}_1, \hat{v}_1]$ by the formula

$$N_1(\hat{w}_1, \hat{v}_1) = \sum_{j=1}^{t_1} v_j r_{1,j}(\hat{w}_1)$$

and observe that

$$N_1(\hat{w}_1(x), \hat{v}_1(x)) = u_1(x)^{k_1-1} u_2(x)^{k_{2,2}} \cdot \ldots \cdot u_p(x)^{k_{p,2}} R_1(x),$$

where $R_1 \in \mathcal{O}(D)$ satisfies $A_l \nsubseteq R_1^{-1}(0)$ for $l = 1, \dots, p$.

Let $V_1 \subset \mathbf{C}^{k+t_1+p+1}$ be the algebraic variety defined by the system of equations (in the variables $\hat{w}_1, \hat{v}_1, u_1, \dots, u_p, R_1$):

(E,1) $\qquad N_1(\hat{w}_1, \hat{v}_1) = u_1^{k_1-1} u_2^{k_{2,2}} \dots u_p^{k_{p,2}} R_1,$

(F,j) $\qquad q_j(\hat{w}_1) = v_j u_1, \quad \text{for } j = 1, \dots, t_1.$

Put $\hat{w}_2(x) = (\hat{w}_1(x), \hat{v}_1(x))$, and define $g_1 \in \mathcal{O}(D)$ by

$$g_1(x) = (\hat{w}_2(x), u_1(x), \dots, u_p(x), R_1(x)).$$

If $k_1 = 1$ then $\tilde{f}_D = g_1, \tilde{V} = \overline{g_1(E)}^z \subseteq V_1$ satisfy the requirements (see Claims 4.5, 4.6). Otherwise we go to Step 2.

Step 2. Define $C_2 = \overline{\hat{w}_2(A_1)}^z$. Then $C_2 \subsetneq \mathbf{C}^{k+t_1}$ is irreducible because $C_2 \subseteq N_1^{-1}(0)$ and A_1 is irreducible. Then by Lemma 3.4 there are $\delta_2, q_{t_1+1}, \dots, q_{t_2} \in \mathbf{C}[\hat{w}_2]$, where $t_2 = k + t_1 - dim C_2$, such that

$$C_2 \setminus \delta_2^{-1}(0) = \{\hat{w}_2 \in \mathbf{C}^{k+t_1} \setminus \delta_2^{-1}(0) : q_1(\hat{w}_2) = \ldots = q_{t_2}(\hat{w}_2) = 0\},$$

and $\delta_2|_{C_2} \neq 0$, and for every $a \in C_2 \setminus \delta_2^{-1}(0)$ the map $(q_1, \dots, q_{t_2}) : \mathbf{C}^{k+t_1} \to \mathbf{C}^{t_2}$ is a submersion in some neighborhood of a in $\mathbf{C}^{k+t_1}$. Moreover, $\delta_2 I(C_2) \subseteq I(q_1, \dots, q_{t_2})$. Every irreducible component Z of $\overline{\bigcup_{j=1}^{p}(u_j^{-1}(0)) \setminus Z_s}$ satisfies $\hat{w}_2(Z) \nsubseteq C_2$ because $C_2 \subseteq C_1 \times \mathbf{C}^{t_1}$ and $\hat{w}_1(Z) \nsubseteq C_1$. Therefore, in view of Remark 3.5, we may assume that every such component Z satisfies $\hat{w}_2(Z) \nsubseteq \delta_2^{-1}(0)$.

The inclusion $C_2 \subseteq N_1^{-1}(0)$ implies $\delta_2 N_1 = \sum_{j=1}^{t_2} q_j r_{2,j}$, where $r_{2,j} \in \mathbf{C}[\hat{w}_2]$, and the fact that u_1 is a minimal defining function implies that there is $v_j \in \mathcal{O}(D)$ such that $q_j(\hat{w}_2(x)) = v_j(x) u_1(x)$ for $j = t_1 + 1, \dots, t_2$.

Let $\hat{v}_2$ denote the tuple $(v_{t_1+1}, \dots, v_{t_2})$ of $t_2 - t_1$ variables. Define $N_2 \in \mathbf{C}[\hat{w}_2, \hat{v}_2]$ by the formula

$$N_2(\hat{w}_2, \hat{v}_2) = \sum_{j=1}^{t_2} v_j r_{2,j}(\hat{w}_2)$$

and observe that

$$N_2(\hat{w}_2(x), \hat{v}_2(x)) = u_1(x)^{k_1-2} u_2(x)^{k_{2,3}} \cdot \ldots \cdot u_p(x)^{k_{p,3}} R_2(x),$$

where $R_2 \in \mathcal{O}(D)$ satisfies $A_l \nsubseteq R_2^{-1}(0)$ for $l = 1, \dots, p$.

Let $V_2 \subset \mathbf{C}^{k+t_2+p+1}$ be the algebraic variety defined by the system of equations (in the variables $\hat{w}_2, \hat{v}_2, u_1, \ldots, u_p, R_2$):

$$N_2(\hat{w}_2, \hat{v}_2) = u_1^{k_1-2} u_2^{k_{2,3}} \ldots u_p^{k_{p,3}} R_2, \tag{E,2}$$
$$q_j(\hat{w}_2) = v_j u_1, \quad \text{for } j = 1, \ldots, t_2. \tag{F,j}$$

(For $j = 1, \ldots, t_1$, the polynomial q_j is precisely the one from Step 1 and it does not really depend on $\hat{v}_1$.) Put $\hat{w}_3(x) = (\hat{w}_2(x), \hat{v}_2(x))$ and define $g_2 \in \mathcal{O}(D)$ by

$$g_2(x) = (\hat{w}_3(x), u_1(x), \ldots, u_p(x), R_2(x)).$$

If $k_1 = 2$ then $\tilde{f}_D = g_2, \tilde{V} = \overline{g_2(E)}^z \subseteq V_2$ satisfy the requirements (see Claims 4.5, 4.6). Otherwise we go to Step 3.

Let us describe Step i+1, assuming that $k_1 > i$ and we have completed Step i (i$\geq$ 2) after which there are an algebraic subvariety $V_i \subseteq \mathbf{C}^{k+t_i+p+1}$ and a holomorphic map $g_i : D \to V_i$ such that the following hold:

-
$$g_i(x) = (\hat{w}_{i+1}(x), u_1(x), \ldots, u_p(x), R_i(x)),$$

$\hat{w}_{i+1}(x) = (\hat{w}_i(x), \hat{v}_i(x)) \in \mathbf{C}^{k+t_i}$, and $\hat{v}_i(x) = (v_{t_{i-1}+1}(x), \ldots, v_{t_i}(x))$. Moreover, $A_l \nsubseteq R_i^{-1}(0)$ for $l = 1, \ldots, p$.

- V_i is defined by the equations (in the variables $\hat{w}_i, \hat{v}_i, u_1, \ldots, u_p, R_i$):

$$N_i(\hat{w}_i, \hat{v}_i) = u_1^{k_1-i} u_2^{k_{2,i+1}} \ldots u_p^{k_{p,i+1}} R_i, \tag{E,i}$$
$$q_j(\hat{w}_i) = v_j u_1, \quad \text{for } j = 1, \ldots, t_i, \tag{F,j}$$

where $0 \neq N_i \in \mathbf{C}[\hat{w}_i, \hat{v}_i]$, and $q_j \in \mathbf{C}[\hat{w}_i]$ for $j = 1, \ldots, t_i$.

- There is $\delta_i \in \mathbf{C}[\hat{w}_i]$ such that for $C_i = \overline{\hat{w}_i(A_1)}^z \subsetneq \mathbf{C}^{k+t_{i-1}}$ the following hold:

$$C_i \setminus \delta_i^{-1}(0) = \{\hat{w}_i \in \mathbf{C}^{k+t_{i-1}} \setminus \delta_i^{-1}(0) : q_j(\hat{w}_i) = 0 \text{ for } j = 1, \ldots, t_i\},$$

and $\delta_i|_{C_i} \neq 0$, and for every $a \in C_i \setminus \delta_i^{-1}(0)$ the map $(q_1, \ldots, q_{t_i}) : \mathbf{C}^{k+t_{i-1}} \to \mathbf{C}^{t_i}$ is a submersion in some neighborhood of a in $\mathbf{C}^{k+t_{i-1}}$, and $\delta_i I(C_i) \subseteq I(q_1, \ldots, q_{t_i})$. Furthermore, $\hat{w}_i(Z) \nsubseteq C_i$ and $\hat{w}_i(Z) \nsubseteq \delta_i^{-1}(0)$ for every irreducible component Z of $\overline{\bigcup_{j=1}^p (u_j^{-1}(0)) \setminus Z_s}$.

Step $i+1$. Define $C_{i+1} = \overline{\hat{w}_{i+1}(A_1)}^z$. Then $C_{i+1} \subsetneq \mathbf{C}^{k+t_i}$ is irreducible because $C_{i+1} \subseteq N_i^{-1}(0)$ and A_1 is irreducible. Then by Lemma 3.4 there are $\delta_{i+1}, q_{t_i+1}, \ldots, q_{t_{i+1}} \in \mathbf{C}[\hat{w}_{i+1}]$, where $t_{i+1} = k + t_i - dim C_{i+1}$, such that

$$C_{i+1} \setminus \delta_{i+1}^{-1}(0) = \{\hat{w}_{i+1} \in \mathbf{C}^{k+t_i} \setminus \delta_{i+1}^{-1}(0) : q_1(\hat{w}_{i+1}) = \ldots = q_{t_{i+1}}(\hat{w}_{i+1}) = 0\},$$

and $\delta_{i+1}|_{C_{i+1}} \neq 0$, and for every $a \in C_{i+1} \setminus \delta_{i+1}^{-1}(0)$ the map $(q_1, \ldots, q_{t_{i+1}}) : \mathbf{C}^{k+t_i} \to \mathbf{C}^{t_{i+1}}$ is a submersion in some neighborhood of a in $\mathbf{C}^{k+t_i}$. Moreover, $\delta_{i+1} I(C_{i+1}) \subseteq I(q_1, \ldots, q_{t_{i+1}})$. Every irreducible component Z of the variety $\overline{\bigcup_{j=1}^p (u_j^{-1}(0)) \setminus Z_s}$ satisfies $\hat{w}_{i+1}(Z) \nsubseteq C_{i+1}$ because $C_{i+1} \subseteq C_i \times \mathbf{C}^{t_i - t_{i-1}}$ and

$\hat{w}_i(Z) \nsubseteq C_i$. Therefore, in view of Remark 3.5, we may assume that every such component Z satisfies $\hat{w}_{i+1}(Z) \nsubseteq \delta_{i+1}^{-1}(0)$.

The inclusion $C_{i+1} \subseteq N_i^{-1}(0)$ implies $\delta_{i+1} N_i = \sum_{j=1}^{t_{i+1}} q_j r_{i+1,j}$, where $r_{i+1,j} \in \mathbf{C}[\hat{w}_{i+1}]$, and the fact that u_1 is a minimal defining function implies that there is $v_j \in \mathcal{O}(D)$ such that $q_j(\hat{w}_{i+1}(x)) = v_j(x) u_1(x)$ for $j = t_i + 1, \ldots, t_{i+1}$.

Let $\hat{v}_{i+1}$ denote the tuple $(v_{t_i+1}, \ldots, v_{t_{i+1}})$ of $t_{i+1} - t_i$ variables. Define $N_{i+1} \in \mathbf{C}[\hat{w}_{i+1}, \hat{v}_{i+1}]$ by the formula

$$N_{i+1}(\hat{w}_{i+1}, \hat{v}_{i+1}) = \sum_{j=1}^{t_{i+1}} v_j r_{i+1,j}(\hat{w}_{i+1})$$

and observe that

$$N_{i+1}(\hat{w}_{i+1}(x), \hat{v}_{i+1}(x)) = u_1(x)^{k_1 - i - 1} u_2(x)^{k_{2,i+2}} \cdot \ldots \cdot u_p(x)^{k_{p,i+2}} R_{i+1}(x),$$

where $R_{i+1} \in \mathcal{O}(D)$ satisfies $A_l \nsubseteq R_{i+1}^{-1}(0)$ for $l = 1, \ldots, p$.

Let $V_{i+1} \subset \mathbf{C}^{k+t_{i+1}+p+1}$ be the algebraic variety defined by the system of equations (in the variables $\hat{w}_{i+1}, \hat{v}_{i+1}, u_1, \ldots, u_p, R_{i+1}$):

$$\begin{array}{ll} (\mathrm{E},i+1) & N_{i+1}(\hat{w}_{i+1}, \hat{v}_{i+1}) = u_1^{k_1-i-1} u_2^{k_{2,i+2}} \ldots u_p^{k_{p,i+2}} R_{i+1}, \\ (\mathrm{F},j) & q_j(\hat{w}_{i+1}) = v_j u_1, \quad \text{for } j = 1, \ldots, t_{i+1}. \end{array}$$

(For $j = 1, \ldots, t_i$, the polynomial q_j was defined in previous steps.) Put $\hat{w}_{i+2}(x) = (\hat{w}_{i+1}(x), \hat{v}_{i+1}(x))$ and define $g_{i+1} \in \mathcal{O}(D)$ by

$$g_{i+1}(x) = (\hat{w}_{i+2}(x), u_1(x), \ldots, u_p(x), R_{i+1}(x)).$$

If $k_1 = i+1$ then $\tilde{f}_D = g_{i+1}, \tilde{V} = \overline{g_{i+1}(E)}^z \subseteq V_{i+1}$ satisfy the requirements (see Claims 4.5, 4.6). Otherwise we go to Step i+2.

Claim 4.5 *The following hold:*
(1) $\overline{g_{k_1}(E)}^z$ *is an irreducible component of* V_{k_1}. *In particular,*

$$\mathrm{Sing}(\overline{g_{k_1}(E)}^z) \subseteq \mathrm{Sing}(V_{k_1}).$$

(2) *Every* $(n-1)$*–dimensional irreducible component* S *of* $g_{k_1}^{-1}(\mathrm{Sing}(V_{k_1}))$ *with* $S \cap E \neq \emptyset$, *satisfies*

$$S \subseteq \overline{G_D^{-1}(\mathrm{Sing}(N^{-1}(0))) \setminus A_1}.$$

In particular, after shrinking D *if necessary, we have*

$$g_{k_1}^{-1}(\mathrm{Sing}(V_{k_1}))_{(n-1)} \subseteq \overline{G_D^{-1}(\mathrm{Sing}(N^{-1}(0))) \setminus A_1}.$$

Proof. Recall that $Z_s = G_D^{-1}(\mathrm{Sing}(N^{-1}(0)))_{(n-1)}$ and $Z_r = \overline{G_D^{-1}(N^{-1}(0)) \setminus Z_s}$ and suppose that there is an $(n-1)$–dimensional irreducible component S of $g_{k_1}^{-1}(\mathrm{Sing}(V_{k_1}))$ with $S \cap E \neq \emptyset$ such that $S \nsubseteq \overline{Z_s \setminus A_1}$. We consider two cases:

(a) $S \subseteq \bigcup_{j=2}^{p} u_j^{-1}(0)$,

(b) $S \nsubseteq \bigcup_{j=2}^{p} u_j^{-1}(0)$,

to show that there is $a \in S$ such that $g_{k_1}(a) \in \mathrm{Reg}(V_{k_1})$, which contradicts the hypothesis.

Let us begin with (a). The properties of $u_1(x), \ldots, u_p(x)$ imply that for the generic $a \in S$, $u_1(a) \neq 0$ hence (in a neighborhood of $g_{k_1}(a)$) the system (F,j), $j = 1, \ldots, t_{k_1}$, depicts the graph of the rational map $(u_1, \hat{w}_1) \mapsto (v_1(u_1, \hat{w}_1), \ldots, v_{t_{k_1}}(u_1, \hat{w}_1))$. Moreover, the definition of N_i and the equations (E,k_1), (F,1),..., (F,t_{k_1}) and $\delta_i N_{i-1} = \sum_{j=1}^{t_i} q_j r_{i,j}$, for $i = 1, \ldots, k_1$ (where $N_0 = N$), imply that for the generic $a \in S$, (in a neighborhood of $g_{k_1}(a)$) the variety V_{k_1} is described by: (F,j), $j = 1, \ldots, t_{k_1}$,

(z) $$N(\hat{w}_1) \prod_{i=1}^{k_1} \delta_i(\hat{w}_i) = u_1^{k_1} u_2^{k_{2,k_1+1}} \cdot \ldots \cdot u_p^{k_{p,k_1+1}} R_{k_1}.$$

Now using (F,j) we can eliminate the variables $v_1, \ldots, v_{t_{k_1}}$ from (z) to obtain

(*) $$F(u_1, \hat{w}_1) N(\hat{w}_1) = u_2^{k_{2,k_1+1}} \ldots u_p^{k_{p,k_1+1}} R_{k_1},$$

where F is rational, $(u_1(a), G_D(a)) \in \mathrm{dom} F$, and $F(u_1(a), G_D(a)) \neq 0$ for the generic $a \in S$. (The last property due to $S \subseteq \overline{\bigcup_{j=1}^{p} u_j^{-1}(0) \setminus Z_s}$ which implies $\hat{w}_i(S) \nsubseteq \delta_i^{-1}(0)$ for $i = 1, \ldots, k_1$.)

Let $\hat{V}$ denote the set defined by (*). To complete the case (a) it is sufficient to show that $\hat{g}(a) \in \mathrm{Reg}(\hat{V})$, for the generic $a \in S$, where $\hat{g}$ is the map consisting of those components of g_{k_1} which correspond to the variables appearing in (*). By assumptions $N(G_D(a)) = 0$ for every $a \in S$. Moreover, by the facts that $S \nsubseteq Z_s$ and N is reduced, there is $j \in \{1, \ldots, k\}$ such that $\frac{\partial N}{\partial w_j}|_{G_D(S)} \neq 0$ which clearly implies that $\hat{g}(a) \in \mathrm{Reg}(\hat{V})$ for the generic $a \in S$.

Let us turn to (b). Let $\tilde{g}$ be the map consisting of those components of g_{k_1} which correspond to the variables appearing in (E,k_1) and let $\bar{g}$ be the map consisting of those components of g_{k_1} which correspond to the variables appearing in (F,j) for $j = 1, \ldots, t_{k_1}$. For the generic $a \in S$, the equation (E,k_1) depicts, in a neighborhood of $\tilde{g}(a)$, the graph of the rational function $R_{k_1} = R_{k_1}(\hat{w}_{k_1}, \hat{v}_{k_1}, u_2, \ldots, u_p)$, and the variable R_{k_1} does not appear in any of (F,j), $j = 1, \ldots, t_{k_1}$. Hence if we show that for every a in an open dense subset of S, the system of equations (F,j), for $j = 1, \ldots, t_{k_1}$, defines a manifold in a neighborhood of $\bar{g}(a)$, then we obtain a contradiction with the assumption that $g_{k_1}(S) \subseteq \mathrm{Sing}(V_{k_1})$.

We have two cases. If $S \nsubseteq u_1^{-1}(0)$, there is nothing to prove because each of the considered equations can be divided by u_1. If $S \subseteq u_1^{-1}(0)$, then for the generic $a \in S$, the map $(u_1, \hat{w}_{k_1}, \hat{v}_{k_1}) \mapsto (q_1(\hat{w}_{k_1}) - v_1 u_1, \ldots, q_{t_{k_1}}(\hat{w}_{k_1}) - v_{t_{k_1}} u_1)$ is a submersion in a neighborhood of $\bar{g}(a)$. This is because $(q_1, \ldots, q_{t_{k_1}})$ is a submersion in a neighborhood of $\hat{w}_{k_1}(a)$, and $u_1(a) = q_j(\hat{w}_{k_1}(a)) = 0$, for $j = 1, \ldots, t_{k_1}$.

It remains to check that $\overline{g_{k_1}(E)}^z$ is an irreducible component of V_{k_1}. We know that $\dim(\overline{g_{k_1}(E)}^z) \geq k + p$ because $u_1, \ldots, u_p$ has been chosen in such a way that $\overline{(G, u_1, \ldots, u_p)(E)}^z = \mathbf{C}^{k+p}$. So it is sufficient to check that there is $a \in E$ such that V_{k_1} is a $(k+p)$–dimensional manifold in some neighborhood of $g_{k_1}(a)$. But this holds for $a \in E$ with $u_1(a) \cdot \ldots \cdot u_p(a) \neq 0$. Indeed, then in a neighborhood of $g_{k_1}(a)$, V_{k_1} is the graph of the rational map $(u_1, \ldots, u_p, w_1, \ldots, w_k) \mapsto (v_1, \ldots, v_{t_{k_1}}, R_{k_1})$.■

Claim 4.6 *If there is a sequence $g_{k_1,\nu} \in \mathcal{O}(E, V_{k_1})$ of Nash maps converging uniformly to $g_{k_1}|_E$ then there are a sequence $G_\nu \in \mathcal{O}(E, \mathbf{C}^k)$ of Nash maps converging uniformly to G and an open neighborhood D' of E such that $\{(N \circ G_{\nu,D'})^{-1}(0)\}$ converges to $(N \circ G_{D'})^{-1}(0)$ in the sense of chains.*

Proof. Let $g_{k_1,\nu,D'} \in \mathcal{O}(D', V_{k_1})$ be a sequence of Nash maps converging uniformly to $g_{k_1,D'}$, where D' is an open neighborhood of E in D such that $(N \circ G_{D'})^{-1}(0) = (A_1 \cup \ldots \cup A_p) \cap D'$, for $A_1, \ldots, A_p$ introduced in the proof of Proposition 4.3.

The map $g_{k_1,\nu,D'}$ is of the form:

$$g_{k_1,\nu,D'}(x) = (\hat{w}_{k_1+1,\nu}(x), u_{1,\nu}(x), \ldots, u_{p,\nu}(x), R_{k_1,\nu}(x)),$$

where $\hat{w}_{i+1,\nu}(x) = (\hat{w}_{i,\nu}(x), \hat{v}_{i,\nu}(x))$, $\hat{v}_{i,\nu}(x) = (v_{t_{i-1}+1,\nu}(x), \ldots, v_{t_i,\nu}(x))$, for $i = 1, \ldots, k_1$, where $t_0 = 0$. We check that $G_{\nu,D'}(x) = \hat{w}_{1,\nu}(x)$ satisfies the requirements.

Clearly, it is sufficient to show that for every $l \in \{1, \ldots, p\}$ and for the generic point $a \in A_l \cap D'$ there is a neighborhood U of a in D' such that $\{(N \circ G_{\nu,D'})^{-1}(0) \cap U\}$ converges to $A_l \cap U$ in the sense of chains. Fix $l \in \{1, \ldots, p\}$.

The components of $g_{k_1,\nu,D'}$ satisfy the equation (E,k_1) therefore

$$(\mathrm{E},k_1,\nu) \qquad N_{k_1}(\hat{w}_{k_1+1,\nu}(x)) = u_{2,\nu}(x)^{k_{2,k_1+1}} \ldots u_{p,\nu}(x)^{k_{p,k_1+1}} R_{k_1,\nu}(x),$$

for every $x \in D', \nu \in \mathbf{N}$.

By the definition of N_{i+1} and by the fact that the components of $g_{k_1,\nu,D'}$ satisfy the equations $\delta_{i+1} N_i = \sum_{j=1}^{t_{i+1}} q_j r_{i+1,j}$, $(F,1), \ldots, (F, t_{k_1})$, we have

$$N_{i+1}(\hat{w}_{i+2,\nu}(x)) u_{1,\nu}(x) = \delta_{i+1}(\hat{w}_{i+1,\nu}(x)) N_i(\hat{w}_{i+1,\nu}(x)),$$

for $i = 0, \ldots, k_1 - 1$, $x \in D'$, where $N_0(\hat{w}_{1,\nu}(x)) = N(\hat{w}_{1,\nu}(x))$. This implies that

$$N_{k_1}(\hat{w}_{k_1+1,\nu}(x)) u_{1,\nu}(x)^{k_1} = \tilde{T}_\nu(x) N(\hat{w}_{1,\nu}(x)),$$

for some $\tilde{T}_\nu \in \mathcal{O}(D')$, which combined with (E,k_1,ν) gives

$$(\alpha) \qquad \tilde{T}_\nu(x) N(\hat{w}_{1,\nu}(x)) = u_{1,\nu}(x)^{k_1} u_{2,\nu}(x)^{k_{2,k_1+1}} \ldots u_{p,\nu}(x)^{k_{p,k_1+1}} R_{k_1,\nu}(x),$$

for every $x \in D', \nu \in \mathbf{N}$.

The facts that $A_l \subseteq \overline{u_l^{-1}(0) \setminus R_{k_1}^{-1}(0)}$ and that $\dim(u_l^{-1}(0) \cap u_t^{-1}(0)) < n-1$, for every $t \neq l$, clearly imply that for the generic $a \in A_l \cap D'$ there is an open neighborhood $U \Subset D'$ such that $(\bigcup_{j \neq l} u_j^{-1}(0) \cup R_{k_1}^{-1}(0)) \cap \overline{U} = \emptyset$. Consequently, for sufficiently large ν, by (α), we have

$$(\tilde{T}_\nu(x) N(\hat{w}_{1,\nu}(x)))^{-1}(0) \cap U = u_{l,\nu}^{-1}(0) \cap U.$$

Now by the fact that u_l is a minimal defining function, $\{u_{l,\nu}^{-1}(0) \cap U\}$ converges to $A_l \cap U$ in the sense of chains. On the other hand, $\{(N(\hat{w}_{1,\nu}(x)))^{-1}(0) \cap U\}$ converges to $A_l \cap U$ locally uniformly and, in view of the last equation, for ν large enough, $(N(\hat{w}_{1,\nu}(x)))^{-1}(0) \cap U \subseteq u_{l,\nu}^{-1}(0) \cap U$ so $\{(N(\hat{w}_{1,\nu}(x)))^{-1}(0) \cap U\}$ converges to $A_l \cap U$ in the sense of chains.∎

Once we have proved Claims 4.5, 4.6, the proof of Proposition 4.3 is also completed.∎

5 Generalization of Theorem 2.1

Let $f : U \to V$ be as in Theorem 2.1. As already mentioned, without loss of generality, we can additionally assume in Theorem 2.1 that $\overline{f(U)}^z = V$ and $V \subset \mathbf{C}^m \times \mathbf{C}^k$ has proper projection onto $\mathbf{C}^m$ where $m = \dim(V)$. Write $f = (\tilde{f}, \hat{f})$, where $\tilde{f}, \hat{f}$ denote the first m and the last k components of f, respectively. Then $\tilde{f}(U) \nsubseteq \Sigma_V$.

Let $\mathcal{V}(\tilde{f})$ denote the pull-back of V by $\tilde{f}$, i. e. $\mathcal{V}(\tilde{f}) = (\tilde{f} \times \mathrm{id}_{\mathbf{C}^k})^{-1}(V)$. Then the fact that $f(U) \subseteq V$ can be equivalently stated as $\mathrm{graph}(\hat{f}) \subset \mathcal{V}(\tilde{f})$. Under these assumptions Theorem 2.1 can be reformulated as follows.

Theorem 2.1' *For every open $\tilde{U} \Subset U$ there are a sequence $\tilde{f}_\nu : \tilde{U} \to \mathbf{C}^m$ of Nash maps converging uniformly to $\tilde{f}|_{\tilde{U}}$ and a sequence M_ν of Nash sets of pure dimension n converging to* $\mathrm{graph}(\hat{f}) \cap (\tilde{U} \times \mathbf{C}^k)$ *in the sense of chains such that $M_\nu \subset \mathcal{V}(\tilde{f}_\nu)$ for every ν.*

Indeed, in this case, for ν large, (shrinking $\tilde{U}$ slightly we obtain that) M_ν is the graph of a map that defines the second part $\hat{f}_\nu$ of $f_\nu = (\tilde{f}_\nu, \hat{f}_\nu)$, cf. the proof of Corollary 3.8. But the method of the proof gives that $\tilde{f}$ can be approximated by Nash maps $\tilde{f}_\nu$ in such a way that all purely n–dimensional analytic sets (in particular all graphs of maps holomorphic on U) contained in $\mathcal{V}(\tilde{f})$ can be simultaneously approximated in the sense of chains by Nash sets contained in $\mathcal{V}(\tilde{f}_\nu)$. More precisely, the following generalization of Theorem 2.1' holds.

Theorem 5.1 *Let U be an open polydisc in $\mathbf{C}^n$ and let $V \subset \mathbf{C}^m \times \mathbf{C}^k$ be an algebraic variety of pure dimension m with proper projection onto $\mathbf{C}^m$. Let $\tilde{f} : U \to \mathbf{C}^m$ be a holomorphic map such that $\tilde{f}(U) \nsubseteq \Sigma_V$. Then for every open $\tilde{U} \Subset U$ there is a sequence $\tilde{f}_\nu : \tilde{U} \to \mathbf{C}^m$ of Nash maps converging uniformly to $\tilde{f}|_{\tilde{U}}$ such that for every analytic set $M \subset \mathcal{V}(\tilde{f})$ of pure dimension n there is a sequence M_ν of Nash sets of pure dimension n converging to $M \cap (\tilde{U} \times \mathbf{C}^k)$ in the sense of chains such that $M_\nu \subset \mathcal{V}(\tilde{f}_\nu)$ for every ν.*

Proof. Let $N_1, \ldots, N_s$ be polynomials in m complex variables such that $\Sigma_V = \{N_1 = \ldots = N_s = 0\}$. Shrinking U if needed we can assume that $\tilde{f}^{-1}(\Sigma_V)$ has finitely many, say p, $(n-1)$-dimensional irreducible components. Denote these components by $C_1, \ldots, C_p$. Let $u_1, \ldots, u_p$ be minimal defining functions for $C_1, \ldots, C_p$, respectively. Then there are $R_j \in \mathcal{O}(U)$ and $k_{j,i} \in \mathbf{N}$, for

$j = 1, \ldots, s$ and $i = 1, \ldots, p$, such that $N_j \circ \tilde{f} = R_j u_1^{k_{j,1}} \cdot \ldots \cdot u_p^{k_{j,p}}$, and every R_j does not vanish identically on any C_i.

Fix open $\tilde{U}, \hat{U}$ with $\tilde{U} \Subset \hat{U} \Subset U$. By Theorem 2.1, there are Nash maps $\tilde{f}_\nu, R_{j,\nu}, u_{i,\nu}$ approximating $\tilde{f}, R_j, u_i$, respectively, on $\hat{U}$ and such that $N_j \circ \tilde{f}_\nu = R_{j,\nu} u_{1,\nu}^{k_{j,1}} \cdot \ldots \cdot u_{p,\nu}^{k_{j,p}}$, for $j = 1, \ldots, s$. Now $X = \mathcal{V}(\tilde{f}) \cap (\hat{U} \times \mathbf{C}^k)$, $X_\nu = \mathcal{V}(\tilde{f}_\nu)$ satisfy the assumptions of Theorem 3.7. Application of this theorem completes the proof.■

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