An Analytic Model for Left-Invertible Weighted Shifts on Directed Trees

Sameer Chavan and Shailesh Trivedi

ABSTRACT. Let \mathscr{T} be a rooted directed tree with finite branching index $k_{\mathscr{T}}$ and let $S_{\lambda} \in B(l^2(V))$ be a left-invertible weighted shift on \mathscr{T} . We show that S_{λ} can be modelled as a multiplication operator \mathscr{M}_z on a reproducing kernel Hilbert space \mathscr{H} of E-valued holomorphic functions on a disc centered at the origin, where $E := \ker S_{\lambda}^*$. The reproducing kernel associated with \mathscr{H} is multi-diagonal and of bandwidth $k_{\mathscr{T}}$. Moreover, \mathscr{H} admits an orthonormal basis consisting of polynomials in z with at most $k_{\mathscr{T}} + 1$ non-zero coefficients. As one of the applications of this model, we give a complete spectral picture of S_{λ} . Unlike the case dim E = 1, the approximate point spectrum of S_{λ} could be disconnected. We also obtain an analytic model for left-invertible weighted shifts on rootless directed trees with finite branching index.

1. Preliminaries

The implementation of methods of graph theory into operator theory gives rise to a new class of operators known as *weighted shifts on directed trees*. These operators are generalization of adjacency operators of the directed trees. Although, the study of adjacency operators of the directed graphs was initiated by Fujii, Sasaoka and Watatani in [15], it was first observed by Jabłoński, Jung and Stochel in [20] that replacing the directed graphs by directed trees not just gives a successful theory of weighted shifts but also provides a rich source of examples and counterexamples in operator theory [21], [22]. Several questions related to boundedness, adjoints, normality, subnormality, hyponormality etc. of weighted shifts on directed trees have been studied in depth in [20].

In the present paper, we discuss a rich interplay between the discrete structures (directed trees) and analytic structures (analytic kernels of finite bandwidth). The starting point of this text is the observation that any left-invertible weighted shift on a rooted directed tree can be realized as the operator of multiplication by the co-ordinate function on a reproducing kernel Hilbert space \mathscr{H} of vector-valued holomorphic functions defined on a disc in the complex plane. In case the directed tree has finite branching index, this analytic model takes a concrete form. In particular, the reproducing kernel associated with \mathscr{H} turns out to be multi-diagonal.

²⁰¹⁰ Mathematics Subject Classification. Primary 47B37, 47A10; Secondary 46E22, 47B38. Key words and phrases. weighted shift, directed tree, multiplication operator, reproducing kernel of finite bandwidth, Hilbert space of holomorphic functions.

Also, the space \mathscr{H} may not be obtained by tensoring a Hilbert space of scalarvalued holomorphic functions with another Hilbert space. In this course, we arrive at a couple of interesting invariants, namely, branching index of a directed tree and radius of convergence for the weighted shift. Importantly, these invariants can be computed explicitly in various situations.

Let \mathbb{Z}_+ , \mathbb{Z} , \mathbb{R} and \mathbb{C} stand for the sets of non-negative integers, integers, real numbers and complex numbers, respectively. The complex conjugate of a complex number w will be denoted by \overline{w} . We use \mathbb{D}_r to denote the open disc $\{z \in \mathbb{C} : |z| < r\}$ of radius r > 0. In case r = 1, we denote the unit disc \mathbb{D}_1 by a simpler notation \mathbb{D} . For a subset A of a non-empty set X, card(A) denotes the cardinality of A.

Let \mathcal{H} be a complex separable Hilbert space. The inner-product on \mathcal{H} will be denoted by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. If no confusion is likely then we suppress the suffix, and simply write the inner-product as $\langle \cdot, \cdot \rangle$. By a *subspace*, we mean a closed linear manifold. Let W be a subset of \mathcal{H} . Then span W stands for the smallest linear manifold generated by W. In case W is singleton $\{w\}$, we use the convenient notation $\langle w \rangle$ in place of span $\{w\}$. By $\bigvee \{w : w \in W\}$, we understand the subspace generated by W. For a subspace \mathcal{M} of \mathcal{H} , we use $\mathcal{P}_{\mathcal{M}}$ to denote the orthogonal projection of \mathcal{H} onto \mathcal{M} . For vectors $x, y \in \mathcal{H}$, we use the notation $x \otimes y$ to denote the rank one operator given by

$$x \otimes y(h) = \langle h, y \rangle x, \ h \in \mathcal{H}.$$

Unless stated otherwise, all the Hilbert spaces occurring below are complex infinite-dimensional separable and for any such Hilbert space \mathcal{H} , $B(\mathcal{H})$ denotes the Banach algebra of bounded linear operators on \mathcal{H} . For $T \in B(\mathcal{H})$, the symbols ker Tand ran T will stand for the kernel and the range of T respectively. The Hilbert space adjoint of T will be denoted by T^* . In what follows, we denote the spectrum, approximate point spectrum, essential spectrum and the point spectrum of T by $\sigma(T)$, $\sigma_{ap}(T)$, $\sigma_e(T)$ and $\sigma_p(T)$ respectively. We reserve the notation r(T) for the spectral radius of T.

Let $T \in B(\mathcal{H})$. We say that T is *left-invertible* if there exists $S \in B(\mathcal{H})$ such that ST = I. Note that T is *left-invertible* if and only if there exists a constant $\alpha > 0$ such that $T^*T \ge \alpha I$. In this case, T^*T is invertible and T admits the left-inverse $(T^*T)^{-1}T^*$. Following [29], we refer to the operator T' given by $T' := T(T^*T)^{-1}$ as the *Cauchy dual* of the left-invertible operator T. Further, we say that T is *analytic* if $\bigcap_{n\ge 0} T^n(\mathcal{H}) = \{0\}$. If \mathscr{H} is a reproducing kernel Hilbert space of holomorphic functions defined on a disc in \mathbb{C} , then the multiplication operator \mathscr{M}_z defined on \mathscr{H} provides an example of an analytic operator. It is interesting to note that almost all analytic operators arise in this way. Indeed, a result of S. Shimorin [29] asserts that any left-invertible analytic operator is unitarily equivalent to the operator of multiplication by z on a reproducing kernel Hilbert space of vector-valued holomorphic functions defined on a disc. Since the proof of this fact, as given in [29, Sections 1 and 2], plays a major role in the proof of the main result, we outline it in the following discussion (cf. [31, Theorem 2.13]).

Let $T \in B(\mathcal{H})$ be a left-invertible analytic operator and let $E := \ker T^*$. For each $x \in \mathcal{H}$, define an *E*-valued holomorphic function U_x as

$$U_x(z) = \sum_{n \ge 0} (P_E T'^{*n} x) z^n,$$

where T' is the Cauchy dual of T. A simple application of the spectral radius formula [11] shows that the function $U_x(z) = P_E(I - zT'^*)^{-1}x$ is well-defined and holomorphic on the disc \mathbb{D}_r , where $r := \frac{1}{r(T')}$. Let \mathscr{H} denote the vector space of Evalued holomorphic functions of the form $U_x, x \in \mathcal{H}$. Consider the map $U : \mathcal{H} \to \mathscr{H}$ defined by $Ux = U_x$. By [29, Lemma 2.2], the kernel of U is precisely $\bigcap_{n\geq 0} T^n(\mathcal{H})$, and hence by the assumption, U is injective. In particular, we may equip the space \mathscr{H} with the norm induced from \mathcal{H} , so that U is unitary. It turns out that \mathscr{H} is a z-invariant reproducing kernel Hilbert space with $UT = \mathscr{M}_z U$, where \mathscr{M}_z is the operator of multiplication by z. Also, the reproducing kernel $\kappa_{\mathscr{H}} : \mathbb{D}_r \times \mathbb{D}_r \to B(E)$ is given by

$$\kappa_{\mathscr{H}}(z,w) = \sum_{j,k\geq 0} P_E T'^{*j} T'^k |_E z^j \overline{w}^k, \tag{1}$$

which satisfies the following:

(i) for any $x \in E$ and $\lambda \in \mathbb{D}_r$,

$$\kappa_{\mathscr{H}}(\cdot,\lambda)x \in \mathscr{H};$$

(ii) for any $x \in E$, $h \in \mathscr{H}$ and $\lambda \in \mathbb{D}_r$,

$$\langle h(\lambda), x \rangle_E = \langle h, \kappa_{\mathscr{H}}(\cdot, \lambda) x \rangle_{\mathscr{H}}.$$

Conditions (i) and (ii) may be rephrased by saying that the set of bounded point evaluations (for short, bpe) for \mathscr{H} contains the disc \mathbb{D}_r . We see in the context of weighted shifts on rooted directed trees that indeed (analytic) bpe contains the disc $\mathbb{D}_{r_{\lambda}}$ of larger radius r_{λ} (see Definition 2.5). This occupies the major part of the proof of the main result.

In the remaining part of this section, we invoke some basic concepts from the theory of directed trees which will be frequently used in the rest of this paper. The reader is referred to [20] for a detailed exposition on directed trees.

A pair $\mathscr{T} = (V, \mathscr{E})$ is called a *directed graph* if V is a non-empty set and \mathscr{E} is a subset of $V \times V \setminus \{(v, v) : v \in V\}$. An element of V (resp. \mathscr{E}) is called a *vertex* (resp. an *edge*) of \mathscr{T} . A finite sequence $\{v_i\}_{i=1}^n$ of distinct vertices is said to be a *circuit* of \mathscr{T} if $n \geq 2$, $(v_i, v_{i+1}) \in \mathscr{E}$ for all $1 \leq i \leq n-1$ and $(v_n, v_1) \in \mathscr{E}$. A directed graph \mathscr{T} is said to be *connected* if for any two distinct vertices u and v of \mathscr{T} , there exists a finite sequence $\{v_i\}_{i=1}^n$ of vertices of \mathscr{T} $(n \geq 2)$ such that $u = v_1, v_n = v$ and (v_i, v_{i+1}) or $(v_{i+1}, v_i) \in \mathscr{E}$ for all $1 \leq i \leq n-1$. For a subset W of V, define $\operatorname{Chi}(W) = \bigcup_{u \in W} \{v \in V : (u, v) \in \mathscr{E}\}$. One may define inductively $\operatorname{Chi}^{\langle n \rangle}(W)$ for $n \in \mathbb{Z}_+$ as follows: Set $\operatorname{Chi}^{\langle n \rangle}(W) = W$ if n = 0, and $\operatorname{Chi}^{\langle n \rangle}(W) = \operatorname{Chi}(\operatorname{Chi}^{\langle n-1 \rangle}(W))$ if $n \geq 1$. Given $v \in V$, we write $\operatorname{Chi}(v) := \operatorname{Chi}(\{v\})$, $\operatorname{Chi}^{\langle n \rangle}(v) = \operatorname{Chi}^{\langle n \rangle}(\{v\})$. A member of $\operatorname{Chi}(v)$ is called a *child* of v. For a given vertex $v \in V$, if there exists a unique vertex $u \in V$ such that $(u, v) \in \mathscr{E}$, we say that v has a *parent* u and denote it by $\operatorname{par}(v)$. A vertex v of \mathscr{T} is called a *root* of \mathscr{T} , or $v \in \operatorname{Root}(\mathscr{T})$, if there is no vertex u of \mathscr{T} such that (u, v) is an edge of \mathscr{T} . If $\operatorname{Root}(\mathscr{T})$.

A directed graph $\mathscr{T} = (V, \mathcal{E})$ is called a *directed tree* if

- (i) \mathscr{T} has no circuits,
- (ii) \mathscr{T} is connected and
- (iii) each vertex $v \in V^{\circ}$ has a parent.

REMARK 1.1. Any directed tree has at most one root [20, Proposition 2.1.1].

A directed tree ${\mathscr T}$ is said to be

- (i) *rooted* if it has a (unique) root.
- (ii) *rootless* if it has no root.
- (iii) *locally finite* if card(Chi(u)) is finite for all $u \in V$.
- (iv) *leafless* if every vertex has at least one child.

In what follows, $l^2(V)$ stands for the Hilbert space of square summable complex functions on V equipped with the standard inner product. Note that the set $\{e_u\}_{u\in V}$ is an orthonormal basis of $l^2(V)$, where $e_u \in l^2(V)$ is the indicator function $\chi_{\{u\}}$ of $\{u\}$. Given a system $\lambda = \{\lambda_v\}_{v\in V^\circ}$ of non-negative real numbers, we define the weighted shift operator S_{λ} on \mathscr{T} with weights λ by

$$\mathcal{D}(S_{\lambda}) := \{ f \in l^2(V) \colon \Lambda_{\mathscr{T}} f \in l^2(V) \}, \\ S_{\lambda} f := \Lambda_{\mathscr{T}} f, \quad f \in \mathcal{D}(S_{\lambda}),$$

where $\Lambda_{\mathscr{T}}$ is the mapping defined on complex functions f on V by

$$(\Lambda_{\mathscr{T}}f)(v) := \begin{cases} \lambda_v \cdot f(\mathsf{par}(v)) & \text{if } v \in V^\circ, \\ 0 & \text{if } v \text{ is a root of } \mathscr{T}. \end{cases}$$

Unless stated otherwise, $\{\lambda_v\}_{v \in V^\circ}$ consists of positive numbers and S_λ belongs to $B(l^2(V))$. It may be concluded from [20, Proposition 3.1.7] that S_λ is an injective weighted shift on \mathscr{T} if and only if \mathscr{T} is leafless. In what follows, we always assume that all the directed trees considered in this text are countably infinite and leafless.

In the proof of the main result, we frequently use the following elementary facts pertaining to the weighted shifts on directed trees.

LEMMA 1.2. If $S_{\lambda} \in B(l^2(V))$, then for any $u \in V$ and positive integer k,

- (i) $S_{\lambda}^{k}e_{u} = \sum_{v \in \mathsf{Chi}^{\langle k \rangle}(u)} \lambda_{v} \lambda_{\mathsf{par}(v)} \cdots \lambda_{\mathsf{par}^{\langle k-1 \rangle}(v)} e_{v}$ and $\|S_{\lambda}^{k}e_{u}\|^{2} = \sum_{v \in \mathsf{Chi}^{\langle k \rangle}(u)} (\lambda_{v} \lambda_{\mathsf{par}(v)} \cdots \lambda_{\mathsf{par}^{\langle k-1 \rangle}(v)})^{2}.$
- (ii) $S_{\lambda}^{*k} e_u = \lambda_u \lambda_{\mathsf{par}(u)} \cdots \lambda_{\mathsf{par}^{\langle k-1 \rangle}(u)} e_{\mathsf{par}^{\langle k \rangle}(u)}$ and $\|S_{\lambda}^{*k} e_u\|^2 = (\lambda_u \lambda_{\mathsf{par}(u)} \cdots \lambda_{\mathsf{par}^{\langle k-1 \rangle}(u)})^2$, where $e_{\mathsf{par}^{\langle n \rangle}(v)}$ is understood to be the zero vector in case $\mathsf{par}^{\langle n \rangle}(v) = \emptyset$. (iii) $S_{\lambda}^{*k} S_{\lambda}^k e_u = \|S_{\lambda}^k e_u\|^2 e_u$.

PROOF. The part (i) has been established in [20, Lemma 6.1.1], whereas (ii) and (iii) can be obtained by a straightforward mathematical induction using [20, Lemma 3.4.1(iii)]. \Box

Let S_{λ} be a left-invertible weighted shift on a rooted directed tree with weights $\{\lambda_v\}_{v\in V^\circ}$. It can be easily seen from (i) and (iii) above that the Cauchy dual S'_{λ} of S_{λ} is given by

$$S'_{\lambda}e_u := \sum_{v \in \mathsf{Chi}(u)} \frac{\lambda_v}{\|S_{\lambda}e_{\mathsf{par}(v)}\|^2} e_v \text{ for all } v \in V^\circ.$$

Note that $S'_{\lambda} \in B(l^2(V))$ is a weighted shift with weights $\{\lambda'_v\}_{v \in V^\circ}$, where

$$\lambda'_{v} := \frac{\lambda_{v}}{\|S_{\lambda}e_{\mathsf{par}(v)}\|^{2}} \text{ for all } v \in V^{\circ}.$$
(2)

4

This also shows that $\{\lambda'_v\}_{v \in V^\circ}$ is a bounded subset of positive real line. Throughout this text, we find it convenient to use the notation $S_{\lambda'}$ in place of S'_{λ} .

It turns out that any weighted shift S_{λ} on a rooted directed tree \mathscr{T} is analytic (see Lemma 3.4 below). Hence by Shimorin's construction as described above, any left-invertible S_{λ} admits an analytic model $(\mathscr{M}_z, \kappa_{\mathscr{H}}, \mathscr{H})$. It turns out that this model can be significantly improved upon provided the underlying directed tree has finite branching index (see Definition 2.1). In this case, the analytic model takes a concrete form with *multi-diagonal* kernel $\kappa_{\mathscr{H}}$ defined on a disc $\mathbb{D}_{r_{\lambda}}$, where r_{λ} is a positive number such that $\frac{1}{r(S_{\lambda'})} \leq r_{\lambda} \leq r(S_{\lambda})$ (see (5)). Moreover, the reproducing kernel Hilbert space admits an orthonormal basis consisting of vectorvalued analytic polynomials. One of the interesting aspects of our model is a handy formula for r_{λ} depending on \mathscr{T} and S_{λ} .

Although the motivation for the present work comes mainly from the theory of weighted shifts on directed trees as expounded in [20], it is closely related to some of the recent developments in the function theoretic operator theory. In particular, the reader is referred to the study of analytic reproducing kernels of finite bandwidth carried out in a series of papers by G. Adams et al [2], [3], [1], [4] (refer also to [6] for the general theory of reproducing kernels). It is also worth noting that the class of weighted shifts on rooted directed trees has some resemblance with the class of adjoints of abstract weighted shifts (in the context of complex Hilbert spaces) [7], [8], [26] and also with the class of operator-valued weighted shifts [25], [23], [18], [30] studied extensively in the literature.

Here is the sketch of the paper. Section 2 is devoted to the statement of the main theorem and some of its immediate consequences. The proof of main theorem is presented in Section 3. In Section 4, we present several examples illustrating the rich interplay between the directed trees and reproducing kernels of finite bandwidth. In Section 5, we use the main theorem to describe various spectral parts of S_{λ} . It turns out that weighted shifts on directed trees with disconnected approximate point spectra are in abundance. In the final section, we introduce a notion of branching index for rootless directed trees and use it to obtain an analytic model for a left-invertible weighted shift S_{λ} in this setting. It turns out that S_{λ} is an extension of a weighted shift operator on a rooted directed tree.

2. Main Result: Statement and Consequences

Let $\mathscr{T} = (V, \mathcal{E})$ be a rooted directed tree with root root. Then

$$V = \bigsqcup_{n=0}^{\infty} \operatorname{Chi}^{\langle n \rangle}(\operatorname{root}) \text{ (disjoint union)}$$
(3)

(see [20, Proposition 2.1.2]). For each $u \in V$, let n_u denote the unique non-negative integer such that $u \in Chi^{\langle n_u \rangle}(root)$. We use the convention that $Chi^{\langle j \rangle}(root) = \emptyset$ if j < 0. Similar convention holds for par.

The statement of the main theorem involves an invariant (to be referred to as the branching index) associated with a rooted directed tree.

DEFINITION 2.1. Let \mathscr{T} be a rooted directed tree and let

$$V_{\prec} := \{ u \in V : \operatorname{card}(\operatorname{Chi}(u)) \ge 2 \}$$

be the set of branching vertices of \mathscr{T} . Define

$$k_{\mathscr{T}} := \begin{cases} 1 + \sup\{n_w : w \in V_{\prec}\}, & \text{if } V_{\prec} \text{ is non-empty} \\ 0, & \text{if } V_{\prec} \text{ is empty.} \end{cases}$$

We refer to $k_{\mathscr{T}} \in \mathbb{Z}_+ \cup \{\infty\}$ as the branching index of \mathscr{T} .

REMARK 2.2. If $\operatorname{card}(V_{\prec})$ is finite then so is $k_{\mathscr{T}}$. On the other hand, directed trees \mathscr{T} with infinite $\operatorname{card}(V_{\prec})$ and finite $k_{\mathscr{T}}$ can be constructed easily.

The condition (i) in the following proposition says precisely that \mathscr{T} is Fredholm (refer to [20, Section 3.6] for more details related to Fredholm directed trees).

PROPOSITION 2.3. Let $S_{\lambda} \in B(l^2(V))$ be a weighted shift on a rooted directed tree \mathscr{T} with root root. Let V_{\prec} be the set of branching vertices of \mathscr{T} and let $k_{\mathscr{T}}$ be the branching index of \mathscr{T} . Then the following statements are equivalent:

- (i) \mathscr{T} is locally finite such that $card(V_{\prec})$ is finite.
- (ii) \mathscr{T} is locally finite such that $k_{\mathscr{T}}$ is finite.
- (iii) The dimension of $E := \ker S_{\lambda}^*$ is finite.

PROOF. That (i) implies (ii) is obvious. Suppose that (ii) holds. If V_{\prec} is not finite, then that $k_{\mathscr{T}}$ is finite implies that there exists an infinite subset W of V_{\prec} such that n_w is constant for all $w \in W$. Clearly, $W \subseteq \operatorname{Chi}^{\langle n_w \rangle}(\operatorname{root})$. Therefore, there exists a vertex $v \in V$ with $n_v < n_w$ such that $\operatorname{card}(\operatorname{Chi}(v))$ is infinite. This contradicts the assumption that \mathscr{T} is locally finite. Thus (ii) implies (i).

By [20, Proposition 3.5.1(ii)],

$$E = \ker S^*_{\lambda} = \langle e_{\mathsf{root}} \rangle \oplus \bigoplus_{v \in V} \left(l^2(\mathsf{Chi}(v)) \ominus \langle \boldsymbol{\lambda}^v \rangle \right),$$

where $\lambda^{v} : \operatorname{Chi}(v) \to \mathbb{C}$ is defined by $\lambda^{v}(u) = \lambda_{u}$, and $\langle f \rangle$ denotes the span of $\{f\}$. Observe now that $l^{2}(\operatorname{Chi}(v)) \ominus \langle \lambda^{v} \rangle \neq \{0\}$ if and only if $v \in V_{\prec}$. Therefore,

$$E = \langle e_{\mathsf{root}} \rangle \oplus \bigoplus_{v \in V_{\prec}} \left(l^2(\mathsf{Chi}(v)) \ominus \langle \boldsymbol{\lambda}^v \rangle \right).$$
(4)

It now follows from (4) that dim E is finite if and only if $\operatorname{card}(\operatorname{Chi}(v))$ is finite for every $v \in V_{\prec}$ and $\operatorname{card}(V_{\prec})$ is finite. This gives the equivalence of (i) and (iii). \Box

REMARK 2.4. It may happen that $k_{\mathcal{T}} < \infty$ and dim $E = \infty$ (see Example 4.6).

DEFINITION 2.5. Let \mathscr{T} be a rooted directed tree with root root and let $k_{\mathscr{T}}$ be the branching index of \mathscr{T} . For any integer n, consider the set

$$W_n := igcup_{j=n}^{k_{\mathscr{T}}+n} \operatorname{Chi}^{\langle j
angle}(\operatorname{root}).$$

Let $S_{\lambda} \in B(l^2(V))$ be a left-invertible weighted shift with weights $\{\lambda_v\}_{v \in V^{\circ}}$ and let $S_{\lambda'}$ be the Cauchy dual of S_{λ} . The radius of convergence for S_{λ} is defined as the non-negative number r_{λ} given by

$$r_{\lambda} := \liminf_{n \to \infty} \left(\sum_{v \in W_n} \left(\lambda'_v \lambda'_{\mathsf{par}(v)} \cdots \lambda'_{\mathsf{par}^{\langle n-1 \rangle}(v)} \right)^2 \right)^{-\frac{1}{2n}}.$$
 (5)

We will see later that r_{λ} is positive whenever $k_{\mathcal{T}}$ is finite (see Lemma 3.2). Let us compute r_{λ} in the case in which S_{λ} is a unilateral weighted shift. EXAMPLE 2.6 ((Diagonal)). Consider the directed tree \mathscr{T}_1 with the set of vertices $V := \mathbb{Z}_+$ and root = 0. We further require that $\mathsf{Chi}(n) = \{n+1\}$ for all $n \ge 0$. For future reference, we note that $V_{\prec} = \emptyset$, and hence $k_{\mathscr{T}_1} = 0$. The weighted shift S_{λ} on the directed tree \mathscr{T}_1 (to be referred to as *unilateral weighted shift*) is given by

$$S_{\lambda}e_n = \lambda_{n+1}e_{n+1}$$
 for all $n \ge 0$.

(Caution: This differs from the standard definition $S_{\lambda}e_n = \lambda_n e_{n+1}$ for all $n \geq 0$ of the unilateral weighted shift.) It is well-known that S_{λ} is unitarily equivalent to the operator \mathscr{M}_z of multiplication by z on the reproducing kernel Hilbert space \mathscr{H} associated with the kernel

$$\kappa_{\mathscr{H}}(z,w) = 1 + \sum_{j \ge 1} C_{j,j} z^j \overline{w}^j \ (z,w \in \mathbb{D}_r),$$

where $r := \liminf_{n \to \infty} (\lambda_n \lambda_{n-1} \cdots \lambda_1)^{\frac{1}{n}}$ and $\{C_{j,j}\}_{j \ge 0}$ is a sequence of positive numbers (refer to [28]). Since $W_n = \{n\}$ for every $n \in \mathbb{Z}_+$, the radius of convergence r_{λ} for (a left-invertible) S_{λ} is precisely r. Moreover, one can verify that (the rank one operator) $C_{j,j}$ is (multiplication by) $\frac{1}{\lambda_1^2 \cdots \lambda_j^2}$ for all $j \ge 1$. Clearly, the reproducing kernel $\kappa_{\mathscr{H}}(\cdot, \cdot)$ is diagonal in this case.

We are now in a position to state the main result of this paper.

THEOREM 2.7. Let \mathscr{T} be a rooted directed tree with finite branching index $k_{\mathscr{T}}$. Let $S_{\lambda} \in B(l^2(V))$ be a left-invertible weighted shift and let $S_{\lambda'}$ be the Cauchy dual of S_{λ} . Set $E := \ker S_{\lambda}^*$. Then there exist a z-invariant reproducing kernel Hilbert space \mathscr{H} of E-valued holomorphic functions defined on the disc $\mathbb{D}_{r_{\lambda}}$ and a unitary mapping $U : l^2(V) \longrightarrow \mathscr{H}$ such that $\mathscr{M}_z U = US_{\lambda}$, where \mathscr{M}_z denotes the operator of multiplication by z on \mathscr{H} and r_{λ} is the radius of convergence for S_{λ} . Moreover, $r_{\lambda}r(S_{\lambda'}) \geq 1$, where $r(S_{\lambda'})$ is the spectral radius of $S_{\lambda'}$. Further, U maps E onto the subspace \mathscr{E} of E-valued constant functions in \mathscr{H} such that Ug = g for every $g \in E$. Furthermore, we have the following:

- (i) The reproducing kernel $\kappa_{\mathscr{H}} : \mathbb{D}_{r_{\lambda}} \times \mathbb{D}_{r_{\lambda}} \to B(E)$ associated with \mathscr{H} satisfies $\kappa_{\mathscr{H}}(\cdot, w)g \in \mathscr{H}$ and $\langle Uf, \kappa_{\mathscr{H}}(\cdot, w)g \rangle_{\mathscr{H}} = \langle (Uf)(w), g \rangle_{E}$ for every $f \in l^{2}(V)$ and $g \in E$.
- (ii) $\kappa_{\mathscr{H}}$ is given by

$$\kappa_{\mathscr{H}}(z,w) = I_E + \sum_{\substack{j,k \ge 1\\|j-k| \le k \le \varepsilon}} C_{j,k} z^j \overline{w}^k \ (z,w \in \mathbb{D}_{r_{\lambda}}), \tag{6}$$

where I_E denotes the identity operator on E, and $C_{j,k}$ are bounded linear operators on E given by

$$C_{j,k} = P_E S_{\lambda'}^{*j} S_{\lambda'}^k |_E \ (j,k=1,2,\cdots)$$

with P_E being the orthogonal projection of $l^2(V)$ onto E. (iii) The *E*-valued polynomials in *z* are dense in \mathcal{H} . In fact,

$$\mathscr{H} = \bigvee \{ z^n f : f \in \mathscr{E}, \ n \ge 0 \}.$$

(iv) \mathscr{H} admits an orthonormal basis consisting of polynomials in z with at most $k_{\mathscr{T}} + 1$ non-zero coefficients.

REMARK 2.8. Let $S_{\lambda} \in B(l^2(V))$ be a left-invertible weighted shift with nonnegative weights $\{\lambda_v\}_{v\in V^{\circ}}$. Let $u_0 \in V$ be such that $par(u_0) \in V_{\prec}$. Suppose that $\lambda_{u_0} = 0$ and $\lambda_u > 0$ for all $u \in V \setminus \{u_0\}$. Then by [20, Proposition 3.1.6], S_{λ} can be decomposed as an orthogonal direct sum of two weighted shifts $S_{\lambda,1}, S_{\lambda,2}$ on directed trees with positive weights. Since S_{λ} is left-invertible, so are $S_{\lambda,1}$ and $S_{\lambda,2}$. By the theorem above, there exist multiplication operators $\mathscr{M}_z^{(i)}$ on reproducing kernel Hilbert spaces $\mathscr{H}^{(i)}$ for i = 1, 2 such that S_{λ} is unitarily equivalent to $\mathscr{M}_z^{(1)} \oplus \mathscr{M}_z^{(2)}$. Note that $\mathscr{H}_1 \oplus \mathscr{H}_2$ is the reproducing kernel Hilbert space associated with the reproducing kernel $\kappa_{\mathscr{H}^{(1)}} + \kappa_{\mathscr{H}^{(2)}}$ (refer to [6]).

The inequality $r_{\lambda}r(S_{\lambda'}) \geq 1$ in Theorem 2.7 may be strict in general.

EXAMPLE 2.9. Consider the weighted shift S_{λ} on the directed tree \mathscr{T}_1 (as discussed in Example 2.6) with weights λ_n given by

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1, \text{ and } \lambda_k = \begin{cases} \frac{1}{2}, & \text{if } 2^n + 1 \le k \le 3.2^{n-1}, n \ge 2\\ 1, & \text{otherwise.} \end{cases}$$

Note that $\inf_{n\geq 1} \lambda_n = \frac{1}{2}$. Thus S_{λ} is left-invertible and hence $S_{\lambda'}$ is bounded. Further, for any $n \geq 1$, total number of $\frac{1}{2}$'s occurring in first 2^n places is equal to $2^{n-1} - 2^{n-2} + 2^{n-2} - 2^{n-3} + \cdots + 4 - 2 = 2^{n-1} - 2$. Therefore, we get

$$\lambda_1 \lambda_2 \cdots \lambda_{2^n} = \frac{1}{2^{2^{n-1}-2}} = \frac{2^2}{2^{2^{n-1}}}$$

Let n be any positive integer. Then there is a unique positive integer m_n such that $2^{m_n} \leq n < 2^{m_n+1}$. Let $n = 2^{m_n} + k$ for some integer k such that $0 \leq k < 2^{m_n}$. Therefore,

$$\lambda_1 \lambda_2 \cdots \lambda_n \ge \frac{1}{2^{2^{m_n - 1} - 2 + k}} = \frac{2^2}{2^{2^{m_n - 1} + k}}.$$

Since $k < 2^{m_n}$, $\frac{2^{m_n-1}+k}{n} = 1 - \frac{2^{m_n-1}}{2^{m_n+k}} < \frac{3}{4}$. It follows that

$$(\lambda_1 \lambda_2 \cdots \lambda_n)^{\frac{1}{n}} \ge \left(\frac{2^2}{2^{2^{m_n-1}+k}}\right)^{\frac{1}{n}} > 2^{\frac{2}{n}-\frac{3}{4}},$$

and hence $r_{\lambda} = \liminf_{n \to \infty} (\lambda_n \lambda_{n-1} \cdots \lambda_1)^{\frac{1}{n}}$ is at least $2^{-\frac{3}{4}}$. On the other hand, $r(S_{\lambda'}) = 2$. This can be seen as follows. Note that $\lambda'_n = \frac{1}{\lambda_n}$. Therefore,

$$r(S_{\lambda'}) = \lim_{n \to \infty} (\sup_{m \ge 1} \lambda'_{m+1} \cdots \lambda'_{m+n})^{\frac{1}{n}} = (2^n)^{\frac{1}{n}} = 2$$

(since 2's occur in $\{\lambda'_n\}$ consecutively at 2^{n-1} places for $n \ge 2$). Thus we have $r_{\lambda}r(S_{\lambda'}) \ge 2^{\frac{1}{4}}$, which is obviously bigger than 1.

Since the proof of Theorem 2.7 consists of several observations of independent interest, it will be presented in the next section. In the remaining part of this section, we discuss some immediate consequences of the main theorem. First a terminology.

Let $\mathcal{M}_z, \kappa_{\mathscr{H}}$, and \mathscr{H} be as appearing in the statement of Theorem 2.7. For the sake of convenience, we will refer to the triple $(\mathcal{M}_z, \kappa_{\mathscr{H}}, \mathscr{H})$ as the *analytic model* of the left-invertible weighted shift S_λ acting on the directed tree \mathscr{T} .

Except the final section of this paper, we assume that \mathscr{T} is a leafless, rooted directed tree with finite branching index $k_{\mathscr{T}}$.

An operator T in $B(\mathcal{H})$ is said to be *finitely cyclic* if there are a finite number of vectors h_1, \dots, h_m in \mathcal{H} such that

$$\mathcal{H} = \bigvee \{ T^k h_i : k \ge 0, i = 1, \cdots, m \}.$$

In case m = 1, we refer to T as cyclic operator with cyclic vector h_1 . We say that T is *infinitely cyclic* if it is not finitely cyclic.

COROLLARY 2.10. Let $S_{\lambda} \in B(l^2(V))$ be a weighted shift on \mathscr{T} . If $E := \ker S^*_{\lambda}$ is finite dimensional then S_{λ} is finitely cyclic.

PROOF. Since \mathscr{T} is leafless, by [20, Proposition 3.1.7], S_{λ} is injective. If $E := \ker S_{\lambda}^*$ is finite dimensional then the range of S_{λ} is closed, and hence S_{λ} is left-invertible. Now appeal to Theorem 2.7(iii).

In general, the reproducing kernel Hilbert space \mathscr{H} as constructed in the proof of Theorem 2.7 can not be realized as the tensor product $\mathscr{H} \otimes E$, where \mathscr{H} is a Hilbert space of scalar-valued holomorphic functions. We make this explicit in the following result.

COROLLARY 2.11. Let $S_{\lambda} \in B(l^2(V))$ be a left-invertible weighted shift on \mathscr{T} . Let $(\mathscr{M}_z, \kappa_{\mathscr{H}}, \mathscr{H})$ denote the analytic model of S_{λ} and let $E := \ker S_{\lambda}^*$. Suppose that there exist a Hilbert space \mathscr{K} of scalar-valued holomorphic functions and an isometric isomorphism $\Phi : \mathscr{H} \to \mathscr{K} \otimes E$ such that $\Phi(pf) = p \otimes f$ for every polynomial $p \in \mathscr{K}$ and $f \in E$. Then the reproducing kernel $\kappa_{\mathscr{H}}$ associated with \mathscr{H} is the diagonal kernel given by

$$\kappa_{\mathscr{H}}(z,w) = I_E + \sum_{j=1}^{\infty} \left(P_E S^{*j}_{\lambda'} S^j_{\lambda'} |_E \right) z^j \overline{w}^j \ (z,w \in \mathbb{D}_{r_\lambda}).$$

PROOF. Note that for any $f, g \in E$ and $m, n \in \mathbb{Z}_+$,

 $\langle S_{\lambda}^{m}f, S_{\lambda}^{n}g\rangle_{l^{2}(V)} = \langle z^{m}f, z^{n}g\rangle_{\mathscr{H}} = \langle \Phi(z^{m}f), \Phi(z^{n}g)\rangle_{\mathscr{H}\otimes E} = \langle z^{m}, z^{n}\rangle_{\mathscr{H}}\langle f, g\rangle_{E}.$ (7) Since $S_{\lambda}^{k}e_{\text{root}} \in \bigvee \{e_{v} : v \in \operatorname{Chi}^{\langle k \rangle}(\operatorname{root})\}$, by an application of (3), we obtain $\langle z^{m}, z^{n}\rangle_{\mathscr{H}} = 0$ for $m \neq n$ after letting $f = e_{\operatorname{root}} = g$. Hence by (7), we must have $\langle S_{\lambda}^{m}f, S_{\lambda}^{n}g\rangle_{l^{2}(V)} = 0$ for any $f, g \in E$ and non-negative integers $m \neq n$. This shows that the sequence $\{S_{\lambda}^{k}E\}_{k\geq 0}$ of subspaces of $l^{2}(V)$ is mutually orthogonal. It follows immediately that for any $f, g \in E$ and non-negative integers $j \neq k$,

$$\langle P_E S_{\lambda'}^{*j} S_{\lambda'}^k f, g \rangle_E = \langle S_{\lambda'}^{*j} S_{\lambda'}^k f, g \rangle_{l^2(V)} = \langle S_{\lambda'}^k f, S_{\lambda'}^j g \rangle_{l^2(V)} = 0.$$

In particular, $P_E S_{\lambda'}^{*j} S_{\lambda'}^k|_E = 0$ for all non-negative integers $j \neq k$. The desired conclusion now follows from Theorem 2.7(ii).

REMARK 2.12. In view of Shimorin's model (as discussed in Section 1), after replacing r_{λ} by $\frac{1}{r(S_{\lambda'})}$, one may obtain the conclusion of Corollary 2.11 for any directed tree with infinite branching index $k_{\mathscr{T}}$. Thus, even for directed trees \mathscr{T} with infinite $k_{\mathscr{T}}$, the associated reproducing kernel $k_{\mathscr{H}}$ could be multi-diagonal.

Recall that $T \in B(\mathcal{H})$ is an *isometry* if $T^*T = I$.

COROLLARY 2.13. Consider the analytic model $(\mathcal{M}_z, \kappa_{\mathscr{H}}, \mathscr{H})$ of a left-invertible weighted shift S_λ on \mathscr{T} and let $E := \ker S^*_\lambda$. If S_λ is an isometry then $\kappa_{\mathscr{H}}$ is the B(E)-valued Cauchy kernel given by

$$\kappa_{\mathscr{H}}(z,w) = \frac{I_E}{1-z\overline{w}} \ (z,w\in\mathbb{D}).$$

In particular, \mathcal{H} is the E-valued Hardy space of the open unit disc.

PROOF. Assume that S_{λ} is an isometry. Note that $S_{\lambda'}$ is also isometry in view of hypothesis and $S_{\lambda'}^* S_{\lambda'} = (S_{\lambda}^* S_{\lambda})^{-1}$. By the uniqueness of the reproducing kernel, it suffices to check that $C_{j,k} = \delta_{j,k} I_E$, where

$$C_{j,k} := P_E S_{\lambda'}^{*j} S_{\lambda'}^k |_E \ (j,k=1,2,\cdots)$$

and $\delta_{j,k}$ denotes the Kronecker delta. If j = k then obviously $C_{j,k} = I_E$. If j < k then $C_{j,k} = P_E S_{\lambda'}^{k-j}|_E = 0$ since $S_{\lambda'}E \subseteq \operatorname{ran} S_{\lambda} = E^{\perp}$.

One rather striking consequence of the preceding corollary is as follows: If S_{λ} is an isometry then $r_{\lambda} = 1$. Note that this observation is irrespective of the structure of the directed tree \mathscr{T} . On the other hand, the definition of r_{λ} relies on the weight sequence $\{\lambda_v\}_{v\in V^{\circ}}$ and of course on the structure of \mathscr{T} . To see this fact, assume that S_{λ} is an isometry. By Theorem 2.7, we must have $r_{\lambda}r(S_{\lambda'}) \geq 1$. However, $S_{\lambda'}$ being isometry, $r(S_{\lambda'}) = 1$, and hence $r_{\lambda} \geq 1$. Since $\frac{I_E}{1-zw}$ is not defined on $\mathbb{D}_r \times \mathbb{D}_r$ for any r > 1, by Theorem 2.7(ii), r_{λ} can not exceed 1.

3. Proof of the Main Theorem

The proof of Theorem 2.7 involves several lemmas. The first of which collects some facts related to the set W_n . Recall from Definition 2.5 that for any integer n, the set W_n is given by

$$W_n := \bigcup_{j=n}^{k_{\mathscr{T}}+n} \operatorname{Chi}^{\langle j \rangle}(\operatorname{root}).$$
(8)

LEMMA 3.1. Let $S_{\lambda} \in B(l^2(V))$ be a weighted shift on \mathscr{T} and let $E := \ker S_{\lambda}^*$. Then we have the following statements:

- (i) E is a subspace of the (possibly infinite dimensional) space ∨{e_v : v ∈ W₀}.
- (ii) $card(Chi^{\langle n \rangle}(root))$ (possibly countably infinite) is constant for $n \ge k_{\mathcal{T}}$. In particular, $card(W_n)$ is constant for $n \ge k_{\mathcal{T}}$.
- (iii) For every $v \notin W_n$, $e_{\mathsf{par}^{(n)}(v)}$ belongs to the orthogonal complement of E, where $e_{\mathsf{par}^{(n)}(v)}$ is understood to be the zero vector in case $\mathsf{par}^{(n)}(v) = \emptyset$.
- (iv) For non-negative integers m and n, $W_n \cap W_m \neq \emptyset$ if and only if $|n-m| \leq k_{\mathscr{T}}$.

PROOF. Note that $Chi(V_{\prec}) \subseteq W_0$. Hence by (4),

$$E \subseteq \langle e_{\mathsf{root}} \rangle \oplus \bigoplus_{v \in V_{\prec}} l^2(\mathsf{Chi}(v)) \subseteq \bigvee \{ e_v : v \in W_0 \}.$$
(9)

This yields (i). To see (ii), recall that n_u is the unique non-negative integer such that $u \in \operatorname{Chi}^{\langle n_u \rangle}(\operatorname{root})$. Note that $\operatorname{card}(\operatorname{Chi}(u)) = 1$ if $n_u \geq k_{\mathscr{T}}$, where we used the assumption that \mathscr{T} is leafless. Thus $\operatorname{card}(\operatorname{Chi}^{\langle n \rangle}(\operatorname{root}))$ is constant for $n \geq k_{\mathscr{T}}$. This proves (ii).

We now check (iii). Let $v \notin W_n$. Since E is a subspace of $\bigvee \{e_v : v \in W_0\}$ by part (i), it suffices to check that $e_{\mathsf{par}^{(n)}(v)}$ is orthogonal to $\{e_v : v \in W_0\}$. Note that $n_v < n$ or $n_v > k_{\mathscr{T}} + n$, If $n_v < n$ then $e_{\mathsf{par}^{(n)}(v)} = e_{\emptyset} = 0$ by convention. Otherwise, $\mathsf{par}^{(n)}(v) \notin W_0$, and hence $e_{\mathsf{par}^{(n)}(v)}$ is orthogonal to $\{e_v : v \in W_0\}$. To see (iv), let n, m be two non-negative integers such that n < m. If $n + k_{\mathscr{T}} < m$ then clearly,

 $W_n \cap W_m = \emptyset$. So suppose that $n + k_{\mathscr{T}} \ge m$. Then $W_n \cap W_m = \bigcup_{k=m}^{n+k_{\mathscr{T}}} \mathsf{Chi}^{\langle k \rangle}(\mathsf{root}),$ which is obviously non-empty.

Next we prove that the radius of convergence for any left-invertible weighted shift with finite dimensional cokernel is positive.

LEMMA 3.2. Let $S_{\lambda} \in B(l^2(V))$ be a left-invertible weighted shift on \mathscr{T} and let $S_{\lambda'}$ be the Cauchy dual of S_{λ} . If $r(S_{\lambda'})$ denotes the spectral radius of $S_{\lambda'}$ then the radius of convergence r_{λ} for S_{λ} satisfies $r_{\lambda}r(S_{\lambda'}) \geq 1$. In particular, r_{λ} is positive.

PROOF. By Lemma 1.2(i), for any integer $k \ge 0$,

$$\|S_{\lambda'}^k e_{\mathsf{root}}\|^2 = \sum_{v \in \mathsf{Chi}^{\langle k \rangle}(\mathsf{root})} \left(\lambda'_v \lambda'_{\mathsf{par}(v)} \cdots \lambda'_{\mathsf{par}^{\langle k-1 \rangle}(v)}\right)^2$$

It follows from (3) that

$$\sum_{v \in W_n} \left(\lambda'_v \lambda'_{\mathsf{par}(v)} \cdots \lambda'_{\mathsf{par}^{\langle k-1 \rangle}(v)} \right)^2 = \sum_{k=n}^{n+k_{\mathscr{T}}} \sum_{v \in \mathsf{Chi}^{\langle k \rangle}(\mathsf{root})} \left(\lambda'_v \lambda'_{\mathsf{par}(v)} \cdots \lambda'_{\mathsf{par}^{\langle k-1 \rangle}(v)} \right)^2$$
$$= \sum_{k=n}^{n+k_{\mathscr{T}}} \|S_{\lambda'}^k e_{\mathsf{root}}\|^2 = \sum_{k=0}^{k_{\mathscr{T}}} \|S_{\lambda'}^{n+k} e_{\mathsf{root}}\|^2$$
$$\leq \|S_{\lambda'}^n\|^2 \sum_{k=0}^{k_{\mathscr{T}}} \|S_{\lambda'}^k e_{\mathsf{root}}\|^2.$$

If we set $M := \sum_{k=0}^{k_{\mathscr{T}}} \|S_{\lambda'}^k e_{\mathsf{root}}\|^2$ (which is finite since $k_{\mathscr{T}} < \infty$) then by the definition of r_{λ} ,

$$r_{\lambda} \ge \liminf_{n \to \infty} \left(M \| S_{\lambda'}^n \|^2 \right)^{-\frac{1}{2n}} = \left(\lim_{n \to \infty} M^{-\frac{1}{2n}} \right) \left(\liminf_{n \to \infty} \| S_{\lambda'}^n \|^{-\frac{1}{n}} \right) = \frac{1}{r(S_{\lambda'})},$$

which completes the proof of the lemma.

REMARK 3.3. If S_{λ} is an expansion (that is, $S_{\lambda}^*S_{\lambda} \ge I$) then $S_{\lambda'}$ is a contraction (that is, $S_{\lambda}^* S_{\lambda} \leq I$). In this case, r_{λ} is at least 1.

We need one more fact in the proof of the main result (cf. [5, Proposition 4.5]).

LEMMA 3.4. Let $S_{\lambda} \in B(l^2(V))$ be a weighted shift on \mathscr{T} . Then S_{λ} is analytic.

PROOF. Put $V_0 := V$ and $V_k := V \setminus \bigcup_{j=0}^{k-1} \operatorname{Chi}^{\langle j \rangle}(\operatorname{root})$ $(k \ge 1)$. Note that $\{V_k\}_{k\geq 0}$ is a strictly decreasing sequence of sets such that $\cap_{k\geq 0}V_k=\emptyset$. Now, for all $u \in V$ and all integers $k \ge 0$, by Lemma 1.2(i),

$$S^k_\lambda e_u = \sum_{v \in \mathsf{Chi}^{\langle k \rangle}(u)} \lambda_v \lambda_{\mathsf{par}(v)} \cdots \lambda_{\mathsf{par}^{\langle k-1 \rangle}(v)} e_v.$$

It follows that

$$\operatorname{ran} S_{\lambda}^{k} \subseteq \bigvee \{ e_{u} : u \in V_{k} \} := M_{k}, \text{ say}$$

Also, if $f \in M_k$, then f(u) = 0 for $u \in V \setminus V_k = \bigcup_{j=0}^{k-1} \operatorname{Chi}^{\langle j \rangle}(\operatorname{root})$. Thus, if $f \in \bigcap_{k=0}^{\infty} M_k$, then f(u) = 0 for $u \in \bigcup_{j=0}^{\infty} \operatorname{Chi}^{\langle j \rangle}(\operatorname{root}) = V$. That is, f = 0. Hence

$$\{0\} \subseteq \bigcap_{k=0}^{\infty} \operatorname{ran} S_{\lambda}^{k} \subseteq \bigcap_{k=0}^{\infty} M_{k} = \{0\}.$$

This shows that S_{λ} is analytic.

PROOF OF THEOREM 2.7. As mentioned earlier, the proof relies on the ideas developed in [29, Sections 1 and 2]. Let $f = \sum_{v \in V} f(v)e_v \in l^2(V)$. By Lemmas 1.2(ii) and 3.1(iii),

$$P_E S_{\lambda'}^{*n} f = \sum_{v \in V} f(v) \lambda'_v \lambda'_{\mathsf{par}(v)} \cdots \lambda'_{\mathsf{par}^{\langle n-1 \rangle}(v)} P_E e_{\mathsf{par}^{\langle n \rangle}(v)}$$
$$= \sum_{v \in W_n} f(v) \lambda'_v \lambda'_{\mathsf{par}(v)} \cdots \lambda'_{\mathsf{par}^{\langle n-1 \rangle}(v)} P_E e_{\mathsf{par}^{\langle n \rangle}(v)}$$

We claim that the E-valued series

$$U_f(z) := \sum_{n \ge 0} (P_E S_{\lambda'}^{*n} f) z^n \tag{10}$$

converges absolutely in E on the disc $\mathbb{D}_{r_{\lambda}}$ for every $f \in l^2(V)$. By Lemma 3.1(iv), for non-negative integers m and $n, W_n \cap W_m \neq \emptyset$ if and only if $|n - m| \leq k_{\mathscr{T}}$. It follows that

$$\sum_{\substack{v \in W_n \\ n \ge 0}} |f(v)|^2 = \sum_{v \in W_0} |f(v)|^2 + \sum_{v \in W_1} |f(v)|^2 + \cdots$$
$$\leq \sum_{v \in V} (k_{\mathscr{T}} + 1) |f(v)|^2 = (k_{\mathscr{T}} + 1) ||f||^2.$$
(11)

Now by the Cauchy-Schwarz inequality, for any integer $k \ge 0$,

$$\begin{split} \left\| \sum_{n=0}^{k} (P_{E} S_{\lambda'}^{*n} f) z^{n} \right\| &\leq \sum_{\substack{v \in W_{n} \\ n \geq 0}} |f(v)| \lambda_{v}' \lambda_{\mathsf{par}(v)}' \cdots \lambda_{\mathsf{par}^{\langle n-1 \rangle}(v)}' |z|^{n} \\ &\leq \left(\sum_{\substack{v \in W_{n} \\ n \geq 0}} |f(v)|^{2} \right)^{\frac{1}{2}} \left(\sum_{\substack{v \in W_{n} \\ n \geq 0}} \left(\lambda_{v}' \lambda_{\mathsf{par}(v)}' \cdots \lambda_{\mathsf{par}^{\langle n-1 \rangle}(v)}' \right)^{2} |z|^{2n} \right)^{\frac{1}{2}} \\ &\stackrel{(11)}{\leq} \sqrt{k_{\mathcal{F}} + 1} \, \|f\| \Big(\sum_{\substack{v \in W_{n} \\ n \geq 0}} \left(\lambda_{v}' \lambda_{\mathsf{par}(v)}' \cdots \lambda_{\mathsf{par}^{\langle n-1 \rangle}(v)}' \right)^{2} |z|^{2n} \Big)^{\frac{1}{2}}. \end{split}$$

Since the series on the right hand side converges absolutely on $\mathbb{D}_{r_{\lambda}}$, the claim stands verified. Thus U_f is holomorphic in the disk $\mathbb{D}_{r_{\lambda}}$. This allows us to define the map $U: l^2(V) \to \mathscr{H}$ by $Uf = U_f$, where \mathscr{H} denotes the complex vector space of *E*valued holomorphic functions of the form U_f . By Lemma 3.4, S_{λ} is analytic, and hence by [**29**, Lemma 2.2], U is injective. Thus the inner-product given by

$$\langle U_f, U_g \rangle = \langle f, g \rangle_{l^2(V)}$$
 for all $f, g \in l^2(V)$

makes \mathscr{H} an inner-product space. Also, the very definition of the inner-product on \mathscr{H} shows that U is unitary, and hence \mathscr{H} is a Hilbert space.

Note that for each $f \in E$, $U_f(z) = f$. We now show that \mathscr{H} is z-invariant. Let $U_f \in \mathscr{H}$. Since $S^*_{\lambda'}S_{\lambda} = I$ and $P_ES_{\lambda} = 0$, we get

$$zU_f(z) = \sum_{n \ge 0} (P_E S_{\lambda'}^{*n} f) z^{n+1} = \sum_{n \ge 1} (P_E S_{\lambda'}^{*n-1} f) z^n$$
$$= \sum_{n \ge 0} (P_E S_{\lambda'}^{*n} S_\lambda f) z^n = U_{S_\lambda f}(z) \in \mathscr{H}.$$

12

Above expression also verifies that $\mathscr{M}_z U = US_\lambda$, where \mathscr{M}_z is the operator of multiplication by z on \mathscr{H} .

Part (i) has been recorded on [29, Pg 154] (see the discussion following (1)). To see (ii), recall from (1) that

$$\kappa_{\mathscr{H}}(z,w) = \sum_{j,k \ge 0} C_{j,k} z^j \overline{w}^k,$$

where $C_{j,k}$ is a bounded linear operator on E given by $C_{j,k} = P_E S_{\lambda'}^{*j} S_{\lambda'}^k |_E$. Since $\ker S_{\lambda'}^* = \ker S_{\lambda}^* = E$, it follows that $P_E S_{\lambda'}^{*j} |_E = 0$ for all $j \ge 1$. Since $C_{j,k}^* = C_{k,j}$, we get $C_{j,0} = 0 = C_{0,j}$ for all $j \ge 1$. Hence the above expression for $\kappa_{\mathscr{H}}(z, w)$ reduces to

$$\kappa_{\mathscr{H}}(z,w) = I_E + \sum_{j,k \ge 1} C_{j,k} z^j \overline{w}^k.$$

As recorded earlier in (9), $E \subseteq \bigvee \{e_v : v \in W_0\}$ (see also (8) for the definition of W_n). It follows that

$$S_{\lambda'}^{*j}S_{\lambda'}^k E \subseteq \bigvee \left\{ e_v : v \in W_{k-j} \right\},$$

and therefore $S_{\lambda'}^{*j}S_{\lambda'}^{k}E$ is orthogonal to E if $|j-k| > k_{\mathscr{T}}$. Thus $C_{j,k} = 0$ if $|j-k| > k_{\mathscr{T}}$. This proves (ii).

To prove (iii), note that by Lemma 3.4, $S_{\lambda'}$ is analytic. Therefore, by [29, Proposition 2.7],

$$l^2(V) = \bigvee_{n \ge 0} S^n_{\lambda}(E).$$

Since \mathcal{M}_z is unitarily equivalent to S_λ and ker $S^*_\lambda = E$, it follows that

$$\mathscr{H} = \bigvee_{n \ge 0} \mathscr{M}_z^n(\mathscr{E}).$$

This is precisely (iii).

Finally, since U is unitary and $\{e_u : u \in V\}$ is an orthonormal basis of $l^2(V)$, $\{U_{e_u} : u \in V\}$ is an orthonormal basis of \mathcal{H} . Note that

$$U_{e_u}(z) = \sum_{k \ge 0} (P_E S_{\lambda'}^{*k} e_u) z^k = \sum_{0 \le k \le n_u} (P_E S_{\lambda'}^{*k} e_u) z^k$$
(12)

(see the discussion following (3) for the definition of n_u). If $n_u \leq k_{\mathscr{T}}$ then clearly U_{e_u} has at most $k_{\mathscr{T}} + 1$ number of non-zero coefficients. Suppose now that $n_u > k_{\mathscr{T}}$. By Lemma 1.2(ii), $S_{\lambda'}^{*k}e_u = \alpha_{\lambda,k}e_{\mathsf{par}^{\langle k \rangle}(u)}$ for some scalar $\alpha_{\lambda,k}$. Let k be an integer such that $0 \leq k \leq n_u - k_{\mathscr{T}} - 1$. Then $n_{\mathsf{par}^{\langle k \rangle}(u)} = n_u - k \geq k_{\mathscr{T}} + 1$, and hence $\mathsf{par}^{\langle k \rangle}(u) \notin W_0$. By Lemma 3.1(i),

$$P_E S_{\lambda'}^{*k} e_u = \alpha_{\lambda,k} P_E e_{\mathsf{par}^{\langle k \rangle}(u)} = 0.$$

Hence total number of possible non-zero coefficients in above expression of U_{e_u} are $n_u + 1 - (n_u - k_{\mathcal{T}}) = k_{\mathcal{T}} + 1$. Thus for each $u \in V$, U_{e_u} is a polynomial in z with at most $k_{\mathcal{T}} + 1$ non-zero coefficients. This completes the proof of the theorem. \Box

REMARK 3.5. The proof above actually shows that the set of analytic bounded point evaluation of \mathscr{H} contains the disc $\mathbb{D}_{r_{\lambda}}$ (refer to [12, Chapter II, Section 7]).

S. CHAVAN AND S. TRIVEDI

We conclude this section with a brief discussion on a possible line of investigation. Note that the proof of Theorem 2.7 relies on the notion of Cauchy dual operator, present already in Shimorin's construction of an analytic model for a left inertible analytic operator [29]. In case dim E = 1, the analytic model can be replaced by the model in which the weighted shift operator can be realized as the operator of multiplication by z on a Hilbert space of formal power series [28] (refer also to [23]). It would be interesting to find a counter-part of the later model in case dim E > 1. In this regard, the authors would like to draw reader's attention to [31, Theorems 2.12 and 2.13] in which it is shown that the adjoint of an arbitrary cyclic operator can be modelled as a backward shift on a reproducing kernel Hilbert space.

4. Examples

In this section, we illustrate Theorem 2.7 with the help of several interesting examples. In particular, we see that various directed trees (discrete structures) render to analytic multi-diagonal kernels (analytic structures). These include mainly tridiagonal and pentadiagonal kernels (the reader is referred to [2], [3], [1], [4] for a systematic study of scalar-valued and matrix-valued kernels of finite bandwidth; refer also to [24] for a new class of matrix-valued kernels on the unit disc arising in the classification problem of homogeneous operators). All important examples are summarized in the form of a table at the end of this section.

EXAMPLE 4.1 ((Tridiagonal)). Consider the directed tree \mathscr{T}_2 with set of vertices

$$V := \{(0,0)\} \cup \{(1,i), (2,i) : i \ge 1\}$$

and root = (0,0). We further require that $Chi(0,0) = \{(1,1), (2,1)\}$ and

$$Chi(1, i) = \{(1, i+1)\}, Chi(2, i) = \{(2, i+1)\}, \text{ for all } i \ge 1$$

Let S_{λ} be a left-invertible weighted shift on \mathscr{T}_2 . It is easy to see from (4) that

$$E := \ker S_{\lambda}^* = \{ \alpha e_{(0,0)} + \beta (\lambda_{(2,1)} e_{(1,1)} - \lambda_{(1,1)} e_{(2,1)}) : \alpha, \beta \in \mathbb{C} \}.$$

Also, $V_{\prec} = \{(0,0)\}$ and $k_{\mathscr{T}_2} = 1$. Therefore, by Theorem 2.7, $\kappa_{\mathscr{H}}(\cdot, \cdot)$ takes the form

$$\kappa_{\mathscr{H}}(z,w) = I_E + \sum_{\substack{j,k \ge 1\\|j-k| \le 1}} C_{j,k} z^j \overline{w}^k \ (z,w \in \mathbb{D}_{r_{\lambda}}),$$

where $C_{j,k}$ is given by $C_{j,k} = P_E S_{\lambda'}^{*j} S_{\lambda'}^k |_E$ $(j, k = 1, 2, \cdots)$. Let us find an explicit expression for the radius of convergence r_{λ} for S_{λ} . Note that

$$W_n = \{(1, n), (2, n), (1, n+1), (2, n+1)\}$$

for $n \ge 1$, and hence by (5),

$$r_{\lambda} = \liminf_{n \to \infty} \left(\sum_{j=1}^{2} \left[\left(\lambda'_{(j,n)} \cdots \lambda'_{(j,1)} \right)^2 + \left(\lambda'_{(j,n+1)} \cdots \lambda'_{(j,2)} \right)^2 \right] \right)^{-\frac{1}{2n}}, \quad (13)$$

where the sequence $\{\lambda'_{(j,n)}\}_{n\geq 1}$ for j=1,2 is given by

$$\lambda'_{(j,n)} = \begin{cases} \frac{\lambda_{(j,n)}}{\lambda^2_{(1,n)} + \lambda^2_{(2,n)}} & \text{if } n = 1\\ \frac{1}{\lambda_{(j,n)}} & \text{if } n \ge 2. \end{cases}$$
(14)

14

In this case, the reproducing kernel $\kappa_{\mathscr{H}}(\cdot, \cdot)$ is tridiagonal. Finally, we note that the weight sequence λ can be chosen so that $\kappa_{\mathscr{H}}(\cdot, \cdot)$ is not diagonal. In fact, a routine calculation shows that

$$C_{2,1}(\lambda_{(2,1)}e_{(1,1)} - \lambda_{(1,1)}e_{(2,1)}) = \frac{\lambda_{(1,1)}\lambda_{(2,1)}}{\lambda_{(1,1)}^2 + \lambda_{(2,1)}^2} \left(\frac{1}{\lambda_{(1,2)}^2} - \frac{1}{\lambda_{(2,2)}^2}\right)e_{(0,0)}, \quad (15)$$

which is clearly non-zero in case $\lambda_{(1,2)} \neq \lambda_{(2,2)}$

The tridiagonal kernel $\kappa_{\mathscr{H}}$ appearing in Example 4.1 takes a concrete form for a family of weighted shifts S_{λ} .

PROPOSITION 4.2. Let \mathscr{T}_2 and S_{λ} be as discussed in Example 4.1. Let x := $e_{(0,0)}, y := \lambda_{(2,1)}e_{(1,1)} - \lambda_{(1,1)}e_{(2,1)}.$ Assume that the weight sequence $\{\lambda_{(j,i)} : i \geq 0\}$ 1, j = 1, 2 of S_{λ} satisfies the following:

- (i) $\lambda_{(1,1)} = \lambda_{(2,1)}$,

(ii) $\lambda_{(1,2)} \neq \lambda_{(2,2)}$, (iii) $\lambda_{(1,n)} \cdots \lambda_{(1,2)} = \lambda_{(2,n)} \cdots \lambda_{(2,2)}$ for every integer $n \ge 3$. Then the reproducing kernel $\kappa_{\mathscr{H}}$ takes the form

$$\kappa_{\mathscr{H}}(z,w) = I_E + \alpha(x \otimes y \, z^2 \overline{w} + y \otimes x \, z \overline{w}^2) + \sum_{k=1}^{\infty} \left(\alpha_k \, x \otimes x + \alpha_{k+1} \, y \otimes y \right) z^k \overline{w}^k \ (z,w \in \mathbb{D}_{r_{\lambda}}),$$

where $\alpha := \frac{\lambda_{(1,1)}^2}{\|y\|^4} \left(\lambda_{(1,2)}^{-2} - \lambda_{(2,2)}^{-2} \right)$ is a non-zero real number, and

$$\alpha_k := \begin{cases} \|y\|^{-2} & \text{if } k = 1, \\ \lambda_{(1,1)}^2 \|y\|^{-4} \Big(\lambda_{(1,2)}^{-2} + \lambda_{(2,2)}^{-2}\Big) & \text{if } k = 2 \\ \|y\|^{-2} \Big(\lambda_{(1,k)} \cdots \lambda_{(1,2)}\Big)^{-2} & \text{if } k \ge 3. \end{cases}$$

PROOF. As seen in Example 4.1, $\kappa_{\mathscr{H}}(\cdot, \cdot)$ is given by

$$\kappa_{\mathscr{H}}(z,w) = I_E + \sum_{\substack{j,k \ge 1\\|j-k| \le 1}} C_{j,k} z^j \overline{w}^k \ (z,w \in \mathbb{D}_{r_{\lambda}}).$$

Since $C_{2,1}e_{(0,0)} = 0$, by (15) and (i), C_{21} is the rank one operator $\alpha x \otimes y$. Note that $\alpha \neq 0$ in view of (ii). Also, $C_{1,2} = C_{2,1}^* = \alpha y \otimes x$.

We claim that $C_{k,k+1} = 0 = C_{k+1,k}$ for all integers $k \ge 2$. Let us first compute the diagonal operator $S_{\lambda'}^{*k}S_{\lambda'}^{k}$. Fix an integer $k \geq 2$. It may be concluded from [20, Lemma 6.1.1 that

$$S_{\lambda'}^{*k}S_{\lambda'}^{k}e_{u} = \sum_{v\in\mathsf{Chi}^{\langle k \rangle}(u)} \left(\lambda_{v}'\lambda_{\mathsf{par}(v)}' \cdots \lambda_{\mathsf{par}^{\langle k-1 \rangle}(v)}'\right)^{2}e_{u}.$$
(16)

Since $S_{\lambda'}^* x = 0$, it follows from (16) that $C_{k+1,k} x = 0$. Note further that

$$\begin{split} S_{\lambda'}^{*k} S_{\lambda'}^{k} y &= \lambda_{(2,1)} S_{\lambda'}^{*k} S_{\lambda'}^{k} e_{(1,1)} - \lambda_{(1,1)} S_{\lambda'}^{*k} S_{\lambda'}^{k} e_{(2,1)} \\ &= \lambda_{(2,1)} \left(\lambda_{(1,k+1)}' \cdots \lambda_{(1,2)}' \right)^{2} e_{(1,1)} - \lambda_{(1,1)} \left(\lambda_{(2,k+1)}' \cdots \lambda_{(2,2)}' \right)^{2} e_{(2,1)} \\ &\stackrel{\text{(iii)}}{=} \frac{1}{\left(\lambda_{(1,k+1)} \cdots \lambda_{(1,2)} \right)^{2}} y, \end{split}$$

where in the last step we used (14). This immediately yields that $C_{k+1,k}y = 0$. This completes the verification of the claim. The above calculation also shows that $C_{k,k}y = \alpha_{k+1} \|y\|^2 y$ for all integers $k \geq 2$. Since $S_{\lambda'}^* S_{\lambda'}y = \frac{\lambda_{(2,1)}}{\lambda_{(1,2)}^2} e_{(1,1)} - \frac{\lambda_{(1,1)}}{\lambda_{(2,2)}^2} e_{(2,1)}$, we have

$$C_{1,1}y = P_E\left(\frac{\lambda_{(2,1)}}{\lambda_{(1,2)}^2}e_{(1,1)} - \frac{\lambda_{(1,1)}}{\lambda_{(2,2)}^2}e_{(2,1)}\right)$$

$$= \frac{\lambda_{(2,1)}}{\lambda_{(1,2)}^2}\langle e_{(1,1)}, y \rangle \frac{y}{\|y\|^2} - \frac{\lambda_{(1,1)}}{\lambda_{(2,2)}^2}\langle e_{(2,1)}, y \rangle \frac{y}{\|y\|^2}$$

$$= \left(\frac{\lambda_{(2,1)}^2}{\lambda_{(1,2)}^2} + \frac{\lambda_{(1,1)}^2}{\lambda_{(2,2)}^2}\right) \frac{y}{\|y\|^2} \stackrel{(i)}{=} \alpha_2 \|y\|^2 y.$$

To compute the diagonal entry $C_{k,k}$, by (16), for any integer $k \ge 1$,

$$C_{k,k}x = \sum_{j=1}^{2} \left(\lambda'_{(j,k)}\cdots\lambda'_{(j,1)}\right)^2 x = \alpha_k x,$$

where we used (14) and (iii). Now it is easy to see that the rank two operator $C_{k,k}$ is given by $C_{k,k} = \left(\alpha_k \, x \otimes x + \alpha_{k+1} \, y \otimes y\right)$ for all integers $k \ge 1$.

EXAMPLE 4.3. The preceding proposition is applicable to S_{λ} with weights $\lambda_{(1,1)} = \lambda_{(2,1)} = \lambda_{(1,2)} = 1, \lambda_{(2,2)} = \sqrt{2} = \lambda_{(1,3)}$, and $\lambda_{(2,3)} = 1 = \lambda_{(j,i)}$ for $i \geq 4$ and for j = 1, 2. In this case, $\alpha = \frac{1}{8}, \alpha_1 = \frac{1}{2}, \alpha_2 = \frac{3}{8}$ and $\alpha_k = \frac{1}{8}$ for all integers $k \geq 3$. Thus the reproducing kernel $\kappa_{\mathscr{H}}$ takes the form

$$\kappa_{\mathscr{H}}(z,w) = I_E + \frac{1}{8} \left(x \otimes y \, z^2 \overline{w} + y \otimes x \, z \overline{w}^2 \right) + \frac{1}{2} \left(x \otimes x + \frac{3}{4} y \otimes y \right) z \overline{w} \\ + \frac{1}{8} \left(3x \otimes x + y \otimes y \right) z^2 \overline{w}^2 + \frac{1}{8} \sum_{k=3}^{\infty} \left(x \otimes x + y \otimes y \right) z^k \overline{w}^k \ (z,w \in \mathbb{D}_{r_{\lambda}}),$$

where, in view of (13), r_{λ} can be easily seen to be equal to 1. Also, one may easily deduce from the proof of Theorem 2.7(iv) that for all integers $i \geq 1$,

$$U_{e_{(j,i)}}(z) = (P_E S_{\lambda'}^{*i-1} e_{(j,i)}) z^{i-1} + (P_E S_{\lambda'}^{*i} e_{(j,i)}) z^i, \ j = 1, 2.$$

It is now easy to see that the orthonormal basis for the reproducing kernel Hilbert space \mathscr{H} associated with $\kappa_{\mathscr{H}}$ is given by

$$\{x, p(z), zp(z)\} \cup \left\{\frac{1}{\sqrt{2}} z^{k-1} p(z)\right\}_{k \ge 3} \cup \{q(z)\} \cup \left\{\frac{1}{\sqrt{2}} z^{k-1} q(z)\right\}_{k \ge 2}$$

where $p(z) = \frac{1}{2}(y + xz)$ and $q(z) = \frac{1}{2}(xz - y)$ are linear *E*-valued polynomials.

EXAMPLE 4.4 ((Pentadiagonal)). Consider the directed tree \mathscr{T}_3 with set of vertices $V = \{(0,0), (1,1)\} \cup \{(2,i), (3,i) : i \ge 1\}$ and root = (0,0). We further require that $\mathsf{Chi}(0,0) = \{(1,1)\}, \mathsf{Chi}(1,1) = \{(2,1), (3,1)\}$ and

$$Chi(2, i) = \{(2, i+1)\}, Chi(3, i) = \{(3, i+1)\}, \text{ for all } i \ge 1$$

Let S_{λ} be a left-invertible weighted shift on \mathscr{T}_3 . As in the preceding example, one can see that

$$E := \ker S_{\lambda}^* = \{ \alpha e_{(0,0)} + \beta (\lambda_{(3,1)} e_{(2,1)} - \lambda_{(2,1)} e_{(3,1)}) : \alpha, \beta \in \mathbb{C} \}.$$

Also $V_{\prec} = \{(1,1)\}$ and $k_{\mathcal{T}_3} = 2$. Therefore, from Theorem 2.7, $\kappa_{\mathcal{H}}(\cdot, \cdot)$ takes the form

$$\kappa_{\mathcal{H}}(z,w) = I_E + \sum_{\substack{j,k \ge 1\\|j-k| \le 2}} C_{j,k} z^j \overline{w}^k \ (z,w \in \mathbb{D}_{r_{\lambda}}).$$

Moreover, for $j \ge 1$, $S_{\lambda'}^j E \subseteq \text{span}\left\{e_v : v \in \mathsf{Chi}^{\langle j \rangle}\left\{(0,0), (2,1), (3,1)\right\}\right\}$. Therefore,

$$S_{\lambda'}^{*j+1}S_{\lambda'}^{j}E \subseteq \text{span}\left\{e_{v}: v \in \mathsf{par}\left\{(0,0), (2,1), (3,1)\right\}\right\} = \text{span}\left\{e_{(1,1)}\right\},$$

which gives that $P_E S_{\lambda'}^{*j+1} S_{\lambda'}^j|_E = 0$. Thus $C_{j,k} = 0$ if |j - k| = 1. Therefore from above, $\kappa_{\mathcal{H}}(\cdot, \cdot)$ becomes

$$\kappa_{\mathcal{H}}(z,w) = I_E + \sum_{\substack{j,k \ge 1\\|j-k|=0,2}} C_{j,k} z^j \overline{w}^k \ (z,w \in \mathbb{D}_{r_{\lambda}}).$$

Since $W_n = \{(2, n - 1), (3, n - 1), (2, n), (3, n), (2, n + 1), (3, n + 1)\}$ for $n \ge 2$, the radius of convergence r_{λ} for S_{λ} is given by

$$\liminf_{n \to \infty} \left(\sum_{j=2}^{3} \left[\left(\lambda'_{(j,n-1)} \cdots \lambda'_{(1,1)} \right)^2 + \left(\lambda'_{(j,n)} \cdots \lambda'_{(j,1)} \right)^2 + \left(\lambda'_{(j,n+1)} \cdots \lambda'_{(j,2)} \right)^2 \right] \right)^{-\frac{1}{2n}}$$

In this case, the reproducing kernel $\kappa_{\mathcal{H}}(\cdot, \cdot)$ is *pentadiagonal*.

The invariant $k_{\mathscr{T}}$ may be bigger than dim E as shown below.

EXAMPLE 4.5 ((Septadiagonal)). Consider the directed tree \mathscr{T}_4 with set of vertices $V = \{(0,0), (1,1), (2,2)\} \cup \{(3,i), (4,i) : i \ge 1\}$ and root = (0,0). We further require that $Chi(0,0) = \{(1,1)\}, Chi(1,1) = \{(2,2)\}, Chi(2,2) = \{(3,1), (4,1)\}$, and

$$Chi(3, i) = \{(3, i+1)\}, Chi(4, i) = \{(4, i+1)\}, \text{ for all } i \ge 1.$$

It is easy to see that dim ker $S^*_{\lambda} = 2, k_{\mathscr{T}_4} = 3$. The kernel $\kappa_{\mathscr{H}}$ in this example is septadiagonal. We leave the details to the reader.

The main result also applies to a directed tree which is not locally finite.

EXAMPLE 4.6. Consider the directed tree \mathscr{T}_{∞} with set of vertices $V = \{(i, j) : i, j \ge 0\}$, and root = (0, 0). We further require that

$$\mathsf{Chi}(i,j) = \begin{cases} \{(1,k): k \ge 0\} & \text{if } (i,j) = \mathsf{root}, \\ \{(i+1,j)\} & \text{otherwise.} \end{cases}$$

Let S_{λ} be a bounded left-invertible weighted shift on \mathscr{T}_{∞} . Then $E = \ker S_{\lambda}^*$ is of infinite dimension. Also $V_{\prec} = \{ \text{root} \}$, and hence $k_{\mathscr{T}_{\infty}} = 1$. By Theorem 2.7, the reproducing kernel $\kappa_{\mathscr{H}}$ is tridiagonal.

REMARK 4.7. In general, S_{λ} is not unitarily equivalent to orthogonal direct sum of unilateral weighted shifts. To see this, consider the weighted shift S_{λ} on \mathscr{T}_{∞} , and suppose that S_{λ} is unitarily equivalent to direct sum $T := \bigoplus_{i=1}^{\infty} T_i$ of unilateral weighted shifts T_i . Choose weights of S_{λ} such that $\lambda_{(2,0)} \neq \lambda_{(2,1)}$. In this case,

$$\langle S_{\lambda}^2 g_1, S_{\lambda} g_2 \rangle = \lambda_{(1,0)} \lambda_{(1,1)} (\lambda_{(2,0)}^2 - \lambda_{(2,1)}^2) \neq 0,$$

where $g_1 = e_{\text{root}}$ and $g_2 = \lambda_{(1,1)}e_{(1,0)} - \lambda_{(1,0)}e_{(1,1)}$ belong to ker S^*_{λ} . However, $\langle T^m X, T^n Y \rangle = 0$ for any $X, Y \in \ker T^*$ and for any positive integers m, n such that $m \neq n$.

Directed Tree ${\mathscr T}$	Dimension of ker S^*_{λ}	$k_{\mathscr{T}}$	Form of $\kappa_{\mathscr{H}}(z,w)$
\mathscr{T}_1	1	0	diagonal
\mathscr{T}_2	2	1	tridiagonal
\mathscr{T}_3	2	2	pentadiagonal
\mathscr{T}_4	2	3	septadiagonal
\mathscr{T}_{∞}	∞	1	tridiagonal

TABLE 1.

5. Spectral Picture of S_{λ}

In this section, we use analytic model constructed in Sections 2 and 3 to discuss spectral theory of weighted shifts S_{λ} on rooted directed trees. This part has an overlap with [9, Theorems 2.1 and 2.3], where the spectral picture of certain weighted composition operators is described. However, the conclusion of (i)-(iii) of Theorem 5.1 can not be deduced from the aforementioned results of [9] as the directed trees considered in this part need not be locally finite. On the other hand, in the context of rooted directed trees, weighted shifts always have connected spectrum. This is in contrast with [9, Example 5], where a composition operator with disconnected spectrum has been constructed. Positively, the power of analytic model comes into the picture while computing the point spectra of S_{λ} and S_{λ}^* . In this regard, the rather technical proof of [9, Theorem 2.1] should be compared with that of (i) and (ii) of Theorem 5.1.

Before we state the main result of this section, we recall a couple of known facts about S_{λ} .

Any weighted shift S_{λ} on a directed tree is *circular* [20, Theorem 3.3.1]: For every $\theta \in \mathbb{R}$, there exists a unitary U_{θ} on $l^2(V)$ such that $U_{\theta}S_{\lambda} = e^{i\theta}S_{\lambda}U_{\theta}$. An immediate consequence of this shows that all spectral parts of S_{λ} have circular symmetry about 0 [20, Corollary 3.3.2].

Here is the statement of the main result of this section.

THEOREM 5.1. Let $S_{\lambda} \in B(l^2(V))$ be a left-invertible weighted shift on \mathscr{T} and let $E := \ker S_{\lambda}^*$. Then we have the following.

- (i) The point spectrum $\sigma_p(S_{\lambda})$ of S_{λ} is empty.
- (ii) If r_{λ} is the radius of convergence for S_{λ} then

$$\mathbb{D}_{r_{\lambda}} \subseteq \sigma_p(S_{\lambda}^*) \subseteq \sigma(S_{\lambda}) = \overline{\mathbb{D}}_{r(S_{\lambda})}.$$

(iii) $\bigvee \{ \ker(S_{\lambda}^* - w) : w \in \mathbb{D}_{\epsilon} \} = l^2(V) \text{ for every positive number } \epsilon.$

If, in addition, E is finite dimensional then

- (iv) $\sigma_{ap}(S_{\lambda}) = \sigma_e(S_{\lambda})$ is a union of at most dim *E* number of annuli centered at the origin.
- (v) the Fredholm index $ind(S_{\lambda} w)$ of $S_{\lambda} w$ is at least $-\dim E$ on any connected component of $\mathbb{C} \setminus \sigma_e(S_{\lambda})$. Moreover, $ind(S_{\lambda} w)$ is exactly $-\dim E$ on the connected component of $\mathbb{C} \setminus \sigma_e(S_{\lambda})$ that contains 0.
- (vi) for any positive integer k,

$$\dim\left(\ker S_{\lambda}^{*k}/\ker S_{\lambda}^{*k-1}\right) = \dim E.$$

REMARK 5.2. Since $r_{\lambda}r(S_{\lambda'}) \geq 1$ (Theorem 2.7), by the inclusion in (ii), $r(S_{\lambda})r(S_{\lambda'}) \geq 1$. This inequality is sharp. In fact, if S_{λ} is an isometry then

 $r(S_{\lambda}) = 1 = r(S_{\lambda'})$, so that equality holds in $r(S_{\lambda})r(S_{\lambda'}) \ge 1$. Also, if $r(S_{\lambda}) = 1 = r(S_{\lambda'})$ then r_{λ} is necessarily equal to 1. Finally, since S_{λ} is analytic, the part (vi) above precisely says that S_{λ}^* is an abstract backward shift in the sense of [7] and [26].

In the proof of Theorem 5.1, we need the analytic model as well as a number of general facts about S_{λ} . The first of which generalizes a well-known fact that the spectrum of a weighted shift is connected [28, Theorem 4](see also [16, Theorem 8], [26, Theorem 3.5]).

LEMMA 5.3. The spectrum of an analytic operator is connected.

PROOF. Let $T \in B(\mathcal{H})$ be analytic. We adapt the technique of [10, Lemma 3.8] to the present situation. Since T is analytic,

$$\bigvee_{k\geq 0} \ker T^{*k} = \mathcal{H}.$$
(17)

Therefore, $0 \in \sigma(T)$. Let K_1 be the connected component of $\sigma(T^*)$ containing 0 and $K_2 = \sigma(T^*) \setminus K_1$. If possible, suppose that K_2 is non-empty. Then by Riesz Decomposition Theorem [11, Chapter VII, Proposition 4.11], there are closed subspaces \mathcal{H}_1 and \mathcal{H}_2 invariant under T^* such that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, $\sigma(T^*|_{\mathcal{H}_1}) = K_1$ and $\sigma(T^*|_{\mathcal{H}_2}) = K_2$. Let $h \in \ker T^{*k}$. Then h = x + y for $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$. Since $T^{*k}h = 0$, it follows that $T^{*k}x = 0 = T^{*k}y$. If y is non-zero, then $0 \in \sigma_p(T^{*k}|_{\mathcal{H}_2}) \subseteq$ $\sigma(T^{*k}|_{\mathcal{H}_2})$, and hence by spectral mapping property, $0 \in \sigma(T^*|_{\mathcal{H}_2}) = K_2$, which is a contradiction. So y must be zero. Therefore, \mathcal{H}_1 contains ker T^{*k} for all $k \geq 0$. Hence from (17), we get $\mathcal{H}_1 = \mathcal{H}$, and hence K_2 must be empty. This is contrary to the assumption that $K_2 \neq \emptyset$. This shows that $\sigma(T^*)$ is connected. Since $\sigma(T) = \{\overline{z} : z \in \sigma(T^*)\}$ and $z \rightsquigarrow \overline{z}$ is continuous, $\sigma(T)$ is connected.

LEMMA 5.4. Let S_{λ} be a weighted shift on \mathscr{T} and let $d := card(Chi^{\langle k_{\mathscr{T}} \rangle}(root))$ (possibly infinite). Then there exist subspaces \mathcal{M} and \mathcal{H}_i $(i = 1, \dots, d)$ such that

$$S_{\lambda} = \begin{bmatrix} A & 0 & 0 & \cdots & 0\\ A_1 & S_1 & 0 & \cdots & 0\\ A_2 & 0 & S_2 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ A_d & 0 & \cdots & 0 & S_d \end{bmatrix} \quad on \ l^2(V) = \mathcal{M} \oplus \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_d, \tag{18}$$

where $A := P_{\mathcal{M}}S_{\lambda}|_{\mathcal{M}}$, $A_i := P_{\mathcal{H}_i}S_{\lambda}|_{\mathcal{M}}$, and $S_i := S_{\lambda}|_{\mathcal{H}_i}$ for $i = 1, \dots, d$. Moreover, the following statements hold.

- (i) Each S_i is unitarily equivalent to a unilateral weighted shift.
- (ii) Let *T* possess the property that v ∈ Chi^{⟨k_T-1⟩}(root) whenever card(Chi(v)) is infinite for some v ∈ W₀. Then S_λ is a finite rank perturbation of S₁ ⊕ · · · ⊕ S_d.

PROOF. Note that for all $v \in \mathsf{Chi}^{\langle k_{\mathscr{T}} \rangle}(\mathsf{root})$, $\operatorname{card}(\mathsf{Chi}(v)) = 1$. Let W_{-1} be as defined in (8). We relabel the set V of vertices as follows:

$$V = W_{-1} \sqcup \{v_{i,n} : n \ge 0, \ i = 1, \cdots, d\}$$

such that $\operatorname{Chi}^{\langle k_{\mathscr{T}} \rangle}(\operatorname{root}) = \{v_{i,0} : i = 1, \cdots, d\}$, and $\operatorname{Chi}(v_{i,n}) = \{v_{i,n+1}\}$ for all $n \ge 0, i = 1, \cdots, d$. Now consider the subspaces \mathcal{M} and \mathcal{H}_i of $l^2(V)$ given by

$$\mathcal{M} := \bigvee \{ e_v : v \in W_{-1} \}, \ \mathcal{H}_i := \bigvee \{ e_{v_{i,n}} : n \ge 0 \}, \ i = 1, \cdots, d.$$

Note that the subspaces $\mathcal{H}_1, \dots, \mathcal{H}_d$ are invariant under S_{λ} . Then S_{λ} admits the decomposition as given by (18). Since $S_i e_{v_{i,n}} = \lambda_{v_{i,n+1}} e_{v_{i,n+1}}$ $(n \ge 0)$, it is clear that each S_i is unitarily equivalent to a unilateral weighted shift.

To see (ii), note that if $\operatorname{card}(W_{-1})$ is infinite then for some $v \in W_{-1} \subseteq W_0$, we must have $\operatorname{Chi}(v) \subseteq W_{-1}$ and $\operatorname{card}(\operatorname{Chi}(v))$ is infinite. But then by hypothesis, $v \in \operatorname{Chi}^{\langle k_{\mathscr{T}}-1 \rangle}(\operatorname{root})$, which implies that $\operatorname{Chi}(v) \cap W_{-1} = \emptyset$. Thus we arrive at a contradiction. This shows that $\operatorname{card}(W_{-1})$ is finite, and hence \mathcal{M} is finite dimensional. Thus A, A_1, \dots, A_d are finite rank operators, and the conclusion in (ii) is now immediate. \Box

The following is certainly known. We include it for the sake of completeness.

LEMMA 5.5. Let $T \in B(\mathcal{H})$ be finitely cyclic. If $\sigma_p(T)$ is empty then $\sigma_{ap}(T) = \sigma_e(T)$.

PROOF. By [17, Proposition 1(i)], dim ker $(T^* - w)$ is finite for every $w \in \mathbb{C}$. If $\sigma_p(T) = \emptyset$ then it is easy to see that $\sigma_{ap}(T) = \sigma_e(T)$.

We also need exact description of the kernel of positive integral powers of S^*_{λ} in the proof of Theorem 5.1.

LEMMA 5.6. Let $S_{\lambda} \in B(l^2(V))$ be a weighted shift on $\mathscr{T} = (V, \mathcal{E})$. Then, for all integers $k \geq 1$,

$$\ker S_{\lambda}^{*k} = \bigvee \left\{ e_v : v \in \bigcup_{i=0}^{k-1} \operatorname{Chi}^{\langle i \rangle}(\operatorname{root}) \right\} \oplus \bigoplus_{v \in W_{-1}} \left(l^2 \left(\operatorname{Chi}^{\langle k \rangle}(v) \right) \ominus \langle \boldsymbol{\lambda}_k^v \rangle \right), \quad (19)$$

where λ_k^v : Chi^(k)(v) $\to \mathbb{C}$ is defined by $\lambda_k^v(u) = \lambda_u \lambda_{par(u)} \cdots \lambda_{par(k-1)}(u)$, and W_{-1} is given by (8). Consequently,

$$\dim \ker S_{\lambda}^{*k} = \sum_{i=0}^{k-1} card(\operatorname{Chi}^{\langle i \rangle}(\operatorname{root})) + \sum_{v \in W_{-1}} \left(card(\operatorname{Chi}^{\langle k \rangle}(v)) - 1 \right).$$
(20)

PROOF. Following the lines of the proof of [20, Proposition 3.5.1], one can easily deduce that for all integers $k \ge 1$,

$$\ker S_{\lambda}^{*k} = \bigvee \left\{ e_v : v \in \bigcup_{i=0}^{k-1} \operatorname{Chi}^{\langle i \rangle}(\operatorname{root}) \right\} \oplus \bigoplus_{v \in V} \left(l^2 \left(\operatorname{Chi}^{\langle k \rangle}(v) \right) \ominus \langle \lambda_k^v \rangle \right).$$

From Lemma 3.1(ii), we know that for a vertex $v \in V$, $\operatorname{card}(\operatorname{Chi}(v)) = 1$ if $n_v \geq k_{\mathscr{T}}$. This is equivalent to the fact that $\operatorname{card}(\operatorname{Chi}^{\langle m \rangle}(v)) = 1$ for all $m \geq 1$ if $v \notin W_{-1}$. Hence, $l^2(\operatorname{Chi}^{\langle k \rangle}(v)) \ominus \langle \boldsymbol{\lambda}_k^v \rangle = \{0\}$ if $v \notin W_{-1}$. This proves (19). The proof of (20) is obvious in view of (19).

PROOF OF THEOREM 5.1. In view of Theorem 2.7, it is sufficient to work with the analytic model $(\mathcal{M}_z, \kappa_{\mathscr{H}}, \mathscr{H})$ of S_{λ} , where the reproducing kernel Hilbert space \mathscr{H} consists of *E*-valued holomorphic functions U_f on the disc $\mathbb{D}_{r_{\lambda}}$ given by

$$U_f(z) := \sum_{n \ge 0} (P_E S_{\lambda'}^{*n} f) z^n.$$

We check that $\sigma_p(\mathscr{M}_z) = \emptyset$. Let $w \in \mathbb{C}$ and $h = \sum_{k=0}^{\infty} a_n z^n \in \mathscr{H}$ be such that $(\mathscr{M}_z - w)h = 0$, where $\{a_n\}_{n \ge 0} \subseteq E$. Then for any $g \in E$,

$$(z-w)\sum_{k=0}^{\infty} \langle a_n, g \rangle z^n = 0 \text{ for all } z \in \mathbb{D}_{r_{\lambda}}.$$

It follows that $\langle a_n, g \rangle = 0$ for $g \in E$, and hence $a_n = 0$ every $n \ge 0$. This shows that h = 0, which gives (i).

To see (ii), note that for $f \in l^2(V)$, $g \in E$ and $w \in \mathbb{D}_{r_{\lambda}}$, by Theorem 2.7(i),

$$\langle U_f, \mathscr{M}_z^* \kappa_{\mathscr{H}}(\cdot, w)g \rangle = \langle \mathscr{M}_z U_f, \kappa_{\mathscr{H}}(\cdot, w)g \rangle = \langle w U_f(w), g \rangle = \langle U_f, \overline{w} \kappa_{\mathscr{H}}(\cdot, w)g \rangle$$

Thus $\mathscr{M}_z^* \kappa_\mathscr{H}(\cdot, w)g = \overline{w}k_\mathscr{H}(\cdot, w)g$ for all $w \in \mathbb{D}_{r_\lambda}$ and $g \in E$. Hence the point spectrum of \mathscr{M}_z^* contains \mathbb{D}_{r_λ} . As recorded earlier, S_λ is circular, so that $\sigma(S_\lambda^*) = \sigma(S_\lambda)$. The second inclusion in (ii) now follows from $\sigma_p(S_\lambda^*) \subseteq \sigma(S_\lambda^*)$. To complete the proof of (ii), it is only left to check that $\sigma(S_\lambda) = \overline{\mathbb{D}}_{r(S_\lambda)}$. By Lemmas 3.4 and 5.3, $\sigma(S_\lambda^*)$ is connected. Now, suppose that $\sigma(S_\lambda) \neq \overline{\mathbb{D}}_{r(S_\lambda)}$. Since $r(S_\lambda) \in \sigma(S_\lambda)$, there is a $w_0 \in \mathbb{D}_{r(S_\lambda)}$ such that $w_0 \in \rho(S_\lambda) := \mathbb{C} \setminus \sigma(S_\lambda)$. Since $\rho(S_\lambda)$ is open, there is an $\epsilon > 0$ such that $\mathbb{D}_{\epsilon}(w_0) := \{w \in \mathbb{C} : |w - w_0| < \epsilon\} \subseteq \rho(S_\lambda) \cap \mathbb{D}_{r(S_\lambda)}$. Since S_λ is circular and $\sigma(S_\lambda)$ is connected, we arrive at a contradiction. Thus, the spectrum of S_λ is a disk of radius $r(S_\lambda)$ centred at the origin.

Suppose that U_f is orthogonal to $\bigvee \{ \ker(\mathscr{M}_z^* - \overline{w}) : w \in \mathbb{D}_{\epsilon} \}$. By the preceding paragraph, $\kappa_{\mathscr{H}}(\cdot, w)g$ belongs to $\ker(\mathscr{M}_z^* - \overline{w})$ for every $w \in \mathbb{D}_{r_{\lambda}}$. Hence

$$\sum_{n\geq 0} \langle P_E S_{\lambda'}^{*n} f, g \rangle w^n = \langle U_f(w), g \rangle = \langle U_f, \kappa_{\mathscr{H}}(\cdot, w)g \rangle = 0$$

for all $w \in \mathbb{D}_{\epsilon} \cap \mathbb{D}_{r_{\lambda}}$ and $g \in E$. This implies that $\langle P_E S_{\lambda'}^{*n} f, g \rangle = 0$ for all $n \geq 0$ and $g \in E$. In particular, $\langle P_E S_{\lambda'}^{*n} f, P_E S_{\lambda'}^{*n} f \rangle = 0$ for all $n \geq 0$. Thus $U_f = 0$. Therefore, $\bigvee \{ \ker(\mathscr{M}_z^* - \overline{w}) : w \in \mathbb{D}_{\epsilon} \} = \mathscr{H}$.

Assume now that E is finite dimensional. By Corollary 2.10, S_{λ} is finitely cyclic. By (i) above and Lemma 5.5, $\sigma_{ap}(S_{\lambda}) = \sigma_e(S_{\lambda})$. Thus to see (iv), it suffices to check that $\sigma_e(S_{\lambda})$ is a finite union of annuli centered at the origin. Let d :=card(Chi^{$\langle k_{\mathcal{T}} \rangle$}(root)), which is finite in view of Proposition 2.3. Also, since E is finite dimensional, by the same proposition \mathcal{T} is locally finite. Hence by Lemma 5.4(ii), there exist unilateral weighted shifts S_1, \dots, S_d such that S_{λ} is a finite rank perturbation of $S_1 \oplus \dots \oplus S_d$. In particular, the essential spectrum of S_{λ} equals the union of essential spectrum of S_1, \dots, S_d [11]. Another application of Lemma 5.5 shows that $\sigma_{ap}(S_i) = \sigma_e(S_i)$. However, the approximate point spectrum of a unilateral weighted shift is necessarily an annulus centered at the origin [27, Theorem 1]. The desired conclusion in (iv) is now immediate.

Let us now see part (v). For any $w \in \mathbb{C}$, note that

ind
$$(S_{\lambda} - w)$$
 = ind $\oplus_{i=1}^{d} (S_i - w) \stackrel{(i)}{=} - \oplus_{i=1}^{d} \dim \ker(S_i^* - \overline{w}).$

Since dim ker $(S_i^* - \overline{w})$ at most one, ind $(S_{\lambda} - w)$ is at least $-\dim E$. However, dim ker $S_i^* = 1$ for all *i*, and hence $-\dim E = \operatorname{ind} S_{\lambda} = -d$. Note that the proof above shows that card $(\operatorname{Chi}^{\langle k_{\mathscr{T}} \rangle}(\operatorname{root})) = \dim E$, and hence by Lemma 3.1(ii),

$$\operatorname{card}(\operatorname{Chi}^{\langle k \rangle}(\operatorname{root})) = \dim E \text{ for all integers } k \ge k_{\mathscr{T}}.$$
 (21)

To see (vi), fix an integer $k \ge 1$ and let $\mathscr{Q}_k := \ker S_{\lambda}^{*k} / \ker S_{\lambda}^{*k-1}$. Then using (20), we get

$$\dim \mathcal{Q}_{k} = \operatorname{card}(\operatorname{Chi}^{\langle k-1 \rangle}(\operatorname{root})) + \sum_{v \in W_{-1}} \operatorname{card}(\operatorname{Chi}^{\langle k \rangle}(v)) - \sum_{v \in W_{-1}} \operatorname{card}(\operatorname{Chi}^{\langle k-1 \rangle}(v))$$
$$= \sum_{v \in W_{-1}} \operatorname{card}(\operatorname{Chi}^{\langle k \rangle}(v)) - \sum_{v \in W_{-1} \setminus \{\operatorname{root}\}} \operatorname{card}(\operatorname{Chi}^{\langle k-1 \rangle}(v))$$
(22)

Since $\mathsf{Chi}^{\langle l \rangle}(\mathsf{root}) = \mathsf{Chi}^{\langle l-1 \rangle}(\mathsf{Chi}(\mathsf{root}))$, it follows that

$$\operatorname{card}(\operatorname{Chi}^{\langle l \rangle}(\operatorname{root})) = \sum_{v \in \operatorname{Chi}(\operatorname{root})} \operatorname{card}(\operatorname{Chi}^{\langle l-1 \rangle}(v))$$

for any positive integer l. Therefore, $\sum_{v \in W_{-1}} \operatorname{card}(\operatorname{Chi}^{\langle k \rangle}(v))$ is equal to

$$\operatorname{card}(\operatorname{Chi}^{\langle k \rangle}(\operatorname{root})) + \sum_{v \in \operatorname{Chi}(\operatorname{root})} \operatorname{card}(\operatorname{Chi}^{\langle k \rangle}(v)) + \dots + \sum_{v \in \operatorname{Chi}^{\langle k | \mathcal{T}^{-1}\rangle}(\operatorname{root})} \operatorname{card}(\operatorname{Chi}^{\langle k \rangle}(v))$$

$$= \operatorname{card}(\operatorname{Chi}^{\langle k \rangle}(\operatorname{root})) + \operatorname{card}(\operatorname{Chi}^{\langle k+1 \rangle}(\operatorname{root})) + \dots + \operatorname{card}(\operatorname{Chi}^{\langle k+k_{\mathscr{T}}-1 \rangle}(\operatorname{root}))$$

Similarly, $\sum_{v \in W_{-1} \setminus \{\text{root}\}} \operatorname{card}(\operatorname{Chi}^{\langle k-1 \rangle}(v))$ is equal to $\operatorname{card}(\operatorname{Chi}^{\langle k \rangle}(\operatorname{root})) + \operatorname{card}(\operatorname{Chi}^{\langle k+1 \rangle}(\operatorname{root})) + \cdots + \operatorname{card}(\operatorname{Chi}^{\langle k-1+k_{\mathscr{T}}-1 \rangle}(\operatorname{root})).$ Substituting last two identities in (22), we get

$$\dim \mathscr{Q}_k = \operatorname{card} (\operatorname{Chi}^{\langle k+k_{\mathscr{T}}-1 \rangle}(\operatorname{root})) \stackrel{(21)}{=} \dim E.$$

This completes the proof of the theorem.

REMARK 5.7. The identity (21), as established in the proof of Theorem 5.1(v), comes surprisingly as a consequence of index theory. As evident, this identity is otherwise difficult to disclose. Note that the left hand side of (21) is a variant dependent on \mathscr{T} while the right hand side of (21) depends solely on S_{λ} . Further, since dim E is finite, by Proposition 2.3, \mathscr{T} is locally finite and $\operatorname{card}(V_{\prec}) < \infty$. Therefore, using (4) and (21), one gets the following.

$$\operatorname{card}(\operatorname{Chi}^{\langle k_{\mathscr{T}} \rangle}(\operatorname{root})) = 1 - \operatorname{card}(V_{\prec}) + \sum_{v \in V_{\prec}} \operatorname{card}(\operatorname{Chi}(v)).$$

One particular consequence of Theorem 5.1(iv) is that $\sigma_{ap}(S_{\lambda})$ (resp. $\sigma_e(S_{\lambda})$) of a weighted shift S_{λ} on a directed tree could be *disconnected*. For instance, in case dim E = 2, by choosing the weight sequence λ appropriately (so that the approximate point spectra of S_1 and S_2 , as appearing in the proof of Theorem 5.1, are disjoint annuli), we can have two connected components of $\sigma_{ap}(S_{\lambda})$. Moreover, the index of $S_{\lambda} - w$ may vary from -2 to 0 on different components of $\mathbb{C} \setminus \sigma_e(S_{\lambda})$. Again, in the above situation,

$$\operatorname{ind} (S_{\lambda} - w) = \begin{cases} -2 \text{ on a bounded component of } \mathbb{C} \setminus \sigma_e(S_{\lambda}) \text{ containing } 0 \\ -1 \text{ on a bounded component of } \mathbb{C} \setminus \sigma_e(S_{\lambda}) \text{ not containing } 0. \end{cases}$$

This is not possible in case dim E = 1 in view of [27, Theorem 1].

The conclusion of Theorem 5.1(iv) need not be true in case dim E is infinite.

EXAMPLE 5.8. Let \mathscr{T}_{∞} be the directed tree as discussed in Example 4.6 and let S_{λ} be a left-invertible weighted shift on \mathscr{T}_{∞} . For a given $\mu > 0$, choose the weight sequence of S_{λ} such that for each $j \geq 0$, the sequence $\{\lambda_{(i+1,j)}\}_{i\geq 0}$ converges to μ . As seen in the proof of Lemma 5.4, S_{λ} is a rank one perturbation of the direct sum of unilateral weighted shifts S_j on \mathcal{H}_j . Thus $\sigma_e(S_{\lambda}) = \sigma_e(\bigoplus_{j=1}^{\infty} S_j)$. Note that $\sigma_e(S_j) = \sigma_{ap}(S_j)$ is the circle of radius μ centered at the origin [28]. Since $\sigma_p(S_j^*) = \mathbb{D}_{\mu}$, the the essential spectrum of $\bigoplus_{j=1}^{\infty} S_j$ contains \mathbb{D}_{μ} . As essential spectrum is always closed, $\overline{\mathbb{D}}_{\mu} \subseteq \sigma_e(\bigoplus_{j=1}^{\infty} S_j)$. Also,

$$\sigma_e(\bigoplus_{j=1}^{\infty} S_j) \subseteq \sigma(\bigoplus_{j=1}^{\infty} S_j) = \overline{\mathbb{D}}_{\mu}.$$

This shows that $\sigma_e(S_{\lambda}) = \overline{\mathbb{D}}_{\mu}$. On the other hand, $0 \notin \sigma_{ap}(S_{\lambda})$ since S_{λ} is left-invertible. In particular, $\sigma_{ap}(S_{\lambda}) \neq \sigma_e(S_{\lambda})$.

We see below that S_{λ} belongs to the Cowen-Douglas class (refer to [13]; refer also to [14] for the extended definition of $B_n(\Omega)$ in case n is not finite).

COROLLARY 5.9. Let $S_{\lambda} \in B(l^2(V))$ be a left-invertible weighted shift on \mathscr{T} and let $S_{\lambda'}$ denote the Cauchy dual of S_{λ} . Let $E := \ker S_{\lambda}^*$ and $\delta := \frac{1}{\|S_{\lambda'}\|}$. Then S_{λ}^* belongs to Cowen-Douglas class $B_{\dim E}(\mathbb{D}_{\delta})$.

PROOF. Since S_{λ} is left-invertible,

$$S_{\lambda}^* S_{\lambda})^{-1} = S_{\lambda'}^* S_{\lambda'} \le \|S_{\lambda'}^* S_{\lambda'}\|I.$$

That is, $S_{\lambda}^* S_{\lambda} \geq \frac{1}{\|S_{\lambda'}\|^2} I = \delta^2 I$, which gives $\|S_{\lambda}f\| \geq \delta \|f\|$ for all $f \in l^2(V)$. Therefore, $\sigma_{ap}(S_{\lambda}) \cap \mathbb{D}_{\delta} = \emptyset$. It follows that for all $w \in \mathbb{D}_{\delta}$, $\ker(S_{\lambda} - w) = \{0\}$ and $\operatorname{ran}(S_{\lambda} - w)$ is closed. Hence $\operatorname{ran}(S_{\lambda}^* - w)$ is dense in \mathscr{H} for all $w \in \mathbb{D}_{\delta}$. Since $\operatorname{ran}(S_{\lambda} - w)$ is closed, it follows that $\operatorname{ran}(S_{\lambda}^* - \overline{w})$ is closed [11, Chapter XI, Section 6], and hence $\operatorname{ran}(S_{\lambda}^* - \overline{w}) = l^2(V)$ for all $w \in \mathbb{D}_{\delta}$. In case dim $E < \infty$, the desired conclusion follows from (iii) and (v) of Theorem 5.1.

Suppose now the case in which dim E is not finite. Consider the analytic model $(\mathcal{M}_z, \kappa_{\mathscr{H}}, \mathscr{H})$ of S_{λ} . We show that

$$\{\kappa_{\mathscr{H}}(\cdot, w)g_i: i=1,\cdots, k\}$$

is linearly independent in $\ker(\mathscr{M}_z^* - \overline{w})$ whenever $\{g_i : i = 1, \dots, k\}$ is linearly independent in E for every integer $k \ge 1$ and $w \in \mathbb{D}_{r_\lambda}$. To this end, suppose that $\kappa_{\mathscr{H}}(\cdot, w)g = 0$ for some $g \in E$. Then $\langle U_g(w), g \rangle = \langle U_g, \kappa_{\mathscr{H}}(\cdot, w)g \rangle = 0$. However, by (10), $U_g = g$ for any $g \in E$. It follows that g = 0, and dim $\ker(\mathscr{M}_z^* - \overline{w}) = \dim E$ for all $w \in \mathbb{D}_{r_\lambda}$.

6. A model for weighted shifts on rootless directed trees

In this short section, we show a way to generalize the main result of this paper to the setting of rootless directed trees. One interest in the theory of weighted shifts on rootless directed trees is due to the fact that these are composition operators in disguise (see [21, Lemma 4.3.1]).

We begin with a counter-part of branching index for rootless directed trees.

DEFINITION 6.1. Let $\mathscr{T} = (V, \mathcal{E})$ be a rootless directed tree and let V_{\prec} be the set of branching vertices of \mathscr{T} . We say that \mathscr{T} has *finite branching index* if there exists a smallest non-negative integer $m_{\mathscr{T}}$ such that

$$\mathsf{Chi}^{\langle k \rangle}(V_{\prec}) \cap V_{\prec} = \emptyset$$
 for every integer $k \geq m_{\mathscr{T}}$.

The role of **root** in the notion of the branching index of a rooted directed tree is taken by a special vertex in the context of rootless directed trees with finite branching index as shown below.

LEMMA 6.2. Let $\mathscr{T} = (V, \mathcal{E})$ be a rootless directed tree with finite branching index $m_{\mathscr{T}}$. Then there exists a vertex $\omega \in V$ such that

$$card(Chi(par^{\langle k \rangle}(\omega))) = 1 \text{ for all integers } k \ge 1.$$
 (23)

Moreover, if V_{\prec} is non-empty then there exists a unique $\omega \in V_{\prec}$ satisfying (23).

PROOF. In case $V_{\prec} = \emptyset$, then every vertex of V satisfies (23). Therefore, we may assume that V_{\prec} contains at least one vertex, say, u_0 .

On contrary, assume that for every $u \in V_{\prec}$ there exists a positive integer k_u (depending on u) such that

$$\operatorname{card}(\operatorname{Chi}(\operatorname{par}^{\langle k_u \rangle}(u))) = 0 \text{ or } \operatorname{card}(\operatorname{Chi}(\operatorname{par}^{\langle k_u \rangle}(u))) \geq 2.$$

Since \mathscr{T} is rootless, the first case can not occur. Hence $\operatorname{card}(\operatorname{Chi}(\operatorname{par}^{\langle k_u \rangle}(u))) \geq 2$, that is, $\operatorname{par}^{\langle k_u \rangle}(u) \in V_{\prec}$. Define inductively $\{u_n\}_{n \geq 0} \subseteq V_{\prec}$ as follows. By assumption, there exists an integer $k_{u_0} \geq 1$ such that $u_1 := \operatorname{par}^{\langle k_{u_0} \rangle}(u_0) \in V_{\prec}$. By finite induction, there exist integers $k_{u_1}, \cdots, k_{u_{n-1}} \geq 1$ such that

$$u_n := \mathsf{par}^{\langle k_{u_0} + k_{u_1} \cdots + k_{u_{n-1}} \rangle}(u_0) \in V_{\prec}$$

In case $n > m_{\mathscr{T}}, u_0 \in \mathsf{Chi}^{\langle k_{u_0} + k_{u_1} \cdots + k_{u_{n-1}} \rangle}(V_{\prec}) \cap V_{\prec}$. This is not possible since $\mathsf{Chi}^{\langle k \rangle}(V_{\prec}) \cap V_{\prec} = \emptyset$ for all integers $k \ge m_{\mathscr{T}}$.

To see the uniqueness part, suppose that there exist distinct vertices $\{\omega_i\}_{i=1}^N$ in V_{\prec} satisfying (23), where either N is a positive integer bigger than 1 or N is infinite. It is easy to see with the help of (23) that for integers $i \neq j$, $\mathsf{par}^{\langle k_1 \rangle}(\omega_i) \neq \mathsf{par}^{\langle k_2 \rangle}(\omega_j)$ for any non-negative integers k_1 and k_2 . One may now easily verify that \mathscr{T} has the separation $\mathscr{T} = \bigsqcup_{i=1}^N \mathscr{T}_i$, where

$$\mathscr{T}_i = \big(\cup_{k\geq 1} \operatorname{Chi}^{\langle k \rangle}(\omega_i)\big) \cup \big\{\operatorname{par}^{\langle k \rangle}(\omega_i) : k \geq 0\big\}.$$

Since \mathscr{T} is connected, we arrive at a contradiction.

Note that a rootless directed tree \mathscr{T}_0 with empty V_{\prec} is isomorphic to the directed tree with set of vertices \mathbb{Z} and $\operatorname{Chi}(n) = \{n+1\}$ for $n \in \mathbb{Z}$. As it is well-known that any weighted shift on \mathscr{T}_0 (to be referred to as *bilateral weighted shift*) can be modelled as the operator of multiplication by z on a Hilbert space of formal Laurent series [28, Proposition 7], we assume in the remaining part of this section that V_{\prec} is non-empty.

We refer to the vertex $\omega \in V_{\prec}$ appearing in the statement of Lemma 6.2 as the generalized root of \mathscr{T} . The generalized root may not exist in general. For example, consider the directed tree \mathscr{T} with set of vertices $V = \mathbb{Z} \times \mathbb{Z}$ such that

$$\mathsf{Chi}(i,j) = \begin{cases} \{(i,j+1)\} & \text{if } j \neq 0, \\ \{(i,j+1), (i+1,j)\} & \text{if } j = 0 \end{cases}$$

(cf. [19, Example 4.4]). In this case, $V_{\prec} = \{(i,0) : i \in \mathbb{Z}\}$, and hence the set $\mathsf{Chi}(\mathsf{par}^{\langle k \rangle}((i,0)))$ contains precisely two vertices for any integer $k \geq 1$.

With the notion of generalized root, we immediately obtain the following.

LEMMA 6.3. Let $\mathscr{T} = (V, \mathscr{E})$ be a rootless directed tree with finite branching index $m_{\mathscr{T}}$ and generalized root ω . Let $S_{\lambda} \in B(l^2(V))$ be a weighted shift on \mathscr{T} . Let $V^{(2)} := \{v_k := \mathsf{par}^{\langle k \rangle}(\omega) : k \geq 1\}$ and let $V^{(1)} := V \setminus V^{(2)}$. Let $\mathscr{T}^{(1)}$ and $\mathscr{T}^{(2)}$ be the directed subtrees corresponding to the sets of vertices $V^{(1)}$ and $V^{(2)}$ respectively. Then S_{λ} admits the following decomposition:

$$S_{\lambda} = \begin{bmatrix} T_{\lambda} & \lambda_{\omega} e_{\omega} \otimes e_{v_1} \\ 0 & B_{\lambda} \end{bmatrix} \text{ on } l^2(V) = l^2(V^{(1)}) \oplus l^2(V^{(2)}), \tag{24}$$

where $T_{\lambda} \in B(l^2(V^{(1)}))$ is a weighted shift on the rooted directed tree $\mathscr{T}^{(1)}$ with root ω and finite branching index $k_{\mathscr{T}^{(1)}} = m_{\mathscr{T}}$, and $B_{\lambda} \in B(l^2(V^{(2)}))$ is the backward

$$\square$$

unilateral weighted shift given by

$$B_{\lambda}e_{v_k} = \begin{cases} 0 & \text{if } k = 1\\ \lambda_{v_{k-1}}e_{v_{k-1}} & \text{if } k \ge 2. \end{cases}$$

PROOF. Note that $\mathscr{T}^{(1)}$ is a rooted directed tree with root ω . Also, the set V_{\prec} of branching vertices of \mathscr{T} is contained in $V^{(1)}$ as $V^{(2)} \cap V_{\prec} = \emptyset$. It follows that $k_{\mathscr{T}^{(1)}} = 1 + \sup\{n_v : v \in V_{\prec}\}$. Since $m_{\mathscr{T}}$ is the smallest integer such that $\operatorname{Chi}^{\langle m_{\mathscr{T}} \rangle}(V_{\prec}) \cap V_{\prec} = \emptyset$, we must have $\sup\{n_v : v \in V_{\prec}\} = m_{\mathscr{T}} - 1$. This shows that $\mathscr{T}^{(1)}$ has branching index precisely $m_{\mathscr{T}}$.

Since $S_{\lambda}^* e_{v_k} = \lambda_{v_k} e_{v_{k+1}}$, $l^2(V^{(2)})$ is invariant under S_{λ}^* . This gives us the decomposition

$$S_{\lambda} = \begin{bmatrix} S_{\lambda}|_{l^{2}(V^{(1)})} & P_{1}S_{\lambda}|_{l^{2}(V^{(2)})} \\ 0 & P_{2}S_{\lambda}|_{l^{2}(V^{(2)})} \end{bmatrix} \text{ on } l^{2}(V) = l^{2}(V^{(1)}) \oplus l^{2}(V^{(2)}),$$

where P_i denotes the orthogonal projection of $l^2(V)$ onto $l^2(V^{(i)})$ for i = 1, 2. It is easy to see that $P_1S_{\lambda}|_{l^2(V^{(2)})}$ is the rank one operator $\lambda_{\omega}e_{\omega} \otimes e_{v_1}$. That $P_2S_{\lambda}|_{l^2(V^{(2)})} = B_{\lambda}$ is also a routine verification.

REMARK 6.4. Note that every weighted shift on a rootless directed tree with finite branching index is an extension of a weighted shift on a rooted direct tree with finite branching index.

We illustrate the result above with the help of the following simple example.

EXAMPLE 6.5. Consider the directed tree \mathscr{T} with set of vertices

$$V := \{(1, i), (2, i) : i \ge 1\} \cup \{-k : k \ge 0\}.$$

We further require that Chi(-k) = -(k-1) if $k \ge 1$, $Chi(0) = \{(1,1), (2,1)\}$ and

$$Chi(1, i) = \{(1, i+1)\}, Chi(2, i) = \{(2, i+1)\}, \text{ for all } i \ge 1.$$

In this case, the branching index $m_{\mathscr{T}} = 1$ and the generalized root ω is 0. Also, $V^{(1)} = \{(1,i), (2,i) : i \geq 1\} \cup \{0\}$ and $V^{(2)} = \{-k : k \geq 1\}$. The weighted shift T_{λ} on $\mathscr{T}^{(1)}$ as defined in the last lemma can be identified with the weighted shift on the directed tree \mathscr{T}_2 (with root 0) as discussed in Example 4.1. Further, the rank one operator $\lambda_{\omega} e_{\omega} \otimes e_{v_1}$ is precisely $\lambda_0 e_0 \otimes e_{-1}$. Finally, the backward unilateral weighted shift B_{λ} can be identified with the adjoint of the weighted shift on the directed tree \mathscr{T}_1 (with root -1) as discussed in Example 2.6.

We now present a counter-part of Theorem 2.7 for rootless directed trees.

THEOREM 6.6. Let $\mathscr{T} = (V, \mathscr{E})$ be a rootless directed tree with finite branching index and generalized root ω . Let $S_{\lambda} \in B(l^2(V))$ be a left-invertible weighted shift on \mathscr{T} . Then there exist a Hilbert space \mathscr{H} of vector-valued Holomorphic functions in z defined on a disc in \mathbb{C} , and a Hilbert space \mathcal{H} of scalar-valued holomorphic functions in t defined on a disc in \mathbb{C} such that S_{λ} is unitarily equivalent to

$$\begin{bmatrix} \mathscr{M}_z & f \otimes g \\ 0 & M_t^* \end{bmatrix} on \mathscr{H} \oplus \mathcal{H},$$

where \mathcal{M}_z is the operator of multiplication by z on \mathcal{H} , $f \otimes g$ is a rank one operator with $f \in \ker \mathcal{M}_z^* \setminus \{0\}$, $g \in \ker \mathcal{M}_t^* \setminus \{0\}$, and M_t is the operator of multiplication by the co-ordinate function t on \mathcal{H} . PROOF. By Lemma 6.3, S_{λ} admits the decomposition (24). Since S_{λ} is left-invertible, so are T_{λ} and B_{λ}^* . The desired decomposition now follows immediately from Theorem 2.7.

REMARK 6.7. A routine calculation shows that the self-commutator $[S_{\lambda}^*, S_{\lambda}] := S_{\lambda}^* S_{\lambda} - S_{\lambda} S_{\lambda}^*$ of S_{λ} (upto unitary equivalence) is equal to

$$\left[\begin{array}{cc} [\mathscr{M}_z^*,\mathscr{M}_z] - f \otimes f & 0\\ 0 & g \otimes g - [M_t^*, M_t] \end{array}\right].$$

In particular, $[S_{\lambda}^*, S_{\lambda}]$ is compact if and only if so are $[\mathscr{M}_z^*, \mathscr{M}_z]$ and $[M_t^*, M_t]$.

We conclude this paper with one application to the spectral theory of weighted shifts on rootless directed trees (cf. [9, Theorem 2.3]).

COROLLARY 6.8. With the hypotheses and notations of Theorem 6.6, we have

$$\sigma_e(S_{\lambda}) = \sigma_e(\mathscr{M}_z) \cup \sigma_e(M_t^*).$$

If, in addition, S_{λ} is Fredholm then so are \mathcal{M}_z and M_t . In this case,

$$ind S_{\lambda} = ind \mathscr{M}_z + 1.$$

Acknowledgment. We express our sincere thanks to Jan Stochel and Zenon Jan Jabłoński for many helpful suggestions. In particular, we acknowledge drawing our attention to the work [9] on the spectral theory of composition operators. Further, the first author is thankful to the faculty and the administrative unit of School of Mathematics, Harish-Chandra Research Institute, Allahabad for their warm hospitality during the preparation of this paper.

References

- G. Adams, N. Feldman, and P. Macguire, Tridiagonal reproducing kernels and subnormality, J. Operator Theory, 70 (2013), 477-494.
- [2] G. Adams and P. McGuire, Analytic tridiagonal reproducing kernels, J. London Math. Soc. 64 (2001), 722-738.

[3] G. Adams and P. McGuire, A class of tridiagonal reproducing kernels, *Oper. Matrices* 2 (2008), 233-247.

[4] G. Adams, P. McGuire, N. Salinas, and A. Schweinsberg, Analytic finite band width reproducing kernels and operator weighted shifts, J. Operator Theory 51 (2004), 35-48.

[5] A. Anand, S. Chavan, Z. Jabłoński, and J. Stochel, Subnormality of the Cauchy Dual of a 2-isometry, in preparation.

[6] N. Aronszajn, Theory of reproducing kernels, Trans. Amer. Math. Soc. 68 (1950), 337-404.

[7] B. Barnes, The commutant of an abstract backward shift, *Canadian Math. Bull.* 43 (2000), 21-24.

[8] B. Barnes, General shifts and backward shifts, Acta Sci. Math. (Szeged) 70 (2004), 703-713.

[9] J. Carlson, The spectra and commutants of some weighted composition operators, *Trans. Amer. Math. Soc.* 317 (1990), 631-654.

[10] S. Chavan and D. Yakubovich, Spherical tuples of Hilbert space operators, Indiana Univ. Math. J. 64 (2015), 577-612.

[11] J. Conway, A course in functional analysis, Springer Verlag, New York, 1985.

[12] J. Conway, The Theory of Subnormal Operators, Math. Surveys Monographs, 36, Amer. Math. Soc. Providence, RI 1991.

[13] M. Cowen and R. Douglas, Complex geometry and operator theory, Acta Math. 141 (1978), 187-261.

[14] R. Curto and N. Salinas, Generalized Bergman kernels and the Cowen-Douglas theory, Amer. J. Math. 106 (1984), 447-488.

[15] M. Fujii, H. Sasaoka, and Y. Watatani, Adjacency operators of infinite directed graphs, *Math. Japon.* 34 (1989), 727-735.

26

[16] R. Gellar, Operators commuting with a weighted shift. Proc. Amer. Math. Soc. 23 (1969), 538-545.

[17] D. Herrero, On multicyclic operators, Integral Equations Operator Theory 1 (1978), 57-102.

[18] Z. Jabłoński, Hyperexpansive operator-valued unilateral weighted shifts, *Glasg. Math. J.* 46 (2004), 405-416.

[19] Z. Jabłoński, Hyperexpansive composition operators, *Math. Proc. Camb. Phil. Soc.* 135 (2003), 513-526.

[20] Z. Jabłoński, Il Bong Jung, and J. Stochel, Weighted shifts on directed trees, Mem. Amer. Math. Soc. 216 (2012), no. 1017, viii+106.

[21] Z. Jabłoński, Il Bong Jung, and J. Stochel, A non-hyponormal operator generating Stieltjes moment sequences, J. Funct. Anal. 262 (2012), 3946-3980.

[22] Z. Jabłoński, Il Bong Jung, and J. Stochel, A hyponormal weighted shift on a directed tree whose square has trivial domain, *Proc. Amer. Math. Soc.* 142 (2014), 3109-3116.

[23] G. Keough, Roots of invertibly weighted shifts with finite defect, *Proc. Amer. Math. Soc.* 91 (1984), 399-404.

[24] A. Korányi and G. Misra, Homogeneous operators on Hilbert spaces of holomorphic functions, J. Funct. Anal. 254 (2008), 2419-2436.

[25] A. Lambert, Unitary equivalence and reducibility of invertibly weighted shifts, Bull. Austral. Math. Soc. 5 (1971), 157-173.

[26] M. Raney, Abstract backward shifts of finite multiplicity, *Acta Sci. Math. (Szeged)* 70 (2004), 339-359.

[27] W. Ridge, Approximate point spectrum of a weighted shift, *Trans. Amer. Math. Soc.* 147, (1970), 349-356.

[28] A. Shields, Weighted shift operators and analytic function theory, in Topics in Operator Theory, Math. Surveys Monographs, vol. 13, Amer. math. Soc., Providence, RI 1974, 49-128.

[29] S. Shimorin, Wold-type decompositions and wandering subspaces for operators close to isometries, J. Reine Angew. Math. 531 (2001), 147-189.

[30] D. Sievewright, Spectral radius algebras and weighted shifts of finite multiplicity, J. Math. Anal. Appl. 429 (2015), 658-675.

[31] V. Singh, Reproducing kernels and operators with a cyclic vector. I, *Pacific J. Math.* 52 (1974), 567-584.

Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, India

E-mail address: chavan@iitk.ac.in

School of Mathematics, Harish-Chandra Research Institute, Chhatnag Road, Jhu-NSI, Allahabad 211019, India

E-mail address: shaileshtrivedi@hri.res.in