# An Analytic Model for Left-Invertible Weighted Shifts on Directed Trees 

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#### Abstract

Let $\mathscr{T}$ be a rooted directed tree with finite branching index $k_{\mathscr{T}}$ and let $S_{\lambda} \in B\left(l^{2}(V)\right)$ be a left-invertible weighted shift on $\mathscr{T}$. We show that $S_{\lambda}$ can be modelled as a multiplication operator $\mathscr{M}_{z}$ on a reproducing kernel Hilbert space $\mathscr{H}$ of $E$-valued holomorphic functions on a disc centered at the origin, where $E:=\operatorname{ker} S_{\lambda}^{*}$. The reproducing kernel associated with $\mathscr{H}$ is multi-diagonal and of bandwidth $k_{\mathscr{T}}$. Moreover, $\mathscr{H}$ admits an orthonormal basis consisting of polynomials in $z$ with at most $k_{\mathscr{T}}+1$ non-zero coefficients. As one of the applications of this model, we give a complete spectral picture of $S_{\lambda}$. Unlike the case $\operatorname{dim} E=1$, the approximate point spectrum of $S_{\lambda}$ could be disconnected. We also obtain an analytic model for left-invertible weighted shifts on rootless directed trees with finite branching index.


## 1. Preliminaries

The implementation of methods of graph theory into operator theory gives rise to a new class of operators known as weighted shifts on directed trees. These operators are generalization of adjacency operators of the directed trees. Although, the study of adjacency operators of the directed graphs was initiated by Fujii, Sasaoka and Watatani in [15], it was first observed by Jabłoński, Jung and Stochel in $[\mathbf{2 0}]$ that replacing the directed graphs by directed trees not just gives a successful theory of weighted shifts but also provides a rich source of examples and counterexamples in operator theory [21], [22]. Several questions related to boundedness, adjoints, normality, subnormality, hyponormality etc. of weighted shifts on directed trees have been studied in depth in [20].

In the present paper, we discuss a rich interplay between the discrete structures (directed trees) and analytic structures (analytic kernels of finite bandwidth). The starting point of this text is the observation that any left-invertible weighted shift on a rooted directed tree can be realized as the operator of multiplication by the co-ordinate function on a reproducing kernel Hilbert space $\mathscr{H}$ of vector-valued holomorphic functions defined on a disc in the complex plane. In case the directed tree has finite branching index, this analytic model takes a concrete form. In particular, the reproducing kernel associated with $\mathscr{H}$ turns out to be multi-diagonal.

[^0]Also, the space $\mathscr{H}$ may not be obtained by tensoring a Hilbert space of scalarvalued holomorphic functions with another Hilbert space. In this course, we arrive at a couple of interesting invariants, namely, branching index of a directed tree and radius of convergence for the weighted shift. Importantly, these invariants can be computed explicitly in various situations.

Let $\mathbb{Z}_{+}, \mathbb{Z}, \mathbb{R}$ and $\mathbb{C}$ stand for the sets of non-negative integers, integers, real numbers and complex numbers, respectively. The complex conjugate of a complex number $w$ will be denoted by $\bar{w}$. We use $\mathbb{D}_{r}$ to denote the open disc $\{z \in \mathbb{C}:|z|<r\}$ of radius $r>0$. In case $r=1$, we denote the unit disc $\mathbb{D}_{1}$ by a simpler notation $\mathbb{D}$. For a subset $A$ of a non-empty set $X, \operatorname{card}(A)$ denotes the cardinality of $A$.

Let $\mathcal{H}$ be a complex separable Hilbert space. The inner-product on $\mathcal{H}$ will be denoted by $\langle\cdot, \cdot\rangle_{\mathcal{H}}$. If no confusion is likely then we suppress the suffix, and simply write the inner-product as $\langle\cdot, \cdot\rangle$. By a subspace, we mean a closed linear manifold. Let $W$ be a subset of $\mathcal{H}$. Then span $W$ stands for the smallest linear manifold generated by $W$. In case $W$ is singleton $\{w\}$, we use the convenient notation $\langle w\rangle$ in place of $\operatorname{span}\{w\}$. By $\bigvee\{w: w \in W\}$, we understand the subspace generated by $W$. For a subspace $\mathcal{M}$ of $\mathcal{H}$, we use $P_{\mathcal{M}}$ to denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$. For vectors $x, y \in \mathcal{H}$, we use the notation $x \otimes y$ to denote the rank one operator given by

$$
x \otimes y(h)=\langle h, y\rangle x, h \in \mathcal{H}
$$

Unless stated otherwise, all the Hilbert spaces occurring below are complex infinite-dimensional separable and for any such Hilbert space $\mathcal{H}, B(\mathcal{H})$ denotes the Banach algebra of bounded linear operators on $\mathcal{H}$. For $T \in B(\mathcal{H})$, the symbols ker $T$ and $\operatorname{ran} T$ will stand for the kernel and the range of $T$ respectively. The Hilbert space adjoint of $T$ will be denoted by $T^{*}$. In what follows, we denote the spectrum, approximate point spectrum, essential spectrum and the point spectrum of $T$ by $\sigma(T), \sigma_{a p}(T), \sigma_{e}(T)$ and $\sigma_{p}(T)$ respectively. We reserve the notation $r(T)$ for the spectral radius of $T$.

Let $T \in B(\mathcal{H})$. We say that $T$ is left-invertible if there exists $S \in B(\mathcal{H})$ such that $S T=I$. Note that $T$ is left-invertible if and only if there exists a constant $\alpha>0$ such that $T^{*} T \geq \alpha I$. In this case, $T^{*} T$ is invertible and $T$ admits the left-inverse $\left(T^{*} T\right)^{-1} T^{*}$. Following [29], we refer to the operator $T^{\prime}$ given by $T^{\prime}:=T\left(T^{*} T\right)^{-1}$ as the Cauchy dual of the left-invertible operator $T$. Further, we say that $T$ is analytic if $\bigcap_{n \geq 0} T^{n}(\mathcal{H})=\{0\}$. If $\mathscr{H}$ is a reproducing kernel Hilbert space of holomorphic functions defined on a disc in $\mathbb{C}$, then the multiplication operator $\mathscr{M}_{z}$ defined on $\mathscr{H}$ provides an example of an analytic operator. It is interesting to note that almost all analytic operators arise in this way. Indeed, a result of S. Shimorin [29] asserts that any left-invertible analytic operator is unitarily equivalent to the operator of multiplication by $z$ on a reproducing kernel Hilbert space of vectorvalued holomorphic functions defined on a disc. Since the proof of this fact, as given in $[\mathbf{2 9}$, Sections 1 and 2], plays a major role in the proof of the main result, we outline it in the following discussion (cf. [31, Theorem 2.13]).

Let $T \in B(\mathcal{H})$ be a left-invertible analytic operator and let $E:=\operatorname{ker} T^{*}$. For each $x \in \mathcal{H}$, define an $E$-valued holomorphic function $U_{x}$ as

$$
U_{x}(z)=\sum_{n \geq 0}\left(P_{E} T^{* n} x\right) z^{n}
$$

where $T^{\prime}$ is the Cauchy dual of $T$. A simple application of the spectral radius formula [11] shows that the function $U_{x}(z)=P_{E}\left(I-z T^{\prime *}\right)^{-1} x$ is well-defined and holomorphic on the disc $\mathbb{D}_{r}$, where $r:=\frac{1}{r\left(T^{\prime}\right)}$. Let $\mathscr{H}$ denote the vector space of $E$ valued holomorphic functions of the form $U_{x}, x \in \mathcal{H}$. Consider the map $U: \mathcal{H} \rightarrow \mathscr{H}$ defined by $U x=U_{x}$. By [29, Lemma 2.2], the kernel of $U$ is precisely $\bigcap_{n \geq 0} T^{n}(\mathcal{H})$, and hence by the assumption, $U$ is injective. In particular, we may equip the space $\mathscr{H}$ with the norm induced from $\mathcal{H}$, so that $U$ is unitary. It turns out that $\mathscr{H}$ is a $z$-invariant reproducing kernel Hilbert space with $U T=\mathscr{M}_{z} U$, where $\mathscr{M}_{z}$ is the operator of multiplication by $z$. Also, the reproducing kernel $\kappa_{\mathscr{H}}: \mathbb{D}_{r} \times \mathbb{D}_{r} \rightarrow B(E)$ is given by

$$
\begin{equation*}
\kappa_{\mathscr{H}}(z, w)=\left.\sum_{j, k \geq 0} P_{E} T^{* * j} T^{\prime k}\right|_{E} z^{j} \bar{w}^{k} \tag{1}
\end{equation*}
$$

which satisfies the following:
(i) for any $x \in E$ and $\lambda \in \mathbb{D}_{r}$,

$$
\kappa_{\mathscr{H}}(\cdot, \lambda) x \in \mathscr{H}
$$

(ii) for any $x \in E, h \in \mathscr{H}$ and $\lambda \in \mathbb{D}_{r}$,

$$
\langle h(\lambda), x\rangle_{E}=\left\langle h, \kappa_{\mathscr{H}}(\cdot, \lambda) x\right\rangle_{\mathscr{H}}
$$

Conditions (i) and (ii) may be rephrased by saying that the set of bounded point evaluations (for short, bpe) for $\mathscr{H}$ contains the disc $\mathbb{D}_{r}$. We see in the context of weighted shifts on rooted directed trees that indeed (analytic) bpe contains the disc $\mathbb{D}_{r_{\lambda}}$ of larger radius $r_{\lambda}$ (see Definition 2.5). This occupies the major part of the proof of the main result.

In the remaining part of this section, we invoke some basic concepts from the theory of directed trees which will be frequently used in the rest of this paper. The reader is referred to $[\mathbf{2 0}]$ for a detailed exposition on directed trees.

A pair $\mathscr{T}=(V, \mathcal{E})$ is called a directed graph if $V$ is a non-empty set and $\mathcal{E}$ is a subset of $V \times V \backslash\{(v, v): v \in V\}$. An element of $V$ (resp. $\mathcal{E}$ ) is called a vertex (resp. an edge) of $\mathscr{T}$. A finite sequence $\left\{v_{i}\right\}_{i=1}^{n}$ of distinct vertices is said to be a circuit of $\mathscr{T}$ if $n \geq 2,\left(v_{i}, v_{i+1}\right) \in \mathcal{E}$ for all $1 \leq i \leq n-1$ and $\left(v_{n}, v_{1}\right) \in \mathcal{E}$. A directed graph $\mathscr{T}$ is said to be connected if for any two distinct vertices $u$ and $v$ of $\mathscr{T}$, there exists a finite sequence $\left\{v_{i}\right\}_{i=1}^{n}$ of vertices of $\mathscr{T}(n \geq 2)$ such that $u=v_{1}, v_{n}=v$ and $\left(v_{i}, v_{i+1}\right)$ or $\left(v_{i+1}, v_{i}\right) \in \mathcal{E}$ for all $1 \leq i \leq n-1$. For a subset $W$ of $V$, define $\operatorname{Chi}(W)=\bigcup_{u \in W}\{v \in V:(u, v) \in \mathcal{E}\}$. One may define inductively $\mathrm{Chi}^{\langle n\rangle}(W)$ for $n \in \mathbb{Z}_{+}$as follows: Set $\mathrm{Chi}^{\langle n\rangle}(W)=W$ if $n=0$, and $\mathrm{Chi}^{\langle n\rangle}(W)=\operatorname{Chi}\left(\mathrm{Chi}^{\langle n-1\rangle}(W)\right)$ if $n \geqslant 1$. Given $v \in V$, we write $\operatorname{Chi}(v):=\operatorname{Chi}(\{v\}), \operatorname{Chi}^{\langle n\rangle}(v)=\operatorname{Chi}^{i n\rangle}(\{v\})$. A member of $\operatorname{Chi}(v)$ is called a child of $v$. For a given vertex $v \in V$, if there exists a unique vertex $u \in V$ such that $(u, v) \in \mathcal{E}$, we say that $v$ has a parent $u$ and denote it by $\operatorname{par}(v)$. A vertex $v$ of $\mathscr{T}$ is called a root of $\mathscr{T}$, or $v \in \operatorname{Root}(\mathscr{T})$, if there is no vertex $u$ of $\mathscr{T}$ such that $(u, v)$ is an edge of $\mathscr{T}$. If $\operatorname{Root}(\mathscr{T})$ is a singleton then its unique element is denoted by root. We set $V^{\circ}:=V \backslash \operatorname{Root}(\mathscr{T})$.

A directed graph $\mathscr{T}=(V, \mathcal{E})$ is called a directed tree if
(i) $\mathscr{T}$ has no circuits,
(ii) $\mathscr{T}$ is connected and
(iii) each vertex $v \in V^{\circ}$ has a parent.

Remark 1.1. Any directed tree has at most one root [20, Proposition 2.1.1].

A directed tree $\mathscr{T}$ is said to be
(i) rooted if it has a (unique) root.
(ii) rootless if it has no root.
(iii) locally finite if $\operatorname{card}(\operatorname{Chi}(u))$ is finite for all $u \in V$.
(iv) leafless if every vertex has at least one child.

In what follows, $l^{2}(V)$ stands for the Hilbert space of square summable complex functions on $V$ equipped with the standard inner product. Note that the set $\left\{e_{u}\right\}_{u \in V}$ is an orthonormal basis of $l^{2}(V)$, where $e_{u} \in l^{2}(V)$ is the indicator function $\chi_{\{u\}}$ of $\{u\}$. Given a system $\boldsymbol{\lambda}=\left\{\lambda_{v}\right\}_{v \in V^{\circ}}$ of non-negative real numbers, we define the weighted shift operator $S_{\lambda}$ on $\mathscr{T}$ with weights $\boldsymbol{\lambda}$ by

$$
\begin{aligned}
\mathscr{D}\left(S_{\lambda}\right) & :=\left\{f \in l^{2}(V): \Lambda_{\mathscr{T}} f \in l^{2}(V)\right\}, \\
S_{\lambda} f & :=\Lambda_{\mathscr{T}} f, \quad f \in \mathscr{D}\left(S_{\lambda}\right),
\end{aligned}
$$

where $\Lambda_{\mathscr{T}}$ is the mapping defined on complex functions $f$ on $V$ by

$$
\left(\Lambda_{\mathscr{T}} f\right)(v):= \begin{cases}\lambda_{v} \cdot f(\operatorname{par}(v)) & \text { if } v \in V^{\circ} \\ 0 & \text { if } v \text { is a root of } \mathscr{T}\end{cases}
$$

Unless stated otherwise, $\left\{\lambda_{v}\right\}_{v \in V^{\circ}}$ consists of positive numbers and $S_{\lambda}$ belongs to $B\left(l^{2}(V)\right)$. It may be concluded from [20, Proposition 3.1.7] that $S_{\lambda}$ is an injective weighted shift on $\mathscr{T}$ if and only if $\mathscr{T}$ is leafless. In what follows, we always assume that all the directed trees considered in this text are countably infinite and leafless.

In the proof of the main result, we frequently use the following elementary facts pertaining to the weighted shifts on directed trees.

Lemma 1.2. If $S_{\lambda} \in B\left(l^{2}(V)\right)$, then for any $u \in V$ and positive integer $k$,
(i) $S_{\lambda}^{k} e_{u}=\sum_{v \in \operatorname{Chi}^{\langle k\rangle}(u)} \lambda_{v} \lambda_{\operatorname{par}(v)} \cdots \lambda_{\operatorname{par}^{\langle k-1\rangle}(v)} e_{v}$ and

$$
\left\|S_{\lambda}^{k} e_{u}\right\|^{2}=\sum_{v \in \operatorname{Chi}^{\langle k\rangle}(u)}\left(\lambda_{v} \lambda_{\operatorname{par}(v)} \cdots \lambda_{\operatorname{par}^{\langle k-1\rangle}(v)}\right)^{2}
$$

(ii) $S_{\lambda}^{* k} e_{u}=\lambda_{u} \lambda_{\operatorname{par}(u)} \cdots \lambda_{\operatorname{par}\langle k-1\rangle(u)} e_{\operatorname{par}{ }^{\langle k\rangle}(u)}$ and
$\left\|S_{\lambda}^{* k} e_{u}\right\|^{2}=\left(\lambda_{u} \lambda_{\operatorname{par}(u)} \cdots \lambda_{\operatorname{par}\langle k-1\rangle(u)}\right)^{2}$, where $e_{\operatorname{par}\langle n\rangle(v)}$ is understood to be the zero vector in case $\operatorname{par}^{\langle n\rangle}(v)=\emptyset$.
(iii) $S_{\lambda}^{* k} S_{\lambda}^{k} e_{u}=\left\|S_{\lambda}^{k} e_{u}\right\|^{2} e_{u}$.

Proof. The part (i) has been established in [20, Lemma 6.1.1], whereas (ii) and (iii) can be obtained by a straightforward mathematical induction using [20, Lemma 3.4.1(iii)].

Let $S_{\lambda}$ be a left-invertible weighted shift on a rooted directed tree with weights $\left\{\lambda_{v}\right\}_{v \in V^{\circ}}$. It can be easily seen from (i) and (iii) above that the Cauchy dual $S_{\lambda}^{\prime}$ of $S_{\lambda}$ is given by

$$
S_{\lambda}^{\prime} e_{u}:=\sum_{v \in \operatorname{Chi}(u)} \frac{\lambda_{v}}{\left\|S_{\lambda} e_{\operatorname{par}(v)}\right\|^{2}} e_{v} \text { for all } v \in V^{\circ}
$$

Note that $S_{\lambda}^{\prime} \in B\left(l^{2}(V)\right)$ is a weighted shift with weights $\left\{\lambda_{v}^{\prime}\right\}_{v \in V^{\circ}}$, where

$$
\begin{equation*}
\lambda_{v}^{\prime}:=\frac{\lambda_{v}}{\left\|S_{\lambda} e_{\operatorname{par}(v)}\right\|^{2}} \text { for all } v \in V^{\circ} \tag{2}
\end{equation*}
$$

This also shows that $\left\{\lambda_{v}^{\prime}\right\}_{v \in V^{\circ}}$ is a bounded subset of positive real line. Throughout this text, we find it convenient to use the notation $S_{\lambda^{\prime}}$ in place of $S_{\lambda}^{\prime}$.

It turns out that any weighted shift $S_{\lambda}$ on a rooted directed tree $\mathscr{T}$ is analytic (see Lemma 3.4 below). Hence by Shimorin's construction as described above, any left-invertible $S_{\lambda}$ admits an analytic model $\left(\mathscr{M}_{z}, \kappa_{\mathscr{H}}, \mathscr{H}\right)$. It turns out that this model can be significantly improved upon provided the underlying directed tree has finite branching index (see Definition 2.1). In this case, the analytic model takes a concrete form with multi-diagonal kernel $\kappa_{\mathscr{H}}$ defined on a disc $\mathbb{D}_{r_{\lambda}}$, where $r_{\lambda}$ is a positive number such that $\frac{1}{r\left(S_{\lambda^{\prime}}\right)} \leq r_{\lambda} \leq r\left(S_{\lambda}\right)$ (see (5)). Moreover, the reproducing kernel Hilbert space admits an orthonormal basis consisting of vectorvalued analytic polynomials. One of the interesting aspects of our model is a handy formula for $r_{\lambda}$ depending on $\mathscr{T}$ and $S_{\lambda}$.

Although the motivation for the present work comes mainly from the theory of weighted shifts on directed trees as expounded in [20], it is closely related to some of the recent developments in the function theoretic operator theory. In particular, the reader is referred to the study of analytic reproducing kernels of finite bandwidth carried out in a series of papers by G. Adams et al [2], [3], [1], [4] (refer also to [6] for the general theory of reproducing kernels). It is also worth noting that the class of weighted shifts on rooted directed trees has some resemblance with the class of adjoints of abstract weighted shifts (in the context of complex Hilbert spaces) [7], $[\mathbf{8}],[\mathbf{2 6}]$ and also with the class of operator-valued weighted shifts $[\mathbf{2 5}],[\mathbf{2 3}],[\mathbf{1 8}]$, [30] studied extensively in the literature.

Here is the sketch of the paper. Section 2 is devoted to the statement of the main theorem and some of its immediate consequences. The proof of main theorem is presented in Section 3. In Section 4, we present several examples illustrating the rich interplay between the directed trees and reproducing kernels of finite bandwidth. In Section 5, we use the main theorem to describe various spectral parts of $S_{\lambda}$. It turns out that weighted shifts on directed trees with disconnected approximate point spectra are in abundance. In the final section, we introduce a notion of branching index for rootless directed trees and use it to obtain an analytic model for a left-invertible weighted shift $S_{\lambda}$ in this setting. It turns out that $S_{\lambda}$ is an extension of a weighted shift operator on a rooted directed tree.

## 2. Main Result: Statement and Consequences

Let $\mathscr{T}=(V, \mathcal{E})$ be a rooted directed tree with root root. Then

$$
\begin{equation*}
V=\bigsqcup_{n=0}^{\infty} \mathrm{Chi}^{\langle n\rangle} \text { (root) (disjoint union) } \tag{3}
\end{equation*}
$$

(see [20, Proposition 2.1.2]). For each $u \in V$, let $n_{u}$ denote the unique non-negative integer such that $u \in \mathrm{Chi}^{\left\langle n_{u}\right\rangle}$ (root). We use the convention that $\mathrm{Chi}^{\langle j\rangle}$ (root) $=\emptyset$ if $j<0$. Similar convention holds for par.

The statement of the main theorem involves an invariant (to be referred to as the branching index) associated with a rooted directed tree.

Definition 2.1. Let $\mathscr{T}$ be a rooted directed tree and let

$$
V_{\prec}:=\{u \in V: \operatorname{card}(\operatorname{Chi}(u)) \geq 2\}
$$

be the set of branching vertices of $\mathscr{T}$. Define

$$
k_{\mathscr{T}}:= \begin{cases}1+\sup \left\{n_{w}: w \in V_{\prec}\right\}, & \text { if } V_{\prec} \text { is non-empty } \\ 0, & \text { if } V_{\prec} \text { is empty }\end{cases}
$$

We refer to $k_{\mathscr{T}} \in \mathbb{Z}_{+} \cup\{\infty\}$ as the branching index of $\mathscr{T}$.
REmark 2.2. If $\operatorname{card}\left(V_{\prec}\right)$ is finite then so is $k_{\mathscr{T}}$. On the other hand, directed trees $\mathscr{T}$ with infinite $\operatorname{card}\left(V_{\prec}\right)$ and finite $k_{\mathscr{T}}$ can be constructed easily.

The condition (i) in the following proposition says precisely that $\mathscr{T}$ is Fredholm (refer to [20, Section 3.6] for more details related to Fredholm directed trees).

Proposition 2.3. Let $S_{\lambda} \in B\left(l^{2}(V)\right)$ be a weighted shift on a rooted directed tree $\mathscr{T}$ with root root. Let $V_{\prec}$ be the set of branching vertices of $\mathscr{T}$ and let $k_{\mathscr{T}}$ be the branching index of $\mathscr{T}$. Then the following statements are equivalent:
(i) $\mathscr{T}$ is locally finite such that $\operatorname{card}\left(V_{\prec}\right)$ is finite.
(ii) $\mathscr{T}$ is locally finite such that $k_{\mathscr{T}}$ is finite.
(iii) The dimension of $E:=\operatorname{ker} S_{\lambda}^{*}$ is finite.

Proof. That (i) implies (ii) is obvious. Suppose that (ii) holds. If $V_{\prec}$ is not finite, then that $k_{\mathscr{T}}$ is finite implies that there exists an infinite subset $W$ of $V_{\prec}$ such that $n_{w}$ is constant for all $w \in W$. Clearly, $W \subseteq \mathrm{Chi}^{\left\langle n_{w}\right\rangle}$ (root). Therefore, there exists a vertex $v \in V$ with $n_{v}<n_{w}$ such that $\operatorname{card}(\operatorname{Chi}(v))$ is infinite. This contradicts the assumption that $\mathscr{T}$ is locally finite. Thus (ii) implies (i).

By [20, Proposition 3.5.1(ii)],

$$
E=\operatorname{ker} S_{\lambda}^{*}=\left\langle e_{\text {root }}\right\rangle \oplus \bigoplus_{v \in V}\left(l^{2}(\operatorname{Chi}(v)) \ominus\left\langle\boldsymbol{\lambda}^{v}\right\rangle\right),
$$

where $\boldsymbol{\lambda}^{v}: \operatorname{Chi}(v) \rightarrow \mathbb{C}$ is defined by $\boldsymbol{\lambda}^{v}(u)=\lambda_{u}$, and $\langle f\rangle$ denotes the span of $\{f\}$. Observe now that $l^{2}(\operatorname{Chi}(v)) \ominus\left\langle\boldsymbol{\lambda}^{v}\right\rangle \neq\{0\}$ if and only if $v \in V_{\prec}$. Therefore,

$$
\begin{equation*}
E=\left\langle e_{\text {root }}\right\rangle \oplus \bigoplus_{v \in V_{\prec}}\left(l^{2}(\operatorname{Chi}(v)) \ominus\left\langle\boldsymbol{\lambda}^{v}\right\rangle\right) . \tag{4}
\end{equation*}
$$

It now follows from (4) that $\operatorname{dim} E$ is finite if and only if $\operatorname{card}(\operatorname{Chi}(v))$ is finite for every $v \in V_{\prec}$ and $\operatorname{card}\left(V_{\prec}\right)$ is finite. This gives the equivalence of (i) and (iii).

Remark 2.4. It may happen that $k_{\mathscr{T}}<\infty$ and $\operatorname{dim} E=\infty$ (see Example 4.6).
Definition 2.5. Let $\mathscr{T}$ be a rooted directed tree with root root and let $k_{\mathscr{T}}$ be the branching index of $\mathscr{T}$. For any integer $n$, consider the set

$$
W_{n}:=\bigcup_{j=n}^{k_{\mathscr{O}}^{+n}} \mathrm{Chi}^{\langle j\rangle} \text { (root). }
$$

Let $S_{\lambda} \in B\left(l^{2}(V)\right)$ be a left-invertible weighted shift with weights $\left\{\lambda_{v}\right\}_{v \in V^{\circ}}$ and let $S_{\lambda^{\prime}}$ be the Cauchy dual of $S_{\lambda}$. The radius of convergence for $S_{\lambda}$ is defined as the non-negative number $r_{\lambda}$ given by

$$
\begin{equation*}
r_{\lambda}:=\liminf _{n \rightarrow \infty}\left(\sum_{v \in W_{n}}\left(\lambda_{v}^{\prime} \lambda_{\operatorname{par}(v)}^{\prime} \cdots \lambda_{\text {par }\langle n-1\rangle}^{\prime}(v)\right)^{2}\right)^{-\frac{1}{2 n}} . \tag{5}
\end{equation*}
$$

We will see later that $r_{\lambda}$ is positive whenever $k_{\mathscr{T}}$ is finite (see Lemma 3.2). Let us compute $r_{\lambda}$ in the case in which $S_{\lambda}$ is a unilateral weighted shift.

Example 2.6 ((Diagonal)). Consider the directed tree $\mathscr{T}_{1}$ with the set of vertices $V:=\mathbb{Z}_{+}$and root $=0$. We further require that $\operatorname{Chi}(n)=\{n+1\}$ for all $n \geq 0$. For future reference, we note that $V_{\prec}=\emptyset$, and hence $k_{\mathscr{T}_{1}}=0$. The weighted shift $S_{\lambda}$ on the directed tree $\mathscr{T}_{1}$ (to be referred to as unilateral weighted shift) is given by

$$
S_{\lambda} e_{n}=\lambda_{n+1} e_{n+1} \text { for all } n \geq 0
$$

(Caution: This differs from the standard definition $S_{\lambda} e_{n}=\lambda_{n} e_{n+1}$ for all $n \geq 0$ of the unilateral weighted shift.) It is well-known that $S_{\lambda}$ is unitarily equivalent to the operator $\mathscr{M}_{z}$ of multiplication by $z$ on the reproducing kernel Hilbert space $\mathscr{H}$ associated with the kernel

$$
\kappa_{\mathscr{H}}(z, w)=1+\sum_{j \geq 1} C_{j, j} z^{j} \bar{w}^{j}\left(z, w \in \mathbb{D}_{r}\right)
$$

where $r:=\liminf _{n \rightarrow \infty}\left(\lambda_{n} \lambda_{n-1} \cdots \lambda_{1}\right)^{\frac{1}{n}}$ and $\left\{C_{j, j}\right\}_{j \geq 0}$ is a sequence of positive numbers (refer to [28]). Since $W_{n}=\{n\}$ for every $n \in \mathbb{Z}_{+}$, the radius of convergence $r_{\lambda}$ for (a left-invertible) $S_{\lambda}$ is precisely $r$. Moreover, one can verify that (the rank one operator) $C_{j, j}$ is (multiplication by) $\frac{1}{\lambda_{1}^{2} \cdots \lambda_{j}^{2}}$ for all $j \geq 1$. Clearly, the reproducing kernel $\kappa_{\mathscr{H}}(\cdot, \cdot)$ is diagonal in this case.

We are now in a position to state the main result of this paper.
THEOREM 2.7. Let $\mathscr{T}$ be a rooted directed tree with finite branching index $k_{\mathscr{T}}$. Let $S_{\lambda} \in B\left(l^{2}(V)\right)$ be a left-invertible weighted shift and let $S_{\lambda^{\prime}}$ be the Cauchy dual of $S_{\lambda}$. Set $E:=\operatorname{ker} S_{\lambda}^{*}$. Then there exist a z-invariant reproducing kernel Hilbert space $\mathscr{H}$ of E-valued holomorphic functions defined on the disc $\mathbb{D}_{r_{\lambda}}$ and a unitary mapping $U: l^{2}(V) \longrightarrow \mathscr{H}$ such that $\mathscr{M}_{z} U=U S_{\lambda}$, where $\mathscr{M}_{z}$ denotes the operator of multiplication by $z$ on $\mathscr{H}$ and $r_{\lambda}$ is the radius of convergence for $S_{\lambda}$. Moreover, $r_{\lambda} r\left(S_{\lambda^{\prime}}\right) \geq 1$, where $r\left(S_{\lambda^{\prime}}\right)$ is the spectral radius of $S_{\lambda^{\prime}}$. Further, $U$ maps $E$ onto the subspace $\mathscr{E}$ of $E$-valued constant functions in $\mathscr{H}$ such that $U g=g$ for every $g \in E$. Furthermore, we have the following:
(i) The reproducing kernel $\kappa_{\mathscr{H}}: \mathbb{D}_{r_{\lambda}} \times \mathbb{D}_{r_{\lambda}} \rightarrow B(E)$ associated with $\mathscr{H}$ satisfies $\kappa_{\mathscr{H}}(\cdot, w) g \in \mathscr{H}$ and $\left\langle U f, \kappa_{\mathscr{H}}(\cdot, w) g\right\rangle_{\mathscr{H}}=\langle(U f)(w), g\rangle_{E}$ for every $f \in l^{2}(V)$ and $g \in E$.
(ii) $\kappa_{\mathscr{H}}$ is given by

$$
\begin{equation*}
\kappa_{\mathscr{H}}(z, w)=I_{E}+\sum_{\substack{j, k \geq 1 \\|j-k| \leq k \mathscr{T}}} C_{j, k} z^{j} \bar{w}^{k}\left(z, w \in \mathbb{D}_{r_{\lambda}}\right), \tag{6}
\end{equation*}
$$

where $I_{E}$ denotes the identity operator on $E$, and $C_{j, k}$ are bounded linear operators on $E$ given by

$$
C_{j, k}=\left.P_{E} S_{\lambda^{\prime}}^{* j} S_{\lambda^{\prime}}^{k}\right|_{E}(j, k=1,2, \cdots)
$$

with $P_{E}$ being the orthogonal projection of $l^{2}(V)$ onto $E$.
(iii) The E-valued polynomials in $z$ are dense in $\mathscr{H}$. In fact,

$$
\mathscr{H}=\bigvee\left\{z^{n} f: f \in \mathscr{E}, n \geq 0\right\}
$$

(iv) $\mathscr{H}$ admits an orthonormal basis consisting of polynomials in $z$ with at most $k_{\mathscr{T}}+1$ non-zero coefficients.

REmARK 2.8. Let $S_{\lambda} \in B\left(l^{2}(V)\right)$ be a left-invertible weighted shift with nonnegative weights $\left\{\lambda_{v}\right\}_{v \in V^{\circ}}$. Let $u_{0} \in V$ be such that $\operatorname{par}\left(u_{0}\right) \in V_{\prec}$. Suppose that $\lambda_{u_{0}}=0$ and $\lambda_{u}>0$ for all $u \in V \backslash\left\{u_{0}\right\}$. Then by [20, Proposition 3.1.6], $S_{\lambda}$ can be decomposed as an orthogonal direct sum of two weighted shifts $S_{\lambda, 1}, S_{\lambda, 2}$ on directed trees with positive weights. Since $S_{\lambda}$ is left-invertible, so are $S_{\lambda, 1}$ and $S_{\lambda, 2}$. By the theorem above, there exist multiplication operators $\mathscr{M}_{z}^{(i)}$ on reproducing kernel Hilbert spaces $\mathscr{H}^{(i)}$ for $i=1,2$ such that $S_{\lambda}$ is unitarily equivalent to $\mathscr{M}_{z}^{(1)} \oplus \mathscr{M}_{z}^{(2)}$. Note that $\mathscr{H}_{1} \oplus \mathscr{H}_{2}$ is the reproducing kernel Hilbert space associated with the reproducing kernel $\kappa_{\mathscr{H}^{(1)}}+\kappa_{\mathscr{H}^{(2)}}($ refer to $[\mathbf{6}])$.

The inequality $r_{\lambda} r\left(S_{\lambda^{\prime}}\right) \geq 1$ in Theorem 2.7 may be strict in general.
Example 2.9. Consider the weighted shift $S_{\lambda}$ on the directed tree $\mathscr{T}_{1}$ (as discussed in Example 2.6) with weights $\lambda_{n}$ given by

$$
\lambda_{1}=\lambda_{2}=\lambda_{3}=\lambda_{4}=1, \text { and } \lambda_{k}= \begin{cases}\frac{1}{2}, & \text { if } 2^{n}+1 \leq k \leq 3.2^{n-1}, n \geq 2 \\ 1, & \text { otherwise }\end{cases}
$$

Note that $\inf _{n \geq 1} \lambda_{n}=\frac{1}{2}$. Thus $S_{\lambda}$ is left-invertible and hence $S_{\lambda^{\prime}}$ is bounded. Further, for any $n \geq 1$, total number of $\frac{1}{2}$ 's occurring in first $2^{n}$ places is equal to $2^{n-1}-2^{n-2}+2^{n-2}-2^{n-3}+\cdots+4-2=2^{n-1}-2$. Therefore, we get

$$
\lambda_{1} \lambda_{2} \cdots \lambda_{2^{n}}=\frac{1}{2^{2^{n-1}-2}}=\frac{2^{2}}{2^{2^{n-1}}}
$$

Let $n$ be any positive integer. Then there is a unique positive integer $m_{n}$ such that $2^{m_{n}} \leq n<2^{m_{n}+1}$. Let $n=2^{m_{n}}+k$ for some integer $k$ such that $0 \leq k<2^{m_{n}}$. Therefore,

$$
\lambda_{1} \lambda_{2} \cdots \lambda_{n} \geq \frac{1}{2^{2^{m_{n}-1}-2+k}}=\frac{2^{2}}{2^{2_{n}-1}+k} .
$$

Since $k<2^{m_{n}}, \frac{2^{m_{n}-1}+k}{n}=1-\frac{2^{m_{n}-1}}{2^{m_{n}+k}}<\frac{3}{4}$. It follows that

$$
\left(\lambda_{1} \lambda_{2} \cdots \lambda_{n}\right)^{\frac{1}{n}} \geq\left(\frac{2^{2}}{2^{2^{m_{n}-1}+k}}\right)^{\frac{1}{n}}>2^{\frac{2}{n}-\frac{3}{4}}
$$

and hence $r_{\lambda}=\liminf _{n \rightarrow \infty}\left(\lambda_{n} \lambda_{n-1} \cdots \lambda_{1}\right)^{\frac{1}{n}}$ is at least $2^{-\frac{3}{4}}$. On the other hand, $r\left(S_{\lambda^{\prime}}\right)=2$. This can be seen as follows. Note that $\lambda_{n}^{\prime}=\frac{1}{\lambda_{n}}$. Therefore,

$$
r\left(S_{\lambda^{\prime}}\right)=\lim _{n \rightarrow \infty}\left(\sup _{m \geq 1} \lambda_{m+1}^{\prime} \cdots \lambda_{m+n}^{\prime}\right)^{\frac{1}{n}}=\left(2^{n}\right)^{\frac{1}{n}}=2
$$

(since 2's occur in $\left\{\lambda_{n}^{\prime}\right\}$ consecutively at $2^{n-1}$ places for $n \geq 2$ ). Thus we have $r_{\lambda} r\left(S_{\lambda^{\prime}}\right) \geq 2^{\frac{1}{4}}$, which is obviously bigger than 1 .

Since the proof of Theorem 2.7 consists of several observations of independent interest, it will be presented in the next section. In the remaining part of this section, we discuss some immediate consequences of the main theorem. First a terminology.

Let $\mathscr{M}_{z}, \kappa_{\mathscr{H}}$, and $\mathscr{H}$ be as appearing in the statement of Theorem 2.7. For the sake of convenience, we will refer to the triple $\left(\mathscr{M}_{z}, \kappa_{\mathscr{H}}, \mathscr{H}\right)$ as the analytic model of the left-invertible weighted shift $S_{\lambda}$ acting on the directed tree $\mathscr{T}$.

Except the final section of this paper, we assume that $\mathscr{T}$ is a leafless, rooted directed tree with finite branching index $k_{\mathscr{T}}$.

An operator $T$ in $B(\mathcal{H})$ is said to be finitely cyclic if there are a finite number of vectors $h_{1}, \cdots, h_{m}$ in $\mathcal{H}$ such that

$$
\mathcal{H}=\bigvee\left\{T^{k} h_{i}: k \geq 0, i=1, \cdots, m\right\}
$$

In case $m=1$, we refer to $T$ as cyclic operator with cyclic vector $h_{1}$. We say that $T$ is infinitely cyclic if it is not finitely cyclic.

Corollary 2.10. Let $S_{\lambda} \in B\left(l^{2}(V)\right)$ be a weighted shift on $\mathscr{T}$. If $E:=\operatorname{ker} S_{\lambda}^{*}$ is finite dimensional then $S_{\lambda}$ is finitely cyclic.

Proof. Since $\mathscr{T}$ is leafless, by [20, Proposition 3.1.7], $S_{\lambda}$ is injective. If $E:=$ $\operatorname{ker} S_{\lambda}^{*}$ is finite dimensional then the range of $S_{\lambda}$ is closed, and hence $S_{\lambda}$ is leftinvertible. Now appeal to Theorem 2.7(iii).

In general, the reproducing kernel Hilbert space $\mathscr{H}$ as constructed in the proof of Theorem 2.7 can not be realized as the tensor product $\mathscr{K} \otimes E$, where $\mathscr{K}$ is a Hilbert space of scalar-valued holomorphic functions. We make this explicit in the following result.

Corollary 2.11. Let $S_{\lambda} \in B\left(l^{2}(V)\right)$ be a left-invertible weighted shift on $\mathscr{T}$. Let $\left(\mathscr{M}_{z}, \kappa_{\mathscr{H}}, \mathscr{H}\right)$ denote the analytic model of $S_{\lambda}$ and let $E:=\operatorname{ker} S_{\lambda}^{*}$. Suppose that there exist a Hilbert space $\mathscr{K}$ of scalar-valued holomorphic functions and an isometric isomorphism $\Phi: \mathscr{H} \rightarrow \mathscr{K} \otimes E$ such that $\Phi(p f)=p \otimes f$ for every polynomial $p \in \mathscr{K}$ and $f \in E$. Then the reproducing kernel $\kappa_{\mathscr{H}}$ associated with $\mathscr{H}$ is the diagonal kernel given by

$$
\kappa_{\mathscr{H}}(z, w)=I_{E}+\sum_{j=1}^{\infty}\left(P_{E} S_{\lambda^{\prime}}^{* j} S_{\lambda^{\prime}}^{j} \mid E\right) z^{j} \bar{w}^{j}\left(z, w \in \mathbb{D}_{r_{\lambda}}\right)
$$

Proof. Note that for any $f, g \in E$ and $m, n \in \mathbb{Z}_{+}$,
$\left\langle S_{\lambda}^{m} f, S_{\lambda}^{n} g\right\rangle_{l^{2}(V)}=\left\langle z^{m} f, z^{n} g\right\rangle_{\mathscr{H}}=\left\langle\Phi\left(z^{m} f\right), \Phi\left(z^{n} g\right)\right\rangle_{\mathscr{K} \otimes E}=\left\langle z^{m}, z^{n}\right\rangle_{\mathscr{K}}\langle f, g\rangle_{E}$.
Since $S_{\lambda}^{k} e_{\text {root }} \in \bigvee\left\{e_{v}: v \in \mathrm{Chi}^{\langle k\rangle}\right.$ (root) $\}$, by an application of (3), we obtain $\left\langle z^{m}, z^{n}\right\rangle_{\mathscr{K}}=0$ for $m \neq n$ after letting $f=e_{\text {root }}=g$. Hence by (7), we must have $\left\langle S_{\lambda}^{m} f, S_{\lambda}^{n} g\right\rangle_{l^{2}(V)}=0$ for any $f, g \in E$ and non-negative integers $m \neq n$. This shows that the sequence $\left\{S_{\lambda}^{k} E\right\}_{k \geq 0}$ of subspaces of $l^{2}(V)$ is mutually orthogonal. It follows immediately that for any $f, g \in E$ and non-negative integers $j \neq k$,

$$
\left\langle P_{E} S_{\lambda^{\prime}}^{* j} S_{\lambda^{\prime}}^{k} f, g\right\rangle_{E}=\left\langle S_{\lambda^{\prime}}^{* j} S_{\lambda^{\prime}}^{k} f, g\right\rangle_{l^{2}(V)}=\left\langle S_{\lambda^{\prime}}^{k} f, S_{\lambda^{\prime}}^{j} g\right\rangle_{l^{2}(V)}=0
$$

In particular, $\left.P_{E} S_{\lambda^{\prime}}^{* j} S_{\lambda^{\prime}}^{k}\right|_{E}=0$ for all non-negative integers $j \neq k$. The desired conclusion now follows from Theorem 2.7(ii).

Remark 2.12. In view of Shimorin's model (as discussed in Section 1), after replacing $r_{\lambda}$ by $\frac{1}{r\left(S_{\lambda^{\prime}}\right)}$, one may obtain the conclusion of Corollary 2.11 for any directed tree with infinite branching index $k_{\mathscr{T}}$. Thus, even for directed trees $\mathscr{T}$ with infinite $k_{\mathscr{T}}$, the associated reproducing kernel $k_{\mathscr{H}}$ could be multi-diagonal.

Recall that $T \in B(\mathcal{H})$ is an isometry if $T^{*} T=I$.
Corollary 2.13. Consider the analytic model $\left(\mathscr{M}_{z}, \kappa_{\mathscr{H}}, \mathscr{H}\right)$ of a left-invertible weighted shift $S_{\lambda}$ on $\mathscr{T}$ and let $E:=\operatorname{ker} S_{\lambda}^{*}$. If $S_{\lambda}$ is an isometry then $\kappa_{\mathscr{H}}$ is the $B(E)$-valued Cauchy kernel given by

$$
\kappa_{\mathscr{H}}(z, w)=\frac{I_{E}}{1-z \bar{w}}(z, w \in \mathbb{D}) .
$$

In particular, $\mathscr{H}$ is the E-valued Hardy space of the open unit disc.
Proof. Assume that $S_{\lambda}$ is an isometry. Note that $S_{\lambda^{\prime}}$ is also isometry in view of hypothesis and $S_{\lambda^{\prime}}^{*} S_{\lambda^{\prime}}=\left(S_{\lambda}^{*} S_{\lambda}\right)^{-1}$. By the uniqueness of the reproducing kernel, it suffices to check that $C_{j, k}=\delta_{j, k} I_{E}$, where

$$
C_{j, k}:=\left.P_{E} S_{\lambda^{\prime}}^{* j} S_{\lambda^{\prime}}^{k}\right|_{E}(j, k=1,2, \cdots)
$$

and $\delta_{j, k}$ denotes the Kronecker delta. If $j=k$ then obviously $C_{j, k}=I_{E}$. If $j<k$ then $C_{j, k}=\left.P_{E} S_{\lambda^{\prime}}^{k-j}\right|_{E}=0$ since $S_{\lambda^{\prime}} E \subseteq \operatorname{ran} S_{\lambda}=E^{\perp}$.

One rather striking consequence of the preceding corollary is as follows: If $S_{\lambda}$ is an isometry then $r_{\lambda}=1$. Note that this observation is irrespective of the structure of the directed tree $\mathscr{T}$. On the other hand, the definition of $r_{\lambda}$ relies on the weight sequence $\left\{\lambda_{v}\right\}_{v \in V^{\circ}}$ and of course on the structure of $\mathscr{T}$. To see this fact, assume that $S_{\lambda}$ is an isometry. By Theorem 2.7, we must have $r_{\lambda} r\left(S_{\lambda^{\prime}}\right) \geq 1$. However, $S_{\lambda^{\prime}}$ being isometry, $r\left(S_{\lambda^{\prime}}\right)=1$, and hence $r_{\lambda} \geq 1$. Since $\frac{I_{E}}{1-z \bar{w}}$ is not defined on $\mathbb{D}_{r} \times \mathbb{D}_{r}$ for any $r>1$, by Theorem 2.7(ii), $r_{\lambda}$ can not exceed 1 .

## 3. Proof of the Main Theorem

The proof of Theorem 2.7 involves several lemmas. The first of which collects some facts related to the set $W_{n}$. Recall from Definition 2.5 that for any integer $n$, the set $W_{n}$ is given by

$$
\begin{equation*}
W_{n}:=\bigcup_{j=n}^{k_{\mathscr{T}}^{+n}} \mathrm{Chi}^{\langle j\rangle}(\text { root }) . \tag{8}
\end{equation*}
$$

Lemma 3.1. Let $S_{\lambda} \in B\left(l^{2}(V)\right)$ be a weighted shift on $\mathscr{T}$ and let $E:=\operatorname{ker} S_{\lambda}^{*}$. Then we have the following statements:
(i) $E$ is a subspace of the (possibly infinite dimensional) space $\bigvee\left\{e_{v}: v \in\right.$ $\left.W_{0}\right\}$.
(ii) $\operatorname{card}\left(\mathrm{Chi}^{\langle n\rangle}(\right.$ root $\left.)\right)$ (possibly countably infinite) is constant for $n \geq k_{\mathscr{T}}$. In particular, $\operatorname{card}\left(W_{n}\right)$ is constant for $n \geq k_{\mathscr{T}}$.
(iii) For every $v \notin W_{n}, e_{\operatorname{par}\langle n\rangle(v)}$ belongs to the orthogonal complement of $E$, where $e_{\operatorname{par}\langle n\rangle(v)}$ is understood to be the zero vector in case $\operatorname{par}^{\langle n\rangle}(v)=\emptyset$.
(iv) For non-negative integers $m$ and $n, W_{n} \cap W_{m} \neq \emptyset$ if and only if $|n-m| \leq$ $k_{\mathscr{T}}$.

Proof. Note that $\mathrm{Chi}\left(V_{\prec}\right) \subseteq W_{0}$. Hence by (4),

$$
\begin{equation*}
E \subseteq\left\langle e_{\text {root }}\right\rangle \oplus \bigoplus_{v \in V_{\prec}} l^{2}(\operatorname{Chi}(v)) \subseteq \bigvee\left\{e_{v}: v \in W_{0}\right\} \tag{9}
\end{equation*}
$$

This yields (i). To see (ii), recall that $n_{u}$ is the unique non-negative integer such that $u \in \operatorname{Chi}^{\left\langle n_{u}\right\rangle}($ root $)$. Note that $\operatorname{card}(\operatorname{Chi}(u))=1$ if $n_{u} \geq k_{\mathscr{T}}$, where we used the assumption that $\mathscr{T}$ is leafless. Thus $\operatorname{card}\left(\mathrm{Chi}^{\langle n\rangle}(\right.$ root $\left.)\right)$ is constant for $n \geq k_{\mathscr{T}}$. This proves (ii).

We now check (iii). Let $v \notin W_{n}$. Since $E$ is a subspace of $\bigvee\left\{e_{v}: v \in W_{0}\right\}$ by part (i), it suffices to check that $e_{\operatorname{par}\langle n\rangle(v)}$ is orthogonal to $\left\{e_{v}: v \in W_{0}\right\}$. Note that $n_{v}<n$ or $n_{v}>k_{\mathscr{T}}+n$, If $n_{v}<n$ then $e_{\text {par }}{ }^{\langle n\rangle}(v)=e_{\emptyset}=0$ by convention. Otherwise, $\operatorname{par}^{\langle n\rangle}(v) \notin W_{0}$, and hence $e_{\text {par }\langle n\rangle(v)}$ is orthogonal to $\left\{e_{v}: v \in W_{0}\right\}$. To see (iv), let $n, m$ be two non-negative integers such that $n<m$. If $n+k_{\mathscr{T}}<m$ then clearly,
$W_{n} \cap W_{m}=\emptyset$. So suppose that $n+k_{\mathscr{T}} \geq m$. Then $W_{n} \cap W_{m}=\cup_{k=m}^{n+k_{\mathscr{F}}} \mathrm{Chi}^{\langle k\rangle}$ (root), which is obviously non-empty.

Next we prove that the radius of convergence for any left-invertible weighted shift with finite dimensional cokernel is positive.

Lemma 3.2. Let $S_{\lambda} \in B\left(l^{2}(V)\right)$ be a left-invertible weighted shift on $\mathscr{T}$ and let $S_{\lambda^{\prime}}$ be the Cauchy dual of $S_{\lambda}$. If $r\left(S_{\lambda^{\prime}}\right)$ denotes the spectral radius of $S_{\lambda^{\prime}}$ then the radius of convergence $r_{\lambda}$ for $S_{\lambda}$ satisfies $r_{\lambda} r\left(S_{\lambda^{\prime}}\right) \geq 1$. In particular, $r_{\lambda}$ is positive.

Proof. By Lemma 1.2(i), for any integer $k \geq 0$,

$$
\left\|S_{\lambda^{\prime}}^{k} e_{\text {root }}\right\|^{2}=\sum_{v \in \mathrm{Chi}^{\langle k\rangle}(\text { root })}\left(\lambda_{v}^{\prime} \lambda_{\mathrm{par}(v)}^{\prime} \cdots \lambda_{\text {par }\langle k-1\rangle}^{\prime}(v)\right)^{2} .
$$

It follows from (3) that

$$
\begin{aligned}
\sum_{v \in W_{n}}\left(\lambda_{v}^{\prime} \lambda_{\mathrm{par}(v)}^{\prime} \cdots \lambda_{\mathrm{par}^{\langle k-1\rangle}(v)}^{\prime}\right)^{2} & =\sum_{k=n}^{n+k_{\mathscr{T}}} \sum_{v \in \mathrm{Chi}^{\langle k\rangle}(\mathrm{root})}\left(\lambda_{v}^{\prime} \lambda_{\mathrm{par}(v)}^{\prime} \cdots \lambda_{\mathrm{par}^{\langle k-1\rangle}(v)}^{\prime}\right)^{2} \\
& =\sum_{k=n}^{n+k_{\mathscr{F}}}\left\|S_{\lambda^{\prime}}^{k} e_{\mathrm{root}}\right\|^{2}=\sum_{k=0}^{k_{\mathscr{F}}}\left\|S_{\lambda^{\prime}}^{n+k} e_{\mathrm{root}}\right\|^{2} \\
& \leq\left\|S_{\lambda^{\prime}}^{n}\right\|^{2} \sum_{k=0}^{k \mathscr{T}}\left\|S_{\lambda^{\prime}}^{k} e_{\text {root }}\right\|^{2} .
\end{aligned}
$$

If we set $M:=\sum_{k=0}^{k_{\mathscr{O}}}\left\|S_{\lambda^{\prime}}^{k} e_{\text {root }}\right\|^{2}$ (which is finite since $k_{\mathscr{T}}<\infty$ ) then by the definition of $r_{\lambda}$,

$$
r_{\lambda} \geq \liminf _{n \rightarrow \infty}\left(M\left\|S_{\lambda^{\prime}}^{n}\right\|^{2}\right)^{-\frac{1}{2 n}}=\left(\lim _{n \rightarrow \infty} M^{-\frac{1}{2 n}}\right)\left(\liminf _{n \rightarrow \infty}\left\|S_{\lambda^{\prime}}^{n}\right\|^{-\frac{1}{n}}\right)=\frac{1}{r\left(S_{\lambda^{\prime}}\right)}
$$

which completes the proof of the lemma.
REMARK 3.3. If $S_{\lambda}$ is an expansion (that is, $S_{\lambda}^{*} S_{\lambda} \geq I$ ) then $S_{\lambda^{\prime}}$ is a contraction (that is, $S_{\lambda}^{*} S_{\lambda} \leq I$ ). In this case, $r_{\lambda}$ is at least 1 .

We need one more fact in the proof of the main result (cf. [5, Proposition 4.5]).
Lemma 3.4. Let $S_{\lambda} \in B\left(l^{2}(V)\right)$ be a weighted shift on $\mathscr{T}$. Then $S_{\lambda}$ is analytic.
Proof. Put $V_{0}:=V$ and $V_{k}:=V \backslash \cup_{j=0}^{k-1} \mathrm{Chi}^{\langle j\rangle}$ (root) $(k \geq 1)$. Note that $\left\{V_{k}\right\}_{k \geq 0}$ is a strictly decreasing sequence of sets such that $\cap_{k \geq 0} V_{k}=\emptyset$. Now, for all $u \in V$ and all integers $k \geq 0$, by Lemma 1.2(i),

$$
S_{\lambda}^{k} e_{u}=\sum_{v \in \operatorname{Chi}^{\langle k\rangle}(u)} \lambda_{v} \lambda_{\operatorname{par}(v)} \cdots \lambda_{\operatorname{par}\langle k-1\rangle}(v) e_{v}
$$

It follows that

$$
\operatorname{ran} S_{\lambda}^{k} \subseteq \bigvee\left\{e_{u}: u \in V_{k}\right\}:=M_{k}, \text { say }
$$

Also, if $f \in M_{k}$, then $f(u)=0$ for $u \in V \backslash V_{k}=\cup_{j=0}^{k-1} \mathrm{Chi}^{\langle j\rangle}$ (root). Thus, if $f \in \cap_{k=0}^{\infty} M_{k}$, then $f(u)=0$ for $u \in \cup_{j=0}^{\infty} \mathrm{Chi}^{\langle j\rangle}$ (root) $=V$. That is, $f=0$. Hence

$$
\{0\} \subseteq \cap_{k=0}^{\infty} \operatorname{ran} S_{\lambda}^{k} \subseteq \cap_{k=0}^{\infty} M_{k}=\{0\}
$$

This shows that $S_{\lambda}$ is analytic.

Proof of Theorem 2.7. As mentioned earlier, the proof relies on the ideas developed in [29, Sections 1 and 2]. Let $f=\sum_{v \in V} f(v) e_{v} \in l^{2}(V)$. By Lemmas 1.2 (ii) and 3.1(iii),

$$
\begin{aligned}
P_{E} S_{\lambda^{\prime}}^{* n} f & =\sum_{v \in V} f(v) \lambda_{v}^{\prime} \lambda_{\operatorname{par}(v)}^{\prime} \cdots \lambda_{\mathrm{par}}\langle n-1\rangle(v) \\
& =P_{E} e_{\mathrm{par}\langle n\rangle(v)} \\
& f(v) \lambda_{v}^{\prime} \lambda_{\operatorname{par}(v)}^{\prime} \cdots \lambda_{\mathrm{par}}{ }^{\langle n-1\rangle}(v)
\end{aligned} P_{E} e_{\mathrm{par}\langle n\rangle}(v) .
$$

We claim that the $E$-valued series

$$
\begin{equation*}
U_{f}(z):=\sum_{n \geq 0}\left(P_{E} S_{\lambda^{\prime}}^{* n} f\right) z^{n} \tag{10}
\end{equation*}
$$

converges absolutely in $E$ on the disc $\mathbb{D}_{r_{\lambda}}$ for every $f \in l^{2}(V)$. By Lemma 3.1(iv), for non-negative integers $m$ and $n, W_{n} \cap W_{m} \neq \emptyset$ if and only if $|n-m| \leq k_{\mathscr{T}}$. It follows that

$$
\begin{align*}
\sum_{\substack{v \in W_{n} \\
n \geq 0}}|f(v)|^{2} & =\sum_{v \in W_{0}}|f(v)|^{2}+\sum_{v \in W_{1}}|f(v)|^{2}+\cdots \\
& \leq \sum_{v \in V}\left(k_{\mathscr{T}}+1\right)|f(v)|^{2}=\left(k_{\mathscr{T}}+1\right)\|f\|^{2} \tag{11}
\end{align*}
$$

Now by the Cauchy-Schwarz inequality, for any integer $k \geq 0$,

$$
\begin{aligned}
\left\|\sum_{n=0}^{k}\left(P_{E} S_{\lambda^{\prime}}^{* n} f\right) z^{n}\right\| & \leq \sum_{\substack{v \in W_{n} \\
n \geq 0}}|f(v)| \lambda_{v}^{\prime} \lambda_{\operatorname{par}(v)}^{\prime} \cdots \lambda_{\operatorname{par}\langle n-1\rangle}^{\prime}(v)|z|^{n} \\
& \leq\left(\sum_{\substack{v \in W_{n} \\
n \geq 0}}|f(v)|^{2}\right)^{\frac{1}{2}}\left(\sum_{\substack{v \in W_{n} \\
n \geq 0}}\left(\lambda_{v}^{\prime} \lambda_{\operatorname{par}(v)}^{\prime} \cdots \lambda_{\operatorname{par}\langle n-1\rangle}^{\prime}(v)\right)^{2}|z|^{2 n}\right)^{\frac{1}{2}} \\
& \stackrel{(11)}{\leq} \sqrt{k_{\mathscr{T}}+1}\|f\|\left(\sum_{\substack{v \in W_{n} n \\
n \geq 0}}\left(\lambda_{v}^{\prime} \lambda_{\operatorname{par}(v)}^{\prime} \cdots \lambda_{\operatorname{par}\langle n-1\rangle}^{\prime}(v)\right)^{2}|z|^{2 n}\right)^{\frac{1}{2}}
\end{aligned}
$$

Since the series on the right hand side converges absolutely on $\mathbb{D}_{r_{\lambda}}$, the claim stands verified. Thus $U_{f}$ is holomorphic in the disk $\mathbb{D}_{r_{\lambda}}$. This allows us to define the map $U: l^{2}(V) \rightarrow \mathscr{H}$ by $U f=U_{f}$, where $\mathscr{H}$ denotes the complex vector space of $E$ valued holomorphic functions of the form $U_{f}$. By Lemma 3.4, $S_{\lambda}$ is analytic, and hence by [29, Lemma 2.2], $U$ is injective. Thus the inner-product given by

$$
\left\langle U_{f}, U_{g}\right\rangle=\langle f, g\rangle_{l^{2}(V)} \text { for all } f, g \in l^{2}(V)
$$

makes $\mathscr{H}$ an inner-product space. Also, the very definition of the inner-product on $\mathscr{H}$ shows that $U$ is unitary, and hence $\mathscr{H}$ is a Hilbert space.

Note that for each $f \in E, U_{f}(z)=f$. We now show that $\mathscr{H}$ is $z$-invariant. Let $U_{f} \in \mathscr{H}$. Since $S_{\lambda^{\prime}}^{*} S_{\lambda}=I$ and $P_{E} S_{\lambda}=0$, we get

$$
\begin{aligned}
z U_{f}(z) & =\sum_{n \geq 0}\left(P_{E} S_{\lambda^{\prime}}^{* n} f\right) z^{n+1}=\sum_{n \geq 1}\left(P_{E} S_{\lambda^{\prime}}^{* n-1} f\right) z^{n} \\
& =\sum_{n \geq 0}\left(P_{E} S_{\lambda^{\prime}}^{* n} S_{\lambda} f\right) z^{n}=U_{S_{\lambda} f}(z) \in \mathscr{H}
\end{aligned}
$$

Above expression also verifies that $\mathscr{M}_{z} U=U S_{\lambda}$, where $\mathscr{M}_{z}$ is the operator of multiplication by z on $\mathscr{H}$.

Part (i) has been recorded on $[\mathbf{2 9}, \mathrm{Pg} 154]$ (see the discussion following (1)). To see (ii), recall from (1) that

$$
\kappa_{\mathscr{H}}(z, w)=\sum_{j, k \geq 0} C_{j, k} z^{j} \bar{w}^{k}
$$

where $C_{j, k}$ is a bounded linear operator on $E$ given by $C_{j, k}=\left.P_{E} S_{\lambda^{\prime}}^{* j} S_{\lambda^{\prime}}^{k}\right|_{E}$. Since $\operatorname{ker} S_{\lambda^{\prime}}^{*}=\operatorname{ker} S_{\lambda}^{*}=E$, it follows that $\left.P_{E} S_{\lambda^{\prime}}^{* j}\right|_{E}=0$ for all $j \geq 1$. Since $C_{j, k}^{*}=C_{k, j}$, we get $C_{j, 0}=0=C_{0, j}$ for all $j \geq 1$. Hence the above expression for $\kappa_{\mathscr{H}}(z, w)$ reduces to

$$
\kappa_{\mathscr{H}}(z, w)=I_{E}+\sum_{j, k \geq 1} C_{j, k} z^{j} \bar{w}^{k}
$$

As recorded earlier in (9), $E \subseteq \bigvee\left\{e_{v}: v \in W_{0}\right\}$ (see also (8) for the definition of $\left.W_{n}\right)$. It follows that

$$
S_{\lambda^{\prime}}^{* j} S_{\lambda^{\prime}}^{k} E \subseteq \bigvee\left\{e_{v}: v \in W_{k-j}\right\}
$$

and therefore $S_{\lambda^{\prime}}^{* j} S_{\lambda^{\prime}}^{k} E$ is orthogonal to $E$ if $|j-k|>k_{\mathscr{T}}$. Thus $C_{j, k}=0$ if $|j-k|>k_{\mathscr{T}}$. This proves (ii).

To prove (iii), note that by Lemma 3.4, $S_{\lambda^{\prime}}$ is analytic. Therefore, by $[\mathbf{2 9}$, Proposition 2.7],

$$
l^{2}(V)=\bigvee_{n \geq 0} S_{\lambda}^{n}(E)
$$

Since $\mathscr{M}_{z}$ is unitarily equivalent to $S_{\lambda}$ and $\operatorname{ker} S_{\lambda}^{*}=E$, it follows that

$$
\mathscr{H}=\bigvee_{n \geq 0} \mathscr{M}_{z}^{n}(\mathscr{E})
$$

This is precisely (iii).
Finally, since $U$ is unitary and $\left\{e_{u}: u \in V\right\}$ is an orthonormal basis of $l^{2}(V)$, $\left\{U_{e_{u}}: u \in V\right\}$ is an orthonormal basis of $\mathscr{H}$. Note that

$$
\begin{equation*}
U_{e_{u}}(z)=\sum_{k \geq 0}\left(P_{E} S_{\lambda^{\prime}}^{* k} e_{u}\right) z^{k}=\sum_{0 \leq k \leq n_{u}}\left(P_{E} S_{\lambda^{\prime}}^{* k} e_{u}\right) z^{k} \tag{12}
\end{equation*}
$$

(see the discussion following (3) for the definition of $n_{u}$ ). If $n_{u} \leq k_{\mathscr{T}}$ then clearly $U_{e_{u}}$ has at most $k_{\mathscr{T}}+1$ number of non-zero coefficients. Suppose now that $n_{u}>k_{\mathscr{T}}$. By Lemma 1.2(ii), $S_{\lambda^{\prime}}^{* k} e_{u}=\alpha_{\lambda, k} e_{\operatorname{par}^{\langle k\rangle}(u)}$ for some scalar $\alpha_{\lambda, k}$. Let $k$ be an integer such that $0 \leq k \leq n_{u}-k_{\mathscr{T}}-1$. Then $n_{\operatorname{par}^{\langle k\rangle}(u)}=n_{u}-k \geq k_{\mathscr{T}}+1$, and hence $\operatorname{par}^{\langle k\rangle}(u) \notin W_{0}$. By Lemma 3.1(i),

$$
P_{E} S_{\lambda^{\prime}}^{* k} e_{u}=\alpha_{\lambda, k} P_{E} e_{\operatorname{par}\langle k\rangle(u)}=0
$$

Hence total number of possible non-zero coefficients in above expression of $U_{e_{u}}$ are $n_{u}+1-\left(n_{u}-k_{\mathscr{T}}\right)=k_{\mathscr{T}}+1$. Thus for each $u \in V, U_{e_{u}}$ is a polynomial in $z$ with at most $k_{\mathscr{T}}+1$ non-zero coefficients. This completes the proof of the theorem.

REMARK 3.5. The proof above actually shows that the set of analytic bounded point evaluation of $\mathscr{H}$ contains the disc $\mathbb{D}_{r_{\lambda}}$ (refer to [12, Chapter II, Section 7]).

We conclude this section with a brief discussion on a possible line of investigation. Note that the proof of Theorem 2.7 relies on the notion of Cauchy dual operator, present already in Shimorin's construction of an analytic model for a left inertible analytic operator $[\mathbf{2 9}]$. In case $\operatorname{dim} E=1$, the analytic model can be replaced by the model in which the weighted shift operator can be realized as the operator of multiplication by $z$ on a Hilbert space of formal power series [28] (refer also to $[\mathbf{2 3}])$. It would be interesting to find a counter-part of the later model in case $\operatorname{dim} E>1$. In this regard, the authors would like to draw reader's attention to [31, Theorems 2.12 and 2.13] in which it is shown that the adjoint of an arbitrary cyclic operator can be modelled as a backward shift on a reproducing kernel Hilbert space.

## 4. Examples

In this section, we illustrate Theorem 2.7 with the help of several interesting examples. In particular, we see that various directed trees (discrete structures) render to analytic multi-diagonal kernels (analytic structures). These include mainly tridiagonal and pentadiagonal kernels (the reader is referred to [2], [3], [1], [4] for a systematic study of scalar-valued and matrix-valued kernels of finite bandwidth; refer also to $[\mathbf{2 4}]$ for a new class of matrix-valued kernels on the unit disc arising in the classification problem of homogeneous operators). All important examples are summarized in the form of a table at the end of this section.

Example 4.1 ((Tridiagonal)). Consider the directed tree $\mathscr{T}_{2}$ with set of vertices

$$
V:=\{(0,0)\} \cup\{(1, i),(2, i): i \geq 1\}
$$

and root $=(0,0)$. We further require that $\operatorname{Chi}(0,0)=\{(1,1),(2,1)\}$ and

$$
\operatorname{Chi}(1, i)=\{(1, i+1)\}, \operatorname{Chi}(2, i)=\{(2, i+1)\}, \text { for all } i \geq 1
$$

Let $S_{\lambda}$ be a left-invertible weighted shift on $\mathscr{T}_{2}$. It is easy to see from (4) that

$$
E:=\operatorname{ker} S_{\lambda}^{*}=\left\{\alpha e_{(0,0)}+\beta\left(\lambda_{(2,1)} e_{(1,1)}-\lambda_{(1,1)} e_{(2,1)}\right): \alpha, \beta \in \mathbb{C}\right\}
$$

Also, $V_{\prec}=\{(0,0)\}$ and $k_{\mathscr{T}_{2}}=1$. Therefore, by Theorem 2.7, $\kappa_{\mathscr{H}}(\cdot, \cdot)$ takes the form

$$
\kappa_{\mathscr{H}}(z, w)=I_{E}+\sum_{\substack{j, k \geq 1 \\|j-k| \leq 1}} C_{j, k} z^{j} \bar{w}^{k}\left(z, w \in \mathbb{D}_{r_{\lambda}}\right)
$$

where $C_{j, k}$ is given by $C_{j, k}=\left.P_{E} S_{\lambda^{\prime}}^{* j} S_{\lambda^{\prime}}^{k}\right|_{E}(j, k=1,2, \cdots)$. Let us find an explicit expression for the radius of convergence $r_{\lambda}$ for $S_{\lambda}$. Note that

$$
W_{n}=\{(1, n),(2, n),(1, n+1),(2, n+1)\}
$$

for $n \geq 1$, and hence by (5),

$$
\begin{equation*}
r_{\lambda}=\liminf _{n \rightarrow \infty}\left(\sum_{j=1}^{2}\left[\left(\lambda_{(j, n)}^{\prime} \cdots \lambda_{(j, 1)}^{\prime}\right)^{2}+\left(\lambda_{(j, n+1)}^{\prime} \cdots \lambda_{(j, 2)}^{\prime}\right)^{2}\right]\right)^{-\frac{1}{2 n}} \tag{13}
\end{equation*}
$$

where the sequence $\left\{\lambda_{(j, n)}^{\prime}\right\}_{n \geq 1}$ for $j=1,2$ is given by

$$
\lambda_{(j, n)}^{\prime}=\left\{\begin{array}{l}
\frac{\lambda_{(j, n)}}{\lambda_{(1, n)}^{2}+\lambda_{(2, n)}^{2}} \text { if } n=1  \tag{14}\\
\frac{1}{\lambda_{(j, n)}} \text { if } n \geq 2 .
\end{array}\right.
$$

In this case, the reproducing kernel $\kappa_{\mathscr{H}}(\cdot, \cdot)$ is tridiagonal. Finally, we note that the weight sequence $\lambda$ can be chosen so that $\kappa_{\mathscr{H}}(\cdot, \cdot)$ is not diagonal. In fact, a routine calculation shows that

$$
\begin{equation*}
C_{2,1}\left(\lambda_{(2,1)} e_{(1,1)}-\lambda_{(1,1)} e_{(2,1)}\right)=\frac{\lambda_{(1,1)} \lambda_{(2,1)}}{\lambda_{(1,1)}^{2}+\lambda_{(2,1)}^{2}}\left(\frac{1}{\lambda_{(1,2)}^{2}}-\frac{1}{\lambda_{(2,2)}^{2}}\right) e_{(0,0)}, \tag{15}
\end{equation*}
$$

which is clearly non-zero in case $\lambda_{(1,2)} \neq \lambda_{(2,2)}$.
The tridiagonal kernel $\kappa_{\mathscr{H}}$ appearing in Example 4.1 takes a concrete form for a family of weighted shifts $S_{\lambda}$.

Proposition 4.2. Let $\mathscr{T}_{2}$ and $S_{\lambda}$ be as discussed in Example 4.1. Let $x:=$ $e_{(0,0)}, y:=\lambda_{(2,1)} e_{(1,1)}-\lambda_{(1,1)} e_{(2,1)}$. Assume that the weight sequence $\left\{\lambda_{(j, i)}: i \geq\right.$ $1, j=1,2\}$ of $S_{\lambda}$ satisfies the following:
(i) $\lambda_{(1,1)}=\lambda_{(2,1)}$,
(ii) $\lambda_{(1,2)} \neq \lambda_{(2,2)}$,
(iii) $\lambda_{(1, n)} \cdots \lambda_{(1,2)}=\lambda_{(2, n)} \cdots \lambda_{(2,2)}$ for every integer $n \geq 3$.

Then the reproducing kernel $\kappa_{\mathscr{H}}$ takes the form

$$
\begin{aligned}
\kappa_{\mathscr{H}}(z, w) & =I_{E}+\alpha\left(x \otimes y z^{2} \bar{w}+y \otimes x z \bar{w}^{2}\right) \\
& +\sum_{k=1}^{\infty}\left(\alpha_{k} x \otimes x+\alpha_{k+1} y \otimes y\right) z^{k} \bar{w}^{k}\left(z, w \in \mathbb{D}_{r_{\lambda}}\right)
\end{aligned}
$$

where $\alpha:=\frac{\lambda_{(1,1)}^{2}}{\|y\|^{4}}\left(\lambda_{(1,2)}^{-2}-\lambda_{(2,2)}^{-2}\right)$ is a non-zero real number, and

$$
\alpha_{k}:=\left\{\begin{array}{l}
\|y\|^{-2} \text { if } k=1 \\
\lambda_{(1,1)}^{2}\|y\|^{-4}\left(\lambda_{(1,2)}^{-2}+\lambda_{(2,2)}^{-2}\right) \text { if } k=2 \\
\|y\|^{-2}\left(\lambda_{(1, k)} \cdots \lambda_{(1,2)}\right)^{-2} \text { if } k \geq 3
\end{array}\right.
$$

Proof. As seen in Example 4.1, $\kappa_{\mathscr{H}}(\cdot, \cdot)$ is given by

$$
\kappa_{\mathscr{H}}(z, w)=I_{E}+\sum_{\substack{j, k \geq 1 \\|j-k| \leq 1}} C_{j, k} z^{j} \bar{w}^{k}\left(z, w \in \mathbb{D}_{r_{\lambda}}\right) .
$$

Since $C_{2,1} e_{(0,0)}=0$, by (15) and (i), $C_{21}$ is the rank one operator $\alpha x \otimes y$. Note that $\alpha \neq 0$ in view of (ii). Also, $C_{1,2}=C_{2,1}^{*}=\alpha y \otimes x$.

We claim that $C_{k, k+1}=0=C_{k+1, k}$ for all integers $k \geq 2$. Let us first compute the diagonal operator $S_{\lambda^{\prime}}^{* k} S_{\lambda^{\prime}}^{k}$. Fix an integer $k \geq 2$. It may be concluded from $[\mathbf{2 0}$, Lemma 6.1.1] that

$$
\begin{equation*}
S_{\lambda^{\prime}}^{* k} S_{\lambda^{\prime}}^{k} e_{u}=\sum_{v \in \mathrm{Chi}^{\langle k\rangle}(u)}\left(\lambda_{v}^{\prime} \lambda_{\mathrm{par}(v)}^{\prime} \cdots \lambda_{\mathrm{par}^{\langle k-1\rangle}(v)}^{\prime}\right)^{2} e_{u} . \tag{16}
\end{equation*}
$$

Since $S_{\lambda^{\prime}}^{*} x=0$, it follows from (16) that $C_{k+1, k} x=0$. Note further that

$$
\begin{aligned}
S_{\lambda^{\prime}}^{* k} S_{\lambda^{\prime}}^{k} y & =\lambda_{(2,1)} S_{\lambda^{\prime}}^{* k} S_{\lambda^{\prime}}^{k} e_{(1,1)}-\lambda_{(1,1)} S_{\lambda^{\prime}}^{* k} S_{\lambda^{\prime}}^{k} e_{(2,1)} \\
& =\lambda_{(2,1)}\left(\lambda_{(1, k+1)}^{\prime} \cdots \lambda_{(1,2)}^{\prime}\right)^{2} e_{(1,1)}-\lambda_{(1,1)}\left(\lambda_{(2, k+1)}^{\prime} \cdots \lambda_{(2,2)}^{\prime}\right)^{2} e_{(2,1)} \\
& \stackrel{(\text { iii })}{=} \frac{1}{\left(\lambda_{(1, k+1)} \cdots \lambda_{(1,2)}\right)^{2}} y
\end{aligned}
$$

where in the last step we used (14). This immediately yields that $C_{k+1, k} y=0$. This completes the verification of the claim. The above calculation also shows that $C_{k, k} y=\alpha_{k+1}\|y\|^{2} y$ for all integers $k \geq 2$. Since $S_{\lambda^{\prime}}^{*} S_{\lambda^{\prime}} y=\frac{\lambda_{(2,1)}}{\lambda_{(1,2)}^{2}} e_{(1,1)}-\frac{\lambda_{(1,1)}}{\lambda_{(2,2)}^{2}} e_{(2,1)}$, we have

$$
\begin{aligned}
C_{1,1} y & =P_{E}\left(\frac{\lambda_{(2,1)}}{\lambda_{(1,2)}^{2}} e_{(1,1)}-\frac{\lambda_{(1,1)}}{\lambda_{(2,2)}^{2}} e_{(2,1)}\right) \\
& =\frac{\lambda_{(2,1)}}{\lambda_{(1,2)}^{2}}\left\langle e_{(1,1)}, y\right\rangle \frac{y}{\|y\|^{2}}-\frac{\lambda_{(1,1)}}{\lambda_{(2,2)}^{2}}\left\langle e_{(2,1)}, y\right\rangle \frac{y}{\|y\|^{2}} \\
& =\left(\frac{\lambda_{(2,1)}^{2}}{\lambda_{(1,2)}^{2}}+\frac{\lambda_{(1,1)}^{2}}{\lambda_{(2,2)}^{2}}\right) \frac{y}{\|y\|^{2}} \stackrel{(\mathrm{i})}{=} \alpha_{2}\|y\|^{2} y .
\end{aligned}
$$

To compute the diagonal entry $C_{k, k}$, by (16), for any integer $k \geq 1$,

$$
C_{k, k} x=\sum_{j=1}^{2}\left(\lambda_{(j, k)}^{\prime} \cdots \lambda_{(j, 1)}^{\prime}\right)^{2} x=\alpha_{k} x
$$

where we used (14) and (iii). Now it is easy to see that the rank two operator $C_{k, k}$ is given by $C_{k, k}=\left(\alpha_{k} x \otimes x+\alpha_{k+1} y \otimes y\right)$ for all integers $k \geq 1$.

Example 4.3. The preceding proposition is applicable to $S_{\lambda}$ with weights $\lambda_{(1,1)}=\lambda_{(2,1)}=\lambda_{(1,2)}=1, \lambda_{(2,2)}=\sqrt{2}=\lambda_{(1,3)}$, and $\lambda_{(2,3)}=1=\lambda_{(j, i)}$ for $i \geq 4$ and for $j=1,2$. In this case, $\alpha=\frac{1}{8}, \alpha_{1}=\frac{1}{2}, \alpha_{2}=\frac{3}{8}$ and $\alpha_{k}=\frac{1}{8}$ for all integers $k \geq 3$. Thus the reproducing kernel $\kappa_{\mathscr{H}}$ takes the form

$$
\begin{aligned}
\kappa_{\mathscr{H}}(z, w) & =I_{E}+\frac{1}{8}\left(x \otimes y z^{2} \bar{w}+y \otimes x z \bar{w}^{2}\right)+\frac{1}{2}\left(x \otimes x+\frac{3}{4} y \otimes y\right) z \bar{w} \\
& +\frac{1}{8}(3 x \otimes x+y \otimes y) z^{2} \bar{w}^{2}+\frac{1}{8} \sum_{k=3}^{\infty}(x \otimes x+y \otimes y) z^{k} \bar{w}^{k}\left(z, w \in \mathbb{D}_{r_{\lambda}}\right),
\end{aligned}
$$

where, in view of (13), $r_{\lambda}$ can be easily seen to be equal to 1 . Also, one may easily deduce from the proof of Theorem 2.7(iv) that for all integers $i \geq 1$,

$$
U_{e_{(j, i)}}(z)=\left(P_{E} S_{\lambda^{\prime}}^{* i-1} e_{(j, i)}\right) z^{i-1}+\left(P_{E} S_{\lambda^{\prime}}^{* i} e_{(j, i)}\right) z^{i}, j=1,2 .
$$

It is now easy to see that the orthonormal basis for the reproducing kernel Hilbert space $\mathscr{H}$ associated with $\kappa_{\mathscr{H}}$ is given by

$$
\{x, p(z), z p(z)\} \cup\left\{\frac{1}{\sqrt{2}} z^{k-1} p(z)\right\}_{k \geq 3} \cup\{q(z)\} \cup\left\{\frac{1}{\sqrt{2}} z^{k-1} q(z)\right\}_{k \geq 2}
$$

where $p(z)=\frac{1}{2}(y+x z)$ and $q(z)=\frac{1}{2}(x z-y)$ are linear $E$-valued polynomials.
Example 4.4 ((Pentadiagonal)). Consider the directed tree $\mathscr{T}_{3}$ with set of vertices $V=\{(0,0),(1,1)\} \cup\{(2, i),(3, i): i \geq 1\}$ and root $=(0,0)$. We further require that $\operatorname{Chi}(0,0)=\{(1,1)\}, \operatorname{Chi}(1,1)=\{(2,1),(3,1)\}$ and

$$
\operatorname{Chi}(2, i)=\{(2, i+1)\}, \operatorname{Chi}(3, i)=\{(3, i+1)\}, \text { for all } i \geq 1 .
$$

Let $S_{\lambda}$ be a left-invertible weighted shift on $\mathscr{T}_{3}$. As in the preceding example, one can see that

$$
E:=\operatorname{ker} S_{\lambda}^{*}=\left\{\alpha e_{(0,0)}+\beta\left(\lambda_{(3,1)} e_{(2,1)}-\lambda_{(2,1)} e_{(3,1)}\right): \alpha, \beta \in \mathbb{C}\right\}
$$

Also $V_{\prec}=\{(1,1)\}$ and $k_{\mathscr{T}_{3}}=2$. Therefore, from Theorem 2.7, $\kappa_{\mathcal{H}}(\cdot, \cdot)$ takes the form

$$
\kappa_{\mathcal{H}}(z, w)=I_{E}+\sum_{\substack{j, k \geq 1 \\|j-k| \leq 2}} C_{j, k} z^{j} \bar{w}^{k}\left(z, w \in \mathbb{D}_{r_{\lambda}}\right)
$$

Moreover, for $j \geq 1, S_{\lambda^{\prime}}^{j} E \subseteq \operatorname{span}\left\{e_{v}: v \in \operatorname{Chi}^{\langle j\rangle}\{(0,0),(2,1),(3,1)\}\right\}$. Therefore,

$$
S_{\lambda^{\prime}}^{* j+1} S_{\lambda^{\prime}}^{j} E \subseteq \operatorname{span}\left\{e_{v}: v \in \operatorname{par}\{(0,0),(2,1),(3,1)\}\right\}=\operatorname{span}\left\{e_{(1,1)}\right\}
$$

which gives that $\left.P_{E} S_{\lambda^{\prime}}^{* j+1} S_{\lambda^{\prime}}^{j}\right|_{E}=0$. Thus $C_{j, k}=0$ if $|j-k|=1$. Therefore from above, $\kappa_{\mathcal{H}}(\cdot, \cdot)$ becomes

$$
\kappa_{\mathcal{H}}(z, w)=I_{E}+\sum_{\substack{j, k \geq 1 \\|j-k|=0,2}} C_{j, k} z^{j} \bar{w}^{k}\left(z, w \in \mathbb{D}_{r_{\lambda}}\right)
$$

Since $W_{n}=\{(2, n-1),(3, n-1),(2, n),(3, n),(2, n+1),(3, n+1)\}$ for $n \geq 2$, the radius of convergence $r_{\lambda}$ for $S_{\lambda}$ is given by
$\liminf _{n \rightarrow \infty}\left(\sum_{j=2}^{3}\left[\left(\lambda_{(j, n-1)}^{\prime} \cdots \lambda_{(1,1)}^{\prime}\right)^{2}+\left(\lambda_{(j, n)}^{\prime} \cdots \lambda_{(j, 1)}^{\prime}\right)^{2}+\left(\lambda_{(j, n+1)}^{\prime} \cdots \lambda_{(j, 2)}^{\prime}\right)^{2}\right]\right)^{-\frac{1}{2 n}}$.
In this case, the reproducing kernel $\kappa_{\mathcal{H}}(\cdot, \cdot)$ is pentadiagonal.
The invariant $k_{\mathscr{T}}$ may be bigger than $\operatorname{dim} E$ as shown below.
Example 4.5 ((Septadiagonal)). Consider the directed tree $\mathscr{T}_{4}$ with set of vertices $V=\{(0,0),(1,1),(2,2)\} \cup\{(3, i),(4, i): i \geq 1\}$ and root $=(0,0)$. We further require that $\operatorname{Chi}(0,0)=\{(1,1)\}, \operatorname{Chi}(1,1)=\{(2,2)\}, \operatorname{Chi}(2,2)=\{(3,1),(4,1)\}$, and

$$
\operatorname{Chi}(3, i)=\{(3, i+1)\}, \operatorname{Chi}(4, i)=\{(4, i+1)\}, \text { for all } i \geq 1
$$

It is easy to see that $\operatorname{dim} \operatorname{ker} S_{\lambda}^{*}=2, k_{\mathscr{T}_{4}}=3$. The kernel $\kappa_{\mathscr{H}}$ in this example is septadiagonal. We leave the details to the reader.

The main result also applies to a directed tree which is not locally finite.
Example 4.6. Consider the directed tree $\mathscr{T}_{\infty}$ with set of vertices $V=\{(i, j)$ : $i, j \geq 0\}$, and root $=(0,0)$. We further require that

$$
\operatorname{Chi}(i, j)= \begin{cases}\{(1, k): k \geq 0\} & \text { if }(i, j)=\text { root } \\ \{(i+1, j)\} & \text { otherwise }\end{cases}
$$

Let $S_{\lambda}$ be a bounded left-invertible weighted shift on $\mathscr{T}_{\infty}$. Then $E=\operatorname{ker} S_{\lambda}^{*}$ is of infinite dimension. Also $V_{\prec}=\{$ root $\}$, and hence $k_{\mathscr{T}_{\infty}}=1$. By Theorem 2.7, the reproducing kernel $\kappa_{\mathscr{H}}$ is tridiagonal.

REmARK 4.7. In general, $S_{\lambda}$ is not unitarily equivalent to orthogonal direct sum of unilateral weighted shifts. To see this, consider the weighted shift $S_{\lambda}$ on $\mathscr{T}_{\infty}$, and suppose that $S_{\lambda}$ is unitarily equivalent to direct sum $T:=\oplus_{i=1}^{\infty} T_{i}$ of unilateral weighted shifts $T_{i}$. Choose weights of $S_{\lambda}$ such that $\lambda_{(2,0)} \neq \lambda_{(2,1)}$. In this case,

$$
\left\langle S_{\lambda}^{2} g_{1}, S_{\lambda} g_{2}\right\rangle=\lambda_{(1,0)} \lambda_{(1,1)}\left(\lambda_{(2,0)}^{2}-\lambda_{(2,1)}^{2}\right) \neq 0
$$

where $g_{1}=e_{\text {root }}$ and $g_{2}=\lambda_{(1,1)} e_{(1,0)}-\lambda_{(1,0)} e_{(1,1)}$ belong to ker $S_{\lambda}^{*}$. However, $\left\langle T^{m} X, T^{n} Y\right\rangle=0$ for any $X, Y \in \operatorname{ker} T^{*}$ and for any positive integers $m, n$ such that $m \neq n$.

Table 1.

| Directed Tree $\mathscr{T}$ | Dimension of ker $S_{\lambda}^{*}$ | $k_{\mathscr{T}}$ | Form of $\kappa_{\mathscr{H}}(z, w)$ |
| :---: | :---: | :---: | :---: |
| $\mathscr{T}_{1}$ | 1 | 0 | diagonal |
| $\mathscr{T}_{2}$ | 2 | 1 | tridiagonal |
| $\mathscr{T}_{3}$ | 2 | 2 | pentadiagonal |
| $\mathscr{T}_{4}$ | 2 | 3 | septadiagonal |
| $\mathscr{T}_{\infty}$ | $\infty$ | 1 | tridiagonal |

## 5. Spectral Picture of $S_{\lambda}$

In this section, we use analytic model constructed in Sections 2 and 3 to discuss spectral theory of weighted shifts $S_{\lambda}$ on rooted directed trees. This part has an overlap with [ $\mathbf{9}$, Theorems 2.1 and 2.3], where the spectral picture of certain weighted composition operators is described. However, the conclusion of (i)-(iii) of Theorem 5.1 can not be deduced from the aforementioned results of [9] as the directed trees considered in this part need not be locally finite. On the other hand, in the context of rooted directed trees, weighted shifts always have connected spectrum. This is in contrast with [9, Example 5], where a composition operator with disconnected spectrum has been constructed. Positively, the power of analytic model comes into the picture while computing the point spectra of $S_{\lambda}$ and $S_{\lambda}^{*}$. In this regard, the rather technical proof of [ $\mathbf{9}$, Theorem 2.1] should be compared with that of (i) and (ii) of Theorem 5.1.

Before we state the main result of this section, we recall a couple of known facts about $S_{\lambda}$.

Any weighted shift $S_{\lambda}$ on a directed tree is circular [20, Theorem 3.3.1]: For every $\theta \in \mathbb{R}$, there exists a unitary $U_{\theta}$ on $l^{2}(V)$ such that $U_{\theta} S_{\lambda}=e^{i \theta} S_{\lambda} U_{\theta}$. An immediate consequence of this shows that all spectral parts of $S_{\lambda}$ have circular symmetry about 0 [ $\mathbf{2 0}$, Corollary 3.3.2].

Here is the statement of the main result of this section.
ThEOREM 5.1. Let $S_{\lambda} \in B\left(l^{2}(V)\right)$ be a left-invertible weighted shift on $\mathscr{T}$ and let $E:=\operatorname{ker} S_{\lambda}^{*}$. Then we have the following.
(i) The point spectrum $\sigma_{p}\left(S_{\lambda}\right)$ of $S_{\lambda}$ is empty.
(ii) If $r_{\lambda}$ is the radius of convergence for $S_{\lambda}$ then

$$
\mathbb{D}_{r_{\lambda}} \subseteq \sigma_{p}\left(S_{\lambda}^{*}\right) \subseteq \sigma\left(S_{\lambda}\right)=\overline{\mathbb{D}}_{r\left(S_{\lambda}\right)}
$$

(iii) $\bigvee\left\{\operatorname{ker}\left(S_{\lambda}^{*}-w\right): w \in \mathbb{D}_{\epsilon}\right\}=l^{2}(V)$ for every positive number $\epsilon$.

If, in addition, $E$ is finite dimensional then
(iv) $\sigma_{a p}\left(S_{\lambda}\right)=\sigma_{e}\left(S_{\lambda}\right)$ is a union of at most $\operatorname{dim} E$ number of annuli centered at the origin.
(v) the Fredholm index $\operatorname{ind}\left(S_{\lambda}-w\right)$ of $S_{\lambda}-w$ is at least $-\operatorname{dim} E$ on any connected component of $\mathbb{C} \backslash \sigma_{e}\left(S_{\lambda}\right)$. Moreover, ind $\left(S_{\lambda}-w\right)$ is exactly $-\operatorname{dim} E$ on the connected component of $\mathbb{C} \backslash \sigma_{e}\left(S_{\lambda}\right)$ that contains 0.
(vi) for any positive integer $k$,

$$
\operatorname{dim}\left(\operatorname{ker} S_{\lambda}^{* k} / \operatorname{ker} S_{\lambda}^{* k-1}\right)=\operatorname{dim} E
$$

REMARK 5.2. Since $r_{\lambda} r\left(S_{\lambda^{\prime}}\right) \geq 1$ (Theorem 2.7), by the inclusion in (ii), $r\left(S_{\lambda}\right) r\left(S_{\lambda^{\prime}}\right) \geq 1$. This inequality is sharp. In fact, if $S_{\lambda}$ is an isometry then
$r\left(S_{\lambda}\right)=1=r\left(S_{\lambda^{\prime}}\right)$, so that equality holds in $r\left(S_{\lambda}\right) r\left(S_{\lambda^{\prime}}\right) \geq 1$. Also, if $r\left(S_{\lambda}\right)=1=$ $r\left(S_{\lambda^{\prime}}\right)$ then $r_{\lambda}$ is necessarily equal to 1 . Finally, since $S_{\lambda}$ is analytic, the part (vi) above precisely says that $S_{\lambda}^{*}$ is an abstract backward shift in the sense of [7] and [26].

In the proof of Theorem 5.1, we need the analytic model as well as a number of general facts about $S_{\lambda}$. The first of which generalizes a well-known fact that the spectrum of a weighted shift is connected [28, Theorem 4] (see also [16, Theorem 8], [26, Theorem 3.5]).

Lemma 5.3. The spectrum of an analytic operator is connected.
Proof. Let $T \in B(\mathcal{H})$ be analytic. We adapt the technique of [10, Lemma 3.8] to the present situation. Since $T$ is analytic,

$$
\begin{equation*}
\bigvee_{k \geq 0} \operatorname{ker} T^{* k}=\mathcal{H} \tag{17}
\end{equation*}
$$

Therefore, $0 \in \sigma(T)$. Let $K_{1}$ be the connected component of $\sigma\left(T^{*}\right)$ containing 0 and $K_{2}=\sigma\left(T^{*}\right) \backslash K_{1}$. If possible, suppose that $K_{2}$ is non-empty. Then by Riesz Decomposition Theorem [11, Chapter VII, Proposition 4.11], there are closed subspaces $\mathcal{H}_{1}$ and $\mathcal{H}_{2}$ invariant under $T^{*}$ such that $\mathcal{H}=\mathcal{H}_{1} \oplus \mathcal{H}_{2}, \sigma\left(\left.T^{*}\right|_{\mathcal{H}_{1}}\right)=K_{1}$ and $\sigma\left(\left.T^{*}\right|_{\mathcal{H}_{2}}\right)=K_{2}$. Let $h \in \operatorname{ker} T^{* k}$. Then $h=x+y$ for $x \in \mathcal{H}_{1}$ and $y \in \mathcal{H}_{2}$. Since $T^{* k} h=0$, it follows that $T^{* k} x=0=T^{* k} y$. If $y$ is non-zero, then $0 \in \sigma_{p}\left(\left.T^{* k}\right|_{\mathcal{H}_{2}}\right) \subseteq$ $\sigma\left(\left.T^{* k}\right|_{\mathcal{H}_{2}}\right)$, and hence by spectral mapping property, $0 \in \sigma\left(\left.T^{*}\right|_{\mathcal{H}_{2}}\right)=K_{2}$, which is a contradiction. So $y$ must be zero. Therefore, $\mathcal{H}_{1}$ contains $\operatorname{ker} T^{* k}$ for all $k \geq 0$. Hence from (17), we get $\mathcal{H}_{1}=\mathcal{H}$, and hence $K_{2}$ must be empty. This is contrary to the assumption that $K_{2} \neq \emptyset$. This shows that $\sigma\left(T^{*}\right)$ is connected. Since $\sigma(T)=\left\{\bar{z}: z \in \sigma\left(T^{*}\right)\right\}$ and $z \rightsquigarrow \bar{z}$ is continuous, $\sigma(T)$ is connected.

Lemma 5.4. Let $S_{\lambda}$ be a weighted shift on $\mathscr{T}$ and let $d:=\operatorname{card}\left(\mathrm{Chi}^{\left\langle{ }^{〔} \boldsymbol{T}\right\rangle}\right.$ (root)) (possibly infinite). Then there exist subspaces $\mathcal{M}$ and $\mathcal{H}_{i}(i=1, \cdots, d)$ such that

$$
S_{\lambda}=\left[\begin{array}{ccccc}
A & 0 & 0 & \cdots & 0  \tag{18}\\
A_{1} & S_{1} & 0 & \cdots & 0 \\
A_{2} & 0 & S_{2} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
A_{d} & 0 & \cdots & 0 & S_{d}
\end{array}\right] \quad \text { on } l^{2}(V)=\mathcal{M} \oplus \mathcal{H}_{1} \oplus \cdots \oplus \mathcal{H}_{d}
$$

where $A:=\left.P_{\mathcal{M}} S_{\lambda}\right|_{\mathcal{M}}, A_{i}:=\left.P_{\mathcal{H}_{i}} S_{\lambda}\right|_{\mathcal{M}}$, and $S_{i}:=\left.S_{\lambda}\right|_{\mathcal{H}_{i}}$ for $i=1, \cdots, d$. Moreover, the following statements hold.
(i) Each $S_{i}$ is unitarily equivalent to a unilateral weighted shift.
(ii) Let $\mathscr{T}$ possess the property that $v \in \mathrm{Chi}^{\left\langle{ }^{\langle\mathscr{T}-1\rangle}(\text { root }) \text { whenever } \operatorname{card}(\operatorname{Chi}(v)) ~\right.}$ is infinite for some $v \in W_{0}$. Then $S_{\lambda}$ is a finite rank perturbation of $S_{1} \oplus \cdots \oplus S_{d}$.
Proof. Note that for all $v \in \mathrm{Chi}^{\left\langle{ }_{\mathscr{G}}\right\rangle}($ root $), \operatorname{card}(\operatorname{Chi}(v))=1$. Let $W_{-1}$ be as defined in (8). We relabel the set $V$ of vertices as follows:

$$
V=W_{-1} \sqcup\left\{v_{i, n}: n \geq 0, i=1, \cdots, d\right\}
$$

such that $\operatorname{Chi}^{\left\langle{ }^{\langle\mathscr{T}\rangle}(\text { root })\right.}=\left\{v_{i, 0}: i=1, \cdots, d\right\}$, and $\operatorname{Chi}\left(v_{i, n}\right)=\left\{v_{i, n+1}\right\}$ for all $n \geq 0, i=1, \cdots, d$. Now consider the subspaces $\mathcal{M}$ and $\mathcal{H}_{i}$ of $l^{2}(V)$ given by

$$
\mathcal{M}:=\bigvee\left\{e_{v}: v \in W_{-1}\right\}, \mathcal{H}_{i}:=\bigvee\left\{e_{v_{i, n}}: n \geq 0\right\}, i=1, \cdots, d
$$

Note that the subspaces $\mathcal{H}_{1}, \cdots, \mathcal{H}_{d}$ are invariant under $S_{\lambda}$. Then $S_{\lambda}$ admits the decomposition as given by (18). Since $S_{i} e_{v_{i, n}}=\lambda_{v_{i, n+1}} e_{v_{i, n+1}}(n \geq 0)$, it is clear that each $S_{i}$ is unitarily equivalent to a unilateral weighted shift.

To see (ii), note that if $\operatorname{card}\left(W_{-1}\right)$ is infinite then for some $v \in W_{-1} \subseteq W_{0}$, we must have $\operatorname{Chi}(v) \subseteq W_{-1}$ and $\operatorname{card}(\operatorname{Chi}(v))$ is infinite. But then by hypothesis, $v \in \operatorname{Chi}^{\left\langle k_{\mathscr{G}-1\rangle}\right.}$ (root), which implies that $\operatorname{Chi}(v) \cap W_{-1}=\emptyset$. Thus we arrive at a contradiction. This shows that $\operatorname{card}\left(W_{-1}\right)$ is finite, and hence $\mathcal{M}$ is finite dimensional. Thus $A, A_{1}, \cdots, A_{d}$ are finite rank operators, and the conclusion in (ii) is now immediate.

The following is certainly known. We include it for the sake of completeness.
Lemma 5.5. Let $T \in B(\mathcal{H})$ be finitely cyclic. If $\sigma_{p}(T)$ is empty then $\sigma_{a p}(T)=$ $\sigma_{e}(T)$.

Proof. By $\left[\mathbf{1 7}\right.$, Proposition 1(i)], $\operatorname{dim} \operatorname{ker}\left(T^{*}-w\right)$ is finite for every $w \in \mathbb{C}$. If $\sigma_{p}(T)=\emptyset$ then it is easy to see that $\sigma_{a p}(T)=\sigma_{e}(T)$.

We also need exact description of the kernel of positive integral powers of $S_{\lambda}^{*}$ in the proof of Theorem 5.1.

Lemma 5.6. Let $S_{\lambda} \in B\left(l^{2}(V)\right)$ be a weighted shift on $\mathscr{T}=(V, \mathcal{E})$. Then, for all integers $k \geq 1$,

$$
\begin{equation*}
\operatorname{ker} S_{\lambda}^{* k}=\bigvee\left\{e_{v}: v \in \cup_{i=0}^{k-1} \mathrm{Chi}^{\langle i\rangle}(\mathrm{root})\right\} \oplus \bigoplus_{v \in W_{-1}}\left(l^{2}\left(\mathrm{Chi}^{\langle k\rangle}(v)\right) \ominus\left\langle\boldsymbol{\lambda}_{k}^{v}\right\rangle\right) \tag{19}
\end{equation*}
$$

where $\boldsymbol{\lambda}_{k}^{v}: \mathrm{Chi}^{\langle k\rangle}(v) \rightarrow \mathbb{C}$ is defined by $\boldsymbol{\lambda}_{k}^{v}(u)=\lambda_{u} \lambda_{\operatorname{par}(u)} \cdots \lambda_{\operatorname{par}}{ }^{\langle k-1\rangle}(u)$, and $W_{-1}$ is given by (8). Consequently,

$$
\begin{equation*}
\operatorname{dim} \operatorname{ker} S_{\lambda}^{* k}=\sum_{i=0}^{k-1} \operatorname{card}\left(\mathrm{Chi}^{\langle i\rangle}(\mathrm{root})\right)+\sum_{v \in W_{-1}}\left(\operatorname{card}\left(\mathrm{Chi}^{\langle k\rangle}(v)\right)-1\right) \tag{20}
\end{equation*}
$$

Proof. Following the lines of the proof of [20, Proposition 3.5.1], one can easily deduce that for all integers $k \geq 1$,

$$
\operatorname{ker} S_{\lambda}^{* k}=\bigvee\left\{e_{v}: v \in \cup_{i=0}^{k-1} \mathrm{Chi}^{\langle i\rangle}(\text { root })\right\} \oplus \bigoplus_{v \in V}\left(l^{2}\left(\mathrm{Chi}^{\langle k\rangle}(v)\right) \ominus\left\langle\boldsymbol{\lambda}_{k}^{v}\right\rangle\right)
$$

From Lemma 3.1(ii), we know that for a vertex $v \in V, \operatorname{card}(\operatorname{Chi}(v))=1$ if $n_{v} \geq k_{\mathscr{T}}$. This is equivalent to the fact that $\operatorname{card}\left(\operatorname{Chi}^{\langle m\rangle}(v)\right)=1$ for all $m \geq 1$ if $v \notin W_{-1}$. Hence, $l^{2}\left(\mathrm{Chi}^{\langle k\rangle}(v)\right) \ominus\left\langle\boldsymbol{\lambda}_{k}^{v}\right\rangle=\{0\}$ if $v \notin W_{-1}$. This proves (19). The proof of (20) is obvious in view of (19).

Proof of Theorem 5.1. In view of Theorem 2.7, it is sufficient to work with the analytic model $\left(\mathscr{M}_{z}, \kappa_{\mathscr{H}}, \mathscr{H}\right)$ of $S_{\lambda}$, where the reproducing kernel Hilbert space $\mathscr{H}$ consists of $E$-valued holomorphic functions $U_{f}$ on the disc $\mathbb{D}_{r_{\lambda}}$ given by

$$
U_{f}(z):=\sum_{n \geq 0}\left(P_{E} S_{\lambda^{\prime}}^{* n} f\right) z^{n}
$$

We check that $\sigma_{p}\left(\mathscr{M}_{z}\right)=\emptyset$. Let $w \in \mathbb{C}$ and $h=\sum_{k=0}^{\infty} a_{n} z^{n} \in \mathscr{H}$ be such that $\left(\mathscr{M}_{z}-w\right) h=0$, where $\left\{a_{n}\right\}_{n \geq 0} \subseteq E$. Then for any $g \in E$,

$$
(z-w) \sum_{k=0}^{\infty}\left\langle a_{n}, g\right\rangle z^{n}=0 \text { for all } z \in \mathbb{D}_{r_{\lambda}}
$$

It follows that $\left\langle a_{n}, g\right\rangle=0$ for $g \in E$, and hence $a_{n}=0$ every $n \geq 0$. This shows that $h=0$, which gives (i).

To see (ii), note that for $f \in l^{2}(V), g \in E$ and $w \in \mathbb{D}_{r_{\lambda}}$, by Theorem 2.7(i),

$$
\left\langle U_{f}, \mathscr{M}_{z}^{*} \kappa_{\mathscr{H}}(\cdot, w) g\right\rangle=\left\langle\mathscr{M}_{z} U_{f}, \kappa_{\mathscr{H}}(\cdot, w) g\right\rangle=\left\langle w U_{f}(w), g\right\rangle=\left\langle U_{f}, \bar{w} \kappa_{\mathscr{H}}(\cdot, w) g\right\rangle .
$$

Thus $\mathscr{M}_{z}^{*} \kappa_{\mathscr{H}}(\cdot, w) g=\bar{w} k_{\mathscr{H}}(\cdot, w) g$ for all $w \in \mathbb{D}_{r_{\lambda}}$ and $g \in E$. Hence the point spectrum of $\mathscr{M}_{z}^{*}$ contains $\mathbb{D}_{r_{\lambda}}$. As recorded earlier, $S_{\lambda}$ is circular, so that $\sigma\left(S_{\lambda}^{*}\right)=$ $\sigma\left(S_{\lambda}\right)$. The second inclusion in (ii) now follows from $\sigma_{p}\left(S_{\lambda}^{*}\right) \subseteq \sigma\left(S_{\lambda}^{*}\right)$. To complete the proof of (ii), it is only left to check that $\sigma\left(S_{\lambda}\right)=\overline{\mathbb{D}}_{r\left(S_{\lambda}\right)}$. By Lemmas 3.4 and $5.3, \sigma\left(S_{\lambda}^{*}\right)$ is connected. Now, suppose that $\sigma\left(S_{\lambda}\right) \neq \overline{\mathbb{D}}_{r\left(S_{\lambda}\right)}$. Since $r\left(S_{\lambda}\right) \in \sigma\left(S_{\lambda}\right)$, there is a $w_{0} \in \mathbb{D}_{r\left(S_{\lambda}\right)}$ such that $w_{0} \in \rho\left(S_{\lambda}\right):=\mathbb{C} \backslash \sigma\left(S_{\lambda}\right)$. Since $\rho\left(S_{\lambda}\right)$ is open, there is an $\epsilon>0$ such that $\mathbb{D}_{\epsilon}\left(w_{0}\right):=\left\{w \in \mathbb{C}:\left|w-w_{0}\right|<\epsilon\right\} \subseteq \rho\left(S_{\lambda}\right) \cap \mathbb{D}_{r\left(S_{\lambda}\right)}$. Since $S_{\lambda}$ is circular and $\sigma\left(S_{\lambda}\right)$ is connected, we arrive at a contradiction. Thus, the spectrum of $S_{\lambda}$ is a disk of radius $r\left(S_{\lambda}\right)$ centred at the origin.

Suppose that $U_{f}$ is orthogonal to $\bigvee\left\{\operatorname{ker}\left(\mathscr{M}_{z}^{*}-\bar{w}\right): w \in \mathbb{D}_{\epsilon}\right\}$. By the preceding paragraph, $\kappa_{\mathscr{H}}(\cdot, w) g$ belongs to $\operatorname{ker}\left(\mathscr{M}_{z}^{*}-\bar{w}\right)$ for every $w \in \mathbb{D}_{r_{\lambda}}$. Hence

$$
\sum_{n \geq 0}\left\langle P_{E} S_{\lambda^{\prime}}^{* n} f, g\right\rangle w^{n}=\left\langle U_{f}(w), g\right\rangle=\left\langle U_{f}, \kappa_{\mathscr{H}}(\cdot, w) g\right\rangle=0
$$

for all $w \in \mathbb{D}_{\epsilon} \cap \mathbb{D}_{r_{\lambda}}$ and $g \in E$. This implies that $\left\langle P_{E} S_{\lambda^{\prime}}^{* n} f, g\right\rangle=0$ for all $n \geq 0$ and $g \in E$. In particular, $\left\langle P_{E} S_{\lambda^{\prime}}^{* n} f, P_{E} S_{\lambda^{\prime}}^{* n} f\right\rangle=0$ for all $n \geq 0$. Thus $U_{f}=0$. Therefore, $\bigvee\left\{\operatorname{ker}\left(\mathscr{M}_{z}^{*}-\bar{w}\right): w \in \mathbb{D}_{\epsilon}\right\}=\mathscr{H}$.

Assume now that $E$ is finite dimensional. By Corollary $2.10, S_{\lambda}$ is finitely cyclic. By (i) above and Lemma 5.5, $\sigma_{a p}\left(S_{\lambda}\right)=\sigma_{e}\left(S_{\lambda}\right)$. Thus to see (iv), it suffices to check that $\sigma_{e}\left(S_{\lambda}\right)$ is a finite union of annuli centered at the origin. Let $d:=$ $\operatorname{card}\left(\mathrm{Chi}^{\left\langle{ }_{\mathscr{J}}\right\rangle}\right.$ (root)), which is finite in view of Proposition 2.3. Also, since $E$ is finite dimensional, by the same proposition $\mathscr{T}$ is locally finite. Hence by Lemma 5.4(ii), there exist unilateral weighted shifts $S_{1}, \cdots, S_{d}$ such that $S_{\lambda}$ is a finite rank perturbation of $S_{1} \oplus \cdots \oplus S_{d}$. In particular, the essential spectrum of $S_{\lambda}$ equals the union of essential spectrum of $S_{1}, \cdots, S_{d}[\mathbf{1 1}]$. Another application of Lemma 5.5 shows that $\sigma_{a p}\left(S_{i}\right)=\sigma_{e}\left(S_{i}\right)$. However, the approximate point spectrum of a unilateral weighted shift is necessarily an annulus centered at the origin $[\mathbf{2 7}$, Theorem 1]. The desired conclusion in (iv) is now immediate.

Let us now see part (v). For any $w \in \mathbb{C}$, note that

$$
\operatorname{ind}\left(S_{\lambda}-w\right)=\operatorname{ind} \oplus_{i=1}^{d}\left(S_{i}-w\right) \stackrel{(\mathrm{i})}{=}-\oplus_{i=1}^{d} \operatorname{dim} \operatorname{ker}\left(S_{i}^{*}-\bar{w}\right)
$$

Since $\operatorname{dim} \operatorname{ker}\left(S_{i}^{*}-\bar{w}\right)$ at most one, $\operatorname{ind}\left(S_{\lambda}-w\right)$ is at least $-\operatorname{dim} E$. However, $\operatorname{dim} \operatorname{ker} S_{i}^{*}=1$ for all $i$, and hence $-\operatorname{dim} E=\operatorname{ind} S_{\lambda}=-d$. Note that the proof above shows that $\operatorname{card}\left(\mathrm{Chi}^{\left\langle{ }^{\prime}{ }^{T}\right\rangle}(\right.$ root $\left.)\right)=\operatorname{dim} E$, and hence by Lemma 3.1(ii),

$$
\begin{equation*}
\operatorname{card}\left(\operatorname{Chi}^{\langle k\rangle}(\text { root })\right)=\operatorname{dim} E \text { for all integers } k \geq k_{\mathscr{T}} . \tag{21}
\end{equation*}
$$

To see (vi), fix an integer $k \geq 1$ and let $\mathscr{Q}_{k}:=\operatorname{ker} S_{\lambda}^{* k} / \operatorname{ker} S_{\lambda}^{* k-1}$. Then using (20), we get

$$
\begin{align*}
\operatorname{dim} \mathscr{Q}_{k} & =\operatorname{card}\left(\mathrm{Chi}^{\langle k-1\rangle}(\text { root })\right)+\sum_{v \in W_{-1}} \operatorname{card}\left(\mathrm{Chi}^{\langle k\rangle}(v)\right)-\sum_{v \in W_{-1}} \operatorname{card}\left(\mathrm{Chi}^{\langle k-1\rangle}(v)\right) \\
& =\sum_{v \in W_{-1}} \operatorname{card}\left(\mathrm{Chi}^{\langle k\rangle}(v)\right)-\sum_{v \in W_{-1} \backslash\{\text { root }\}} \operatorname{card}\left(\mathrm{Chi}^{\langle k-1\rangle}(v)\right) \tag{22}
\end{align*}
$$

Since Chi ${ }^{\langle l\rangle}($ root $)=$ Chi $^{\langle l-1\rangle}($ Chi(root $\left.)\right)$, it follows that

$$
\operatorname{card}\left(\mathrm{Chi}^{\langle l\rangle}(\text { root })\right)=\sum_{v \in \mathrm{Chi}(\text { root })} \operatorname{card}\left(\mathrm{Chi}^{\langle l-1\rangle}(v)\right)
$$

for any positive integer $l$. Therefore, $\sum_{v \in W_{-1}} \operatorname{card}\left(\mathrm{Chi}^{\langle k\rangle}(v)\right)$ is equal to

$$
\begin{aligned}
& \operatorname{card}\left(\mathrm{Chi}^{\langle k\rangle}(\text { root })\right)+\sum_{v \in \mathrm{Chi}(\text { root })} \operatorname{card}\left(\mathrm{Chi}^{\langle k\rangle}(v)\right)+\cdots+\sum_{v \in \mathrm{Chi}^{\langle k} \mathscr{F}^{-1\rangle}(\text { root })} \operatorname{card}\left(\mathrm{Chi}^{\langle k\rangle}(v)\right) \\
& \quad=\operatorname{card}\left(\mathrm{Chi}^{\langle k\rangle}(\text { root })\right)+\operatorname{card}\left(\mathrm{Chi}^{\langle k+1\rangle}(\text { root })\right)+\cdots+\operatorname{card}\left(\mathrm{Chi}^{\left\langle k+k_{\mathscr{T}}-1\right\rangle}(\text { root })\right) .
\end{aligned}
$$

Similarly, $\sum_{v \in W_{-1} \backslash\{\text { root }\}} \operatorname{card}\left(\mathrm{Chi}^{\langle k-1\rangle}(v)\right)$ is equal to
$\operatorname{card}\left(\mathrm{Chi}^{\langle k\rangle}(\right.$ root $\left.)\right)+\operatorname{card}\left(\mathrm{Chi}^{\langle k+1\rangle}(\right.$ root $\left.)\right)+\cdots+\operatorname{card}\left(\mathrm{Chi}^{\langle k-1+k \mathscr{T}-1\rangle}\right.$ (root) $)$.
Substituting last two identities in (22), we get

$$
\operatorname{dim} \mathscr{Q}_{k}=\operatorname{card}\left(\mathrm{Chi}^{\left\langle k+k_{\mathscr{F}}-1\right\rangle}(\text { root })\right) \stackrel{(21)}{=} \operatorname{dim} E .
$$

This completes the proof of the theorem.
Remark 5.7. The identity (21), as established in the proof of Theorem 5.1(v), comes surprisingly as a consequence of index theory. As evident, this identity is otherwise difficult to disclose. Note that the left hand side of (21) is a variant dependent on $\mathscr{T}$ while the right hand side of (21) depends solely on $S_{\lambda}$. Further, since $\operatorname{dim} E$ is finite, by Proposition 2.3, $\mathscr{T}$ is locally finite and $\operatorname{card}\left(V_{\prec}\right)<\infty$. Therefore, using (4) and (21), one gets the following.

$$
\operatorname{card}\left(\operatorname{Chi}^{\left\langle k_{\mathscr{T}}\right\rangle}(\text { root })\right)=1-\operatorname{card}\left(V_{\prec}\right)+\sum_{v \in V_{\prec}} \operatorname{card}(\operatorname{Chi}(v))
$$

One particular consequence of Theorem $5.1(\mathrm{iv})$ is that $\sigma_{a p}\left(S_{\lambda}\right)$ (resp. $\sigma_{e}\left(S_{\lambda}\right)$ ) of a weighted shift $S_{\lambda}$ on a directed tree could be disconnected. For instance, in case $\operatorname{dim} E=2$, by choosing the weight sequence $\lambda$ appropriately (so that the approximate point spectra of $S_{1}$ and $S_{2}$, as appearing in the proof of Theorem 5.1, are disjoint annuli), we can have two connected components of $\sigma_{a p}\left(S_{\lambda}\right)$. Moreover, the index of $S_{\lambda}-w$ may vary from -2 to 0 on different components of $\mathbb{C} \backslash \sigma_{e}\left(S_{\lambda}\right)$. Again, in the above situation,
ind $\left(S_{\lambda}-w\right)=\left\{\begin{array}{l}-2 \text { on a bounded component of } \mathbb{C} \backslash \sigma_{e}\left(S_{\lambda}\right) \text { containing } 0 \\ -1 \text { on a bounded component of } \mathbb{C} \backslash \sigma_{e}\left(S_{\lambda}\right) \text { not containing } 0 .\end{array}\right.$
This is not possible in case $\operatorname{dim} E=1$ in view of [27, Theorem 1].
The conclusion of Theorem 5.1(iv) need not be true in case $\operatorname{dim} E$ is infinite.
Example 5.8. Let $\mathscr{T}_{\infty}$ be the directed tree as discussed in Example 4.6 and let $S_{\lambda}$ be a left-invertible weighted shift on $\mathscr{T}_{\infty}$. For a given $\mu>0$, choose the weight sequence of $S_{\lambda}$ such that for each $j \geq 0$, the sequence $\left\{\lambda_{(i+1, j)}\right\}_{i \geq 0}$ converges to $\mu$. As seen in the proof of Lemma 5.4, $S_{\lambda}$ is a rank one perturbation of the direct sum of unilateral weighted shifts $S_{j}$ on $\mathcal{H}_{j}$. Thus $\sigma_{e}\left(S_{\lambda}\right)=\sigma_{e}\left(\oplus_{j=1}^{\infty} S_{j}\right)$. Note that $\sigma_{e}\left(S_{j}\right)=\sigma_{a p}\left(S_{j}\right)$ is the circle of radius $\mu$ centered at the origin [28]. Since $\sigma_{p}\left(S_{j}^{*}\right)=\mathbb{D}_{\mu}$, the the essential spectrum of $\oplus_{j=1}^{\infty} S_{j}$ contains $\mathbb{D}_{\mu}$. As essential spectrum is always closed, $\overline{\mathbb{D}}_{\mu} \subseteq \sigma_{e}\left(\oplus_{j=1}^{\infty} S_{j}\right)$. Also,

$$
\sigma_{e}\left(\oplus_{j=1}^{\infty} S_{j}\right) \subseteq \sigma\left(\oplus_{j=1}^{\infty} S_{j}\right)=\overline{\mathbb{D}}_{\mu}
$$

This shows that $\sigma_{e}\left(S_{\lambda}\right)=\overline{\mathbb{D}}_{\mu}$. On the other hand, $0 \notin \sigma_{a p}\left(S_{\lambda}\right)$ since $S_{\lambda}$ is leftinvertible. In particular, $\sigma_{a p}\left(S_{\lambda}\right) \neq \sigma_{e}\left(S_{\lambda}\right)$.

We see below that $S_{\lambda}$ belongs to the Cowen-Douglas class (refer to [13]; refer also to $[\mathbf{1 4}]$ for the extended definition of $B_{n}(\Omega)$ in case $n$ is not finite).

Corollary 5.9. Let $S_{\lambda} \in B\left(l^{2}(V)\right)$ be a left-invertible weighted shift on $\mathscr{T}$ and let $S_{\lambda^{\prime}}$ denote the Cauchy dual of $S_{\lambda}$. Let $E:=\operatorname{ker} S_{\lambda}^{*}$ and $\delta:=\frac{1}{\left\|S_{\lambda^{\prime}}\right\|}$. Then $S_{\lambda}^{*}$ belongs to Cowen-Douglas class $B_{\operatorname{dim} E}\left(\mathbb{D}_{\delta}\right)$.

Proof. Since $S_{\lambda}$ is left-invertible,

$$
\left(S_{\lambda}^{*} S_{\lambda}\right)^{-1}=S_{\lambda^{\prime}}^{*} S_{\lambda^{\prime}} \leq\left\|S_{\lambda^{\prime}}^{*} S_{\lambda^{\prime}}\right\| I
$$

That is, $S_{\lambda}^{*} S_{\lambda} \geq \frac{1}{\left\|S_{\lambda^{\prime}}\right\|^{2}} I=\delta^{2} I$, which gives $\left\|S_{\lambda} f\right\| \geq \delta\|f\|$ for all $f \in l^{2}(V)$. Therefore, $\sigma_{a p}\left(S_{\lambda}\right) \cap \mathbb{D}_{\delta}=\emptyset$. It follows that for all $w \in \mathbb{D}_{\delta}, \operatorname{ker}\left(S_{\lambda}-w\right)=\{0\}$ and $\operatorname{ran}\left(S_{\lambda}-w\right)$ is closed. Hence $\operatorname{ran}\left(S_{\lambda}^{*}-w\right)$ is dense in $\mathscr{H}$ for all $w \in \mathbb{D}_{\delta}$. Since ran $\left(S_{\lambda}-w\right)$ is closed, it follows that $\operatorname{ran}\left(S_{\lambda}^{*}-\bar{w}\right)$ is closed [11, Chapter XI, Section $6]$, and hence $\operatorname{ran}\left(S_{\lambda}^{*}-\bar{w}\right)=l^{2}(V)$ for all $w \in \mathbb{D}_{\delta}$. In case $\operatorname{dim} E<\infty$, the desired conclusion follows from (iii) and (v) of Theorem 5.1.

Suppose now the case in which $\operatorname{dim} E$ is not finite. Consider the analytic model $\left(\mathscr{M}_{z}, \kappa_{\mathscr{H}}, \mathscr{H}\right)$ of $S_{\lambda}$. We show that

$$
\left\{\kappa_{\mathscr{H}}(\cdot, w) g_{i}: i=1, \cdots, k\right\}
$$

is linearly independent in $\operatorname{ker}\left(\mathscr{M}_{z}^{*}-\bar{w}\right)$ whenever $\left\{g_{i}: i=1, \cdots, k\right\}$ is linearly independent in $E$ for every integer $k \geq 1$ and $w \in \mathbb{D}_{r_{\lambda}}$. To this end, suppose that $\kappa_{\mathscr{H}}(\cdot, w) g=0$ for some $g \in E$. Then $\left\langle U_{g}(w), g\right\rangle=\left\langle U_{g}, \kappa_{\mathscr{H}}(\cdot, w) g\right\rangle=0$. However, by (10), $U_{g}=g$ for any $g \in E$. It follows that $g=0$, and $\operatorname{dim} \operatorname{ker}\left(\mathscr{M}_{z}^{*}-\bar{w}\right)=\operatorname{dim} E$ for all $w \in \mathbb{D}_{r_{\lambda}}$.

## 6. A model for weighted shifts on rootless directed trees

In this short section, we show a way to generalize the main result of this paper to the setting of rootless directed trees. One interest in the theory of weighted shifts on rootless directed trees is due to the fact that these are composition operators in disguise (see [21, Lemma 4.3.1]).

We begin with a counter-part of branching index for rootless directed trees.
Definition 6.1. Let $\mathscr{T}=(V, \mathcal{E})$ be a rootless directed tree and let $V_{\prec}$ be the set of branching vertices of $\mathscr{T}$. We say that $\mathscr{T}$ has finite branching index if there exists a smallest non-negative integer $m_{\mathscr{T}}$ such that

$$
\mathrm{Chi}^{\langle k\rangle}\left(V_{\prec}\right) \cap V_{\prec}=\emptyset \text { for every integer } k \geq m_{\mathscr{T}} .
$$

The role of root in the notion of the branching index of a rooted directed tree is taken by a special vertex in the context of rootless directed trees with finite branching index as shown below.

Lemma 6.2. Let $\mathscr{T}=(V, \mathcal{E})$ be a rootless directed tree with finite branching index $m_{\mathscr{G}}$. Then there exists a vertex $\omega \in V$ such that

$$
\begin{equation*}
\operatorname{card}\left(\operatorname{Chi}\left(\operatorname{par}^{\langle k\rangle}(\omega)\right)\right)=1 \text { for all integers } k \geq 1 \tag{23}
\end{equation*}
$$

Moreover, if $V_{\prec}$ is non-empty then there exists a unique $\omega \in V_{\prec}$ satisfying (23).

Proof. In case $V_{\prec}=\emptyset$, then every vertex of $V$ satisfies (23). Therefore, we may assume that $V_{\prec}$ contains at least one vertex, say, $u_{0}$.

On contrary, assume that for every $u \in V_{\prec}$ there exists a positive integer $k_{u}$ (depending on $u$ ) such that

$$
\operatorname{card}\left(\operatorname{Chi}\left(\operatorname{par}^{\left\langle k_{u}\right\rangle}(u)\right)\right)=0 \text { or } \operatorname{card}\left(\operatorname{Chi}\left(\operatorname{par}^{\left\langle k_{u}\right\rangle}(u)\right)\right) \geq 2
$$

Since $\mathscr{T}$ is rootless, the first case can not occur. Hence $\operatorname{card}\left(\operatorname{Chi}\left(\operatorname{par}^{\left\langle k_{u}\right\rangle}(u)\right)\right) \geq 2$, that is, $\operatorname{par}^{\left\langle k_{u}\right\rangle}(u) \in V_{\prec}$. Define inductively $\left\{u_{n}\right\}_{n \geq 0} \subseteq V_{\prec}$ as follows. By assumption, there exists an integer $k_{u_{0}} \geq 1$ such that $u_{1}:=\operatorname{par}^{\left\langle k_{u_{0}}\right\rangle}\left(u_{0}\right) \in V_{\prec}$. By finite induction, there exist integers $k_{u_{1}}, \cdots, k_{u_{n-1}} \geq 1$ such that

$$
u_{n}:=\operatorname{par}^{\left\langle k_{u_{0}}+k_{u_{1}} \cdots+k_{u_{n-1}}\right\rangle}\left(u_{0}\right) \in V_{\prec .} .
$$

In case $n>m_{\mathscr{T}}, u_{0} \in \mathrm{Chi}{ }^{\left\langle k_{u_{0}}+k_{u_{1}} \cdots+k_{u_{n-1}}\right\rangle}\left(V_{\prec}\right) \cap V_{\prec}$. This is not possible since $\mathrm{Chi}^{\langle k\rangle}\left(V_{\prec}\right) \cap V_{\prec}=\emptyset$ for all integers $k \geq m_{\mathscr{T}}$.

To see the uniqueness part, suppose that there exist distinct vertices $\left\{\omega_{i}\right\}_{i=1}^{N}$ in $V_{\prec}$ satisfying (23), where either $N$ is a positive integer bigger than 1 or $N$ is infinite. It is easy to see with the help of (23) that for integers $i \neq j$, $\operatorname{par}^{\left\langle k_{1}\right\rangle}\left(\omega_{i}\right) \neq \operatorname{par}^{\left\langle k_{2}\right\rangle}\left(\omega_{j}\right)$ for any non-negative integers $k_{1}$ and $k_{2}$. One may now easily verify that $\mathscr{T}$ has the separation $\mathscr{T}=\sqcup_{i=1}^{N} \mathscr{T}_{i}$, where

$$
\mathscr{T}_{i}=\left(\cup_{k \geq 1} \operatorname{Chi}^{\langle k\rangle}\left(\omega_{i}\right)\right) \cup\left\{\operatorname{par}^{\langle k\rangle}\left(\omega_{i}\right): k \geq 0\right\} .
$$

Since $\mathscr{T}$ is connected, we arrive at a contradiction.
Note that a rootless directed tree $\mathscr{T}_{0}$ with empty $V_{\prec}$ is isomorphic to the directed tree with set of vertices $\mathbb{Z}$ and $\operatorname{Chi}(n)=\{n+1\}$ for $n \in \mathbb{Z}$. As it is wellknown that any weighted shift on $\mathscr{T}_{0}$ (to be referred to as bilateral weighted shift) can be modelled as the operator of multiplication by $z$ on a Hilbert space of formal Laurent series [28, Proposition 7], we assume in the remaining part of this section that $V_{\prec}$ is non-empty.

We refer to the vertex $\omega \in V_{\prec}$ appearing in the statement of Lemma 6.2 as the generalized root of $\mathscr{T}$. The generalized root may not exist in general. For example, consider the directed tree $\mathscr{T}$ with set of vertices $V=\mathbb{Z} \times \mathbb{Z}$ such that

$$
\operatorname{Chi}(i, j)= \begin{cases}\{(i, j+1)\} & \text { if } j \neq 0 \\ \{(i, j+1),(i+1, j)\} & \text { if } j=0\end{cases}
$$

(cf. [19, Example 4.4]). In this case, $V_{\prec}=\{(i, 0): i \in \mathbb{Z}\}$, and hence the set Chi $\left(\operatorname{par}^{\langle k\rangle}((i, 0))\right)$ contains precisely two vertices for any integer $k \geq 1$.

With the notion of generalized root, we immediately obtain the following.
Lemma 6.3. Let $\mathscr{T}=(V, \mathcal{E})$ be a rootless directed tree with finite branching index $m_{\mathscr{T}}$ and generalized root $\omega$. Let $S_{\lambda} \in B\left(l^{2}(V)\right)$ be a weighted shift on $\mathscr{T}$. Let $V^{(2)}:=\left\{v_{k}:=\operatorname{par}^{\langle k\rangle}(\omega): k \geq 1\right\}$ and let $V^{(1)}:=V \backslash V^{(2)}$. Let $\mathscr{T}^{(1)}$ and $\mathscr{T}^{(2)}$ be the directed subtrees corresponding to the sets of vertices $V^{(1)}$ and $V^{(2)}$ respectively. Then $S_{\lambda}$ admits the following decomposition:

$$
S_{\lambda}=\left[\begin{array}{cc}
T_{\lambda} & \lambda_{\omega} e_{\omega} \otimes e_{v_{1}}  \tag{24}\\
0 & B_{\lambda}
\end{array}\right] \text { on } l^{2}(V)=l^{2}\left(V^{(1)}\right) \oplus l^{2}\left(V^{(2)}\right)
$$

where $T_{\lambda} \in B\left(l^{2}\left(V^{(1)}\right)\right)$ is a weighted shift on the rooted directed tree $\mathscr{T}^{(1)}$ with root $\omega$ and finite branching index $k_{\mathscr{T}(1)}=m_{\mathscr{T}}$, and $B_{\lambda} \in B\left(l^{2}\left(V^{(2)}\right)\right.$ is the backward
unilateral weighted shift given by

$$
B_{\lambda} e_{v_{k}}= \begin{cases}0 & \text { if } k=1 \\ \lambda_{v_{k-1}} e_{v_{k-1}} & \text { if } k \geq 2\end{cases}
$$

Proof. Note that $\mathscr{T}^{(1)}$ is a rooted directed tree with root $\omega$. Also, the set $V_{\prec}$ of branching vertices of $\mathscr{T}$ is contained in $V^{(1)}$ as $V^{(2)} \cap V_{\prec}=\emptyset$. It follows that $k_{\mathscr{T}^{(1)}}=1+\sup \left\{n_{v}: v \in V_{\prec}\right\}$. Since $m_{\mathscr{T}}$ is the smallest integer such that $\mathrm{Chi}{ }^{\left\langle m_{\mathscr{F}}\right\rangle}\left(V_{\prec}\right) \cap V_{\prec}=\emptyset$, we must have $\sup \left\{n_{v}: v \in V_{\prec}\right\}=m_{\mathscr{T}}-1$. This shows that $\mathscr{T}^{(1)}$ has branching index precisely $m_{\mathscr{T}}$.

Since $S_{\lambda}^{*} e_{v_{k}}=\lambda_{v_{k}} e_{v_{k+1}}, l^{2}\left(V^{(2)}\right)$ is invariant under $S_{\lambda}^{*}$. This gives us the decomposition

$$
S_{\lambda}=\left[\begin{array}{cc}
\left.S_{\lambda}\right|_{l^{2}\left(V^{(1)}\right)} & \left.P_{1} S_{\lambda}\right|_{l^{2}\left(V^{(2)}\right)} \\
0 & \left.P_{2} S_{\lambda}\right|_{l^{2}\left(V^{(2)}\right)}
\end{array}\right] \text { on } l^{2}(V)=l^{2}\left(V^{(1)}\right) \oplus l^{2}\left(V^{(2)}\right),
$$

where $P_{i}$ denotes the orthogonal projection of $l^{2}(V)$ onto $l^{2}\left(V^{(i)}\right)$ for $i=1,2$. It is easy to see that $\left.P_{1} S_{\lambda}\right|_{l^{2}\left(V^{(2)}\right)}$ is the rank one operator $\lambda_{\omega} e_{\omega} \otimes e_{v_{1}}$. That $\left.P_{2} S_{\lambda}\right|_{l^{2}\left(V^{(2)}\right)}=B_{\lambda}$ is also a routine verification.

Remark 6.4. Note that every weighted shift on a rootless directed tree with finite branching index is an extension of a weighted shift on a rooted direct tree with finite branching index.

We illustrate the result above with the help of the following simple example.
Example 6.5. Consider the directed tree $\mathscr{T}$ with set of vertices

$$
V:=\{(1, i),(2, i): i \geq 1\} \cup\{-k: k \geq 0\}
$$

We further require that $\operatorname{Chi}(-k)=-(k-1)$ if $k \geq 1, \operatorname{Chi}(0)=\{(1,1),(2,1)\}$ and

$$
\operatorname{Chi}(1, i)=\{(1, i+1)\}, \operatorname{Chi}(2, i)=\{(2, i+1)\}, \text { for all } i \geq 1
$$

In this case, the branching index $m_{\mathscr{T}}=1$ and the generalized root $\omega$ is 0 . Also, $V^{(1)}=\{(1, i),(2, i): i \geq 1\} \cup\{0\}$ and $V^{(2)}=\{-k: k \geq 1\}$. The weighted shift $T_{\lambda}$ on $\mathscr{T}^{(1)}$ as defined in the last lemma can be identified with the weighted shift on the directed tree $\mathscr{T}_{2}$ (with root 0) as discussed in Example 4.1. Further, the rank one operator $\lambda_{\omega} e_{\omega} \otimes e_{v_{1}}$ is precisely $\lambda_{0} e_{0} \otimes e_{-1}$. Finally, the backward unilateral weighted shift $B_{\lambda}$ can be identified with the adjoint of the weighted shift on the directed tree $\mathscr{T}_{1}$ (with root -1 ) as discussed in Example 2.6.

We now present a counter-part of Theorem 2.7 for rootless directed trees.
Theorem 6.6. Let $\mathscr{T}=(V, \mathcal{E})$ be a rootless directed tree with finite branching index and generalized root $\omega$. Let $S_{\lambda} \in B\left(l^{2}(V)\right)$ be a left-invertible weighted shift on $\mathscr{T}$. Then there exist a Hilbert space $\mathscr{H}$ of vector-valued Holomorphic functions in $z$ defined on a disc in $\mathbb{C}$, and a Hilbert space $\mathcal{H}$ of scalar-valued holomorphic functions in $t$ defined on a disc in $\mathbb{C}$ such that $S_{\lambda}$ is unitarily equivalent to

$$
\left[\begin{array}{cc}
\mathscr{M}_{z} & f \otimes g \\
0 & M_{t}^{*}
\end{array}\right] \text { on } \mathscr{H} \oplus \mathcal{H}
$$

where $\mathscr{M}_{z}$ is the operator of multiplication by $z$ on $\mathscr{H}, f \otimes g$ is a rank one operator with $f \in \operatorname{ker} \mathscr{M}_{z}^{*} \backslash\{0\}, g \in \operatorname{ker} M_{t}^{*} \backslash\{0\}$, and $M_{t}$ is the operator of multiplication by the co-ordinate function $t$ on $\mathcal{H}$.

Proof. By Lemma $6.3, S_{\lambda}$ admits the decomposition (24). Since $S_{\lambda}$ is leftinvertible, so are $T_{\lambda}$ and $B_{\lambda}^{*}$. The desired decomposition now follows immediately from Theorem 2.7.

REMARK 6.7. A routine calculation shows that the self-commutator $\left[S_{\lambda}^{*}, S_{\lambda}\right]:=$ $S_{\lambda}^{*} S_{\lambda}-S_{\lambda} S_{\lambda}^{*}$ of $S_{\lambda}$ (upto unitary equivalence) is equal to

$$
\left[\begin{array}{cc}
{\left[\mathscr{M}_{z}^{*}, \mathscr{M}_{z}\right]-f \otimes f} & 0 \\
0 & g \otimes g-\left[M_{t}^{*}, M_{t}\right]
\end{array}\right] .
$$

In particular, $\left[S_{\lambda}^{*}, S_{\lambda}\right]$ is compact if and only if so are $\left[\mathscr{M}_{z}^{*}, \mathscr{M}_{z}\right]$ and $\left[M_{t}^{*}, M_{t}\right]$.
We conclude this paper with one application to the spectral theory of weighted shifts on rootless directed trees (cf. [9, Theorem 2.3]).

Corollary 6.8. With the hypotheses and notations of Theorem 6.6, we have

$$
\sigma_{e}\left(S_{\lambda}\right)=\sigma_{e}\left(\mathscr{M}_{z}\right) \cup \sigma_{e}\left(M_{t}^{*}\right)
$$

If, in addition, $S_{\lambda}$ is Fredholm then so are $\mathscr{M}_{z}$ and $M_{t}$. In this case,

$$
i n d S_{\lambda}=\operatorname{ind} \mathscr{M}_{z}+1
$$

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