The maximum product of weights of cross-intersecting families

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Abstract

Two families \mathcal{A} and \mathcal{B} of sets are said to be *cross-t-intersecting* if each set in \mathcal{A} intersects each set in \mathcal{B} in at least t elements. An active problem in extremal set theory is to determine the maximum product of sizes of cross-t-intersecting subfamilies of a given family. We prove a cross-t-intersection theorem for *weighted* subsets of a set by means of a new subfamily alteration method, and use the result to provide solutions for three natural families. For $r \in [n] = \{1, 2, \ldots, n\}$, let $\binom{[n]}{r}$ be the family of r-element subsets of [n], and let $\binom{[n]}{\leq r}$ be the family of subsets of [n] that have at most r elements. Let $\mathcal{F}_{n,r,t}$ be the family of sets in $\binom{[n]}{\leq r}$ that contain [t]. We show that if $g: \binom{[m]}{\leq r} \to \mathbb{R}^+$ and $h: \binom{[m]}{\leq s} \to \mathbb{R}^+$ are functions that obey certain conditions, $\mathcal{A} \subseteq \binom{[m]}{\leq r}$, $\mathcal{B} \subseteq \binom{[m]}{\leq s}$, and \mathcal{A} and \mathcal{B} are cross-t-intersecting, then

$$\sum_{A \in \mathcal{A}} g(A) \sum_{B \in \mathcal{B}} h(B) \leq \sum_{C \in \mathcal{F}_{m,r,t}} g(C) \sum_{D \in \mathcal{F}_{n,s,t}} h(D),$$

and equality holds if $\mathcal{A} = \mathcal{F}_{m,r,t}$ and $\mathcal{B} = \mathcal{F}_{n,s,t}$. We prove this in a more general setting and characterise the cases of equality. We use the result to show that the maximum product of sizes of two cross-*t*-intersecting families $\mathcal{A} \subseteq {\binom{[m]}{r}}$ and $\mathcal{B} \subseteq {\binom{[n]}{s}}$ is ${\binom{m-t}{r-t}} {\binom{n-t}{s-t}}$ for min $\{m, n\} \ge n_0(r, s, t)$, where $n_0(r, s, t)$ is close to best possible. We obtain analogous results for families of integer sequences and for families of multisets. The results yield generalisations for $k \ge 2$ cross-*t*-intersecting families, and Erdos–Ko–Rado-type results.

1 Introduction

Unless otherwise stated, we shall use small letters such as x to denote elements of a set or non-negative integers or functions, capital letters such as X to denote sets, and calligraphic letters such as \mathcal{F} to denote *families* (that is, sets whose elements are sets themselves). The set $\{1, 2, \ldots\}$ of all positive integers is denoted by \mathbb{N} . For any $m, n \in \mathbb{N}$, the set $\{i \in \mathbb{N} : m \leq i \leq n\}$ is denoted by [m, n]. We abbreviate [1, n] to [n]. It is to be assumed that arbitrary sets and families are *finite*. We call a set Aan *r*-element set, or simply an *r*-set, if its size |A| is r. For a set X, 2^X denotes the power set of X (that is, the family of all subsets of X), $\binom{X}{r}$ denotes the family of all *r*-element subsets of X, and $\binom{X}{\leq r}$ denotes the family of all subsets of X of size at most r. For a family \mathcal{F} and a set T, we denote the family $\{F \in \mathcal{F} \colon T \subseteq F\}$ by $\mathcal{F}(T)$.

We say that a set A t-intersects a set B if A and B contain at least t common elements. A family \mathcal{A} of sets is said to be t-intersecting if every two sets in \mathcal{A} tintersect. A 1-intersecting family is also simply called an *intersecting family*. If T is a t-element subset of at least one set in a family \mathcal{F} , then we call the family of all the sets in \mathcal{F} that contain T the t-star of \mathcal{F} . A t-star of a family is the simplest example of a t-intersecting subfamily.

One of the most popular endeavours in extremal set theory is that of determining the size of a largest t-intersecting subfamily of a given family \mathcal{F} . This took off with [20], which features the classical result, known as the Erdős-Ko-Rado (EKR) Theorem, that says that if $1 \le r \le n/2$, then the size of a largest intersecting subfamily \mathcal{A} of $\binom{[n]}{r}$ is the size $\binom{n-1}{r-1}$ of every 1-star of $\binom{[n]}{r}$. If r < n/2, then, by the Hilton-Milner Theorem [28], \mathcal{A} attains the bound if and only if \mathcal{A} is a star of $\binom{[n]}{r}$. If $n/2 < r \le n$, then $\binom{[n]}{r}$ itself is intersecting. There are various proofs of the EKR Theorem (see [34, 28, 32, 18]), two of which are particularly short and beautiful: Katona's [32], introducing the elegant cycle method, and Daykin's [18], using the fundamental Kruskal–Katona Theorem [35, 33]. A sequence of results [20, 22, 48, 1] culminated in the solution of the problem for t-intersecting subfamilies of $\binom{[n]}{r}$; the solution particularly tells us that the size of a largest t-intersecting subfamily of $\binom{[n]}{r}$ is the size $\binom{n-t}{r-t}$ of a t-star of $\binom{[n]}{r}$ if and only if $n \ge (t+1)(r-t+1)$. The t-intersection problem for $2^{[n]}$ was solved by Katona [34]. These are among the most prominent results in extremal set theory. The EKR Theorem inspired a wealth of results, including generalisations (see [43, 11]), that establish how large a system of sets can be under certain intersection conditions; see [19, 23, 21, 13, 30, 31].

Two families \mathcal{A} and \mathcal{B} are said to be *cross-t-intersecting* if each set in \mathcal{A} *t*-intersects each set in \mathcal{B} . More generally, k families $\mathcal{A}_1, \ldots, \mathcal{A}_k$ (not necessarily distinct or nonempty) are said to be *cross-t-intersecting* if for every i and j in [k] with $i \neq j$, each set in \mathcal{A}_i *t*-intersects each set in \mathcal{A}_j . Cross-1-intersecting families are also simply called *cross-intersecting families*.

For t-intersecting subfamilies of a given family \mathcal{F} , the natural question to ask is how large they can be. For cross-t-intersecting families, two natural parameters arise: the sum and the product of sizes of the cross-t-intersecting families (note that the product of sizes of k families $\mathcal{A}_1, \ldots, \mathcal{A}_k$ is the number of k-tuples $(\mathcal{A}_1, \ldots, \mathcal{A}_k)$ such that $\mathcal{A}_i \in \mathcal{A}_i$ for each $i \in [k]$). It is therefore natural to consider the problem of maximising the sum or the product of sizes of k cross-t-intersecting subfamilies $\mathcal{A}_1, \ldots, \mathcal{A}_k$ of a given family \mathcal{F} . The paper [15] analyses this problem in general, particularly showing that for k sufficiently large, both the sum and the product are maxima if $\mathcal{A}_1 = \cdots = \mathcal{A}_k = \mathcal{L}$ for some largest t-intersecting subfamily \mathcal{L} of \mathcal{F} . Therefore, this problem incorporates the t-intersection problem. Solutions have been obtained for various families (see [15]), including $\binom{[n]}{r}$ [27, 41, 37, 4, 7, 45, 47, 46, 25], $2^{[n]}$ [36, 15], $\binom{[n]}{\leq r}$ [6], and families of integer sequences [39, 12, 16, 47, 49, 44, 25, 40]. Most of these results tell us that for the family \mathcal{F} under consideration and for certain values of k, the sum or the product is maximum when $\mathcal{A}_1 = \cdots = \mathcal{A}_k = \mathcal{L}$ for some largest t-star \mathcal{L} of \mathcal{F} . In such a case, \mathcal{L} is a largest *t*-intersecting subfamily of \mathcal{F} .

Remark 1.1 In general, if $\mathcal{L} \subseteq \mathcal{F}$, $k \geq 2$, and the sum or the product is maximum when $\mathcal{A}_1 = \cdots = \mathcal{A}_k = \mathcal{L}$, then \mathcal{L} is a largest *t*-intersecting subfamily of \mathcal{F} . Indeed, the cross-*t*-intersection condition implies that every two sets A and B in \mathcal{L} *t*-intersect (as $A \in \mathcal{A}_1$ and $B \in \mathcal{A}_2$), and by taking an arbitrary *t*-intersecting subfamily \mathcal{A} of \mathcal{F} and setting $\mathcal{B}_1 = \cdots = \mathcal{B}_k = \mathcal{A}$, we obtain that $\mathcal{B}_1, \ldots, \mathcal{B}_k$ are cross-*t*-intersecting, and hence $|\mathcal{A}| \leq |\mathcal{L}|$ since $k|\mathcal{A}| = \sum_{i=1}^k |\mathcal{B}_i| \leq \sum_{i=1}^k |\mathcal{A}_i| = k|\mathcal{L}|$ or $|\mathcal{A}|^k = \prod_{i=1}^k |\mathcal{B}_i| \leq \prod_{i=1}^k |\mathcal{A}_i| = |\mathcal{L}|^k$.

Wang and Zhang [47] solved the maximum sum problem for an important class of families that includes $\binom{[n]}{r}$ and families of integer sequences, using a striking combination of the method in [7, 8, 9, 16, 10] and an important lemma that is found in [3, 17] and referred to as the 'no-homomorphism lemma'. The solution for $\binom{[n]}{r}$ with t = 1 had been obtained by Hilton [27] and is the first result of this kind.

In this paper we address the maximum product problem for $\binom{[n]}{r}$ and families of integer sequences. We will actually consider more general problems; one generalisation allows the cross-*t*-intersecting families to come from different families, and another one involves maximising instead the product of *weights* of cross-*t*-intersecting families of subsets of a set. As we explain in the next section, if the product for k = 2 is maximum when the cross-*t*-intersecting families are certain *t*-stars, then this immediately generalises for $k \geq 2$.

The maximum product problem for $\binom{[n]}{r}$ was first addressed by Pyber [41], who proved that for any r, s, and n such that either $r = s \leq n/2$ or r < s and $n \geq 2s+r-2$, if $\mathcal{A} \subseteq \binom{[n]}{r}$ and $\mathcal{B} \subseteq \binom{[n]}{s}$ such that \mathcal{A} and \mathcal{B} are cross-intersecting, then $|\mathcal{A}||\mathcal{B}| \leq \binom{n-1}{r-1}\binom{n-1}{s-1}$. Subsequently, Matsumoto and Tokushige [37] proved this for $r \leq s \leq n/2$. It has been shown in [14] that there exists an integer $n_0(r, s, t)$ such that for $t \leq r \leq s$ and $n \geq n_0(r, s, t)$, if $\mathcal{A} \subseteq \binom{[n]}{r}$, $\mathcal{B} \subseteq \binom{[n]}{s}$, and \mathcal{A} and \mathcal{B} are cross-t-intersecting, then $|\mathcal{A}||\mathcal{B}| \leq \binom{n-t}{r-t}\binom{n-t}{s-t}$. The value of $n_0(r, s, t)$ given in [14] is far from best possible. The special case r = s is treated in [45, 46, 25], which establish values of $n_0(r, r, t)$ that are close to the conjectured smallest value of (t+1)(r-t+1), and which use algebraic methods and Frankl's random walk method [22]; in particular, $n_0(r, r, t) = (t+1)r$ is determined in [25] for $t \geq 14$. Using purely combinatorial arguments, we solve the problem for $n \geq (t+u+2)(s-t)+r-1$, where u can be any non-negative real number satisfying $u > \frac{6-t}{3}$; thus, we can take $n_0(r, s, t) = (t+2)(s-t)+r-1$ for $t \geq 7$, and $n_0(r, s, t) < (t+4)(s-t)+r-1$ for $1 \leq t \leq 6$. We actually prove the following more general result in Section 5.

Theorem 1.2 If $1 \le t \le r \le s$, u is a non-negative real number such that $u > \frac{6-t}{3}$, min $\{m,n\} \ge (t+u+2)(s-t)+r-1$, $\mathcal{A} \subseteq {\binom{[m]}{r}}$, $\mathcal{B} \subseteq {\binom{[n]}{s}}$, and \mathcal{A} and \mathcal{B} are cross-t-intersecting, then

$$|\mathcal{A}||\mathcal{B}| \le \binom{m-t}{r-t}\binom{n-t}{s-t}.$$

Moreover, if u > 0, then the bound is attained if and only if $\mathcal{A} = \left\{ A \in \binom{[m]}{r} : T \subseteq A \right\}$ and $\mathcal{B} = \left\{ B \in \binom{[n]}{s} : T \subseteq B \right\}$ for some t-element subset T of $[\min\{m, n\}]$.

In Section 5, we show that Theorem 1.2 is a consequence of our main result, Theorem 1.3, for which we need some additional definitions and notation.

For any $i, j \in [n]$, let $\delta_{i,j} \colon 2^{[n]} \to 2^{[n]}$ be defined by

$$\delta_{i,j}(A) = \begin{cases} (A \setminus \{j\}) \cup \{i\} & \text{if } j \in A \text{ and } i \notin A; \\ A & \text{otherwise,} \end{cases}$$

and let $\Delta_{i,j} \colon 2^{2^{[n]}} \to 2^{2^{[n]}}$ be the *compression operation* defined by

$$\Delta_{i,j}(\mathcal{A}) = \{ \delta_{i,j}(A) \colon A \in \mathcal{A} \} \cup \{ A \in \mathcal{A} \colon \delta_{i,j}(A) \in \mathcal{A} \}.$$

The compression operation was introduced in the seminal paper [20]. The paper [23] provides a survey on the properties and uses of compression (also called *shifting*) operations in extremal set theory. All our new results make use of compression operations.

If i < j, then we call $\Delta_{i,j}$ a *left-compression*. A family $\mathcal{F} \subseteq 2^{[n]}$ is said to be compressed if $\Delta_{i,j}(\mathcal{F}) = \mathcal{F}$ for every $i, j \in [n]$ with i < j. In other words, \mathcal{F} is compressed if it is invariant under left-compressions. Note that \mathcal{F} is compressed if and only if $(F \setminus \{j\}) \cup \{i\} \in \mathcal{F}$ whenever $i < j \in F \in \mathcal{F}$ and $i \in [n] \setminus F$.

A family \mathcal{H} is said to be *hereditary* if for each $H \in \mathcal{H}$, all the subsets of H are in \mathcal{H} . Thus, a family is hereditary if and only if it is a union of power sets. The family $\binom{[n]}{\leq r}$ (which is $2^{[n]}$ if r = n) is an example of a hereditary family that is compressed.

Let \mathbb{R}^+ denote the set of positive real numbers. With a slight abuse of notation, for any non-empty family \mathcal{F} , any function $w: \mathcal{F} \to \mathbb{R}^+$ (called a *weight function*), and any $\mathcal{A} \subseteq \mathcal{F}$, we denote the sum $\sum_{A \in \mathcal{A}} w(A)$ (of *weights* of sets in \mathcal{A}) by $w(\mathcal{A})$. Note that if \mathcal{A} is empty, then $w(\mathcal{A})$ is the *empty sum*, and we will adopt the convention of taking this to be 0.

In Section 4, we prove the following result.

Theorem 1.3 Let $1 \leq t \leq n$, T = [t], and $u \in \{0\} \cup \mathbb{R}^+$ such that $u > \frac{6-t}{3}$. Let \mathcal{G} and \mathcal{H} be non-empty compressed hereditary subfamilies of $2^{[n]}$. For each $\mathcal{F} \in \{\mathcal{G}, \mathcal{H}\}$, let $w_{\mathcal{F}} \colon \mathcal{F} \to \mathbb{R}^+$ such that

(a) $w_{\mathcal{F}}(A) \geq (t+u)w_{\mathcal{F}}(B)$ for every $A, B \in \mathcal{F}$ with $A \subseteq B$ and $|A| \geq t$, and

(b) $w_{\mathcal{F}}(\delta_{i,j}(C)) \ge w_{\mathcal{F}}(C)$ for every $C \in \mathcal{F}$ and every $i, j \in [n]$ with i < j.

Let $g = w_{\mathcal{G}}$ and $h = w_{\mathcal{H}}$. If $\mathcal{A} \subseteq \mathcal{G}$ and $\mathcal{B} \subseteq \mathcal{H}$ such that \mathcal{A} and \mathcal{B} are cross-t-intersecting, then

$$g(\mathcal{A})h(\mathcal{B}) \leq g(\mathcal{G}(T))h(\mathcal{H}(T)).$$

Moreover, if u > 0 and each of \mathcal{G} and \mathcal{H} has a member of size at least t, then the bound is attained if and only if $\mathcal{A} = \mathcal{G}(T')$ and $\mathcal{B} = \mathcal{H}(T')$ for some $T' \in {\binom{[n]}{t}}$ such that $g(\mathcal{G}(T')) = g(\mathcal{G}(T))$ and $h(\mathcal{H}(T')) = h(\mathcal{H}(T))$.

Remark 1.4 For $u > \frac{6-t}{3}$ to hold, we can always take u = 2, and we can take u = 0 for $t \ge 7$. We conjecture that the inequality $g(\mathcal{A})h(\mathcal{B}) \le g(\mathcal{G}(T))h(\mathcal{H}(T))$ still holds

if the condition $u > \frac{6-t}{3}$ is replaced by u = 0. As we mentioned above, this is true for $t \ge 7$. Also, the proof of Theorem 1.3 shows that for $t \ge 3$, the conjecture is true if it is true for $t + 3 \le n \le t + 6$ (see Remark 4.3). A verification of the conjecture for $t + 3 \le n \le t + 6$ could be obtained through detailed case-checking similar to that used in our proof for the special case $n \le t + 2$; however, the process would be significantly more laborious. The condition on u cannot be relaxed further, because no real number u < 0 with $t + u \ge 1$ guarantees that the result holds. Indeed, if $1 \le x = t + u < t \le n - 2$, $\mathcal{G} = \mathcal{H} = 2^{[n]}$, $g(G) = h(G) = x^{n-|G|}$ for all $G \in 2^{[n]}$, and $\mathcal{A} = \mathcal{B} = \{A \in 2^{[n]} : |A \cap [t+2]| \ge t+1\} = \mathcal{A}^*$, then conditions (a) and (b) of Theorem 1.3 are satisfied, \mathcal{A} and \mathcal{B} are cross-t-intersecting, but

$$\begin{split} (g(\mathcal{A})h(\mathcal{B}))^{1/2} &= g(\mathcal{A}) = \sum_{X \in \binom{[t+2]}{t+1}} \sum_{Y \subseteq [t+3,n]} g(X \cup Y) + \sum_{Y \subseteq [t+3,n]} g([t+2] \cup Y) \\ &= (t+2) \sum_{j=0}^{n-t-2} \binom{n-t-2}{j} x^{n-t-1-j} + \sum_{j=0}^{n-t-2} \binom{n-t-2}{j} x^{n-t-2-j} \\ &= (t+2) x^{n-t-1} \left(1+x^{-1}\right)^{n-t-2} + x^{n-t-2} \left(1+x^{-1}\right)^{n-t-2} \\ &= x^{n-t-2} \left(1+x^{-1}\right)^{n-t-2} (tx+2x+1) = (x+1)^{n-t-2} (tx+2x+1) \\ &> (x+1)^{n-t-2} (x^2+2x+1) \quad (\text{as } 1 \le x < t) \\ &= x^{n-t} \left(1+x^{-1}\right)^{n-t} = \sum_{j=0}^{n-t} \binom{n-t}{j} x^{n-t-j} = \sum_{Y \subseteq [t+1,n]} g([t] \cup Y) \\ &= g(\mathcal{G}(T)) = (g(\mathcal{G}(T))h(\mathcal{H}(T)))^{1/2}, \end{split}$$

and hence $g(\mathcal{A})h(\mathcal{B}) > g(\mathcal{G}(T))h(\mathcal{H}(T))$. It has been shown in [6] that for t = 1, the product of sizes of \mathcal{A} and \mathcal{B} is maximised by taking $\mathcal{A} = \mathcal{G}(T)$ and $\mathcal{B} = \mathcal{H}(T)$; equivalently, for the special case where t = 1 and g(A) = h(A) = 1 for all $A \in \mathcal{G} \cup \mathcal{H}$, the bound in Theorem 1.3 also holds (that is, the conjecture is true). However, this is not true for t > 1, and hence Theorem 1.3 does not imply that the product of sizes is maximised by taking $\mathcal{A} = \mathcal{G}(T)$ and $\mathcal{B} = \mathcal{H}(T)$. Indeed, if $\mathcal{G} = \mathcal{H} = 2^{[n]}$ and $\mathcal{A} = \mathcal{B} = \mathcal{A}^*$ as above, then $|\mathcal{A}||\mathcal{B}| > |\mathcal{G}(T)||\mathcal{H}(T)|$ (take x = 1 above).

The proof of Theorem 1.3 contains the main observations in this paper and is based on induction, compression, a new subfamily alteration method, and double-counting. The alteration method can be regarded as the main new component and appears to have the potential of yielding other intersection results of this kind.

The bound in [25, Theorem 1.3] for product measures of cross-*t*-intersecting subfamilies of $2^{[n]}$ is given by Theorem 1.3 with $\mathcal{G} = \mathcal{H} = 2^{[n]}$, $t \ge 14$, u = 0, and $g(A) = h(A) = p^{|A|}(1-p)^{n-|A|}$ for all $A \in 2^{[n]}$, where $p \in \mathbb{R}^+$ such that $p \le \frac{1}{t+1}$.

The subsequent results in this section and in the next section are also consequences of Theorem 1.3. Our next application is a cross-*t*-intersection result for integer sequences.

We will represent a sequence a_1, \ldots, a_n by an *n*-tuple (a_1, \ldots, a_n) , and we say that it is of *length* n. We call a sequence of positive integers a *positive sequence*. We call (a_1, \ldots, a_n) an *r*-partial sequence if exactly *r* of its entries are positive integers and the rest are all zero. Thus, an *n*-partial sequence of length *n* is positive. A sequence (c_1, \ldots, c_n) is said to be *increasing* if $c_1 \leq \cdots \leq c_n$. We call an increasing positive sequence an *IP sequence*. Note that (c_1, \ldots, c_n) is an *IP* sequence if and only if $1 \leq c_1 \leq \cdots \leq c_n$.

We call $\{(x_1, y_1), \ldots, (x_r, y_r)\}$ a *labeled set* (following [12]) if x_1, \ldots, x_r are distinct. For any IP sequence $\mathbf{c} = (c_1, \ldots, c_n)$ and any $r \in [n]$, let $\mathcal{S}_{\mathbf{c},r}$ be the family of all labeled sets $\{(x_1, y_{x_1}), \ldots, (x_r, y_{x_r})\}$ such that $\{x_1, \ldots, x_r\} \in \binom{[n]}{r}$ and $y_{x_j} \in [c_{x_j}]$ for each $j \in [r]$. For any sets Y_1, \ldots, Y_n , let $Y_1 \times \cdots \times Y_n$ denote the *Cartesian product of* Y_1, \ldots, Y_n , that is, the set of sequences (y_1, \ldots, y_n) such that $y_i \in Y_i$ for each $i \in [n]$. Note that $\mathcal{S}_{\mathbf{c},n} = \{\{(1, y_1), \ldots, (n, y_n)\}: y_i \in [c_i]$ for each $i \in [n]\}$, so $\mathcal{S}_{\mathbf{c},n}$ is isomorphic to $[c_1] \times \cdots \times [c_n]$. Also note that $\mathcal{S}_{\mathbf{c},r}$ is isomorphic to the set of *r*-partial sequences (y_1, \ldots, y_n) such that for some $R \in \binom{[n]}{r}, y_i \in [c_i]$ for each $i \in R$ (and hence $y_j = 0$ for each $j \in [n] \setminus R$). Let $\mathcal{S}_{\mathbf{c},r,t} = \mathcal{S}_{\mathbf{c},r}([t] \times [1]) = \{A \in \mathcal{S}_{\mathbf{c},r}: (x, 1) \in A$ for each $x \in [t]\}$.

In Section 6, we prove the following result.

Theorem 1.5 Let $\mathbf{c} = (c_1, \ldots, c_m)$ and $\mathbf{d} = (d_1, \ldots, d_n)$ be IP sequences. Let $r \in [m]$, $s \in [n]$, $t \in [\min\{r, s\}]$, and $u \in \{0\} \cup \mathbb{R}^+$ such that $u > \frac{6-t}{3}$. If $c_1 \ge t + u + 1$, $d_1 \ge t + u + 1$, $\mathcal{A} \subseteq \mathcal{S}_{\mathbf{c},r}$, $\mathcal{B} \subseteq \mathcal{S}_{\mathbf{d},s}$, and \mathcal{A} and \mathcal{B} are cross-t-intersecting, then

$$|\mathcal{A}||\mathcal{B}| \leq \left(\sum_{I \in \binom{[t+1,m]}{r-t}} \prod_{i \in I} c_i\right) \left(\sum_{J \in \binom{[t+1,n]}{s-t}} \prod_{j \in J} c_j\right).$$

Moreover, if u > 0, then the bound is attained if and only if for some $T \in S_{\mathbf{c},t} \cap S_{\mathbf{d},t}$ with $|S_{\mathbf{c},r}(T)| = |S_{\mathbf{c},r,t}|$ and $|S_{\mathbf{d},s}(T)| = |S_{\mathbf{d},s,t}|$, $\mathcal{A} = S_{\mathbf{c},r}(T)$ and $\mathcal{B} = S_{\mathbf{d},s}(T)$.

Note that this result holds for $c_1 \ge t+1$ and $d_1 \ge t+1$ when $t \ge 7$, for $c_1 \ge t+2$ and $d_1 \ge t+2$ when $4 \le t \le 6$, and for $c_1 \ge t+3$ and $d_1 \ge t+3$ when $1 \le t \le 3$. We conjecture that the result holds for $c_1 \ge t+1$ and $d_1 \ge t+1$, and, as can be seen from the proof of Theorem 1.5, this conjecture is true if the conjecture in Remark 1.4 is true. The result does not hold for $c_1 < t+1$. Indeed, if $r = s = m = n \ge t+2$, $c_1 = \cdots = c_n = x+1 < t+1 = d_1 = \cdots = d_n$, $Z = [n] \times [1], Z_1 = [t+2] \times [1], Z_2 =$ $[t+3, n] \times [1], \mathcal{A} = \{A \in \mathcal{S}_{\mathbf{c},n} : |A \cap Z_1| \ge t+1\}$, and $\mathcal{B} = \{B \in \mathcal{S}_{\mathbf{d},n} : |B \cap Z_1| \ge t+1\}$, then \mathcal{A} and \mathcal{B} are cross-t-intersecting,

$$\begin{aligned} |\mathcal{A}| &= \left| \bigcup_{X \in \binom{Z_1}{t+1} \cup \{Z_1\}} \bigcup_{j=0}^{|Z_2|} \bigcup_{Y \in \binom{Z_2}{j}} \{A \in \mathcal{A} \colon A \cap Z = X \cup Y\} \right| \\ &= (t+2) \sum_{j=0}^{n-t-2} \binom{n-t-2}{j} x^{n-t-1-j} + \sum_{j=0}^{n-t-2} \binom{n-t-2}{j} x^{n-t-2-j} \\ &= (x+1)^{n-t-2} \left(tx+2x+1\right) \quad \text{(as in Remark 1.4)} \\ &> (x+1)^{n-t-2} \left(x^2+2x+1\right) = (x+1)^{n-t} = |\mathcal{S}_{\mathbf{c},n,t}|, \end{aligned}$$

 $|\mathcal{B}| = (t+1)^{n-t-2} (t^2 + 2t + 1) = (t+1)^{n-t} = |\mathcal{S}_{\mathbf{d},n,t}|$ (by a calculation similar to that for $|\mathcal{A}|$), and hence $|\mathcal{A}||\mathcal{B}| > |\mathcal{S}_{\mathbf{c},r,t}||\mathcal{S}_{\mathbf{d},s,t}|$.

Solutions for the special case where $\mathbf{c} = \mathbf{d}$ and r = s = n already exist. The solution for $t + 2 \leq c_1 = c_n$ was first obtained by Moon [39]. Inspired by [49], Pach and Tardos [40] recently generalised Moon's result to include the cases $t + 2 \leq c_1 \leq c_n$ and $8 \leq t + 1 \leq c_1 \leq c_n$. Another proof for $15 \leq t + 1 \leq c_1 = c_n$ is given in [25].

Our last application of Theorem 1.3 in this section is a cross-*t*-intersection result for multisets.

A multiset is a collection A of objects such that each object possibly appears more than once in A. Thus the difference between a multiset and a set is that a multiset may have repetitions of its elements. We can uniquely represent a multiset A of positive integers by an IP sequence (a_1, \ldots, a_r) , where a_1, \ldots, a_r form A. Thus we will take multisets to be IP sequences. For $A = (a_1, \ldots, a_r)$, the support of A is the set $\{a_1, \ldots, a_r\}$ and will be denoted by S_A . For any $n, r \in \mathbb{N}$, let $M_{n,r}$ denote the set of all multisets (a_1, \ldots, a_r) such that $a_1, \ldots, a_r \in [n]$; thus $M_{n,r} = \{(a_1, \ldots, a_r) : a_1 \leq \cdots \leq a_r, a_1, \ldots, a_r \subseteq [n]\}$. An elementary counting result is that

$$|M_{n,r}| = \binom{n+r-1}{r}.$$

With a slight abuse of terminology, we say that a multiset A t-intersects a multiset B if and A and B have at least t distinct common elements, that is, if S_A t-intersects S_B . A set A of multisets is said to be t-intersecting if every two multisets in A t-intersect, and k sets A_1, \ldots, A_k of multisets are said to be cross-t-intersecting if for every $i, j \in [k]$ with $i \neq j$, each multiset in A_i t-intersects each multiset in A_j .

In Section 7, we prove the following result.

Theorem 1.6 If $1 \leq t \leq r \leq s$, $u \in \{0\} \cup \mathbb{R}^+$ such that $u > \frac{6-t}{3}$, $\min\{m, n\} \geq (t+u+1)(s-t)+r-t$, $\mathcal{A} \subseteq M_{m,r}$, $\mathcal{B} \subseteq M_{n,s}$, and \mathcal{A} and \mathcal{B} are cross-t-intersecting, then

$$|\mathcal{A}||\mathcal{B}| \le \binom{m+r-t-1}{r-t} \binom{n+s-t-1}{s-t}.$$

Moreover, if u > 0, then the bound is attained if and only if $\mathcal{A} = \{A \in M_{m,r} : T \subseteq S_A\}$ and $\mathcal{B} = \{B \in M_{n,s} : T \subseteq S_B\}$ for some t-element subset T of $[\min\{m, n\}]$.

The condition $\min\{m, n\} \ge (t + u + 1)(s - t) + r - t$ is close to being sharp, as is evident from the fact that if r = s, m = n < t(r - t) + 2, and $\mathcal{A} = \mathcal{B} = \{A \in M_{n,r} : |S_A \cap [t+2]| \ge t+1\}$, then \mathcal{A} and \mathcal{B} are cross-t-intersecting,

$$\begin{aligned} |\mathcal{A}| &= \sum_{X \in \binom{[t+2]}{t+1} \cup \{[t+2]\}} |\{A \in M_{n,r} \colon S_A \cap [t+2] = X\}| \\ &= \sum_{X \in \binom{[t+2]}{t+1} \cup \{[t+2]\}} |\{(a_1, \dots, a_{r-|X|}) \colon a_1 \leq \dots \leq a_{r-|X|}, a_1, \dots, a_{r-|X|} \in X \cup [t+3, n]\}| \\ &= \sum_{X \in \binom{[t+2]}{t+1} \cup \{[t+2]\}} |M_{|X|+n-t-2, r-|X|}| \\ &= \sum_{X \in \binom{[t+2]}{t+1} \cup \{[t+2]\}} \binom{n+r-t-3}{r-|X|} = (t+2)\binom{n+r-t-3}{r-t-1} + \binom{n+r-t-3}{r-t-2} \\ &= \frac{\binom{n+r-t-1}{r-t}}{(n+r-t-1)(n+r-t-2)} \left((t+2)(r-t)(n-1) + (r-t)(r-t-1)\right) \\ &> \frac{\binom{n+r-t-1}{r-t}}{((t+1)(r-t)+1)((t+1)(r-t))} \left((t+2)(r-t)(t(r-t)+1) + (r-t)(r-t-1)\right) \\ &= \binom{n+r-t-1}{r-t}, \end{aligned}$$

and hence $|\mathcal{A}||\mathcal{B}| > {\binom{m+r-t-1}{r-t}}^2 = {\binom{m+r-t-1}{r-t}} {\binom{n+s-t-1}{s-t}}$. EKR-type results for multisets have been obtained in [38, 26]. To the best of the

author's knowledge, Theorem 1.6 is the first cross-*t*-intersection result for multisets.

In the next section, we show that the above results generalise for $k \ge 2$ families and yield EKR-type results. Section 3 provides basic compression results used in our proofs. Sections 4–7 are dedicated to the proofs of Theorems 1.3, 1.2, 1.5, and 1.6, respectively.

2 Multiple cross-*t*-intersecting families and *t*-intersecting families

Theorem 1.2 generalises as follows.

Theorem 2.1 Let $k \geq 2$, $t \leq r_1 \leq \cdots \leq r_k$, $u \in \{0\} \cup \mathbb{R}^+$ such that $u > \frac{6-t}{3}$, and $\min\{n_1, \ldots, n_k\} \geq (t+u+2)(r_k-t)+r_{k-1}-1$. If $\mathcal{A}_1 \subseteq \binom{[n_1]}{r_1}, \ldots, \mathcal{A}_k \subseteq \binom{[n_k]}{r_k}$, and $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are cross-t-intersecting, then

$$\prod_{i=1}^{k} |\mathcal{A}_i| \le \prod_{i=1}^{k} \binom{n_i - t}{r_i - t}.$$

Moreover, if u > 0, then the bound is attained if and only if for some t-element subset T of $[\min\{n_1, \ldots, n_k\}]$, $\mathcal{A}_i = \{A \in {[n_i] \choose r_i} : T \subseteq A\}$ for each $i \in [k]$.

The line of argument in the proof of [14, Theorem 1.2] yields the result above together with a similar generalisation of Theorem 1.6 and the following generalisations of Theorem 1.3 and Theorem 1.5.

Theorem 2.2 If t, u, and T are as in Theorem 1.3, $\mathcal{H}_1, \ldots, \mathcal{H}_k$ are non-empty compressed hereditary subfamilies of $2^{[n]}, w_{\mathcal{F}} \colon \mathcal{F} \to \mathbb{R}^+$ is a function satisfying (a) and (b) (of Theorem 1.3) for each $\mathcal{F} \in \{\mathcal{H}_1, \ldots, \mathcal{H}_k\}, \mathcal{A}_i \subseteq \mathcal{H}_i$ for each $i \in [k]$, and $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are cross-t-intersecting, then

$$\prod_{i=1}^{k} w_{\mathcal{H}_i}(\mathcal{A}_i) \le \prod_{i=1}^{k} w_{\mathcal{H}_i}(\mathcal{H}_i(T)).$$

Moreover, if u > 0 and each of $\mathcal{H}_1, \ldots, \mathcal{H}_k$ has a member of size at least t, then the bound is attained if and only if for some $T' \in {\binom{[n]}{t}}$ such that $w_{\mathcal{H}_i}(\mathcal{H}_i(T')) = w_{\mathcal{H}_i}(\mathcal{H}_i(T))$ for each $i \in [k]$.

Theorem 2.3 Let $\mathbf{c}_1 = (c_{1,1}, \ldots, c_{1,n_1}), \ldots, \mathbf{c}_k = (c_{k,1}, \ldots, c_{k,n_k})$ be *IP* sequences. Let $r_1 \in [n_1], \ldots, r_k \in [n_k], t \in [\min\{r_1, \ldots, r_k\}], and u \in \{0\} \cup \mathbb{R}^+$ such that $u > \frac{6-t}{3}$. If $c_{1,1} \ge t + u + 1, \ldots, c_{k,1} \ge t + u + 1, \mathcal{A}_1 \subseteq \mathcal{S}_{\mathbf{c}_1, r_1}, \ldots, \mathcal{A}_k \subseteq \mathcal{S}_{\mathbf{c}_k, r_k}, and \mathcal{A}_1, \ldots, \mathcal{A}_k$ are cross-t-intersecting, then

$$\prod_{i=1}^{k} |\mathcal{A}_i| \leq \prod_{i=1}^{k} \left(\sum_{\substack{I \in \binom{[t+1,n_i]}{r_i-t}}} \prod_{j \in I} c_{i,j} \right).$$

Moreover, if u > 0, then the bound is attained if and only if for some $T \in \bigcap_{i=1}^{k} S_{\mathbf{c}_{i},t}$ with $|S_{\mathbf{c}_{i},r_{i}}(T)| = |S_{\mathbf{c}_{i},r_{i},t}|$ for each $i \in [k]$, $\mathcal{A}_{i} = S_{\mathbf{c}_{i},r_{i}}(T)$ for each $i \in [k]$.

We simply observe that $\left(\prod_{i=1}^{k} a_{i}\right)^{k-1} = \prod_{i=1}^{k} \prod_{j \in [k] \setminus [i]} a_{i}a_{j}$ (see also [15, Lemma 5.2] with p = 2) and that if $\mathcal{A}_{1}, \ldots, \mathcal{A}_{k}$ are cross-*t*-intersecting, then any \mathcal{A}_{i} and \mathcal{A}_{j} with $i \neq j$ are cross-*t*-intersecting. Thus, if, for example, $\mathcal{A}_{1}, \ldots, \mathcal{A}_{k}$ are as in Theorem 2.2, $a_{i} = w_{\mathcal{H}_{i}}(\mathcal{A}_{i})$ for each $i \in [k]$, and $b_{i} = w_{\mathcal{H}_{i}}(\mathcal{H}_{i}(T))$ for each $i \in [k]$, then Theorem 1.3 gives us $\prod_{i=1}^{k} \prod_{j \in [k] \setminus [i]} a_{i}a_{j} \leq \prod_{i=1}^{k} \prod_{j \in [k] \setminus [i]} b_{i}b_{j}$, and hence $\left(\prod_{i=1}^{k} a_{i}\right)^{k-1} \leq \left(\prod_{i=1}^{k} b_{i}\right)^{k-1}$ (giving $\prod_{i=1}^{k} a_{i} \leq \prod_{i=1}^{k} b_{i}$, as required).

As in Remark 1.1, Theorem 1.3 immediately implies an EKR-type version for a family \mathcal{H} as in Theorem 1.3. By taking $\mathcal{G} = \mathcal{H}$ in Theorem 1.3 and applying an argument similar to the one in Remark 1.1, we obtain the following new result.

Theorem 2.4 Let t, u, T, H, and h be as in Theorem 1.3. If A is a t-intersecting subfamily of H, then

$$h(\mathcal{A}) \le h(\mathcal{H}(T)).$$

Moreover, if u > 0 and \mathcal{H} has a member of size at least t, then the bound is attained if and only if $\mathcal{A} = \mathcal{H}(T')$ for some t-set T' such that $h(\mathcal{H}(T')) = h(\mathcal{H}(T))$. By taking $\mathbf{c} = \mathbf{d}$ in Theorem 1.5 and applying the argument in Remark 1.1, we obtain the following EKR-type result.

Theorem 2.5 If $1 \le t \le r \le n$, $u \in \{0\} \cup \mathbb{R}^+$ such that $u > \frac{6-t}{3}$, $\mathbf{c} = (c_1, \ldots, c_n)$ is an IP sequence, $c_1 \ge t + u + 1$, and \mathcal{A} is a t-intersecting subfamily of $\mathcal{S}_{\mathbf{c},r}$, then

$$|\mathcal{A}| \leq \left(\sum_{I \in \binom{[t+1,n]}{r-t}} \prod_{i \in I} c_i\right).$$

Moreover, if u > 0, then the bound is attained if and only if $\mathcal{A} = \mathcal{S}_{\mathbf{c},r}(T)$ for some $T \in \mathcal{S}_{\mathbf{c},t}$ with $|\mathcal{S}_{\mathbf{c},r}(T)| = |\mathcal{S}_{\mathbf{c},r,t}|$.

The EKR problem for $S_{\mathbf{c},r}$ attracted much attention and has been dealt with extensively (see, for example, [13]). In particular, for $c_1 = c_n$, it was solved for r = n in [2, 24], and for $n \ge \left\lfloor \frac{(r-t+c_1)(t+1)}{c_1} \right\rfloor$ in [5]. Similarly to Theorem 1.5, Theorem 2.5 does not hold for $c_1 < t + 1$.

By taking m = n and r = s in Theorem 1.6, and applying the argument in Remark 1.1, we obtain the following EKR-type result.

Theorem 2.6 If $1 \leq t \leq r$, $u \in \{0\} \cup \mathbb{R}^+$ such that $u > \frac{6-t}{3}$, $n \geq (t+u+2)(r-t)$, $\mathcal{A} \subseteq M_{n,r}$, and \mathcal{A} is t-intersecting, then

$$|\mathcal{A}| \le \binom{n+r-t-1}{r-t}.$$

Moreover, if u > 0, then the bound is attained if and only if $\mathcal{A} = \{A \in M_{n,r} : T \subseteq S_A\}$ for some $T \in {[n] \choose t}$.

The condition $n \ge (t+u+2)(r-t)$ is close to being sharp. Indeed, as shown in Section 1, if n < t(r-t) + 2 and $\mathcal{A} = \{A \in M_{n,r} : |S_A \cap [t+2]| \ge t+1\}$, then $|\mathcal{A}| > {n+r-t-1 \choose r-t}$.

The EKR problem for $M_{n,r}$ and t = 1 is solved in [38]. Generalising this result, Füredi, Gerbner, and Vizer [26] solved the EKR problem of maximising the size of a largest subset \mathcal{A} of $M_{n,r}$ such that for every $(a_1, \ldots, a_r), (b_1, \ldots, b_r) \in \mathcal{A}$, there exist t distinct elements i_1, \ldots, i_t of [r] and t distinct elements j_1, \ldots, j_t of [r] such that $a_{i_p} = b_{j_p}$ for each $p \in [t]$.

3 The compression operation

Compression operations have various useful properties. It is straightforward that for $i, j \in [n]$ and $\mathcal{A} \subseteq 2^{[n]}$,

$$|\Delta_{i,j}(\mathcal{A})| = |\mathcal{A}|.$$

We will also need the following well-known basic result (see, for example, [14, Lemma 2.1]).

Lemma 3.1 Let \mathcal{A} and \mathcal{B} be cross-t-intersecting subfamilies of $2^{[n]}$. (i) For any $i, j \in [n]$, $\Delta_{i,j}(\mathcal{A})$ and $\Delta_{i,j}(\mathcal{B})$ are cross-t-intersecting subfamilies of $2^{[n]}$. (ii) If $1 \leq t \leq r \leq s \leq n$, $\mathcal{A} \subseteq {[n] \choose \leq r}$, $\mathcal{B} \subseteq {[n] \choose \leq s}$, and \mathcal{A} and \mathcal{B} are compressed, then

$$|A \cap B \cap [r+s-t]| \ge t$$

for any $A \in \mathcal{A}$ and any $B \in \mathcal{B}$.

The only difference between Lemma 3.1 and [14, Lemma 2.1] is that the latter is for $\mathcal{A} \subseteq {\binom{[n]}{r}}$ and $\mathcal{B} \subseteq {\binom{[n]}{s}}$; however, the former follows by the argument for the latter.

Suppose that a subfamily \mathcal{A} of $2^{[n]}$ is not compressed. Then \mathcal{A} can be transformed to a compressed family through left-compressions as follows. Since \mathcal{A} is not compressed, we can find a left-compression that changes \mathcal{A} , and we apply it to \mathcal{A} to obtain a new subfamily of $2^{[n]}$. We keep on repeating this (always applying a left-compression to the last family obtained) until we obtain a subfamily of $2^{[n]}$ that is invariant under any left-compression (such a point is indeed reached, because if $\Delta_{i,j}(\mathcal{F}) \neq \mathcal{F} \subseteq 2^{[n]}$ and i < j, then $0 < \sum_{G \in \Delta_{i,j}(\mathcal{F})} \sum_{b \in G} b < \sum_{F \in \mathcal{F}} \sum_{a \in F} a$. Now consider $\mathcal{A}, \mathcal{B} \subseteq 2^{[n]}$ such that \mathcal{A} and \mathcal{B} are cross-*t*-intersecting. Then, by

Now consider $\mathcal{A}, \mathcal{B} \subseteq 2^{[n]}$ such that \mathcal{A} and \mathcal{B} are cross-*t*-intersecting. Then, by Lemma 3.1, we can obtain $\mathcal{A}^*, \mathcal{B}^* \subseteq 2^{[n]}$ such that \mathcal{A}^* and \mathcal{B}^* are compressed and cross*t*-intersecting, $|\mathcal{A}^*| = |\mathcal{A}|$, and $|\mathcal{B}^*| = |\mathcal{B}|$. Indeed, similarly to the above procedure, if we can find a left-compression that changes at least one of \mathcal{A} and \mathcal{B} , then we apply it to both \mathcal{A} and \mathcal{B} , and we keep on repeating this (always performing this on the last two families obtained) until we obtain $\mathcal{A}^*, \mathcal{B}^* \subseteq 2^{[n]}$ such that both \mathcal{A}^* and \mathcal{B}^* are invariant under any left-compression.

4 Proof of the main result

This section is dedicated to the proof of Theorem 1.3.

For the extremal cases of Theorem 1.3, we shall use the following two lemmas.

Lemma 4.1 Let $1 \leq t \leq n$ and T = [t]. Let \mathcal{H} be a compressed subfamily of $2^{[n]}$. Let $w: \mathcal{H} \to \mathbb{R}^+$ such that $w(\delta_{i,j}(H)) \geq w(H)$ for every $H \in \mathcal{H}$ and every $i, j \in [n]$ with i < j. Then $w(\mathcal{H}(T')) \leq w(\mathcal{H}(T))$ for each $T' \in {[n] \choose t}$.

Proof. Let $T' \in {\binom{[n]}{t}}$, and let a_1, \ldots, a_t be the elements of T'. Let $\mathcal{D}_0 = \mathcal{H}(T')$. Let $\mathcal{D}_1 = \Delta_{1,a_1}(\mathcal{D}_0), \ldots, \mathcal{D}_t = \Delta_{t,a_t}(\mathcal{D}_{t-1})$. Since \mathcal{H} is compressed, $\mathcal{D}_i \subseteq \mathcal{H}$ for each $i \in [t]$. It follows from the properties of w and of left-compressions that $w(\mathcal{D}_0) \leq w(\mathcal{D}_1) \leq \cdots \leq w(\mathcal{D}_t)$. Thus the result follows if we show that $\mathcal{D}_t \subseteq \mathcal{H}(T)$.

Let $D_1 \in \mathcal{D}_1$. If $D_1 \notin \mathcal{D}_0$, then $D_1 = \delta_{1,a_1}(D) \neq D$ for some $D \in \mathcal{D}_0$, and hence $1 \in D_1$. Suppose $D_1 \in \mathcal{D}_0$, so $a_1 \in D_1$ by definition of \mathcal{D}_0 . Since D_1 is also in \mathcal{D}_1 , $\delta_{1,a_1}(D_1) \in \mathcal{D}_0$. Thus $a_1 \in \delta_{1,a_1}(D_1)$ by definition of \mathcal{D}_0 . Since $a_1 \in D_1$, it follows that $1 \in D_1$.

Therefore, $1 \in H$ for each $H \in \mathcal{D}_1$, that is, $\mathcal{D}_1 \subseteq \mathcal{H}(\{1\})$. If t = 1, then we have $w(\mathcal{D}_0) \leq w(\mathcal{D}_1) \leq w(\mathcal{H}(\{1\})) = w(\mathcal{H}(T))$, as required.

Suppose $t \geq 2$. Since $\mathcal{D}_1 \subseteq \mathcal{H}(\{1\})$, we clearly have $1 \in H$ for each $H \in \mathcal{D}_2$. By an argument similar to that for \mathcal{D}_1 , we also obtain that $2 \in H$ for each $H \in \mathcal{D}_2$. Continuing this way, we obtain that $1, \ldots, t \in H$ for each $H \in \mathcal{D}_t$. Thus $\mathcal{D}_t \subseteq \mathcal{H}(T)$, as required.

Lemma 4.2 Let $n, t, T, \mathcal{G}, \mathcal{H}, g$, and h be as in Theorem 1.3. If $U \in \mathcal{A} \subseteq \mathcal{G}$, $V \in \mathcal{B} \subseteq \mathcal{H}, |U| = |V| = t$, and \mathcal{A} and \mathcal{B} are cross-t-intersecting, then

$$g(\mathcal{A})h(\mathcal{B}) \leq g(\mathcal{G}(T))h(\mathcal{H}(T)),$$

and equality holds if and only if $\mathcal{A} = \mathcal{G}(V)$, $\mathcal{B} = \mathcal{H}(V)$, $g(\mathcal{G}(V)) = g(\mathcal{G}(T))$, and $h(\mathcal{H}(V)) = h(\mathcal{H}(T))$.

Proof. Since \mathcal{A} and \mathcal{B} are cross-*t*-intersecting, we have U = V, $\mathcal{A} \subseteq \mathcal{G}(V)$, and $\mathcal{B} \subseteq \mathcal{H}(V)$. By Lemma 4.1, $g(\mathcal{G}(V)) \leq g(\mathcal{G}(T))$ and $h(\mathcal{H}(V)) \leq h(\mathcal{H}(T))$. Hence the result.

Proof of Theorem 1.3. We prove the result by induction on n.

Consider the base case n = t. If $g(\mathcal{A})h(\mathcal{B}) \neq 0$, then $\mathcal{A} \neq \emptyset \neq \mathcal{B}$, and hence, since \mathcal{A} and \mathcal{B} are cross-*t*-intersecting, $\mathcal{A} = \{T\} = \mathcal{B}$.

Now consider $n \geq t + 1$. Let $\mathcal{A} \subseteq \mathcal{G}$ and $\mathcal{B} \subseteq \mathcal{H}$ such that $g(\mathcal{A})h(\mathcal{B})$ is maximum under the condition that \mathcal{A} and \mathcal{B} are cross-*t*-intersecting. If \mathcal{G} does not have a member of size at least *t*, then $\mathcal{A} = \emptyset$ or $\mathcal{B} = \emptyset$ (since \mathcal{A} and \mathcal{B} are cross-*t*-intersecting), and hence $g(\mathcal{A})h(\mathcal{B}) = 0 = g(\mathcal{G}(T))h(\mathcal{H}(T))$. Similarly, $g(\mathcal{A})h(\mathcal{B}) = 0 = g(\mathcal{G}(T))h(\mathcal{H}(T))$ if \mathcal{H} does not have a member of size at least *t*. Therefore, we will assume that each of \mathcal{G} and \mathcal{H} has a member of size at least *t*. Since \mathcal{G} and \mathcal{H} are hereditary and compressed, we clearly have $T \in \mathcal{G}$ and $T \in \mathcal{H}$. Thus $g(\mathcal{G}(T)) > 0$ and $h(\mathcal{H}(T)) > 0$. Since $\mathcal{G}(T)$ and $\mathcal{H}(T)$ are cross-*t*-intersecting, it follows by the choice of \mathcal{A} and \mathcal{B} that

$$g(\mathcal{A})h(\mathcal{B}) \ge g(\mathcal{G}(T))h(\mathcal{H}(T)) > 0.$$
(1)

It follows that $\mathcal{A} \neq \emptyset \neq \mathcal{B}$. It also follows that no member of \mathcal{A} is of size less than t, because otherwise $\mathcal{B} = \emptyset$, contradicting (1). Similarly, no member of \mathcal{B} is of size less than t.

As explained in Section 3, we apply left-compressions to \mathcal{A} and \mathcal{B} simultaneously until we obtain two compressed cross-*t*-intersecting families \mathcal{A}^* and \mathcal{B}^* , respectively. Thus $|\mathcal{A}^*| = |\mathcal{A}|$ and $|\mathcal{B}^*| = |\mathcal{B}|$. Since \mathcal{G} and \mathcal{H} are compressed, $\mathcal{A}^* \subseteq \mathcal{G}$ and $\mathcal{B}^* \subseteq \mathcal{H}$. By (b), $g(\mathcal{A}) \leq g(\mathcal{A}^*)$ and $h(\mathcal{B}) \leq h(\mathcal{B}^*)$. By the choice of \mathcal{A} and \mathcal{B} , we actually have $g(\mathcal{A}) = g(\mathcal{A}^*)$ and $h(\mathcal{B}) = h(\mathcal{B}^*)$.

Suppose that $\mathcal{A}^* = \mathcal{G}(U)$ and $\mathcal{B}^* = \mathcal{H}(U)$ for some $U \in {\binom{[n]}{t}}$ such that $g(\mathcal{G}(U)) = g(\mathcal{G}(T))$ and $h(\mathcal{H}(U)) = h(\mathcal{H}(T))$. Then $g(\mathcal{G}(U)) > 0$ and $h(\mathcal{H}(U)) > 0$, so $\mathcal{G}(U) \neq \emptyset$ and $\mathcal{H}(U) \neq \emptyset$. Thus, since \mathcal{G} and \mathcal{H} are hereditary, $U \in \mathcal{A}^*$ and $U \in \mathcal{B}^*$. Hence $V \in \mathcal{A}$ for some $V \in {\binom{[n]}{t}}$, and $V' \in \mathcal{B}$ for some $V' \in {\binom{[n]}{t}}$. By Lemma 4.2, the result follows.

Therefore, we may assume that \mathcal{A} and \mathcal{B} are compressed.

We first consider $t+1 \le n \le t+2$. If \mathcal{A} has a member of size t and \mathcal{B} has a member of size t, then the result follows by Lemma 4.2. Thus, without loss of generality, we may assume that no member of \mathcal{A} is of size t.

Suppose n = t + 1. Then $\mathcal{A} = \{[t+1]\} \subseteq \mathcal{G}(T) \setminus \{T\}$ (since $\mathcal{A} \neq \emptyset$ and $\mathcal{A} \cap {\binom{[n]}{t}} = \emptyset$) and $\mathcal{B} \subseteq \mathcal{H} \cap \left({\binom{[t+1]}{t}} \cup \{[t+1]\} \right)$. Thus we have

$$\begin{split} g(\mathcal{A})h(\mathcal{B}) &\leq \left(g(\mathcal{G}(T)) - g(T)\right) \left(h(\mathcal{H}(T)) + \sum_{H \in \binom{[t+1]}{t} \cap \mathcal{H} \setminus \{T\}} h(H)\right) \\ &\leq \left(g(\mathcal{G}(T)) - g(T)\right) \left(h(\mathcal{H}(T)) + th(T)\right) \quad (\text{by (b)}) \\ &= g(\mathcal{G}(T))h(\mathcal{H}(T)) + th(T)g(\mathcal{G}(T)) - g(T)\left(h(\mathcal{H}(T)) + th(T)\right) \\ &\leq g(\mathcal{G}(T))h(\mathcal{H}(T)) + th(T)(g(T) + g([t+1])) - (t+1)g(T)h(T) \\ &\leq g(\mathcal{G}(T))h(\mathcal{H}(T)) + th(T)\left(g(T) + \frac{g(T)}{t+u}\right) - (t+1)g(T)h(T) \quad (\text{by (a)}). \end{split}$$

Therefore, $g(\mathcal{A})h(\mathcal{B}) \leq g(\mathcal{G}(T))h(\mathcal{H}(T))$, and equality holds only if u = 0.

Suppose n = t + 2. This case requires a number of observations followed by the separate treatment of a few sub-cases.

Let $T_1 = [t+1], T'_1 = T \cup \{t+2\}$, and $T_2 = [t+2]$. For each $i \in \{t, t+1, t+2\}$ and each $\mathcal{F} \in \{\mathcal{A}, \mathcal{B}, \mathcal{G}, \mathcal{H}\}$, let $\mathcal{F}^{(i)} = \mathcal{F} \cap {[t+2] \choose i}$. Thus $\mathcal{A} = \mathcal{A}^{(t)} \cup \mathcal{A}^{(t+1)} \cup \mathcal{A}^{(t+2)}$ and $\mathcal{B} = \mathcal{B}^{(t)} \cup \mathcal{B}^{(t+1)} \cup \mathcal{B}^{(t+2)}$. Recall that \mathcal{A} has no t-set, so $\mathcal{A}^{(t)} = \emptyset$. Since $\mathcal{A}^{(t+2)}, \mathcal{B}^{(t+2)} \subseteq {[t+2] \choose t+2} = \{[t+2]\}$, we have $\mathcal{A}^{(t+2)} \subseteq \mathcal{G}(T)$ and $\mathcal{B}^{(t+2)} \subseteq \mathcal{H}(T)$. Let

$$\mathcal{A}_T = \mathcal{A} \cap \mathcal{G}(T), \quad \mathcal{A}_{\overline{T}} = \mathcal{A} \setminus \mathcal{A}_T, \quad \mathcal{B}_T = \mathcal{B} \cap \mathcal{H}(T), \quad \mathcal{B}_{\overline{T}} = \mathcal{B} \setminus \mathcal{B}_T.$$

We have

$$\mathcal{A}_T \subseteq \mathcal{G}(T) \setminus \{T\}, \quad \mathcal{A}_{\overline{T}} \subseteq \mathcal{G}^{(t+1)} \setminus \{T_1, T_1'\}, \quad \mathcal{B}_{\overline{T}} \subseteq \mathcal{H}^{(t)} \cup \mathcal{H}^{(t+1)} \setminus \{T_1, T_1'\}.$$
(2)

Since $T \subsetneq T_1 \subsetneq T_2$, we have $g(T_1) \leq \frac{g(T)}{t+u}$, $g(T_2) \leq \frac{g(T_1)}{t+u} \leq \frac{g(T)}{(t+u)^2}$, $h(T_1) \leq \frac{h(T)}{t+u}$, and $h(T_2) \leq \frac{h(T_1)}{t+u} \leq \frac{h(T)}{(t+u)^2}$. Clearly, for each $U \in \mathcal{G}^{(t)}$, there is a composition of leftcompressions that gives T when applied to U, and hence $g(U) \leq g(T)$ by (b). Similarly, $h(V) \leq h(T)$ for each $V \in \mathcal{H}^{(t)}$, $g(U) \leq g(T_1)$ for each $U \in \mathcal{G}^{(t+1)}$, and $h(V) \leq h(T_1)$ for each $V \in \mathcal{H}^{(t+1)}$.

Suppose $\mathcal{A}^{(t+1)} = \emptyset$. Then $\mathcal{A} = \{T_2\}$. Since $T_2 \in \mathcal{G}$ and \mathcal{G} is hereditary, we have $T, T_1, T'_1, T_2 \in \mathcal{G}(T)$. Thus

$$g(\mathcal{G}(T)) \ge g(T) + g(T_1) + g(T_1') + g(T_2)$$

$$\ge (t+u)^2 g(T_2) + 2(t+u)g(T_2) + g(T_2)$$

$$\ge ((t+u)+1)^2 g(T_2) = (t+u+1)^2 g(\mathcal{A}),$$

and hence $g(\mathcal{A}) \leq \frac{g(\mathcal{G}(T))}{(t+u+1)^2}$. Now

$$h(\mathcal{B}) = h(\mathcal{B}_T) + h(\mathcal{B}_{\overline{T}}) \le h(\mathcal{H}(T)) + \left(\binom{t+2}{t} - 1\right) h(T) + \left(\binom{t+2}{t+1} - 2\right) h(T_1)$$

$$\le h(\mathcal{H}(T)) + \left(\frac{(t+2)(t+1)}{2} - 1\right) h(T) + t\frac{h(T)}{t} = h(\mathcal{H}(T)) + \frac{t^2 + 3t + 2}{2} h(T)$$

$$\le h(\mathcal{H}(T)) + \frac{t^2 + 3t + 2}{2} h(\mathcal{H}(T)) = \frac{t^2 + 3t + 4}{2} h(\mathcal{H}(T)).$$

Thus

$$g(\mathcal{A})h(\mathcal{B}) \le \frac{t^2 + 3t + 4}{2(t+u+1)^2} g(\mathcal{G}(T))h(\mathcal{H}(T)) \le \frac{t^2 + 3t + 4}{2(t+1)^2} g(\mathcal{G}(T))h(\mathcal{H}(T)).$$

Hence $g(\mathcal{A})h(\mathcal{B}) \leq g(\mathcal{G}(T))h(\mathcal{H}(T))$, and equality holds only if u = 0.

Suppose that $\mathcal{A}^{(t+1)}$ has at least 3 sets. Let U_1, U_2 , and U_3 be 3 distinct sets in $\mathcal{A}^{(t+1)}$. Since $U_1, U_2, U_3 \in {[t+2] \choose t+1}$, no *t*-set is a subset of each of U_1, U_2 , and U_3 . Thus no *t*-set *t*-intersects each of U_1, U_2 , and U_3 , and hence $\mathcal{B}^{(t)} = \emptyset$. We have

$$g(\mathcal{A}) \le g(\mathcal{G}(T)) - g(T) + \left(\begin{pmatrix} t+2\\t+1 \end{pmatrix} - 2 \right) g(T_1) \le g(\mathcal{G}(T)) - g(T) + t \frac{g(T)}{t+u} \le g(\mathcal{G}(T)).$$

Similarly, $h(\mathcal{B}) \leq h(\mathcal{H}(T))$. Thus $g(\mathcal{A})h(\mathcal{B}) \leq g(\mathcal{G}(T))h(\mathcal{H}(T))$, and equality holds only if u = 0.

We still need to consider $1 \leq |\mathcal{A}^{(t+1)}| \leq 2$, for which we need more detailed observations. Let $\mathcal{C}_0 = \mathcal{B}_{\overline{T}} \cap {\binom{[t+2]}{t}}, \mathcal{C}_1 = \mathcal{B}_{\overline{T}} \cap {\binom{[t+2]}{t+1}}, \mathcal{D}_0 = \mathcal{H}(T) \cap {\binom{[t+2]}{t}}$, and $\mathcal{D}_1 = \mathcal{H}(T) \cap {\binom{[t+2]}{t+1}}$. By (2), $\mathcal{B}_{\overline{T}} = \mathcal{C}_0 \cup \mathcal{C}_1$. If $\mathcal{H}^{(t+1)} \setminus \{T_1, T_1'\}$ has a set V, then $t+2 \in V$, and hence there is a composition of left-compressions that gives T_1' when applied to V. Thus, if $\mathcal{H}^{(t+1)} \setminus \{T_1, T_1'\}$ is non-empty, then $T_1, T_1' \in \mathcal{H}(T)$ (as \mathcal{H} is compressed, $T \subset T_1$, and $T \subset T_1'$), and hence we have

$$h(\mathcal{C}_{1}) \leq \sum_{V \in \mathcal{H}^{(t+1)} \setminus \{T_{1}, T_{1}'\}} h(V) \quad (by (2))$$

$$\leq \sum_{V \in \mathcal{H}^{(t+1)} \setminus \{T_{1}, T_{1}'\}} h(T_{1}') \leq th(T_{1}') \leq t\frac{h(T_{1}) + h(T_{1}')}{2} = \frac{t}{2} |\mathcal{D}_{1}|.$$

If $\mathcal{H}^{(t+1)} \setminus \{T_1, T_1'\} = \emptyset$, then $\mathcal{C}_1 = \emptyset$, and hence we also have $h(\mathcal{C}_1) \leq \frac{t}{2}h(\mathcal{D}_1)$. With a slight abuse of notation, we set $g(T_1') = 0$ if $T_1' \notin \mathcal{G}$, and we set $g(T_2) = 0$ if $T_2 \notin \mathcal{G}$. Since \mathcal{G} is hereditary, $T_1' \in \mathcal{G}$ if $T_2 \in \mathcal{G}$. Thus $g(T_1') \geq (t+u)g(T_2)$.

Suppose that $\mathcal{A}^{(t+1)}$ has exactly one set. Since \mathcal{A} is compressed, $\mathcal{A}^{(t+1)} = \{T_1\}$. Thus $\mathcal{A} \subseteq \{T_1, T_2\}$, and hence $g(\mathcal{A}) \leq g(T_1) + g(T_2) = g(\mathcal{G}(T)) - g(T) - g(T'_1)$. The *t*-sets that *t*-intersect T_1 are those in $\binom{T_1}{t}$, so $\mathcal{B}^{(t)} \subseteq \binom{T_1}{t}$, and hence

$$h(\mathcal{C}_0) \leq \sum_{V \in (\mathcal{H}^{(t)} \setminus \{T\}) \cap \binom{T_1}{t}} h(V) \leq \left(\binom{t+1}{t} - 1 \right) h(T) = th(\mathcal{D}_0).$$

We have

$$\begin{split} g(\mathcal{A})h(\mathcal{B}) &\leq (g(\mathcal{G}(T)) - g(T) - g(T_{1}'))(h(\mathcal{H}(T)) + h(\mathcal{B}_{\overline{T}})) \\ &= g(\mathcal{G}(T))h(\mathcal{H}(T)) + g(\mathcal{G}(T))h(\mathcal{B}_{\overline{T}}) - (g(T) + g(T_{1}'))(h(\mathcal{H}(T)) + h(\mathcal{B}_{\overline{T}})) \\ &= g(\mathcal{G}(T))h(\mathcal{H}(T)) + (g(T_{1}) + g(T_{2}))h(\mathcal{B}_{\overline{T}}) - (g(T) + g(T_{1}'))h(\mathcal{H}(T)) \\ &\leq g(\mathcal{G}(T))h(\mathcal{H}(T)) + \left(\frac{g(T)}{t+u} + \frac{g(T_{1}')}{t+u}\right)h(\mathcal{B}_{\overline{T}}) - (g(T) + g(T_{1}'))h(\mathcal{H}(T)) \\ &= g(\mathcal{G}(T))h(\mathcal{H}(T)) + (g(T) + g(T_{1}'))\left(\frac{h(\mathcal{C}_{0}) + h(\mathcal{C}_{1})}{t+u} - h(\mathcal{H}(T))\right) \\ &\leq g(\mathcal{G}(T))h(\mathcal{H}(T)) + (g(T) + g(T_{1}'))\left(\frac{th(\mathcal{D}_{0}) + \frac{t}{2}h(\mathcal{D}_{1})}{t+u} - (h(\mathcal{D}_{0}) + h(\mathcal{D}_{1}))\right). \end{split}$$

Thus $g(\mathcal{A})h(\mathcal{B}) \leq g(\mathcal{G}(T))h(\mathcal{H}(T))$, and equality holds only if u = 0.

Suppose that $\mathcal{A}^{(t+1)}$ has exactly 2 sets. Since \mathcal{A} is compressed, $\mathcal{A}^{(t+1)} = \{T_1, T_1'\}$. The only t-set that t-intersects each of T_1 and T_1' is T, so $\mathcal{B}^{(t)} \subseteq \{T\}$. Thus $\mathcal{C}_0 = \emptyset$, and hence $\mathcal{B}_{\overline{T}} = \mathcal{C}_1$. Since $\mathcal{D}_1 \subseteq \{T_1, T_1'\}$, $h(\mathcal{D}_1) \leq 2\frac{h(T)}{t+u} = 2\frac{h(\mathcal{D}_0)}{t+u}$. Since $h(\mathcal{H}(T)) \geq h(\mathcal{D}_0) + h(\mathcal{D}_1)$, $h(\mathcal{H}(T)) \geq \frac{t+u}{2}h(\mathcal{D}_1) + h(\mathcal{D}_1) = (\frac{t+u}{2}+1)h(\mathcal{D}_1)$. We have

$$\begin{split} g(\mathcal{A})h(\mathcal{B}) &\leq (g(\mathcal{G}(T)) - g(T))(h(\mathcal{H}(T)) + h(\mathcal{C}_1)) \\ &= g(\mathcal{G}(T))h(\mathcal{H}(T)) + g(\mathcal{G}(T))h(\mathcal{C}_1) - g(T)(h(\mathcal{H}(T)) + h(\mathcal{C}_1)) \\ &= g(\mathcal{G}(T))h(\mathcal{H}(T)) + (g(T_1) + g(T_1') + g(T_2))h(\mathcal{C}_1) - g(T)h(\mathcal{H}(T)) \\ &\leq g(\mathcal{G}(T))h(\mathcal{H}(T)) + \left(\frac{2g(T)}{t+u} + \frac{g(T)}{(t+u)^2}\right)\frac{t}{2}h(\mathcal{D}_1) - g(T)\left(\frac{t+u}{2} + 1\right)h(\mathcal{D}_1) \\ &= g(\mathcal{G}(T))h(\mathcal{H}(T)) + g(T)h(\mathcal{D}_1)\left(\frac{t}{t+u} + \frac{t}{2(t+u)^2} - \frac{t+u}{2} - 1\right). \end{split}$$

Thus $g(\mathcal{A})h(\mathcal{B}) \leq g(\mathcal{G}(T))h(\mathcal{H}(T))$, and equality holds only if u = 0.

Now consider $n \ge t+3$.

Define $\mathcal{H}_0 = \{H \in \mathcal{H} : n \notin H\}$ and $\mathcal{H}_1 = \{H \setminus \{n\} : n \in H \in \mathcal{H}\}$. Define $\mathcal{G}_0, \mathcal{G}_1, \mathcal{A}_0, \mathcal{A}_1, \mathcal{B}_0, \mathcal{A}_1, \mathcal{B}_0, \mathcal{B}_1, \mathcal{G}_0, \mathcal{G}_1, \mathcal{H}_0, \mathcal{A}_1, \mathcal{B}, \mathcal{G}, \mathcal{G}_1, \mathcal{H}_0, \mathcal{H}_1$ are compressed. Since \mathcal{G} and \mathcal{H} are hereditary, we clearly have that $\mathcal{G}_0, \mathcal{G}_1, \mathcal{H}_0, \mathcal{H}_1$ and \mathcal{H}_1 are hereditary, $\mathcal{G}_1 \subseteq \mathcal{G}_0$, and $\mathcal{H}_1 \subseteq \mathcal{H}_0$. Obviously, we have $\mathcal{A}_0 \subseteq \mathcal{G}_0 \subseteq 2^{[n-1]}, \mathcal{A}_1 \subseteq \mathcal{G}_1 \subseteq 2^{[n-1]}, \mathcal{B}_0 \subseteq \mathcal{H}_0 \subseteq 2^{[n-1]}, \text{and } \mathcal{B}_1 \subseteq \mathcal{H}_1 \subseteq 2^{[n-1]}.$

Let $h_0 : \mathcal{H}_0 \to \mathbb{R}^+$ such that $h_0(H) = h(H)$ for each $H \in \mathcal{H}_0$. Let $h_1 : \mathcal{H}_1 \to \mathbb{R}^+$ such that $h_1(H) = h(H \cup \{n\})$ for each $H \in \mathcal{H}_1$ (note that $H \cup \{n\} \in \mathcal{H}$ by definition of \mathcal{H}_1). By (a) and (b), we have the following consequences. For any $A, B \in \mathcal{H}_0$ with $A \subsetneq B$ and $|A| \ge t$,

$$h_0(A) = h(A) \ge (t+u)h(B) = (t+u)h_0(B).$$
 (3)

For any $C \in \mathcal{H}_0$ and any $i, j \in [n-1]$ with i < j,

$$h_0(\delta_{i,j}(C)) = h(\delta_{i,j}(C)) \ge h(C) = h_0(C).$$
 (4)

For any $A, B \in \mathcal{H}_1$ with $A \subsetneq B$ and $|A| \ge t$,

$$h_1(A) = h(A \cup \{n\}) \ge (t+u)h(B \cup \{n\}) = (t+u)h_1(B).$$
(5)

For any $C \in \mathcal{H}_1$ and any $i, j \in [n-1]$ with i < j,

$$h_1(\delta_{i,j}(C)) = h(\delta_{i,j}(C) \cup \{n\}) = h(\delta_{i,j}(C \cup \{n\})) \ge h(C \cup \{n\}) = h_1(C).$$
(6)

Therefore, we have shown that properties (a) and (b) are inherited by h_0 and h_1 .

Since $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}(\{n\}), \mathcal{B}_0 \cap \mathcal{B}(\{n\}) = \emptyset$, and $\mathcal{B}(\{n\}) = \{B \cup \{n\} : B \in \mathcal{B}_1\}$, we have

$$h(\mathcal{B}) = h(\mathcal{B}_0) + h(\mathcal{B}(\{n\})) = h_0(\mathcal{B}_0) + h_1(\mathcal{B}_1).$$
(7)

Along the same lines,

$$h(\mathcal{H}(T)) = h(\mathcal{H}_0(T)) + h(\{H \in \mathcal{H} \colon T \cup \{n\} \subseteq H\})$$

= $h_0(\mathcal{H}_0(T)) + h(\{H \cup \{n\} \colon H \in \mathcal{H}_1(T)\})$
= $h_0(\mathcal{H}_0(T)) + h_1(\mathcal{H}_1(T)).$ (8)

Suppose $\mathcal{G}_1 = \emptyset$. Clearly, \mathcal{A} and \mathcal{B}_0 are cross-*t*-intersecting. Since $\mathcal{G}_1 = \emptyset$, no set in \mathcal{A} contains *n*, and hence \mathcal{A} and \mathcal{B}_1 are cross-*t*-intersecting. Thus, by the induction hypothesis,

$$g(\mathcal{A})h_j(\mathcal{B}_j) \le g(\mathcal{G}(T))h_j(\mathcal{H}_j(T)) \quad \text{for each } j \in \{0, 1\}.$$
(9)

Together with (7) and (8), this gives us

$$g(\mathcal{A})h(\mathcal{B}) = g(\mathcal{A})h_0(\mathcal{B}_0) + g(\mathcal{A})h_1(\mathcal{B}_1)$$

$$\leq g(\mathcal{G}(T))h_0(\mathcal{H}_0(T)) + g(\mathcal{G}(T))h_1(\mathcal{H}_1(T))$$

$$= g(\mathcal{G}(T))h(\mathcal{H}(T)).$$

By (1), equality holds throughout, and hence $g(\mathcal{A})h(\mathcal{B}) = g(\mathcal{G}(T))h(\mathcal{H}(T))$. Thus, in (9), we actually have equality. Suppose u > 0. By the induction hypothesis, for each $j \in \{0, 1\}$, we have $\mathcal{A} = \mathcal{G}(V_j)$ and $\mathcal{B}_j = \mathcal{H}_j(V_j)$ for some $V_j \in \binom{[n-1]}{t}$ such that $g(\mathcal{G}(V_j)) = g(\mathcal{G}(T))$ and $h_j(\mathcal{H}_j(V_j)) = h_j(\mathcal{H}_j(T))$. Thus $g(\mathcal{G}(V_0)) > 0$, and hence $\mathcal{G}(V_0) \neq \emptyset$. Thus, since \mathcal{G} is hereditary and $\mathcal{A} = \mathcal{G}(V_0)$, $V_0 \in \mathcal{A}$. Since \mathcal{A} and \mathcal{B} are cross-*t*-intersecting, $\mathcal{B} \subseteq \mathcal{H}(V_0)$. Since $\mathcal{A} = \mathcal{G}(V_0)$, and since $\mathcal{G}(V_0)$ and $\mathcal{H}(V_0)$ are cross-*t*-intersecting, it follows by the choice of \mathcal{A} and \mathcal{B} that $\mathcal{B} = \mathcal{H}(V_0)$. By (1), $\mathcal{H}(V_0) \neq \emptyset$. Since \mathcal{H} is hereditary, $V_0 \in \mathcal{B}$. By Lemma 4.2, the result follows.

Now suppose that \mathcal{G}_1 is non-empty. If $\mathcal{H}_1 = \emptyset$, then the result follows by an argument similar to that for the case $\mathcal{G}_1 = \emptyset$ above. Thus we assume that \mathcal{H}_1 is non-empty. Since $\mathcal{G}_1 \subseteq \mathcal{G}_0$ and $\mathcal{H}_1 \subseteq \mathcal{H}_0$, \mathcal{G}_0 and \mathcal{H}_0 are non-empty too.

Similarly to h_0 and h_1 , let $g_0 : \mathcal{G}_0 \to \mathbb{R}^+$ such that $g_0(G) = g(G)$ for each $G \in \mathcal{G}_0$, and let $g_1 : \mathcal{G}_1 \to \mathbb{R}^+$ such that $g_1(G) = g(G \cup \{n\})$ for each $G \in \mathcal{G}_1$ (note that $G \cup \{n\} \in \mathcal{G}$ by definition of \mathcal{G}_1). Then properties (a) and (b) are inherited by g_0 and g_1 in the same way they are inherited by h_0 and h_1 , as shown above; that is, similarly to (3)–(6), we have the following. For any $A, B \in \mathcal{G}_0$ with $A \subsetneq B$ and $|A| \ge t$,

$$g_0(A) \ge (t+u)g_0(B).$$
 (10)

For any $C \in \mathcal{G}_0$ and any $i, j \in [n-1]$ with i < j,

$$g_0(\delta_{i,j}(C)) \ge g_0(C). \tag{11}$$

For any $A, B \in \mathcal{G}_1$ with $A \subsetneq B$ and $|A| \ge t$,

$$g_1(A) \ge (t+u)g_1(B).$$
 (12)

For any $C \in \mathcal{G}_1$ and any $i, j \in [n-1]$ with i < j,

$$g_1(\delta_{i,j}(C)) \ge g_1(C). \tag{13}$$

Similarly to (7) and (8), we have

$$g(\mathcal{A}) = g_0(\mathcal{A}_0) + g_1(\mathcal{A}_1), \tag{14}$$

$$g(\mathcal{G}(T)) = g_0(\mathcal{G}_0(T)) + g_1(\mathcal{G}_1(T)).$$
(15)

Clearly, \mathcal{A}_0 and \mathcal{B}_0 are cross-*t*-intersecting, \mathcal{A}_0 and \mathcal{B}_1 are cross-*t*-intersecting, and \mathcal{A}_1 and \mathcal{B}_0 are cross-*t*-intersecting.

Let us first assume that \mathcal{A}_1 and \mathcal{B}_1 are cross-*t*-intersecting too. Then, by the induction hypothesis,

$$g_i(\mathcal{A}_i)h_j(\mathcal{B}_j) \le g_i(\mathcal{G}_i(T))h_j(\mathcal{H}_j(T)) \quad \text{for any } i, j \in \{0, 1\}.$$
(16)

Together with (7), (8), (14), and (15), this gives us

$$g(\mathcal{A})h(\mathcal{B}) = g_0(\mathcal{A}_0)h_0(\mathcal{B}_0) + g_0(\mathcal{A}_0)h_1(\mathcal{B}_1) + g_1(\mathcal{A}_1)h_0(\mathcal{B}_0) + g_1(\mathcal{A}_1)h_1(\mathcal{B}_1) \leq g_0(\mathcal{G}_0(T))h_0(\mathcal{H}_0(T)) + g_0(\mathcal{G}_0(T))h_1(\mathcal{H}_1(T)) + g_1(\mathcal{G}_1(T))h_0(\mathcal{H}_0(T)) + g_1(\mathcal{G}_1(T))h_1(\mathcal{H}_1(T)) = g(\mathcal{G}(T))h(\mathcal{H}(T)).$$

By (1), equality holds throughout, and hence $g(\mathcal{A})h(\mathcal{B}) = g(\mathcal{G}(T))h(\mathcal{H}(T))$. Thus, in (16), we actually have equality. Suppose u > 0. By the induction hypothesis, we particularly have $\mathcal{A}_0 = \mathcal{G}_0(V_0)$ and $\mathcal{B}_0 = \mathcal{H}_0(V_0)$ for some $V_0 \in \binom{[n-1]}{t}$ such that $g_0(\mathcal{G}_0(V_0)) = g_0(\mathcal{G}_0(T))$ and $h_0(\mathcal{H}_0(V_0)) = h_0(\mathcal{H}_0(T))$. Recall that $T \in \mathcal{G}$, so $T \in \mathcal{G}_0$, and hence $g_0(\mathcal{G}_0(T)) > 0$. Thus $g_0(\mathcal{G}_0(V_0)) > 0$, and hence $\mathcal{G}_0(V_0) \neq \emptyset$. Since \mathcal{G}_0 is hereditary, it follows that $V_0 \in \mathcal{G}_0(V_0)$, and hence $V_0 \in \mathcal{A}$. Similarly, $V_0 \in \mathcal{B}$. By Lemma 4.2, the result follows.

We will now show that \mathcal{A}_1 and \mathcal{B}_1 are indeed cross-*t*-intersecting. Note that \mathcal{A}_1 and \mathcal{B}_1 are cross-(t-1)-intersecting.

Suppose that \mathcal{A}_1 and \mathcal{B}_1 are not cross-*t*-intersecting. Then there exists $A^* \in \mathcal{A}_1$ such that $|A^* \cap B^*| = t - 1$ for some $B^* \in \mathcal{B}_1$. Let $r = |A^*| + 1$ and s = n - r + t. Let

$$\mathcal{R} = \{ A \in \mathcal{A}_1 \colon |A| = r - 1, |A \cap B| = t - 1 \text{ for some } B \in \mathcal{B}_1 \}.$$
$$\mathcal{S} = \{ B \in \mathcal{B}_1 \colon |B| = s - 1, |A \cap B| = t - 1 \text{ for some } A \in \mathcal{A}_1 \}$$

We have $A^* \in \mathcal{R}$.

Consider any $R \in \mathcal{R}$ and $B \in \mathcal{B}_1$ such that $|R \cap B| < t$. Since \mathcal{A}_1 and \mathcal{B}_1 are cross-(t-1)-intersecting, $|R \cap B| = t-1$. We have

$$\begin{split} |B| &= |B \cap R| + |B \backslash R| = t - 1 + |B \backslash R| \\ &\leq t - 1 + |[n - 1] \backslash R| = t - 1 + (n - 1) - (r - 1) = s - 1. \end{split}$$

Suppose $B \notin S$. Then |B| < s - 1. Thus we have

$$|R \cup B| = |R| + |B| - |R \cap B| \le r - 1 + s - 2 - t + 1 = n - 2,$$

and hence $R \cup B \subsetneq [n-1]$. Let $c \in [n-1] \setminus (R \cup B)$. Since $B \in \mathcal{B}_1, B \cup \{n\} \in \mathcal{B}$. Let $C = \delta_{c,n}(B \cup \{n\})$. Since $c \notin B \cup \{n\}, C = B \cup \{c\}$. Since \mathcal{B} is compressed, $C \in \mathcal{B}$. However, since $c \notin R \cup \{n\}$ and $|R \cap B| = t-1$, we have $|(R \cup \{n\}) \cap C| = t-1$, which is a contradiction as \mathcal{A} and \mathcal{B} are cross-t-intersecting, $R \cup \{n\} \in \mathcal{A}$, and $C \in \mathcal{B}$.

We have therefore shown that

for each
$$B \in \mathcal{B}_1$$
 such that $|R \cap B| < t$ for some $R \in \mathcal{R}, B \in \mathcal{S}$. (17)

By a similar argument,

for each
$$A \in \mathcal{A}_1$$
 such that $|A \cap S| < t$ for some $S \in \mathcal{S}, A \in \mathcal{R}$. (18)

For each $A \in \mathcal{A}_1 \cup \mathcal{B}_1$, let $A' = A \cup \{n\}$. Let $\mathcal{R}' = \{R' : R \in \mathcal{R}\}$ and $\mathcal{S}' = \{S' : S \in \mathcal{S}\}$. Since $\mathcal{R} \subseteq \mathcal{A}_1$ and $\mathcal{S} \subseteq \mathcal{B}_1, \mathcal{R}' \subseteq \mathcal{A}(\{n\})$ and $\mathcal{S}' \subseteq \mathcal{B}(\{n\})$. Let

$$\mathcal{A}' = \mathcal{A} \cup \mathcal{R}, \quad \mathcal{A}'' = \mathcal{A} ackslash \mathcal{R}', \quad \mathcal{B}' = \mathcal{B} ackslash \mathcal{S}', \quad \mathcal{B}'' = \mathcal{B} \cup \mathcal{S}.$$

By (17), \mathcal{A}' and \mathcal{B}' are cross-*t*-intersecting. By (18), \mathcal{A}'' and \mathcal{B}'' are cross-*t*-intersecting. Since \mathcal{G} and \mathcal{H} are hereditary, and since $\mathcal{R}' \subseteq \mathcal{A} \subseteq \mathcal{G}$ and $\mathcal{S}' \subseteq \mathcal{B} \subseteq \mathcal{H}$, we have $\mathcal{R} \subseteq \mathcal{G}$ and $\mathcal{S} \subseteq \mathcal{H}$, and hence $\mathcal{A}', \mathcal{A}'' \subseteq \mathcal{G}$ and $\mathcal{B}', \mathcal{B}'' \subseteq \mathcal{H}$.

Let $x = g(\mathcal{A}), x_1 = g(\mathcal{R}'), y = h(\mathcal{B}), \text{ and } y_1 = h(\mathcal{S}')$. We use a double-counting argument to obtain $x \ge nx_1/r$ and $y \ge ny_1/s$. For any $R \in \mathcal{R}'$ and any set A such that $A = \delta_{i,n}(R)$ for some $i \in [n] \setminus R$, we write A < R. If $A < R \in \mathcal{R}'$, then, since \mathcal{A} is compressed and $n \in R \in \mathcal{A}$, we have $A \in \mathcal{A}_0$. For any $A \in \mathcal{A}_0$ and any $R \in \mathcal{R}'$, let

$$\chi(A, R) = \begin{cases} 1 & \text{if } A < R; \\ 0 & \text{otherwise.} \end{cases}$$

Then $\sum_{A \in \mathcal{A}_0} \chi(A, R) = n - r$ for each $R \in \mathcal{R}'$. For each $A \in \mathcal{A}_0$, $\chi(A, R) = 1$ only if |A| = |R| and $R = (A \setminus \{i\}) \cup \{n\}$ for some $i \in A$. Thus $\sum_{R \in \mathcal{R}'} \chi(A, R) \leq r$ for each $A \in \mathcal{A}_0$. We have

$$(n-r)x_{1} = \sum_{R \in \mathcal{R}'} (n-r)g(R) = \sum_{R \in \mathcal{R}'} \sum_{A \in \mathcal{A}_{0}} \chi(A, R)g(R) = \sum_{A \in \mathcal{A}_{0}} \sum_{R \in \mathcal{R}'} \chi(A, R)g(R)$$
$$\leq \sum_{A \in \mathcal{A}_{0}} \sum_{R \in \mathcal{R}'} \chi(A, R)g(A) \quad (by (b))$$
$$\leq \sum_{A \in \mathcal{A}_{0}} rg(A) = rg(\mathcal{A}_{0}) = r(x - g(\mathcal{A}(\{n\}))) \leq r(x - x_{1}),$$

so $x \ge nx_1/r$. Similarly, $y \ge ny_1/s$.

Since $t - 1 = |A^* \cap B^*| \le |A^*| = r - 1$, $r \ge t$. By (17), $B^* \in \mathcal{S}$. Since $t - 1 = |A^* \cap B^*| \le |B^*| = s - 1$, $s \ge t$.

Suppose r = t. Then s = n. Thus $\mathcal{S}' = \{[n]\}$ and $\mathcal{S} = \{[n-1]\}$. Let $C^* = [t-1] \cup \{n\}$. Since $A^* \in \mathcal{A}_1$, \mathcal{A}_1 is compressed, and $|A^*| = r - 1 = t - 1$, we have $[t-1] \in \mathcal{A}_1$, and hence $C^* \in \mathcal{A}$.

Suppose that there exists $D^* \in \mathcal{B}$ such that $D^* \neq [n]$. Since \mathcal{A} and \mathcal{B} are cross-*t*-intersecting, we have $|C^* \cap D^*| \geq t$. Thus $C^* \subseteq D^*$ as $|C^*| = t$. Since $D^* \neq [n]$, there exists $c \in [n]$ such that $c \notin D^*$. Thus $c \notin C^*$. Since \mathcal{A} is compressed, $\delta_{c,n}(C^*) \in \mathcal{A}$. However, $|\delta_{c,n}(C^*) \cap D^*| = |C^* \setminus \{n\}| = t - 1$, which is a contradiction as \mathcal{A} and \mathcal{B} are cross-*t*-intersecting.

Therefore, $\mathcal{B} = \{[n]\}$. Since n-1 > t, $h([n-1]) \ge (t+u)h([n]) \ge th([n])$. We have

$$h(\mathcal{B}'') = h([n]) + h([n-1]) \ge h([n]) + th([n]) = (t+1)h(\mathcal{B}) = (t+1)y.$$

Since $x \ge nx_1/r = nx_1/t \ge (t+3)x_1/t$, $x_1 \le tx/(t+3)$. We have

$$g(\mathcal{A}'') = x - x_1 \ge x - \frac{tx}{t+3} = \frac{3x}{t+3}$$

Thus we obtain

$$g(\mathcal{A}'')h(\mathcal{B}'') \ge \frac{3(t+1)xy}{t+3} > xy = g(\mathcal{A})h(\mathcal{B}),$$

contradicting the choice of \mathcal{A} and \mathcal{B} .

Therefore, $r \ge t+1$. Similarly, $s \ge t+1$. Since $r-1 \ge t$ and each set in \mathcal{R} is of size r-1, $g(\mathcal{R}) \ge (t+u)g(\mathcal{R}')$. Similarly, $h(\mathcal{S}) \ge (t+u)g(\mathcal{S}')$.

Consider any $R \in \mathcal{R}$. By definition of \mathcal{R} , there exists $B_R \in \mathcal{B}_1$ such that $|R \cap B_R| = t-1$. Thus $|R \cap B_R'| = t-1$. Since $B_R' \in \mathcal{B}$, and since \mathcal{A} and \mathcal{B} are cross-*t*-intersecting, $R \notin \mathcal{A}$. Therefore, $\mathcal{A} \cap \mathcal{R} = \emptyset$. Similarly, $\mathcal{B} \cap \mathcal{S} = \emptyset$.

We have

$$g(\mathcal{A}') = x + g(\mathcal{R}) \ge x + (t+u)g(\mathcal{R}') = x + (t+u)x_1, g(\mathcal{A}'') = x - x_1, h(\mathcal{B}') = y - y_1, h(\mathcal{B}'') = y + h(\mathcal{S}) \ge y + (t+u)h(\mathcal{S}') = y + (t+u)y_1.$$

By the choice of \mathcal{A} and \mathcal{B} ,

$$g(\mathcal{A}')h(\mathcal{B}') \leq g(\mathcal{A})h(\mathcal{B}) \text{ and } g(\mathcal{A}'')h(\mathcal{B}'') \leq g(\mathcal{A})h(\mathcal{B}).$$

Thus we have

$$\begin{aligned} &(x + (t+u)x_1)(y - y_1) \leq xy \quad \text{and} \quad (x - x_1)(y + (t+u)y_1) \leq xy \\ \Rightarrow & (t+u)x_1y \leq xy_1 + (t+u)x_1y_1 \quad \text{and} \quad (t+u)xy_1 \leq x_1y + (t+u)x_1y_1 \\ \Rightarrow & (t+u)x_1y + (t+u)xy_1 \leq (xy_1 + (t+u)x_1y_1) + (x_1y + (t+u)x_1y_1) \\ \Rightarrow & (t+u-1)(x_1y + xy_1) \leq 2(t+u)x_1y_1. \\ \Rightarrow & (t+u-1)\left(x_1\frac{ny_1}{s} + \frac{nx_1}{r}y_1\right) \leq 2(t+u)x_1y_1 \\ \Rightarrow & (t+u-1)(r+s)x_1y_1n \leq 2(t+u)rsx_1y_1 \\ \Rightarrow & (t+u-1)(n+t)n \leq 2(t+u)r(n-r+t). \end{aligned}$$

Using differentiation, we find that the maximum value of the function f(z) = z(n-z+t) occurs at $z = \frac{n+t}{2}$. Thus $r(n-r+t) \leq \frac{n+t}{2} \left(n - \frac{n+t}{2} + t\right) = (n+t)^2/4$, and hence

$$(t+u-1)(n+t)n \le 2(t+u)(n+t)^2/4 \Rightarrow 2(t+u-1)n \le (t+u)(n+t) \Rightarrow n \le \frac{(t+u)t}{t+u-2}.$$
(19)

Since $u > \frac{6-t}{3}$, $\frac{(t+u)t}{t+u-2} < t+3$. Thus we have n < t+3, which is a contradiction. Hence the result.

Remark 4.3 Note that the proof for the special case $n \le t + 2$ actually verifies the conjecture in Remark 1.4 for $n \le t + 2$. Also note that for $t \ge 3$, if we also settle the conjecture for $t + 3 \le n \le t + 6$, then we can take u = 0 and proceed for $n \ge t + 7$ in exactly the same way we did for $n \ge t + 3$, because again we obtain a contradiction to (19); thus, as mentioned in Remark 1.4, this would settle the conjecture for $t \ge 3$.

5 Proof of Theorem 1.2

In this section, we use Theorem 1.3 to prove Theorem 1.2.

For a family \mathcal{F} and an integer $r \geq 0$, we denote the families $\{F \in \mathcal{F} : |F| = r\}$ and $\{F \in \mathcal{F} : |F| \leq r\}$ by $\mathcal{F}^{(r)}$ and $\mathcal{F}^{(\leq r)}$, respectively.

We will need the following lemma only when dealing with the characterisation of the extremal structures in the proof of Theorem 1.2.

Lemma 5.1 Let t, r, s, u, m, and n be as in Theorem 1.2. Let $i, j \in [\max\{m, n\}]$ with i < j. Let $\mathcal{G} = 2^{[m]}$ and $\mathcal{H} = 2^{[n]}$. Let $\mathcal{A} \subseteq \mathcal{G}^{(r)}$ and $\mathcal{B} \subseteq \mathcal{H}^{(s)}$ such that \mathcal{A} and \mathcal{B} are cross-t-intersecting. Suppose that $\Delta_{i,j}(\mathcal{A}) = \mathcal{G}^{(r)}(T)$ and $\Delta_{i,j}(\mathcal{B}) = \mathcal{H}^{(s)}(T)$ for some t-element subset T of $[\min\{m, n\}]$. Then $\mathcal{A} = \mathcal{G}^{(r)}(T')$ and $\mathcal{B} = \mathcal{H}^{(s)}(T')$ for some t-element subset T' of $[\min\{m, n\}]$.

We prove the above lemma using the following special case of [11, Lemma 5.6].

Lemma 5.2 Let $t \ge 1$, $r \ge t+1$, $n \ge 2r-t+2$, and $i, j \in [n]$. Let $\mathcal{H} = 2^{[n]}$, and let \mathcal{A} be a t-intersecting subfamily of $\mathcal{H}^{(r)}$. If $\Delta_{i,j}(\mathcal{A})$ is a largest t-star of $\mathcal{H}^{(r)}$, then \mathcal{A} is a largest t-star of $\mathcal{H}^{(r)}$.

Proof of Lemma 5.1. We are given that $t \leq r \leq s$.

Suppose r = t. Then $\Delta_{i,j}(\mathcal{A}) = \{T\}$. Thus $\mathcal{A} = \{T'\} = \mathcal{H}^{(r)}(T')$ for some $T' \in \binom{[m]}{t}$. Since \mathcal{A} and \mathcal{B} are cross-*t*-intersecting, $T' \subseteq B$ for all $B \in \mathcal{B}$. Thus $\mathcal{B} \subseteq \mathcal{H}^{(s)}(T')$. Since $\binom{n-t}{s-t} = |\mathcal{H}^{(s)}(T)| = |\Delta_{i,j}(\mathcal{B})| = |\mathcal{B}| \leq |\mathcal{H}^{(s)}(T')| = \binom{n-t}{s-t}, |\mathcal{B}| = \binom{n-t}{s-t}$. Hence $\mathcal{B} = \mathcal{H}^{(s)}(T')$.

Now suppose $r \ge t+1$. Note that $T \setminus \{i\} \subseteq E$ for all $E \in \mathcal{A} \cup \mathcal{B}$.

Suppose that \mathcal{A} is not *t*-intersecting. Then there exist $A_1, A_2 \in \mathcal{A}$ such that $|A_1 \cap A_2| \leq t-1$, and hence $T \notin A_l$ for some $l \in \{1,2\}$; we may assume that l = 1. Thus, since $\Delta_{i,j}(\mathcal{A}) = \mathcal{H}^{(r)}(T)$, we have $A_1 \notin \Delta_{i,j}(\mathcal{A}), A_1 \neq \delta_{i,j}(A_1) \in \Delta_{i,j}(\mathcal{A}), \delta_{i,j}(A_1) \notin \mathcal{A}$ (because otherwise $A_1 \in \Delta_{i,j}(\mathcal{A})$), $i \in T$, $j \notin T$, $j \in A_1$, and $A_1 \cap T = T \setminus \{i\}$. Since $T \setminus \{i\} \subseteq A_1 \cap A_2$ and $|A_1 \cap A_2| \leq t-1$, $A_1 \cap A_2 = T \setminus \{i\}$. Thus $j \notin A_2$, and hence $A_2 = \delta_{i,j}(A_2)$. Since $\delta_{i,j}(A_2) \in \Delta_{i,j}(\mathcal{A}) = \mathcal{H}^{(r)}(T), T \subseteq A_2$. Let $X = [n] \setminus (A_1 \cup A_2)$. We have

$$\begin{aligned} |X| &= n - |A_1 \cup A_2| = n - (|A_1| + |A_2| - |A_1 \cap A_2|) = n - 2r + t - 1\\ &\ge (t + u + 2)(s - t) + r - 1 - 2(r - t) - (t + 1) \ge (t + u)(s - t) - 1\\ &> \left(2 + \frac{2t}{3}\right)(s - t) - 1 = \left(1 + \frac{2t}{3}\right)(s - t) + s - (t + 1).\end{aligned}$$

Since $t + 1 \leq r \leq s$, we have |X| > s - t, and hence $\binom{X}{s-t} \neq \emptyset$. Let $C \in \binom{X}{s-t}$ and $D = C \cup T$. Then $D \in \mathcal{H}^{(s)}(T)$ and $D \cap A_1 = T \setminus \{i\}$, meaning that $D \in \Delta_{i,j}(\mathcal{B})$ and $|D \cap A_1| = t - 1$. Since \mathcal{A} and \mathcal{B} are cross-t-intersecting, we obtain $D \notin \mathcal{B}$ and $(D \setminus \{i\}) \cup \{j\} \in \mathcal{B}$, which is a contradiction since $|((D \setminus \{i\}) \cup \{j\}) \cap A_2| = |T \setminus \{i\}| = t - 1$.

Therefore, \mathcal{A} is *t*-intersecting. Similarly, \mathcal{B} is *t*-intersecting. Now $\mathcal{H}^{(r)}(T)$ is a largest *t*-star of $\mathcal{H}^{(r)}$, and $\mathcal{H}^{(s)}(T)$ is a largest *t*-star of $\mathcal{H}^{(s)}$. Since $t + 1 \leq r \leq s$ and

$$\max\{m,n\} \ge (t+u+2)(s-t)+r-1 = (t+u)(s-t) + (2s-t+2) + r - (t+1) - 2,$$

we have

$$\max\{m,n\} - (2s - t + 2) > \left(2 + \frac{2t}{3}\right)(s - t) - 2 \ge \frac{2t}{3}(s - t) > 0,$$

and hence $\max\{m, n\} > 2s - t + 2$. By Lemma 5.2, $\mathcal{A} = \mathcal{H}^{(r)}(T')$ for some $T' \in {\binom{[m]}{t}}$, and $\mathcal{B} = \mathcal{H}^{(s)}(T^*)$ for some $T^* \in {\binom{[n]}{t}}$.

Suppose $T' \neq T^*$. Let $z \in T^* \setminus T'$. Since m > 2s - t + 2 > r, we can choose $A' \in \mathcal{H}^{(r)}(T')$ such that $z \notin A'$. Since $n > 2s - t + 2 \ge r + s - t + 2 > r + s - t$ and $z \in T^* \setminus A'$, we can choose $B^* \in \mathcal{H}^{(s)}(T^*)$ such that $|A' \cap B^*| \le t - 1$; however, this is a contradiction since $\mathcal{A} = \mathcal{H}^{(r)}(T')$, $\mathcal{B} = \mathcal{H}^{(s)}(T^*)$, and \mathcal{A} and \mathcal{B} are cross-t-intersecting. Therefore, $T' = T^*$.

Proof of Theorem 1.2. If $\mathcal{A} = \emptyset$ or $\mathcal{B} = \emptyset$, then $|\mathcal{A}||\mathcal{B}| = 0$. Thus we assume that $\mathcal{A} \neq \emptyset$ and $\mathcal{B} \neq \emptyset$. Let $l = \max\{m, n\}$, so $\mathcal{A}, \mathcal{B} \subseteq 2^{[l]}$.

As explained in Section 3, we apply left-compressions to \mathcal{A} and \mathcal{B} simultaneously until we obtain two compressed cross-*t*-intersecting families \mathcal{A}^* and \mathcal{B}^* , respectively. We have $\mathcal{A}^* \subseteq {[m] \choose r}$, $\mathcal{B}^* \subseteq {[n] \choose s}$, $|\mathcal{A}^*| = |\mathcal{A}|$, and $|\mathcal{B}^*| = |\mathcal{B}|$. In view of Lemma 5.1, we may therefore assume that \mathcal{A} and \mathcal{B} are compressed. Thus, by Lemma 3.1(ii),

$$|A \cap B \cap [r+s-t]| \ge t \text{ for any } A \in \mathcal{A} \text{ and any } B \in \mathcal{B}.$$
(20)

Let p = r + s - t. Let $\mathcal{G} = {\binom{[p]}{\leq r}}$ and $\mathcal{H} = {\binom{[p]}{\leq s}}$. Let $g : \mathcal{G} \to \mathbb{N}$ such that $g(G) = {\binom{m-p}{r-|G|}}$ for each $G \in \mathcal{G}$. Let $h : \mathcal{H} \to \mathbb{N}$ such that $h(H) = {\binom{n-p}{s-|H|}}$ for each $H \in \mathcal{H}$.

For every $F, G \in \mathcal{G}$ with $F \subsetneq G$ and $t \leq |F| = |G| - 1$, we have

$$\begin{aligned} \frac{g(F) - (t+u)g(G)}{\binom{m-p}{r-|F|}} &= 1 - \frac{(t+u)\binom{m-p}{r-|F|-1}}{\binom{m-p}{r-|F|}} = 1 - \frac{(t+u)(r-|F|)}{m-p-(r-|F|)+1} \\ &= \frac{m-p - (t+u+1)(r-|F|) + 1}{m-p-(r-|F|)+1} \\ &\geq \frac{m-p - (t+u+1)(r-t) + 1}{m-p-(r-|F|)+1} \\ &= \frac{m-(t+u+2)(r-t) - s + 1}{m-p-(r-|F|)+1} \\ &\geq \frac{(t+u+2)(s-t) + r - 1 - ((t+u+2)(r-t) + s - 1)}{m-p-(r-|F|)+1} \geq 0, \end{aligned}$$

and hence $g(F) \ge (t+u)g(G)$. It follows that $g(F) \ge (t+u)g(G)$ for every $F, G \in \mathcal{G}$ with $F \subsetneq G$ and $|F| \ge t$. Similarly, $h(F) \ge (t+u)g(H)$ for every $F, H \in \mathcal{H}$ with $F \subsetneq H$ and $|F| \ge t$.

We have $g(\delta_{i,j}(G)) = g(G)$ for every $G \in \mathcal{G}$ and every $i, j \in [p]$. Similarly, $h(\delta_{i,j}(H)) = h(H)$ for every $H \in \mathcal{H}$ and every $i, j \in [p]$.

Let $C = \{A \cap [p] : A \in A\}$ and $D = \{B \cap [p] : B \in B\}$. Then $C \subseteq G$, $D \subseteq H$, and, by (20), C and D are cross-t-intersecting. Let T = [t]. By Theorem 1.3,

$$g(\mathcal{C})h(\mathcal{D}) \le g(\mathcal{G}(T))h(\mathcal{H}(T)),$$
(21)

and if u > 0, then equality holds only if $\mathcal{C} = \mathcal{G}(T')$ and $\mathcal{D} = \mathcal{H}(T')$ for some $T' \in {[p] \choose t}$. We have

$$|\mathcal{A}| = \left| \bigcup_{i=0}^{r} \left\{ A \in \mathcal{A} \colon A \cap [p] \in \mathcal{C}^{(i)} \right\} \right| \le \sum_{i=0}^{r} \left| \mathcal{C}^{(i)} \right| \binom{m-p}{r-i} = g(\mathcal{C}), \tag{22}$$

$$|\mathcal{B}| = \left| \bigcup_{j=0}^{s} \left\{ B \in \mathcal{B} \colon B \cap [p] \in \mathcal{D}^{(j)} \right\} \right| \le \sum_{j=0}^{s} \left| \mathcal{D}^{(j)} \right| \binom{n-p}{s-j} = h(\mathcal{D}), \tag{23}$$

and hence, by (21),

$$|\mathcal{A}||\mathcal{B}| \le g(\mathcal{G}(T))h(\mathcal{H}(T)).$$
(24)

Now

$$g(\mathcal{G}(T)) = \sum_{i=t}^{r} \left| \mathcal{G}^{(i)}(T) \right| \binom{m-p}{r-i} = \left| \bigcup_{i=t}^{r} \left\{ A \in \binom{[m]}{r} : T \subseteq A, |A \cap [p]| = i \right\} \right|$$
$$= \left| \left\{ A \in \binom{[m]}{r} : T \subseteq A \right\} \right| = \binom{m-t}{r-t}$$

and, similarly, $h(\mathcal{H}(T)) = \binom{n-t}{s-t}$. Together with (24), this gives us

$$|\mathcal{A}||\mathcal{B}| \le \binom{m-t}{r-t} \binom{n-t}{s-t},$$

as required.

Suppose $|\mathcal{A}||\mathcal{B}| = \binom{m-t}{r-t}\binom{n-t}{s-t}$ and u > 0. Then equality holds throughout in each of (21)–(24), and hence $\mathcal{C} = \mathcal{G}(T')$ and $\mathcal{D} = \mathcal{H}(T')$ for some $T' \in \binom{[p]}{t}$. It follows that $\mathcal{A} \subseteq \left\{A \in \binom{[m]}{r}: T' \subseteq A\right\}$ and $\mathcal{B} \subseteq \left\{B \in \binom{[n]}{s}: T' \subseteq B\right\}$. Since $|\mathcal{A}||\mathcal{B}| = \binom{m-t}{r-t}\binom{n-t}{s-t}$, both inclusion relations are actually equalities, $T' \subseteq [m]$, and $T' \subseteq [n]$.

6 Proof of Theorem 1.5

In this section, we use Theorem 1.3 to prove Theorem 1.5.

We start by defining a compression operation for labeled sets. For any $x, y \in \mathbb{N}$, let

$$\gamma_{x,y}(A) = \begin{cases} (A \setminus \{(x,y)\}) \cup \{(x,1)\} & \text{if } (x,y) \in A; \\ A & \text{otherwise} \end{cases}$$

for any labeled set A, and let

$$\Gamma_{x,y}(\mathcal{A}) = \{\gamma_{x,y}(A) \colon A \in \mathcal{A}\} \cup \{A \in \mathcal{A} \colon \gamma_{x,y}(A) \in \mathcal{A}\}$$

for any family \mathcal{A} of labeled sets.

Note that $|\Gamma_{x,y}(\mathcal{A})| = |\mathcal{A}|$ and that if $\mathcal{A} \subseteq \mathcal{S}_{\mathbf{c},r}$, then $\Gamma_{x,y}(\mathcal{A}) \subseteq \mathcal{S}_{\mathbf{c},r}$. It is well known that if \mathcal{A} and \mathcal{B} are cross-*t*-intersecting families of labeled sets, then so are $\Gamma_{x,y}(\mathcal{A})$ and $\Gamma_{x,y}(\mathcal{B})$. We present a result that gives more than this.

For any IP sequence $\mathbf{c} = (c_1, \ldots, c_n)$ and any $r \in [n]$, let $\mathcal{S}_{\mathbf{c},\leq r}$ denote the union $\bigcup_{i=1}^r \mathcal{S}_{\mathbf{c},i}$.

Lemma 6.1 Let $\mathbf{c} = (c_1, \ldots, c_m)$ and $\mathbf{d} = (d_1, \ldots, d_n)$ be IP sequences. Let $x, y \in \mathbb{N}$, $y \geq 2$. Let $l = \max\{m, n\}$ and $h = \max\{c_m, d_n\}$. Let $V \subseteq [l] \times [2, h]$. Let $\mathcal{A} \subseteq \mathcal{S}_{\mathbf{c}, \leq m}$ and $\mathcal{B} \subseteq \mathcal{S}_{\mathbf{d}, \leq n}$ such that $|(\mathcal{A} \cap \mathcal{B}) \setminus V| \geq t$ for every $\mathcal{A} \in \mathcal{A}$ and every $\mathcal{B} \in \mathcal{B}$. Then $|(\mathcal{C} \cap \mathcal{D}) \setminus (V \cup \{(x, y)\})| \geq t$ for every $\mathcal{C} \in \Gamma_{x,y}(\mathcal{A})$ and every $\mathcal{D} \in \Gamma_{x,y}(\mathcal{B})$.

Proof. Suppose $C \in \Gamma_{x,y}(\mathcal{A})$ and $D \in \Gamma_{x,y}(\mathcal{B})$. We first show that $|(C \cap D) \setminus V| \geq t$. Let $C' = (C \setminus \{(x, 1)\}) \cup \{(x, y)\}$ and $D' = (D \setminus \{(x, 1)\}) \cup \{(x, y)\}$. If $C \in \mathcal{A}$ and $D \in \mathcal{B}$, then $|(C \cap D) \setminus V| \geq t$. If $C \notin \mathcal{A}$ and $D \notin \mathcal{B}$, then $(x, 1) \in C \cap D$, $C' \in \mathcal{A}$, $D' \in \mathcal{B}$, and hence, since $(x, 1) \notin V$, $|(C \cap D) \setminus V| \ge |(C' \cap D') \setminus V| \ge t$. Suppose $C \notin \mathcal{A}$ and $D \in \mathcal{B}$. Then $(x, 1) \in C$ and $C' \in \mathcal{A}$. If $(x, y) \notin D$, then, since $C' \in \mathcal{A}$ and $D \in \mathcal{B}, t \le |(C' \cap D) \setminus V| \le |(C \cap D) \setminus V|$. If $(x, y) \in D$, then $\gamma_{x,y}(D) \in \mathcal{B}$ (because otherwise $D \notin \Gamma_{x,y}(\mathcal{B})$), and hence, since $C' \in \mathcal{A}, t \le |(C' \cap \gamma_{x,y}(D)) \setminus V| = |(C \cap D) \setminus V|$. Similarly, if $C \in \mathcal{A}$ and $D \notin \mathcal{B}$, then $|(C \cap D) \setminus V| \ge t$.

Now suppose $(C \cap D) \setminus (V \cup \{(x, y)\}) < t$. Since $|(C \cap D) \setminus V| \ge t$, $(x, y) \in C \cap D$. Thus $C, \gamma_{x,y}(C) \in \mathcal{A}, D, \gamma_{x,y}(D) \in \mathcal{B}$, and $|(C \cap \gamma_{x,y}(D)) \setminus V| = |(C \cap D) \setminus (V \cup \{(x, y)\})| < t$, a contradiction.

Corollary 6.2 Let $\mathbf{c} = (c_1, \ldots, c_m), \mathbf{d} = (d_1, \ldots, d_n), l$, and h be as in Lemma 6.1. Let $\mathcal{A} \subseteq \mathcal{S}_{\mathbf{c},\leq m}$ and $\mathcal{B} \subseteq \mathcal{S}_{\mathbf{d},\leq n}$ such that \mathcal{A} and \mathcal{B} are cross-t-intersecting. Let

$$\mathcal{A}^* = \Gamma_{l,h} \circ \cdots \circ \Gamma_{l,2} \circ \cdots \circ \Gamma_{2,h} \circ \cdots \circ \Gamma_{2,2} \circ \Gamma_{1,h} \circ \cdots \circ \Gamma_{1,2}(\mathcal{A}),$$
$$\mathcal{B}^* = \Gamma_{l,h} \circ \cdots \circ \Gamma_{l,2} \circ \cdots \circ \Gamma_{2,h} \circ \cdots \circ \Gamma_{2,2} \circ \Gamma_{1,h} \circ \cdots \circ \Gamma_{1,2}(\mathcal{B}).$$

Then $|A \cap B \cap ([l] \times [1])| \ge t$ for every $A \in \mathcal{A}^*$ and every $B \in \mathcal{B}^*$.

Proof. Let $Z = [l] \times [2, h]$. By repeated application of Lemma 6.1, $|(A \cap B) \setminus Z| \ge t$ for every $A \in \mathcal{A}^*$ and every $B \in \mathcal{B}^*$. The result follows since $(A \cap B) \setminus Z = A \cap B \cap ([l] \times [1])$.

The next lemma is needed for the characterisation of the extremal structures in Theorem 1.5.

Lemma 6.3 Let $\mathbf{c} = (c_1, \ldots, c_m), \mathbf{d} = (d_1, \ldots, d_n), l$, and h be as in Lemma 6.1. Suppose $c_1 \geq 3$ and $d_1 \geq 3$. Let $r \in [m], s \in [n]$, and $t \in [\min\{r, s\}]$. Let $\mathcal{A} \subseteq \mathcal{S}_{\mathbf{c},r}$ and $\mathcal{B} \subseteq \mathcal{S}_{\mathbf{d},s}$ such that \mathcal{A} and \mathcal{B} are cross-t-intersecting. Suppose $\Gamma_{x,y}(\mathcal{A}) = \mathcal{S}_{\mathbf{c},r}(T)$ and $\Gamma_{x,y}(\mathcal{B}) = \mathcal{S}_{\mathbf{d},s}(T)$ for some $(x, y) \in [l] \times [h]$ and some labeled set $T \in {[l] \times [h] \choose t}$. Then $\mathcal{A} = \mathcal{S}_{\mathbf{c},r}(T')$ and $\mathcal{B} = \mathcal{S}_{\mathbf{d},s}(T')$ for some labeled set $T' \in {[l] \times [h] \choose t}$.

Proof. The result is immediate if $\mathcal{A} = \Gamma_{x,y}(\mathcal{A})$ and $\mathcal{B} = \Gamma_{x,y}(\mathcal{B})$. Suppose $\mathcal{A} \neq \Gamma_{x,y}(\mathcal{A})$ or $\mathcal{B} \neq \Gamma_{x,y}(\mathcal{B})$. We may assume that $\mathcal{A} \neq \Gamma_{x,y}(\mathcal{A})$. Thus there exists $A_1 \in \mathcal{A} \setminus \Gamma_{x,y}(\mathcal{A})$ such that $\gamma_{x,y}(A_1) \in \Gamma_{x,y}(\mathcal{A}) \setminus \mathcal{A}$. Then $(x, 1) \neq (x, y) \in A_1$ and $\gamma_{x,y}(A_1) = (A_1 \setminus \{(x, y)\}) \cup \{(x, 1)\}.$

Suppose $(x, 1) \notin T$. Together with $\gamma_{x,y}(A_1) \in \Gamma_{x,y}(\mathcal{A}) = \mathcal{S}_{\mathbf{c},r}(T)$, this gives us $T \subseteq A_1$, which contradicts $A_1 \notin \Gamma_{x,y}(\mathcal{A})$.

Therefore, $(x, 1) \in T$. Let $(a_1, b_1), \ldots, (a_t, b_t)$ be the elements of T, where $(a_t, b_t) = (x, 1)$. Let $T' = (T \setminus \{(x, 1)\}) \cup \{(x, y)\}$. Since $\gamma_{x,y}(A_1) \in \mathcal{S}_{\mathbf{c},r}(T)$, we have $T' \subseteq A_1$, and hence $\mathcal{S}_{\mathbf{c},r}(T') \neq \emptyset$. Note that $|\mathcal{S}_{\mathbf{c},r}(T')| = |\mathcal{S}_{\mathbf{c},r}(T)|$.

Let $A^* \in \mathcal{S}_{\mathbf{c},r}(T')$. If s > t, then let x_1, \ldots, x_{s-t} be distinct elements of $[n] \setminus \{a_1, \ldots, a_t\}$. For each $i \in [n]$, let $D_i = \{i\} \times [d_i]$. We are given that $3 \leq d_1 \leq \cdots \leq d_n$. By definition of a labeled set, for each $i \in [n]$, we have $|A \cap D_i| \leq 1$ for all $A \in \mathcal{S}_{\mathbf{c},r}$. Thus $|D_i \setminus (A_1 \cup A^*)| \geq d_i - 2 \geq 1$ for each $i \in [n]$. If s > t, then let $(x_i, y_i) \in D_{x_i} \setminus (A_1 \cup A^*)$ for each $i \in [s-t]$, and let $B^* = T' \cup \{(x_1, y_1), \ldots, (x_{s-t}, y_{s-t})\}$. If s = t, then let $B^* = T'$. Thus $B^* \in \mathcal{S}_{\mathbf{d},s}(T')$. Since $\Gamma_{x,y}(\mathcal{B}) = \mathcal{S}_{\mathbf{d},s}(T)$, we have $B^* \in \mathcal{B}$ or $\gamma_{x,y}(B^*) \in \mathcal{B}$. However, $|\gamma_{x,y}(B^*) \cap A_1| = |T \cap A_1| = |T \setminus \{(x,1)\}| = t - 1$, so $B^* \in \mathcal{B}$. Since $\Gamma_{x,y}(\mathcal{A}) = \mathcal{S}_{\mathbf{c},r}(T)$, we have $A^* \in \mathcal{A}$ or $\gamma_{x,y}(A^*) \in \mathcal{A}$. However, $|\gamma_{x,y}(A^*) \cap B^*| = t - 1$, so $A^* \in \mathcal{A}$.

We have therefore shown that $\mathcal{S}_{\mathbf{c},r}(T') \subseteq \mathcal{A}$. Since $|\mathcal{A}| = |\Gamma_{x,y}(\mathcal{A})| = |\mathcal{S}_{\mathbf{c},r}(T)| = |\mathcal{S}_{\mathbf{c},r}(T')|$, we actually have $\mathcal{A} = \mathcal{S}_{\mathbf{c},r}(T')$. Clearly, for each $L \in \mathcal{S}_{\mathbf{d},s}$ with $T' \notin L$, there exists $L' \in \mathcal{S}_{\mathbf{c},r}(T')$ such that $|L \cap L'| = |L \cap T'| < |T'| = t$. Thus, since $\mathcal{A} = \mathcal{S}_{\mathbf{c},r}(T')$, each set in \mathcal{B} contains T'. Hence $\mathcal{B} \subseteq \mathcal{S}_{\mathbf{d},s}(T')$. Since $|\mathcal{B}| = |\Gamma_{x,y}(\mathcal{B})| = |\mathcal{S}_{\mathbf{d},s}(T)| = |\mathcal{S}_{\mathbf{d},s}(T)|$ and $|\mathcal{S}_{\mathbf{d},s}(T')|$, we actually have $\mathcal{B} = \mathcal{S}_{\mathbf{d},s}(T')$.

The next lemma allows us to translate the setting in Theorem 1.5 to one given by Theorem 1.3.

Lemma 6.4 Let **c** be an IP sequence (c_1, \ldots, c_n) . Let $r \in [n]$. Let $w: \binom{[n]}{\leq r} \to \mathbb{N}$ such that for each $A \in \binom{[n]}{\leq r}$,

$$w(A) = |\{L \in \mathcal{S}_{\mathbf{c},r} : L \cap ([n] \times [1]) = A \times [1]\}|.$$

Then:

(i) $w(A) \ge (c_1 - 1)w(B)$ for every $A, B \in {[n] \\ \leq r}$ with $A \subsetneq B$. (ii) $w(\delta_{i,j}(A)) \ge w(A)$ for every $A \in {[n] \\ \leq r}$ and every $i, j \in [n]$ with i < j.

Proof. (i) Let $A, B \in {\binom{[n]}{\leq r}}$ with $A \subsetneq B$. Let $B' = B \setminus A$. Thus $|B'| \ge 1$. For each $L \in \mathcal{S}_{\mathbf{c},r}$, let $\sigma(L) = \{x \in [n] : (x, y) \in L \text{ for some } y \in [c_i]\}$. We have

$$w(A) \ge |\{L \in \mathcal{S}_{\mathbf{c},r} \colon L \cap ([n] \times [1]) = A \times [1], B' \subseteq \sigma(L)\}|$$

= $\sum_{E \in \binom{[n] \setminus (A \cup B')}{r-|A|-|B'|}} \prod_{b \in B'} (c_b - 1) \prod_{e \in E} (c_e - 1)$
= $\prod_{b \in B'} (c_b - 1) \left(\sum_{E \in \binom{[n] \setminus B}{r-|B|}} \prod_{e \in E} (c_e - 1) \right)$
= $w(B) \prod_{b \in B'} (c_b - 1) \ge (c_1 - 1)^{|B'|} w(B) \ge (c_1 - 1) w(B)$

(ii) Let $A \in {[n] \choose \leq r}$, and let $i, j \in [n]$ with i < j. Suppose $\delta_{i,j}(A) \neq A$. Then $j \in A$, $i \notin A$, and $\delta_{i,j}(A) = (A \setminus \{j\}) \cup \{i\}$. Let $B = A \setminus \{j\}$. Let

$$\mathcal{E}_{0} = \binom{[n] \setminus (B \cup \{i, j\})}{r - |A|},$$
$$\mathcal{E}_{1} = \left\{ E \in \binom{[n] \setminus (B \cup \{i\})}{r - |A|} : j \in E \right\},$$
$$\mathcal{E}_{2} = \left\{ E \in \binom{[n] \setminus (B \cup \{j\})}{r - |A|} : i \in E \right\}.$$

We have

$$w(B \cup \{i\}) = \sum_{E \in \binom{[n] \setminus (B \cup \{i\})}{r-|A|}} \prod_{e \in E} (c_e - 1)$$

= $\sum_{D \in \mathcal{E}_0} \prod_{d \in D} (c_d - 1) + \sum_{F \in \mathcal{E}_1} \prod_{f \in F} (c_f - 1)$
 $\geq \sum_{D \in \mathcal{E}_0} \prod_{d \in D} (c_d - 1) + \sum_{F \in \mathcal{E}_1} \prod_{f \in F} (c_f - 1) \frac{c_i - 1}{c_j - 1}$ (since $c_i \le c_j$)
 $= \sum_{D \in \mathcal{E}_0} \prod_{d \in D} (c_d - 1) + \sum_{F \in \mathcal{E}_2} \prod_{f \in F} (c_f - 1)$
 $= \sum_{E \in \binom{[n] \setminus (B \cup \{j\})}{r-|A|}} \prod_{e \in E} (c_e - 1) = w(B \cup \{j\}),$

and hence $w(\delta_{i,j}(A)) \ge w(A)$.

Proof of Theorem 1.5. Let $\mathcal{G} = \binom{[m]}{\leq r}$. Let $v : \mathcal{G} \to \mathbb{N}$ such that for each $G \in \mathcal{G}$,

$$v(G) = |\{L \in \mathcal{S}_{\mathbf{c},r} \colon L \cap ([m] \times [1]) = G \times [1]\}|.$$

Let $\mathcal{H} = {\binom{[n]}{\leq s}}$. Let $w \colon \mathcal{H} \to \mathbb{N}$ such that for each $H \in \mathcal{H}$,

$$\mathcal{V}(H) = |\{L \in \mathcal{S}_{\mathbf{d},s} \colon L \cap ([n] \times [1]) = H \times [1]\}|.$$

Let $l = \max\{m, n\}$ and $h = \max\{c_m, d_n\}$. Let

$$\mathcal{A}^* = \Gamma_{l,h} \circ \cdots \circ \Gamma_{l,2} \circ \cdots \circ \Gamma_{2,h} \circ \cdots \circ \Gamma_{2,2} \circ \Gamma_{1,h} \circ \cdots \circ \Gamma_{1,2}(\mathcal{A}),$$
$$\mathcal{B}^* = \Gamma_{l,h} \circ \cdots \circ \Gamma_{l,2} \circ \cdots \circ \Gamma_{2,h} \circ \cdots \circ \Gamma_{2,2} \circ \Gamma_{1,h} \circ \cdots \circ \Gamma_{1,2}(\mathcal{B}).$$

Now let

$$\mathcal{C} = \{ G \in \mathcal{G} \colon E \cap ([m] \times [1]) = G \times [1] \text{ for some } E \in \mathcal{A}^* \}, \\ \mathcal{D} = \{ H \in \mathcal{H} \colon F \cap ([n] \times [1]) = H \times [1] \text{ for some } F \in \mathcal{B}^* \}.$$

Then $C \subseteq G \subseteq 2^{[l]}$, $D \subseteq H \subseteq 2^{[l]}$, and, by Corollary 6.2, C and D are cross-*t*-intersecting. We have

$$\mathcal{A}^* \subseteq \bigcup_{C \in \mathcal{C}} \{ L \in \mathcal{S}_{\mathbf{c},r} \colon L \cap ([m] \times [1]) = C \times [1] \},$$
(25)

$$\mathcal{B}^* \subseteq \bigcup_{D \in \mathcal{D}} \{ L \in \mathcal{S}_{\mathbf{d},s} \colon L \cap ([n] \times [1]) = D \times [1] \}.$$
(26)

Thus

$$|\mathcal{A}^*| \le \sum_{C \in \mathcal{C}} v(C) = v(\mathcal{C}), \tag{27}$$

$$|\mathcal{B}^*| \le \sum_{D \in \mathcal{D}} w(D) = w(\mathcal{D}).$$
(28)

Since $|\mathcal{A}| = |\mathcal{A}^*|$ and $|\mathcal{B}| = |\mathcal{B}^*|$, we therefore have

$$|\mathcal{A}| \le v(\mathcal{C}),\tag{29}$$

$$|\mathcal{B}| \le w(\mathcal{D}). \tag{30}$$

Let $T_0 = [t]$. Let $\mathcal{I} = \mathcal{G}(T_0)$, $\mathcal{J} = \mathcal{H}(T_0)$, $\mathcal{X} = \mathcal{S}_{\mathbf{c},r}(T_0 \times [1])$, and $\mathcal{Y} = \mathcal{S}_{\mathbf{d},s}(T_0 \times [1])$. By Lemma 6.4 and Theorem 1.3,

$$v(\mathcal{C})w(\mathcal{D}) \le v(\mathcal{I})w(\mathcal{J}).$$
(31)

Now

$$v(\mathcal{I}) = \left(\sum_{I \in \mathcal{I}} v(I)\right) = \left(\sum_{I \in \mathcal{I}} |\{L \in \mathcal{S}_{\mathbf{c},r} \colon L \cap ([m] \times [1]) = I \times [1]\}|\right)$$
$$= \left|\bigcup_{I \in \mathcal{I}} \{L \in \mathcal{S}_{\mathbf{c},r} \colon L \cap ([m] \times [1]) = I \times [1]\}\right| = |\mathcal{X}|$$

and, similarly, $w(\mathcal{J}) = |\mathcal{Y}|$. Together with (29)–(31), this gives us $|\mathcal{A}||\mathcal{B}| \leq |\mathcal{X}||\mathcal{Y}|$, which establishes the first part of the theorem.

We now prove the second part of the theorem. The sufficiency condition is trivial, so we prove the necessary condition.

Suppose $|\mathcal{A}||\mathcal{B}| = |\mathcal{X}||\mathcal{Y}|$ and u > 0. Then all the inequalities in (27)–(31) are equalities. Having equality throughout in each of (27) and (28) implies that equality holds in each of (25) and (26). By Theorem 1.3, equality in (31) gives us that $\mathcal{C} = \mathcal{G}(T_1)$ and $\mathcal{D} = \mathcal{H}(T_1)$ for some $T_1 \in {[l] \choose t}$. Together with equality in each of (25) and (26), this gives us that $\mathcal{A}^* = \mathcal{S}_{\mathbf{c},r}(T_2)$ and $\mathcal{B}^* = \mathcal{S}_{\mathbf{d},s}(T_2)$, where $T_2 = T_1 \times [1]$. By Lemma 6.3, $\mathcal{A} = \mathcal{S}_{\mathbf{c},r}(T_3)$ and $\mathcal{B} = \mathcal{S}_{\mathbf{d},s}(T_3)$ for some $T_3 \in {[l] \times [h] \choose t}$. Since $|\mathcal{A}||\mathcal{B}| = |\mathcal{X}||\mathcal{Y}| > 0$, we clearly have $T_3 \in \mathcal{S}_{\mathbf{c},t} \cap \mathcal{S}_{\mathbf{d},t}$.

7 Proof of Theorem 1.6

In this section, we use Theorem 1.3 to prove Theorem 1.6.

As in Section 5, for any family $\mathcal{F}, \mathcal{F}^{(r)}$ denotes $\{F \in \mathcal{F} : |F| = r\}$. For any $n, r \in \mathbb{N}$ and any family \mathcal{A} , let $M_{n,r,\mathcal{A}}$ denote the set $\{A \in M_{n,r} : S_A \in \mathcal{A}\}$.

Lemma 7.1 If $n, r \in \mathbb{N}$, $i, j \in [n]$, and $\mathcal{A} \subseteq 2^{[n]}$, then $|M_{n,r,\Delta_{i,j}(\mathcal{A})}| = |M_{n,r,\mathcal{A}}|$.

Proof. Let $\mathcal{B} = \Delta_{i,j}(\mathcal{A})$. Clearly, $|\mathcal{B}^{(p)}| = |\mathcal{A}^{(p)}|$ for each $p \in [n]$. We have

$$|M_{n,r,\mathcal{B}}| = \sum_{B \in \mathcal{B}} |M_{n,r,\{B\}}| = \sum_{p=1}^{n} \sum_{B \in \mathcal{B}^{(p)}} |M_{n,r,\{B\}}| = \sum_{p=1}^{n} |\mathcal{B}^{(p)}| |M_{n,r,\{[p]\}}|$$
$$= \sum_{p=1}^{n} |\mathcal{A}^{(p)}| |M_{n,r,\{[p]\}}| = \sum_{p=1}^{n} \sum_{A \in \mathcal{A}^{(p)}} |M_{n,r,\{A\}}| = \sum_{A \in \mathcal{A}} |M_{n,r,\{A\}}| = |M_{n,r,\mathcal{A}}|,$$

as required.

Proof of Theorem 1.6. Let $\mathcal{C} = \{S_A : A \in \mathcal{A}\}$ and $\mathcal{D} = \{S_B : B \in \mathcal{B}\}$. Clearly, $\mathcal{A} \subseteq M_{m,r,\mathcal{C}}, \mathcal{B} \subseteq M_{n,s,\mathcal{D}}, \text{ and, since } \mathcal{A} \text{ and } \mathcal{B} \text{ are cross-}t\text{-intersecting}, M_{m,r,\mathcal{C}} \text{ and } M_{n,s,\mathcal{D}}$ are cross-*t*-intersecting. Thus we assume that

$$\mathcal{A} = M_{m,r,\mathcal{C}} \quad \text{and} \quad \mathcal{B} = M_{n,s,\mathcal{D}}.$$
 (32)

As explained in Section 3, we apply left-compressions to \mathcal{C} and \mathcal{D} simultaneously until we obtain two compressed cross-t-intersecting families \mathcal{C}^* and \mathcal{D}^* , respectively. Since $\mathcal{C} \subseteq {\binom{[m]}{\leq r}}$ and $\mathcal{D} \subseteq {\binom{[n]}{\leq s}}$, we have $\mathcal{C}^* \subseteq {\binom{[m]}{\leq r}}$ and $\mathcal{D}^* \subseteq {\binom{[n]}{\leq s}}$. By Lemma 3.1(ii),

$$|C \cap D \cap [r+s-t]| \ge t \text{ for any } C \in \mathcal{C}^* \text{ and any } D \in \mathcal{D}^*.$$
(33)

Let p = r + s - t. Let $\mathcal{G} = {[p] \choose \leq r}$ and $\mathcal{H} = {[p] \choose \leq s}$. Let $g : \mathcal{G} \to \mathbb{N}$ such that $g(G) = {m+r-p-1 \choose r-|G|}$ for each $G \in \mathcal{G}$. Let $h : \mathcal{H} \to \mathbb{N}$ such that $h(H) = {n+s-p-1 \choose s-|H|}$ for each $H \in \mathcal{H}$. (33)

For every $F, G \in \mathcal{G}$ with $F \subsetneq G$ and $t \leq |F| = |G| - 1$, we have

$$\frac{g(F) - (t+u)g(G)}{\binom{m+r-p-1}{r-|F|}} = 1 - \frac{(t+u)\binom{m+r-p-1}{r-|F|-1}}{\binom{m+r-p-1}{r-|F|}} = 1 - \frac{(t+u)(r-|F|)}{m-p+|F|}$$

$$= \frac{m-p+|F| - (t+u)(r-|F|)}{m-p+|F|}$$

$$\geq \frac{m-p+t - (t+u)(r-t)}{m-p+|F|} = \frac{m-(t+u+1)(r-t) - s+t}{m-p+|F|}$$

$$\geq \frac{(t+u+1)(s-t) + r - t - ((t+u+1)(r-t) + s-t)}{m-p+|F|} \ge 0,$$

and hence $g(F) \ge (t+u)g(G)$. It follows that $g(F) \ge (t+u)g(G)$ for every $F, G \in \mathcal{G}$ with $F \subsetneq G$ and $|F| \ge t$. Similarly, $h(F) \ge (t+u)g(H)$ for every $F, H \in \mathcal{H}$ with $F \subsetneq H$ and $|F| \ge t$.

We have $g(\delta_{i,j}(G)) = g(G)$ for every $G \in \mathcal{G}$ and every $i, j \in [p]$. Similarly, $h(\delta_{i,j}(H)) = h(H)$ for every $H \in \mathcal{H}$ and every $i, j \in [p]$.

Let $\mathcal{E} = \{C \cap [p] : C \in \mathcal{C}^*\}$ and $\mathcal{F} = \{D \cap [p] : D \in \mathcal{D}^*\}$. Then $\mathcal{E} \subseteq \mathcal{G}, \mathcal{F} \subseteq \mathcal{H}$, and, by (33), \mathcal{E} and \mathcal{F} are cross-t-intersecting. Let T = [t]. By Theorem 1.3,

$$g(\mathcal{E})h(\mathcal{F}) \le g(\mathcal{G}(T))h(\mathcal{H}(T)), \tag{34}$$

and if u > 0, then equality holds only if $\mathcal{E} = \mathcal{G}(T')$ and $\mathcal{F} = \mathcal{H}(T')$ for some $T' \in \binom{[p]}{t}$. By (32) and Lemma 7.1,

$$\mathcal{A}| = |M_{m,r,\mathcal{C}^*}| \leq |\{A \in M_{m,r} \colon S_A \cap [p] = E \text{ for some } E \in \mathcal{E}\}|$$

$$= \sum_{E \in \mathcal{E}} |\{A \in M_{m,r} \colon S_A \cap [p] = E\}$$

$$= \sum_{E \in \mathcal{E}} |\{(a_1, \dots, a_{r-|E|}) \colon a_1 \leq \dots \leq a_{r-|E|}, a_1, \dots, a_{r-|E|} \in E \cup [p+1,m]\}|$$

$$= \sum_{E \in \mathcal{E}} |M_{|E|+m-p,r-|E|}| = \sum_{E \in \mathcal{E}} \binom{m+r-p-1}{r-|E|} = g(\mathcal{E}).$$
(35)

Similarly,

$$|\mathcal{B}| \le h(\mathcal{F}). \tag{36}$$

By (34)-(36),

$$|\mathcal{A}||\mathcal{B}| \le g(\mathcal{G}(T))h(\mathcal{H}(T)). \tag{37}$$

Now, similarly to (35),

$$g(\mathcal{G}(T)) = |\{A \in M_{n,r} \colon S_A \cap [p] = E \text{ for some } E \in \mathcal{G}(T)\}|$$
$$= |\{A \in M_{n,r} \colon T \subseteq S_A\}| = \binom{m+r-t-1}{r-t}.$$

Similarly, $h(\mathcal{H}(T)) = \binom{n+s-t-1}{s-t}$. By (37), it follows that

$$|\mathcal{A}||\mathcal{B}| \le \binom{m+r-t-1}{r-t} \binom{n+s-t-1}{s-t},$$

as required.

Suppose $|\mathcal{A}||\mathcal{B}| = \binom{m+r-t-1}{r-t} \binom{n+s-t-1}{s-t}$ and u > 0. Then equality holds throughout in each of (34)–(37), and hence $\mathcal{E} = \mathcal{G}(T')$ and $\mathcal{F} = \mathcal{H}(T')$ for some $T' \in \binom{[p]}{t}$. Having equality throughout in (35) implies that $M_{m,r,\mathcal{C}^*} = \{A \in M_{m,r} : S_A \cap [p] = E$ for some $E \in \mathcal{E}\} = \{A \in M_{m,r} : T' \subseteq S_A\}$. Thus $T' \in \mathcal{C}^*$, and hence there exists $T_1 \in \binom{[m]}{t}$ such that $T_1 \in \mathcal{C}$. Similarly, there exists $T_2 \in \binom{[n]}{t}$ such that $T_2 \in \mathcal{D}$. Since \mathcal{C} and \mathcal{D} are cross-*t*-intersecting, we have $T_1 = T_2, \mathcal{C} \subseteq \{C \in \binom{[m]}{\leq r}: T_1 \subseteq C\}$, and $\mathcal{D} \subseteq \{D \in \binom{[n]}{\leq s}: T_1 \subseteq D\}$. Consequently, $\mathcal{A} \subseteq \{A \in M_{m,r}: T_1 \subseteq S_A\}$ and $\mathcal{B} \subseteq \{B \in M_{n,s}: T_1 \subseteq S_B\}$. Since $|\mathcal{A}||\mathcal{B}| = \binom{m+r-t-1}{r-t}\binom{n+s-t-1}{s-t}$, both inclusion relations are actually equalities.

8 The remaining cases

Each of Theorems 1.2, 1.5, and 1.6 solves the particular cross-*t*-intersection problem under consideration for all cases where the ground sets are not smaller than a certain value dependent on r, s, and t. Solving any of these problems completely appears to be very difficult and would take this area of study to a significantly deeper level. We conjecture that the complete solutions are (38)-(40) below.

For any $n \in \mathbb{N}$ and any $r, t, i, j \in \{0\} \cup [n]$ with $1 \leq t \leq r$ and $t + i + j \leq n$, let $\mathcal{M}_{n,r,t,i,j} = \{A \in \binom{[n]}{r}: |A \cap [t+i+j]| \geq t+i\}$. In [22], Frankl conjectured that the size of a largest *t*-intersecting subfamily of $\binom{[n]}{r}$ is max $\{|\mathcal{M}_{m,r,t,i,i}|: i, j \in \{0\} \cup \mathbb{N}, t+2i \leq n\}$, and this was verified in [1]. Hirschorn suggested an analogous conjecture [29, Conjecture 4] for cross-*t*-intersecting families \mathcal{A} and \mathcal{B} with $\mathcal{A} \subseteq \binom{[n]}{r}$ and $\mathcal{B} \subseteq \binom{[n]}{s}$. Generalising Hirschorn's conjecture, we conjecture that if $m, n \in \mathbb{N}, r \in [m], s \in [n], t \in [\min\{r, s\}], \mathcal{A} \subseteq \binom{[m]}{r}, \mathcal{B} \subseteq \binom{[n]}{s}$, and \mathcal{A} and \mathcal{B} are cross-*t*-intersecting, then

$$|\mathcal{A}||\mathcal{B}| \le \max\{|\mathcal{M}_{m,r,t,i,j}||\mathcal{M}_{n,s,t,j,i}|: i, j \in \{0\} \cup \mathbb{N}, t+i+j \le \min\{m,n\}\}.$$
 (38)

For any IP sequence $\mathbf{c} = (c_1, \ldots, c_n)$ and any $r, t, i, j \in \{0\} \cup [n]$ with $1 \leq t \leq r$ and $t + i + j \leq n$, let $\mathcal{S}_{\mathbf{c},r,t,i,j} = \{A \in \mathcal{S}_{\mathbf{c},r} \colon |A \cap ([t + i + j] \times [1])| \geq t + i\}$. We conjecture that if $\mathbf{c} = (c_1, \ldots, c_m)$ and $\mathbf{d} = (d_1, \ldots, d_n)$ are IP sequences, $c_1 \geq 2, d_1 \geq 2, r \in [m], s \in [n], t \in [\min\{r, s\}], \mathcal{A} \subseteq \mathcal{S}_{\mathbf{c},r}, \mathcal{B} \subseteq \mathcal{S}_{\mathbf{d},s}$, and \mathcal{A} and \mathcal{B} are cross-t-intersecting, then

$$|\mathcal{A}||\mathcal{B}| \le \max\{|\mathcal{S}_{\mathbf{c},r,t,i,j}||\mathcal{S}_{\mathbf{d},s,t,j,i}|: i, j \in \{0\} \cup \mathbb{N}, t+i+j \le \min\{m,n\}\}.$$
(39)

This generalises [40, Conjecture 3], which is a conjecture for the case r = s = m = n.

For any $n \in \mathbb{N}$ and any $r, t, i, j \in \{0\} \cup [n]$ with $1 \leq t \leq r$ and $t + i + j \leq n$, let $M_{n,r,t,i,j} = \{A \in M_{n,r} : |S_A \cap [t + i + j]| \geq t + i\}$. We conjecture that if $m, n, r, s \in \mathbb{N}$, $t \in [\min\{r, s\}], A \subseteq M_{m,r}, B \subseteq M_{n,s}$, and A and B are cross-t-intersecting, then

$$|\mathcal{A}||\mathcal{B}| \le \max\{|M_{m,r,t,i,j}||M_{n,s,t,j,i}|: i, j \in \{0\} \cup \mathbb{N}, t+i+j \le \min\{m,n\}\}.$$
 (40)

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