# The maximum product of weights of cross-intersecting families 

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#### Abstract

Two families $\mathcal{A}$ and $\mathcal{B}$ of sets are said to be cross-t-intersecting if each set in $\mathcal{A}$ intersects each set in $\mathcal{B}$ in at least $t$ elements. An active problem in extremal set theory is to determine the maximum product of sizes of cross- $t$-intersecting subfamilies of a given family. We prove a cross- $t$-intersection theorem for weighted subsets of a set by means of a new subfamily alteration method, and use the result to provide solutions for three natural families. For $r \in[n]=\{1,2, \ldots, n\}$, let $\binom{[n]}{r}$ be the family of $r$-element subsets of $[n]$, and let $\binom{[n]}{\leq r}$ be the family of subsets of $[n]$ that have at most $r$ elements. Let $\mathcal{F}_{n, r, t}$ be the family of sets in $\binom{[n]}{\leq r}$ that contain $[t]$. We show that if $g:\binom{[m]}{\leq r} \rightarrow \mathbb{R}^{+}$ and $h:\binom{[n]}{\leq s} \rightarrow \mathbb{R}^{+}$are functions that obey certain conditions, $\mathcal{A} \subseteq\binom{[m]}{\leq r}, \mathcal{B} \subseteq\binom{[n]}{\leq s}$, and $\mathcal{A}$ and $\mathcal{B}$ are cross- $t$-intersecting, then $$
\sum_{A \in \mathcal{A}} g(A) \sum_{B \in \mathcal{B}} h(B) \leq \sum_{C \in \mathcal{F}_{m, r, t}} g(C) \sum_{D \in \mathcal{F}_{n, s, t}} h(D),
$$ and equality holds if $\mathcal{A}=\mathcal{F}_{m, r, t}$ and $\mathcal{B}=\mathcal{F}_{n, s, t}$. We prove this in a more general setting and characterise the cases of equality. We use the result to show that the maximum product of sizes of two cross- $t$-intersecting families $\mathcal{A} \subseteq\binom{[m]}{r}$ and $\mathcal{B} \subseteq\binom{[n]}{s}$ is $\binom{m-t}{r-t}\binom{n-t}{s-t}$ for $\min \{m, n\} \geq n_{0}(r, s, t)$, where $n_{0}(r, s, t)$ is close to best possible. We obtain analogous results for families of integer sequences and for families of multisets. The results yield generalisations for $k \geq 2$ cross- $t$-intersecting families, and Erdos-Ko-Rado-type results.


## 1 Introduction

Unless otherwise stated, we shall use small letters such as $x$ to denote elements of a set or non-negative integers or functions, capital letters such as $X$ to denote sets, and calligraphic letters such as $\mathcal{F}$ to denote families (that is, sets whose elements are sets themselves). The set $\{1,2, \ldots\}$ of all positive integers is denoted by $\mathbb{N}$. For any $m, n \in \mathbb{N}$, the set $\{i \in \mathbb{N}: m \leq i \leq n\}$ is denoted by $[m, n]$. We abbreviate $[1, n]$ to $[n]$. It is to be assumed that arbitrary sets and families are finite. We call a set $A$ an $r$-element set, or simply an $r$-set, if its size $|A|$ is $r$. For a set $X, 2^{X}$ denotes the power set of $X$ (that is, the family of all subsets of $X$ ) , $\binom{X}{r}$ denotes the family of all
$r$-element subsets of $X$, and $\binom{X}{\leq r}$ denotes the family of all subsets of $X$ of size at most $r$. For a family $\mathcal{F}$ and a set $T$, we denote the family $\{F \in \mathcal{F}: T \subseteq F\}$ by $\mathcal{F}(T)$.

We say that a set $A$ t-intersects a set $B$ if $A$ and $B$ contain at least $t$ common elements. A family $\mathcal{A}$ of sets is said to be $t$-intersecting if every two sets in $\mathcal{A} t$ intersect. A 1 -intersecting family is also simply called an intersecting family. If $T$ is a $t$-element subset of at least one set in a family $\mathcal{F}$, then we call the family of all the sets in $\mathcal{F}$ that contain $T$ the $t$-star of $\mathcal{F}$. A $t$-star of a family is the simplest example of a $t$-intersecting subfamily.

One of the most popular endeavours in extremal set theory is that of determining the size of a largest $t$-intersecting subfamily of a given family $\mathcal{F}$. This took off with [20], which features the classical result, known as the Erdős-Ko-Rado (EKR) Theorem, that says that if $1 \leq r \leq n / 2$, then the size of a largest intersecting subfamily $\mathcal{A}$ of $\binom{[n]}{r}$ is the size $\binom{n-1}{r-1}$ of every 1-star of $\binom{[n]}{r}$. If $r<n / 2$, then, by the Hilton-Milner Theorem [28], $\mathcal{A}$ attains the bound if and only if $\mathcal{A}$ is a star of $\binom{[n]}{r}$. If $n / 2<r \leq n$, then $\binom{[n]}{r}$ itself is intersecting. There are various proofs of the EKR Theorem (see [34, 28, [32, 18]), two of which are particularly short and beautiful: Katona's [32], introducing the elegant cycle method, and Daykin's [18], using the fundamental Kruskal-Katona Theorem [35, 33]. A sequence of results [20, 22, 48, 1] culminated in the solution of the problem for $t$-intersecting subfamilies of $\binom{[n]}{r}$; the solution particularly tells us that the size of a largest $t$-intersecting subfamily of $\binom{[n]}{r}$ is the size $\binom{n-t}{r-t}$ of a $t$-star of $\binom{[n]}{r}$ if and only if $n \geq(t+1)(r-t+1)$. The $t$-intersection problem for $2^{[n]}$ was solved by Katona [34]. These are among the most prominent results in extremal set theory. The EKR Theorem inspired a wealth of results, including generalisations (see [43, 11), that establish how large a system of sets can be under certain intersection conditions; see [19, 23, 21, 13, 30, 31].

Two families $\mathcal{A}$ and $\mathcal{B}$ are said to be cross-t-intersecting if each set in $\mathcal{A} t$-intersects each set in $\mathcal{B}$. More generally, $k$ families $\mathcal{A}_{1}, \ldots, \mathcal{A}_{k}$ (not necessarily distinct or nonempty) are said to be cross-t-intersecting if for every $i$ and $j$ in $[k]$ with $i \neq j$, each set in $\mathcal{A}_{i} t$-intersects each set in $\mathcal{A}_{j}$. Cross-1-intersecting families are also simply called cross-intersecting families.

For $t$-intersecting subfamilies of a given family $\mathcal{F}$, the natural question to ask is how large they can be. For cross- $t$-intersecting families, two natural parameters arise: the sum and the product of sizes of the cross-t-intersecting families (note that the product of sizes of $k$ families $\mathcal{A}_{1}, \ldots, \mathcal{A}_{k}$ is the number of $k$-tuples $\left(A_{1}, \ldots, A_{k}\right)$ such that $A_{i} \in \mathcal{A}_{i}$ for each $i \in[k]$ ). It is therefore natural to consider the problem of maximising the sum or the product of sizes of $k$ cross- $t$-intersecting subfamilies $\mathcal{A}_{1}, \ldots, \mathcal{A}_{k}$ of a given family $\mathcal{F}$. The paper [15] analyses this problem in general, particularly showing that for $k$ sufficiently large, both the sum and the product are maxima if $\mathcal{A}_{1}=\cdots=\mathcal{A}_{k}=\mathcal{L}$ for some largest $t$-intersecting subfamily $\mathcal{L}$ of $\mathcal{F}$. Therefore, this problem incorporates the $t$-intersection problem. Solutions have been obtained for various families (see [15]), including $\binom{[n]}{r}$ [27, 41, 37, 4, 4, 45, 47, 46, 25], $2^{[n]}$ [36, 15], $\binom{[n]}{\leq r}$ [6], and families of integer sequences [39, 12, 16, 47, 49, 44, 25, 40]. Most of these results tell us that for the family $\mathcal{F}$ under consideration and for certain values of $k$, the sum or the product is maximum when $\mathcal{A}_{1}=\cdots=\mathcal{A}_{k}=\mathcal{L}$ for some largest $t$-star $\mathcal{L}$ of $\mathcal{F}$. In such a case,
$\mathcal{L}$ is a largest $t$-intersecting subfamily of $\mathcal{F}$.
Remark 1.1 In general, if $\mathcal{L} \subseteq \mathcal{F}, k \geq 2$, and the sum or the product is maximum when $\mathcal{A}_{1}=\cdots=\mathcal{A}_{k}=\mathcal{L}$, then $\mathcal{L}$ is a largest $t$-intersecting subfamily of $\mathcal{F}$. Indeed, the cross- $t$-intersection condition implies that every two sets $A$ and $B$ in $\mathcal{L} t$-intersect (as $A \in \mathcal{A}_{1}$ and $B \in \mathcal{A}_{2}$ ), and by taking an arbitrary $t$-intersecting subfamily $\mathcal{A}$ of $\mathcal{F}$ and setting $\mathcal{B}_{1}=\cdots=\mathcal{B}_{k}=\mathcal{A}$, we obtain that $\mathcal{B}_{1}, \ldots, \mathcal{B}_{k}$ are cross- $t$-intersecting, and hence $|\mathcal{A}| \leq|\mathcal{L}|$ since $k|\mathcal{A}|=\sum_{i=1}^{k}\left|\mathcal{B}_{i}\right| \leq \sum_{i=1}^{k}\left|\mathcal{A}_{i}\right|=k|\mathcal{L}|$ or $|\mathcal{A}|^{k}=\prod_{i=1}^{k}\left|\mathcal{B}_{i}\right| \leq$ $\prod_{i=1}^{k}\left|\mathcal{A}_{i}\right|=|\mathcal{L}|^{k}$.

Wang and Zhang 47] solved the maximum sum problem for an important class of families that includes $\binom{[n]}{r}$ and families of integer sequences, using a striking combination of the method in [7, 8, 9, 16, 10] and an important lemma that is found in [3, 17] and referred to as the 'no-homomorphism lemma'. The solution for $\binom{[n]}{r}$ with $t=1$ had been obtained by Hilton [27] and is the first result of this kind.

In this paper we address the maximum product problem for $\binom{[n]}{r}$ and families of integer sequences. We will actually consider more general problems; one generalisation allows the cross- $t$-intersecting families to come from different families, and another one involves maximising instead the product of weights of cross- $t$-intersecting families of subsets of a set. As we explain in the next section, if the product for $k=2$ is maximum when the cross- $t$-intersecting families are certain $t$-stars, then this immediately generalises for $k \geq 2$.

The maximum product problem for $\binom{[n]}{r}$ was first addressed by Pyber [41], who proved that for any $r, s$, and $n$ such that either $r=s \leq n / 2$ or $r<s$ and $n \geq 2 s+r-2$, if $\mathcal{A} \subseteq\binom{[n]}{r}$ and $\mathcal{B} \subseteq\binom{[n]}{s}$ such that $\mathcal{A}$ and $\mathcal{B}$ are cross-intersecting, then $|\mathcal{A}||\mathcal{B}| \leq$ $\binom{n-1}{r-1}\binom{n-1}{s-1}$. Subsequently, Matsumoto and Tokushige [37] proved this for $r \leq s \leq n / 2$. It has been shown in [14] that there exists an integer $n_{0}(r, s, t)$ such that for $t \leq r \leq s$ and $n \geq n_{0}(r, s, t)$, if $\mathcal{A} \subseteq\binom{[n]}{r}, \mathcal{B} \subseteq\binom{[n]}{s}$, and $\mathcal{A}$ and $\mathcal{B}$ are cross-t-intersecting, then $|\mathcal{A}||\mathcal{B}| \leq\binom{ n-t}{r-t}\binom{n-t}{s-t}$. The value of $n_{0}(r, s, t)$ given in [14] is far from best possible. The special case $r=s$ is treated in [45, 46, 25], which establish values of $n_{0}(r, r, t)$ that are close to the conjectured smallest value of $(t+1)(r-t+1)$, and which use algebraic methods and Frankl's random walk method [22]; in particular, $n_{0}(r, r, t)=(t+1) r$ is determined in [25] for $t \geq 14$. Using purely combinatorial arguments, we solve the problem for $n \geq(t+u+2)(s-t)+r-1$, where $u$ can be any non-negative real number satisfying $u>\frac{6-t}{3}$; thus, we can take $n_{0}(r, s, t)=(t+2)(s-t)+r-1$ for $t \geq 7$, and $n_{0}(r, s, t)<(t+4)(s-t)+r-1$ for $1 \leq t \leq 6$. We actually prove the following more general result in Section 5 .

Theorem 1.2 If $1 \leq t \leq r \leq s$, $u$ is a non-negative real number such that $u>\frac{6-t}{3}$, $\min \{m, n\} \geq(t+u+2)(s-t)+r-1, \mathcal{A} \subseteq\binom{[m]}{r}, \mathcal{B} \subseteq\binom{[n]}{s}$, and $\mathcal{A}$ and $\mathcal{B}$ are cross-t-intersecting, then

$$
|\mathcal{A}||\mathcal{B}| \leq\binom{ m-t}{r-t}\binom{n-t}{s-t}
$$

Moreover, if $u>0$, then the bound is attained if and only if $\mathcal{A}=\left\{A \in\binom{[m]}{r}: T \subseteq A\right\}$ and $\mathcal{B}=\left\{B \in\binom{[n]}{s}: T \subseteq B\right\}$ for some $t$-element subset $T$ of $[\min \{m, n\}]$.

In Section 5, we show that Theorem 1.2 is a consequence of our main result, Theorem 1.3, for which we need some additional definitions and notation.

For any $i, j \in[n]$, let $\delta_{i, j}: 2^{[n]} \rightarrow 2^{[n]}$ be defined by

$$
\delta_{i, j}(A)= \begin{cases}(A \backslash\{j\}) \cup\{i\} & \text { if } j \in A \text { and } i \notin A ; \\ A & \text { otherwise },\end{cases}
$$

and let $\Delta_{i, j}: 2^{2^{[n]}} \rightarrow 2^{2^{[n]}}$ be the compression operation defined by

$$
\Delta_{i, j}(\mathcal{A})=\left\{\delta_{i, j}(A): A \in \mathcal{A}\right\} \cup\left\{A \in \mathcal{A}: \delta_{i, j}(A) \in \mathcal{A}\right\} .
$$

The compression operation was introduced in the seminal paper [20]. The paper [23] provides a survey on the properties and uses of compression (also called shifting) operations in extremal set theory. All our new results make use of compression operations.

If $i<j$, then we call $\Delta_{i, j}$ a left-compression. A family $\mathcal{F} \subseteq 2^{[n]}$ is said to be compressed if $\Delta_{i, j}(\mathcal{F})=\mathcal{F}$ for every $i, j \in[n]$ with $i<j$. In other words, $\mathcal{F}$ is compressed if it is invariant under left-compressions. Note that $\mathcal{F}$ is compressed if and only if $(F \backslash\{j\}) \cup\{i\} \in \mathcal{F}$ whenever $i<j \in F \in \mathcal{F}$ and $i \in[n] \backslash F$.

A family $\mathcal{H}$ is said to be hereditary if for each $H \in \mathcal{H}$, all the subsets of $H$ are in $\mathcal{H}$. Thus, a family is hereditary if and only if it is a union of power sets. The family $\binom{[n]}{\leq r}$ (which is $2^{[n]}$ if $r=n$ ) is an example of a hereditary family that is compressed.

Let $\mathbb{R}^{+}$denote the set of positive real numbers. With a slight abuse of notation, for any non-empty family $\mathcal{F}$, any function $w: \mathcal{F} \rightarrow \mathbb{R}^{+}$(called a weight function), and any $\mathcal{A} \subseteq \mathcal{F}$, we denote the sum $\sum_{A \in \mathcal{A}} w(A)$ (of weights of sets in $\mathcal{A}$ ) by $w(\mathcal{A})$. Note that if $\mathcal{A}$ is empty, then $w(\mathcal{A})$ is the empty sum, and we will adopt the convention of taking this to be 0 .

In Section 4, we prove the following result.
Theorem 1.3 Let $1 \leq t \leq n$, $T=[t]$, and $u \in\{0\} \cup \mathbb{R}^{+}$such that $u>\frac{6-t}{3}$. Let $\mathcal{G}$ and $\mathcal{H}$ be non-empty compressed hereditary subfamilies of $2^{[n]}$. For each $\mathcal{F} \in\{\mathcal{G}, \mathcal{H}\}$, let $w_{\mathcal{F}}: \mathcal{F} \rightarrow \mathbb{R}^{+}$such that
(a) $w_{\mathcal{F}}(A) \geq(t+u) w_{\mathcal{F}}(B)$ for every $A, B \in \mathcal{F}$ with $A \subsetneq B$ and $|A| \geq t$, and
(b) $w_{\mathcal{F}}\left(\delta_{i, j}(C)\right) \geq w_{\mathcal{F}}(C)$ for every $C \in \mathcal{F}$ and every $i, j \in[n]$ with $i<j$.

Let $g=w_{\mathcal{G}}$ and $h=w_{\mathcal{H}}$. If $\mathcal{A} \subseteq \mathcal{G}$ and $\mathcal{B} \subseteq \mathcal{H}$ such that $\mathcal{A}$ and $\mathcal{B}$ are cross-tintersecting, then

$$
g(\mathcal{A}) h(\mathcal{B}) \leq g(\mathcal{G}(T)) h(\mathcal{H}(T))
$$

Moreover, if $u>0$ and each of $\mathcal{G}$ and $\mathcal{H}$ has a member of size at least $t$, then the bound is attained if and only if $\mathcal{A}=\mathcal{G}\left(T^{\prime}\right)$ and $\mathcal{B}=\mathcal{H}\left(T^{\prime}\right)$ for some $T^{\prime} \in\binom{[n]}{t}$ such that $g\left(\mathcal{G}\left(T^{\prime}\right)\right)=g(\mathcal{G}(T))$ and $h\left(\mathcal{H}\left(T^{\prime}\right)\right)=h(\mathcal{H}(T))$.

Remark 1.4 For $u>\frac{6-t}{3}$ to hold, we can always take $u=2$, and we can take $u=0$ for $t \geq 7$. We conjecture that the inequality $g(\mathcal{A}) h(\mathcal{B}) \leq g(\mathcal{G}(T)) h(\mathcal{H}(T))$ still holds
if the condition $u>\frac{6-t}{3}$ is replaced by $u=0$. As we mentioned above, this is true for $t \geq 7$. Also, the proof of Theorem 1.3 shows that for $t \geq 3$, the conjecture is true if it is true for $t+3 \leq n \leq t+6$ (see Remark 4.3). A verification of the conjecture for $t+3 \leq n \leq t+6$ could be obtained through detailed case-checking similar to that used in our proof for the special case $n \leq t+2$; however, the process would be significantly more laborious. The condition on $u$ cannot be relaxed further, because no real number $u<0$ with $t+u \geq 1$ guarantees that the result holds. Indeed, if $1 \leq x=t+u<t \leq n-2, \mathcal{G}=\mathcal{H}=2^{[n]}, g(G)=h(G)=x^{n-|G|}$ for all $G \in 2^{[n]}$, and $\mathcal{A}=\mathcal{B}=\left\{A \in 2^{[n]}:|A \cap[t+2]| \geq t+1\right\}=\mathcal{A}^{*}$, then conditions (a) and (b) of Theorem 1.3 are satisfied, $\mathcal{A}$ and $\mathcal{B}$ are cross- $t$-intersecting, but

$$
\begin{aligned}
(g(\mathcal{A}) h(\mathcal{B}))^{1 / 2} & =g(\mathcal{A})=\sum_{X \in\binom{[t+2]}{t+1}} \sum_{Y \subseteq[t+3, n]} g(X \cup Y)+\sum_{Y \subseteq[t+3, n]} g([t+2] \cup Y) \\
& =(t+2) \sum_{j=0}^{n-t-2}\binom{n-t-2}{j} x^{n-t-1-j}+\sum_{j=0}^{n-t-2}\binom{n-t-2}{j} x^{n-t-2-j} \\
& =(t+2) x^{n-t-1}\left(1+x^{-1}\right)^{n-t-2}+x^{n-t-2}\left(1+x^{-1}\right)^{n-t-2} \\
& =x^{n-t-2}\left(1+x^{-1}\right)^{n-t-2}(t x+2 x+1)=(x+1)^{n-t-2}(t x+2 x+1) \\
& >(x+1)^{n-t-2}\left(x^{2}+2 x+1\right)(\text { as } 1 \leq x<t) \\
& =x^{n-t}\left(1+x^{-1}\right)^{n-t}=\sum_{j=0}^{n-t}\binom{n-t}{j} x^{n-t-j}=\sum_{Y \subseteq[t+1, n]} g([t] \cup Y) \\
& =g(\mathcal{G}(T))=(g(\mathcal{G}(T)) h(\mathcal{H}(T)))^{1 / 2},
\end{aligned}
$$

and hence $g(\mathcal{A}) h(\mathcal{B})>g(\mathcal{G}(T)) h(\mathcal{H}(T))$. It has been shown in [6] that for $t=1$, the product of sizes of $\mathcal{A}$ and $\mathcal{B}$ is maximised by taking $\mathcal{A}=\mathcal{G}(T)$ and $\mathcal{B}=\mathcal{H}(T)$; equivalently, for the special case where $t=1$ and $g(A)=h(A)=1$ for all $A \in \mathcal{G} \cup \mathcal{H}$, the bound in Theorem 1.3 also holds (that is, the conjecture is true). However, this is not true for $t>1$, and hence Theorem 1.3 does not imply that the product of sizes is maximised by taking $\mathcal{A}=\mathcal{G}(T)$ and $\mathcal{B}=\mathcal{H}(T)$. Indeed, if $\mathcal{G}=\mathcal{H}=2^{[n]}$ and $\mathcal{A}=\mathcal{B}=\mathcal{A}^{*}$ as above, then $|\mathcal{A}||\mathcal{B}|>|\mathcal{G}(T)||\mathcal{H}(T)|$ (take $x=1$ above).

The proof of Theorem 1.3 contains the main observations in this paper and is based on induction, compression, a new subfamily alteration method, and double-counting. The alteration method can be regarded as the main new component and appears to have the potential of yielding other intersection results of this kind.

The bound in [25, Theorem 1.3] for product measures of cross- $t$-intersecting subfamilies of $2^{[n]}$ is given by Theorem 1.3 with $\mathcal{G}=\mathcal{H}=2^{[n]}, t \geq 14, u=0$, and $g(A)=h(A)=p^{|A|}(1-p)^{n-|A|}$ for all $A \in 2^{[n]}$, where $p \in \mathbb{R}^{+}$such that $p \leq \frac{1}{t+1}$.

The subsequent results in this section and in the next section are also consequences of Theorem 1.3, Our next application is a cross- $t$-intersection result for integer sequences.

We will represent a sequence $a_{1}, \ldots, a_{n}$ by an $n$-tuple ( $a_{1}, \ldots, a_{n}$ ), and we say that it is of length $n$. We call a sequence of positive integers a positive sequence. We
call $\left(a_{1}, \ldots, a_{n}\right)$ an $r$-partial sequence if exactly $r$ of its entries are positive integers and the rest are all zero. Thus, an $n$-partial sequence of length $n$ is positive. A sequence $\left(c_{1}, \ldots, c_{n}\right)$ is said to be increasing if $c_{1} \leq \cdots \leq c_{n}$. We call an increasing positive sequence an IP sequence. Note that $\left(c_{1}, \ldots, c_{n}\right)$ is an IP sequence if and only if $1 \leq c_{1} \leq \cdots \leq c_{n}$.

We call $\left\{\left(x_{1}, y_{1}\right), \ldots,\left(x_{r}, y_{r}\right)\right\}$ a labeled set (following [12]) if $x_{1}, \ldots, x_{r}$ are distinct. For any IP sequence $\mathbf{c}=\left(c_{1}, \ldots, c_{n}\right)$ and any $r \in[n]$, let $\mathcal{S}_{\mathbf{c}, r}$ be the family of all labeled sets $\left\{\left(x_{1}, y_{x_{1}}\right), \ldots,\left(x_{r}, y_{x_{r}}\right)\right\}$ such that $\left\{x_{1}, \ldots, x_{r}\right\} \in\binom{[n]}{r}$ and $y_{x_{j}} \in\left[c_{x_{j}}\right]$ for each $j \in[r]$. For any sets $Y_{1}, \ldots, Y_{n}$, let $Y_{1} \times \cdots \times Y_{n}$ denote the Cartesian product of $Y_{1}, \ldots, Y_{n}$, that is, the set of sequences $\left(y_{1}, \ldots, y_{n}\right)$ such that $y_{i} \in Y_{i}$ for each $i \in[n]$. Note that $\mathcal{S}_{\mathbf{c}, n}=\left\{\left\{\left(1, y_{1}\right), \ldots,\left(n, y_{n}\right)\right\}: y_{i} \in\left[c_{i}\right]\right.$ for each $\left.i \in[n]\right\}$, so $\mathcal{S}_{\mathbf{c}, n}$ is isomorphic to $\left[c_{1}\right] \times \cdots \times\left[c_{n}\right]$. Also note that $\mathcal{S}_{\mathbf{c}, r}$ is isomorphic to the set of $r$-partial sequences $\left(y_{1}, \ldots, y_{n}\right)$ such that for some $R \in\binom{[n]}{r}, y_{i} \in\left[c_{i}\right]$ for each $i \in R$ (and hence $y_{j}=0$ for each $j \in[n] \backslash R)$. Let $\mathcal{S}_{\mathbf{c}, r, t}=\mathcal{S}_{\mathbf{c}, r}([t] \times[1])=\left\{A \in \mathcal{S}_{\mathbf{c}, r}:(x, 1) \in A\right.$ for each $\left.x \in[t]\right\}$.

In Section 6, we prove the following result.
Theorem 1.5 Let $\mathbf{c}=\left(c_{1}, \ldots, c_{m}\right)$ and $\mathbf{d}=\left(d_{1}, \ldots, d_{n}\right)$ be IP sequences. Let $r \in[m]$, $s \in[n], t \in[\min \{r, s\}]$, and $u \in\{0\} \cup \mathbb{R}^{+}$such that $u>\frac{6-t}{3}$. If $c_{1} \geq t+u+1$, $d_{1} \geq t+u+1, \mathcal{A} \subseteq \mathcal{S}_{\mathbf{c}, r}, \mathcal{B} \subseteq \mathcal{S}_{\mathbf{d}, s}$, and $\mathcal{A}$ and $\mathcal{B}$ are cross-t-intersecting, then

$$
|\mathcal{A}||\mathcal{B}| \leq\left(\sum_{\substack{\left[\begin{array}{c}
{[t+1, m] \\
--t}
\end{array}\right)}} \prod_{i \in I} c_{i}\right)\left(\sum_{\substack{\left[t\left(\begin{array}{c}
{[t+1, n] \\
s-t}
\end{array}\right)\right.}} \prod_{j \in J} c_{j}\right)
$$

Moreover, if $u>0$, then the bound is attained if and only if for some $T \in \mathcal{S}_{\mathbf{c}, t} \cap \mathcal{S}_{\mathbf{d}, t}$ with $\left|\mathcal{S}_{\mathbf{c}, r}(T)\right|=\left|\mathcal{S}_{\mathbf{c}, r, t}\right|$ and $\left|\mathcal{S}_{\mathbf{d}, s}(T)\right|=\left|\mathcal{S}_{\mathbf{d}, s, t}\right|, \mathcal{A}=\mathcal{S}_{\mathbf{c}, r}(T)$ and $\mathcal{B}=\mathcal{S}_{\mathbf{d}, s}(T)$.

Note that this result holds for $c_{1} \geq t+1$ and $d_{1} \geq t+1$ when $t \geq 7$, for $c_{1} \geq t+2$ and $d_{1} \geq t+2$ when $4 \leq t \leq 6$, and for $c_{1} \geq t+3$ and $d_{1} \geq t+3$ when $1 \leq t \leq 3$. We conjecture that the result holds for $c_{1} \geq t+1$ and $d_{1} \geq t+1$, and, as can be seen from the proof of Theorem [1.5, this conjecture is true if the conjecture in Remark 1.4 is true. The result does not hold for $c_{1}<t+1$. Indeed, if $r=s=m=n \geq t+2$, $c_{1}=\cdots=c_{n}=x+1<t+1=d_{1}=\cdots=d_{n}, Z=[n] \times[1], Z_{1}=[t+2] \times[1], Z_{2}=$ $[t+3, n] \times[1], \mathcal{A}=\left\{A \in \mathcal{S}_{\mathbf{c}, n}:\left|A \cap Z_{1}\right| \geq t+1\right\}$, and $\mathcal{B}=\left\{B \in \mathcal{S}_{\mathbf{d}, n}:\left|B \cap Z_{1}\right| \geq t+1\right\}$, then $\mathcal{A}$ and $\mathcal{B}$ are cross- $t$-intersecting,

$$
\begin{aligned}
|\mathcal{A}| & =\left|\bigcup_{X \in\binom{Z_{1}}{t+1)} \cup\left\{Z_{1}\right\}} \bigcup_{j=0}^{\left|Z_{2}\right|} \bigcup_{Y \in\binom{Z_{2}}{j}}\{A \in \mathcal{A}: A \cap Z=X \cup Y\}\right| \\
& =(t+2) \sum_{j=0}^{n-t-2}\binom{n-t-2}{j} x^{n-t-1-j}+\sum_{j=0}^{n-t-2}\binom{n-t-2}{j} x^{n-t-2-j} \\
& =(x+1)^{n-t-2}(t x+2 x+1) \quad(\text { as in Remark } 1.4) \\
& >(x+1)^{n-t-2}\left(x^{2}+2 x+1\right)=(x+1)^{n-t}=\left|\mathcal{S}_{\mathbf{c}, n, t}\right|
\end{aligned}
$$

$|\mathcal{B}|=(t+1)^{n-t-2}\left(t^{2}+2 t+1\right)=(t+1)^{n-t}=\left|\mathcal{S}_{\mathbf{d}, n, t}\right|$ (by a calculation similar to that for $|\mathcal{A}|)$, and hence $|\mathcal{A}||\mathcal{B}|>\left|\mathcal{S}_{\mathbf{c}, r, t}\right|\left|\mathcal{S}_{\mathbf{d}, s, t}\right|$.

Solutions for the special case where $\mathbf{c}=\mathbf{d}$ and $r=s=n$ already exist. The solution for $t+2 \leq c_{1}=c_{n}$ was first obtained by Moon [39]. Inspired by [49], Pach and Tardos [40] recently generalised Moon's result to include the cases $t+2 \leq c_{1} \leq c_{n}$ and $8 \leq t+1 \leq c_{1} \leq c_{n}$. Another proof for $15 \leq t+1 \leq c_{1}=c_{n}$ is given in [25].

Our last application of Theorem 1.3 in this section is a cross- $t$-intersection result for multisets.

A multiset is a collection $A$ of objects such that each object possibly appears more than once in $A$. Thus the difference between a multiset and a set is that a multiset may have repetitions of its elements. We can uniquely represent a multiset $A$ of positive integers by an IP sequence $\left(a_{1}, \ldots, a_{r}\right)$, where $a_{1}, \ldots, a_{r}$ form $A$. Thus we will take multisets to be IP sequences. For $A=\left(a_{1}, \ldots, a_{r}\right)$, the support of $A$ is the set $\left\{a_{1}, \ldots, a_{r}\right\}$ and will be denoted by $\mathrm{S}_{A}$. For any $n, r \in \mathbb{N}$, let $M_{n, r}$ denote the set of all multisets ( $a_{1}, \ldots, a_{r}$ ) such that $a_{1}, \ldots, a_{r} \in[n] ;$ thus $M_{n, r}=\left\{\left(a_{1}, \ldots, a_{r}\right): a_{1} \leq \cdots \leq a_{r}, a_{1}, \ldots, a_{r} \subseteq[n]\right\}$. An elementary counting result is that

$$
\left|M_{n, r}\right|=\binom{n+r-1}{r}
$$

With a slight abuse of terminology, we say that a multiset $A$ t-intersects a multiset $B$ if and $A$ and $B$ have at least $t$ distinct common elements, that is, if $\mathrm{S}_{A} t$-intersects $\mathrm{S}_{B}$. A set $\mathcal{A}$ of multisets is said to be $t$-intersecting if every two multisets in $\mathcal{A} t$ intersect, and $k$ sets $\mathcal{A}_{1}, \ldots, \mathcal{A}_{k}$ of multisets are said to be cross-t-intersecting if for every $i, j \in[k]$ with $i \neq j$, each multiset in $\mathcal{A}_{i} t$-intersects each multiset in $\mathcal{A}_{j}$.

In Section 7, we prove the following result.
Theorem 1.6 If $1 \leq t \leq r \leq s, u \in\{0\} \cup \mathbb{R}^{+}$such that $u>\frac{6-t}{3}$, $\min \{m, n\} \geq$ $(t+u+1)(s-t)+r-t, \mathcal{A} \subseteq M_{m, r}, \mathcal{B} \subseteq M_{n, s}$, and $\mathcal{A}$ and $\mathcal{B}$ are cross-t-intersecting, then

$$
|\mathcal{A}||\mathcal{B}| \leq\binom{ m+r-t-1}{r-t}\binom{n+s-t-1}{s-t} .
$$

Moreover, if $u>0$, then the bound is attained if and only if $\mathcal{A}=\left\{A \in M_{m, r}: T \subseteq \mathrm{~S}_{A}\right\}$ and $\mathcal{B}=\left\{B \in M_{n, s}: T \subseteq S_{B}\right\}$ for some $t$-element subset $T$ of $[\min \{m, n\}]$.

The condition $\min \{m, n\} \geq(t+u+1)(s-t)+r-t$ is close to being sharp, as is evident from the fact that if $r=s, m=n<t(r-t)+2$, and $\mathcal{A}=\mathcal{B}=\{A \in$ $\left.M_{n, r}:\left|\mathrm{S}_{A} \cap[t+2]\right| \geq t+1\right\}$, then $\mathcal{A}$ and $\mathcal{B}$ are cross- $t$-intersecting,

$$
\begin{aligned}
|\mathcal{A}| & =\sum_{X \in\binom{[t+2]}{t+1)} \cup\{[t+2]\}}\left|\left\{A \in M_{n, r}: \mathrm{S}_{A} \cap[t+2]=X\right\}\right| \\
& =\sum_{X \in\binom{[t+2]}{t+1)} \cup\{[t+2]\}}\left|\left\{\left(a_{1}, \ldots, a_{r-|X|}\right): a_{1} \leq \cdots \leq a_{r-|X|}, a_{1}, \ldots, a_{r-|X|} \in X \cup[t+3, n]\right\}\right| \\
& =\sum_{X \in\binom{[t+2]}{t+1} \cup\{[t+2]\}}\left|M_{|X|+n-t-2, r-|X|}\right| \\
& =\sum_{X \in\binom{[t+2]}{t+1} \cup\{[t+2]\}}\binom{n+r-t-3}{r-|X|}=(t+2)\binom{n+r-t-3}{r-t-1}+\binom{n+r-t-3}{r-t-2} \\
& =\frac{\binom{n+r-t-1}{r-t}}{(n+r-t-1)(n+r-t-2)}((t+2)(r-t)(n-1)+(r-t)(r-t-1)) \\
& >\frac{\binom{n+r-t-1}{r-t}}{((t+1)(r-t)+1)((t+1)(r-t))}((t+2)(r-t)(t(r-t)+1)+(r-t)(r-t-1)) \\
& =\binom{n+r-t-1}{r-t},
\end{aligned}
$$

and hence $|\mathcal{A}||\mathcal{B}|>\binom{m+r-t-1}{r-t}^{2}=\binom{m+r-t-1}{r-t}\binom{n+s-t-1}{s-t}$.
EKR-type results for multisets have been obtained in [38, 26]. To the best of the author's knowledge, Theorem 1.6 is the first cross- $t$-intersection result for multisets.

In the next section, we show that the above results generalise for $k \geq 2$ families and yield EKR-type results. Section 3 provides basic compression results used in our proofs. Sections 477 are dedicated to the proofs of Theorems 1.3, 1.2, 1.5, and 1.6 , respectively.

## 2 Multiple cross- $t$-intersecting families and $t$-intersecting families

Theorem 1.2 generalises as follows.
Theorem 2.1 Let $k \geq 2, t \leq r_{1} \leq \cdots \leq r_{k}, u \in\{0\} \cup \mathbb{R}^{+}$such that $u>\frac{6-t}{3}$, and
 $\mathcal{A}_{1}, \ldots, \mathcal{A}_{k}$ are cross-t-intersecting, then

$$
\prod_{i=1}^{k}\left|\mathcal{A}_{i}\right| \leq \prod_{i=1}^{k}\binom{n_{i}-t}{r_{i}-t}
$$

Moreover, if $u>0$, then the bound is attained if and only if for some $t$-element subset $T$ of $\left[\min \left\{n_{1}, \ldots, n_{k}\right\}\right], \mathcal{A}_{i}=\left\{A \in\binom{\left[n_{i}\right]}{r_{i}}: T \subseteq A\right\}$ for each $i \in[k]$.

The line of argument in the proof of [14, Theorem 1.2] yields the result above together with a similar generalisation of Theorem 1.6 and the following generalisations of Theorem 1.3 and Theorem 1.5 ,

Theorem 2.2 If $t, u$, and $T$ are as in Theorem 1.3, $\mathcal{H}_{1}, \ldots, \mathcal{H}_{k}$ are non-empty compressed hereditary subfamilies of $2^{[n]}, w_{\mathcal{F}}: \mathcal{F} \rightarrow \mathbb{R}^{+}$is a function satisfying (a) and (b) (of Theorem 1.3) for each $\mathcal{F} \in\left\{\mathcal{H}_{1}, \ldots, \mathcal{H}_{k}\right\}, \mathcal{A}_{i} \subseteq \mathcal{H}_{i}$ for each $i \in[k]$, and $\mathcal{A}_{1}, \ldots, \mathcal{A}_{k}$ are cross-t-intersecting, then

$$
\prod_{i=1}^{k} w_{\mathcal{H}_{i}}\left(\mathcal{A}_{i}\right) \leq \prod_{i=1}^{k} w_{\mathcal{H}_{i}}\left(\mathcal{H}_{i}(T)\right)
$$

Moreover, if $u>0$ and each of $\mathcal{H}_{1}, \ldots, \mathcal{H}_{k}$ has a member of size at least $t$, then the bound is attained if and only if for some $T^{\prime} \in\binom{[n]}{t}$ such that $w_{\mathcal{H}_{i}}\left(\mathcal{H}_{i}\left(T^{\prime}\right)\right)=w_{\mathcal{H}_{i}}\left(\mathcal{H}_{i}(T)\right)$ for each $i \in[k], \mathcal{A}_{i}=\mathcal{H}_{i}\left(T^{\prime}\right)$ for each $i \in[k]$.

Theorem 2.3 Let $\mathbf{c}_{1}=\left(c_{1,1}, \ldots, c_{1, n_{1}}\right), \ldots, \mathbf{c}_{k}=\left(c_{k, 1}, \ldots, c_{k, n_{k}}\right)$ be IP sequences. Let $r_{1} \in\left[n_{1}\right], \ldots, r_{k} \in\left[n_{k}\right], t \in\left[\min \left\{r_{1}, \ldots, r_{k}\right\}\right]$, and $u \in\{0\} \cup \mathbb{R}^{+}$such that $u>\frac{6-t}{3}$. If $c_{1,1} \geq t+u+1, \ldots, c_{k, 1} \geq t+u+1, \mathcal{A}_{1} \subseteq \mathcal{S}_{\mathbf{c}_{1}, r_{1}}, \ldots, \mathcal{A}_{k} \subseteq \mathcal{S}_{\mathbf{c}_{k}, r_{k}}$, and $\mathcal{A}_{1}, \ldots, \mathcal{A}_{k}$ are cross-t-intersecting, then

$$
\prod_{i=1}^{k}\left|\mathcal{A}_{i}\right| \leq \prod_{i=1}^{k}\left(\sum_{\substack{\left[\in\left(\begin{array}{c}
{\left[t+1, n_{i}\right] \\
r_{i}-t}
\end{array}\right)\right.}} \prod_{j \in I} c_{i, j}\right)
$$

Moreover, if $u>0$, then the bound is attained if and only if for some $T \in \bigcap_{i=1}^{k} \mathcal{S}_{\mathbf{c}_{i}, t}$ with $\left|\mathcal{S}_{\mathbf{c}_{i}, r_{i}}(T)\right|=\left|\mathcal{S}_{\mathbf{c}_{i}, r_{i}, t}\right|$ for each $i \in[k], \mathcal{A}_{i}=\mathcal{S}_{\mathbf{c}_{i}, r_{i}}(T)$ for each $i \in[k]$.

We simply observe that $\left(\prod_{i=1}^{k} a_{i}\right)^{k-1}=\prod_{i=1}^{k} \prod_{j \in[k] \backslash[i]} a_{i} a_{j}$ (see also [15, Lemma 5.2] with $p=2$ ) and that if $\mathcal{A}_{1}, \ldots, \mathcal{A}_{k}$ are cross- $t$-intersecting, then any $\mathcal{A}_{i}$ and $\mathcal{A}_{j}$ with $i \neq$ $j$ are cross- $t$-intersecting. Thus, if, for example, $\mathcal{A}_{1}, \ldots, \mathcal{A}_{k}$ are as in Theorem 2.2, $a_{i}=$ $w_{\mathcal{H}_{i}}\left(\mathcal{A}_{i}\right)$ for each $i \in[k]$, and $b_{i}=w_{\mathcal{H}_{i}}\left(\mathcal{H}_{i}(T)\right)$ for each $i \in[k]$, then Theorem 1.3 gives us $\prod_{i=1}^{k} \prod_{j \in[k] \backslash i]} a_{i} a_{j} \leq \prod_{i=1}^{k} \prod_{j \in[k] \backslash[i]} b_{i} b_{j}$, and hence $\left(\prod_{i=1}^{k} a_{i}\right)^{k-1} \leq\left(\prod_{i=1}^{k} b_{i}\right)^{k-1}$ (giving $\prod_{i=1}^{k} a_{i} \leq \prod_{i=1}^{k} b_{i}$, as required).

As in Remark 1.1, Theorem 1.3 immediately implies an EKR-type version for a family $\mathcal{H}$ as in Theorem 1.3, By taking $\mathcal{G}=\mathcal{H}$ in Theorem 1.3 and applying an argument similar to the one in Remark 1.1, we obtain the following new result.

Theorem 2.4 Let $t, u, T, \mathcal{H}$, and $h$ be as in Theorem 1.3. If $\mathcal{A}$ is a t-intersecting subfamily of $\mathcal{H}$, then

$$
h(\mathcal{A}) \leq h(\mathcal{H}(T))
$$

Moreover, if $u>0$ and $\mathcal{H}$ has a member of size at least $t$, then the bound is attained if and only if $\mathcal{A}=\mathcal{H}\left(T^{\prime}\right)$ for some $t$-set $T^{\prime}$ such that $h\left(\mathcal{H}\left(T^{\prime}\right)\right)=h(\mathcal{H}(T))$.

By taking $\mathbf{c}=\mathbf{d}$ in Theorem 1.5 and applying the argument in Remark 1.1, we obtain the following EKR-type result.

Theorem 2.5 If $1 \leq t \leq r \leq n, u \in\{0\} \cup \mathbb{R}^{+}$such that $u>\frac{6-t}{3}$, $\mathbf{c}=\left(c_{1}, \ldots, c_{n}\right)$ is an IP sequence, $c_{1} \geq t+u+1$, and $\mathcal{A}$ is a t-intersecting subfamily of $\mathcal{S}_{\mathbf{c}, r}$, then

$$
|\mathcal{A}| \leq\left(\sum_{I \in\binom{[t+1, n]}{r-t}} \prod_{i \in I} c_{i}\right) .
$$

Moreover, if $u>0$, then the bound is attained if and only if $\mathcal{A}=\mathcal{S}_{\mathbf{c}, r}(T)$ for some $T \in \mathcal{S}_{\mathbf{c}, t}$ with $\left|\mathcal{S}_{\mathbf{c}, r}(T)\right|=\left|\mathcal{S}_{\mathbf{c}, r, t}\right|$.

The EKR problem for $\mathcal{S}_{\mathbf{c}, r}$ attracted much attention and has been dealt with extensively (see, for example, [13]). In particular, for $c_{1}=c_{n}$, it was solved for $r=n$ in [2, 24], and for $n \geq\left[\frac{\left(r-t+c_{1}\right)(t+1)}{c_{1}}\right]$ in [5]. Similarly to Theorem 1.5, Theorem 2.5) does not hold for $c_{1}<t+1$.

By taking $m=n$ and $r=s$ in Theorem 1.6, and applying the argument in Remark 1.1, we obtain the following EKR-type result.

Theorem 2.6 If $1 \leq t \leq r, u \in\{0\} \cup \mathbb{R}^{+}$such that $u>\frac{6-t}{3}, n \geq(t+u+2)(r-t)$, $\mathcal{A} \subseteq M_{n, r}$, and $\mathcal{A}$ is $t$-intersecting, then

$$
|\mathcal{A}| \leq\binom{ n+r-t-1}{r-t}
$$

Moreover, if $u>0$, then the bound is attained if and only if $\mathcal{A}=\left\{A \in M_{n, r}: T \subseteq \mathrm{~S}_{A}\right\}$ for some $T \in\binom{[n]}{t}$.

The condition $n \geq(t+u+2)(r-t)$ is close to being sharp. Indeed, as shown in Section [1, if $n<t(r-t)+2$ and $\mathcal{A}=\left\{A \in M_{n, r}:\left|\mathrm{S}_{A} \cap[t+2]\right| \geq t+1\right\}$, then $|\mathcal{A}|>\binom{n+r-t-1}{r-t}$.

The EKR problem for $M_{n, r}$ and $t=1$ is solved in [38]. Generalising this result, Füredi, Gerbner, and Vizer [26] solved the EKR problem of maximising the size of a largest subset $\mathcal{A}$ of $M_{n, r}$ such that for every $\left(a_{1}, \ldots, a_{r}\right),\left(b_{1}, \ldots, b_{r}\right) \in \mathcal{A}$, there exist $t$ distinct elements $i_{1}, \ldots, i_{t}$ of $[r]$ and $t$ distinct elements $j_{1}, \ldots, j_{t}$ of $[r]$ such that $a_{i_{p}}=b_{j_{p}}$ for each $p \in[t]$.

## 3 The compression operation

Compression operations have various useful properties. It is straightforward that for $i, j \in[n]$ and $\mathcal{A} \subseteq 2^{[n]}$,

$$
\left|\Delta_{i, j}(\mathcal{A})\right|=|\mathcal{A}| .
$$

We will also need the following well-known basic result (see, for example, [14, Lemma 2.1]).

Lemma 3.1 Let $\mathcal{A}$ and $\mathcal{B}$ be cross-t-intersecting subfamilies of $2^{[n]}$.
(i) For any $i, j \in[n], \Delta_{i, j}(\mathcal{A})$ and $\Delta_{i, j}(\mathcal{B})$ are cross-t-intersecting subfamilies of $2^{[n]}$.
(ii) If $1 \leq t \leq r \leq s \leq n, \mathcal{A} \subseteq\binom{[n]}{\leq r}, \mathcal{B} \subseteq\binom{[n]}{\leq s}$, and $\mathcal{A}$ and $\mathcal{B}$ are compressed, then

$$
|A \cap B \cap[r+s-t]| \geq t
$$

for any $A \in \mathcal{A}$ and any $B \in \mathcal{B}$.
The only difference between Lemma 3.1 and [14, Lemma 2.1] is that the latter is for


Suppose that a subfamily $\mathcal{A}$ of $2^{[n]}$ is not compressed. Then $\mathcal{A}$ can be transformed to a compressed family through left-compressions as follows. Since $\mathcal{A}$ is not compressed, we can find a left-compression that changes $\mathcal{A}$, and we apply it to $\mathcal{A}$ to obtain a new subfamily of $2^{[n]}$. We keep on repeating this (always applying a left-compression to the last family obtained) until we obtain a subfamily of $2^{[n]}$ that is invariant under any left-compression (such a point is indeed reached, because if $\Delta_{i, j}(\mathcal{F}) \neq \mathcal{F} \subseteq 2^{[n]}$ and $i<j$, then $\left.0<\sum_{G \in \Delta_{i, j}(\mathcal{F})} \sum_{b \in G} b<\sum_{F \in \mathcal{F}} \sum_{a \in F} a\right)$.

Now consider $\mathcal{A}, \mathcal{B} \subseteq 2^{[n]}$ such that $\mathcal{A}$ and $\mathcal{B}$ are cross- $t$-intersecting. Then, by Lemma 3.1, we can obtain $\mathcal{A}^{*}, \mathcal{B}^{*} \subseteq 2^{[n]}$ such that $\mathcal{A}^{*}$ and $\mathcal{B}^{*}$ are compressed and cross-$t$-intersecting, $\left|\mathcal{A}^{*}\right|=|\mathcal{A}|$, and $\left|\mathcal{B}^{*}\right|=|\mathcal{B}|$. Indeed, similarly to the above procedure, if we can find a left-compression that changes at least one of $\mathcal{A}$ and $\mathcal{B}$, then we apply it to both $\mathcal{A}$ and $\mathcal{B}$, and we keep on repeating this (always performing this on the last two families obtained) until we obtain $\mathcal{A}^{*}, \mathcal{B}^{*} \subseteq 2^{[n]}$ such that both $\mathcal{A}^{*}$ and $\mathcal{B}^{*}$ are invariant under any left-compression.

## 4 Proof of the main result

This section is dedicated to the proof of Theorem 1.3 ,
For the extremal cases of Theorem [1.3, we shall use the following two lemmas.
Lemma 4.1 Let $1 \leq t \leq n$ and $T=[t]$. Let $\mathcal{H}$ be a compressed subfamily of $2^{[n]}$. Let $w: \mathcal{H} \rightarrow \mathbb{R}^{+}$such that $w\left(\delta_{i, j}(H)\right) \geq w(H)$ for every $H \in \mathcal{H}$ and every $i, j \in[n]$ with $i<j$. Then $w\left(\mathcal{H}\left(T^{\prime}\right)\right) \leq w(\mathcal{H}(T))$ for each $T^{\prime} \in\binom{[n]}{t}$.

Proof. Let $T^{\prime} \in\binom{[n]}{t}$, and let $a_{1}, \ldots, a_{t}$ be the elements of $T^{\prime}$. Let $\mathcal{D}_{0}=\mathcal{H}\left(T^{\prime}\right)$. Let $\mathcal{D}_{1}=\Delta_{1, a_{1}}\left(\mathcal{D}_{0}\right), \ldots, \mathcal{D}_{t}=\Delta_{t, a_{t}}\left(\mathcal{D}_{t-1}\right)$. Since $\mathcal{H}$ is compressed, $\mathcal{D}_{i} \subseteq \mathcal{H}$ for each $i \in[t]$. It follows from the properties of $w$ and of left-compressions that $w\left(\mathcal{D}_{0}\right) \leq w\left(\mathcal{D}_{1}\right) \leq$ $\cdots \leq w\left(\mathcal{D}_{t}\right)$. Thus the result follows if we show that $\mathcal{D}_{t} \subseteq \mathcal{H}(T)$.

Let $D_{1} \in \mathcal{D}_{1}$. If $D_{1} \notin \mathcal{D}_{0}$, then $D_{1}=\delta_{1, a_{1}}(D) \neq D$ for some $D \in \mathcal{D}_{0}$, and hence $1 \in D_{1}$. Suppose $D_{1} \in \mathcal{D}_{0}$, so $a_{1} \in D_{1}$ by definition of $\mathcal{D}_{0}$. Since $D_{1}$ is also in $\mathcal{D}_{1}$, $\delta_{1, a_{1}}\left(D_{1}\right) \in \mathcal{D}_{0}$. Thus $a_{1} \in \delta_{1, a_{1}}\left(D_{1}\right)$ by definition of $\mathcal{D}_{0}$. Since $a_{1} \in D_{1}$, it follows that $1 \in D_{1}$.

Therefore, $1 \in H$ for each $H \in \mathcal{D}_{1}$, that is, $\mathcal{D}_{1} \subseteq \mathcal{H}(\{1\})$. If $t=1$, then we have $w\left(\mathcal{D}_{0}\right) \leq w\left(\mathcal{D}_{1}\right) \leq w(\mathcal{H}(\{1\}))=w(\mathcal{H}(T))$, as required.

Suppose $t \geq 2$. Since $\mathcal{D}_{1} \subseteq \mathcal{H}(\{1\})$, we clearly have $1 \in H$ for each $H \in \mathcal{D}_{2}$. By an argument similar to that for $\mathcal{D}_{1}$, we also obtain that $2 \in H$ for each $H \in \mathcal{D}_{2}$. Continuing this way, we obtain that $1, \ldots, t \in H$ for each $H \in \mathcal{D}_{t}$. Thus $\mathcal{D}_{t} \subseteq \mathcal{H}(T)$, as required.

Lemma 4.2 Let $n, t, T, \mathcal{G}, \mathcal{H}, g$, and $h$ be as in Theorem 1.3. If $U \in \mathcal{A} \subseteq \mathcal{G}$, $V \in \mathcal{B} \subseteq \mathcal{H},|U|=|V|=t$, and $\mathcal{A}$ and $\mathcal{B}$ are cross-t-intersecting, then

$$
g(\mathcal{A}) h(\mathcal{B}) \leq g(\mathcal{G}(T)) h(\mathcal{H}(T))
$$

and equality holds if and only if $\mathcal{A}=\mathcal{G}(V), \mathcal{B}=\mathcal{H}(V), g(\mathcal{G}(V))=g(\mathcal{G}(T))$, and $h(\mathcal{H}(V))=h(\mathcal{H}(T))$.

Proof. Since $\mathcal{A}$ and $\mathcal{B}$ are cross- $t$-intersecting, we have $U=V, \mathcal{A} \subseteq \mathcal{G}(V)$, and $\mathcal{B} \subseteq \mathcal{H}(V)$. By Lemma 4.1, $g(\mathcal{G}(V)) \leq g(\mathcal{G}(T))$ and $h(\mathcal{H}(V)) \leq h(\mathcal{H}(T))$. Hence the result.

Proof of Theorem 1.3. We prove the result by induction on $n$.
Consider the base case $n=t$. If $g(\mathcal{A}) h(\mathcal{B}) \neq 0$, then $\mathcal{A} \neq \emptyset \neq \mathcal{B}$, and hence, since $\mathcal{A}$ and $\mathcal{B}$ are cross- $t$-intersecting, $\mathcal{A}=\{T\}=\mathcal{B}$.

Now consider $n \geq t+1$. Let $\mathcal{A} \subseteq \mathcal{G}$ and $\mathcal{B} \subseteq \mathcal{H}$ such that $g(\mathcal{A}) h(\mathcal{B})$ is maximum under the condition that $\mathcal{A}$ and $\mathcal{B}$ are cross- $t$-intersecting. If $\mathcal{G}$ does not have a member of size at least $t$, then $\mathcal{A}=\emptyset$ or $\mathcal{B}=\emptyset$ (since $\mathcal{A}$ and $\mathcal{B}$ are cross- $t$-intersecting), and hence $g(\mathcal{A}) h(\mathcal{B})=0=g(\mathcal{G}(T)) h(\mathcal{H}(T))$. Similarly, $g(\mathcal{A}) h(\mathcal{B})=0=g(\mathcal{G}(T)) h(\mathcal{H}(T))$ if $\mathcal{H}$ does not have a member of size at least $t$. Therefore, we will assume that each of $\mathcal{G}$ and $\mathcal{H}$ has a member of size at least $t$. Since $\mathcal{G}$ and $\mathcal{H}$ are hereditary and compressed, we clearly have $T \in \mathcal{G}$ and $T \in \mathcal{H}$. Thus $g(\mathcal{G}(T))>0$ and $h(\mathcal{H}(T))>0$. Since $\mathcal{G}(T)$ and $\mathcal{H}(T)$ are cross- $t$-intersecting, it follows by the choice of $\mathcal{A}$ and $\mathcal{B}$ that

$$
\begin{equation*}
g(\mathcal{A}) h(\mathcal{B}) \geq g(\mathcal{G}(T)) h(\mathcal{H}(T))>0 \tag{1}
\end{equation*}
$$

It follows that $\mathcal{A} \neq \emptyset \neq \mathcal{B}$. It also follows that no member of $\mathcal{A}$ is of size less than $t$, because otherwise $\mathcal{B}=\emptyset$, contradicting (1). Similarly, no member of $\mathcal{B}$ is of size less than $t$.

As explained in Section 3, we apply left-compressions to $\mathcal{A}$ and $\mathcal{B}$ simultaneously until we obtain two compressed cross- $t$-intersecting families $\mathcal{A}^{*}$ and $\mathcal{B}^{*}$, respectively. Thus $\left|\mathcal{A}^{*}\right|=|\mathcal{A}|$ and $\left|\mathcal{B}^{*}\right|=|\mathcal{B}|$. Since $\mathcal{G}$ and $\mathcal{H}$ are compressed, $\mathcal{A}^{*} \subseteq \mathcal{G}$ and $\mathcal{B}^{*} \subseteq \mathcal{H}$. By (b), $g(\mathcal{A}) \leq g\left(\mathcal{A}^{*}\right)$ and $h(\mathcal{B}) \leq h\left(\mathcal{B}^{*}\right)$. By the choice of $\mathcal{A}$ and $\mathcal{B}$, we actually have $g(\mathcal{A})=g\left(\mathcal{A}^{*}\right)$ and $h(\mathcal{B})=h\left(\mathcal{B}^{*}\right)$.

Suppose that $\mathcal{A}^{*}=\mathcal{G}(U)$ and $\mathcal{B}^{*}=\mathcal{H}(U)$ for some $U \in\binom{[n]}{t}$ such that $g(\mathcal{G}(U))=$ $g(\mathcal{G}(T))$ and $h(\mathcal{H}(U))=h(\mathcal{H}(T))$. Then $g(\mathcal{G}(U))>0$ and $h(\mathcal{H}(U))>0$, so $\mathcal{G}(U) \neq \emptyset$ and $\mathcal{H}(U) \neq \emptyset$. Thus, since $\mathcal{G}$ and $\mathcal{H}$ are hereditary, $U \in \mathcal{A}^{*}$ and $U \in \mathcal{B}^{*}$. Hence $V \in \mathcal{A}$ for some $V \in\binom{[n]}{t}$, and $V^{\prime} \in \mathcal{B}$ for some $V^{\prime} \in\binom{[n]}{t}$. By Lemma 4.2, the result follows.

Therefore, we may assume that $\mathcal{A}$ and $\mathcal{B}$ are compressed.

We first consider $t+1 \leq n \leq t+2$. If $\mathcal{A}$ has a member of size $t$ and $\mathcal{B}$ has a member of size $t$, then the result follows by Lemma 4.2. Thus, without loss of generality, we may assume that no member of $\mathcal{A}$ is of size $t$.

Suppose $n=t+1$. Then $\mathcal{A}=\{[t+1]\} \subseteq \mathcal{G}(T) \backslash\{T\}$ (since $\mathcal{A} \neq \emptyset$ and $\left.\mathcal{A} \cap\binom{[n]}{t}=\emptyset\right)$ and $\mathcal{B} \subseteq \mathcal{H} \cap\left(\binom{[t+1]}{t} \cup\{[t+1]\}\right)$. Thus we have

$$
\begin{aligned}
g(\mathcal{A}) h(\mathcal{B}) & \leq(g(\mathcal{G}(T))-g(T))\left(h(\mathcal{H}(T))+\sum_{\substack{[(t+1]) \\
t}} h(H)\right) \\
& \leq(g(\mathcal{G}(T))-g(T))(h(\mathcal{H}(T))+\operatorname{th}(T)) \quad(\mathrm{by}(\mathrm{~b})) \\
& =g(\mathcal{G}(T)) h(\mathcal{H}(T))+\operatorname{th}(T) g(\mathcal{G}(T))-g(T)(h(\mathcal{H}(T))+\operatorname{th}(T)) \\
& \leq g(\mathcal{G}(T)) h(\mathcal{H}(T))+\operatorname{th}(T)(g(T)+g([t+1]))-(t+1) g(T) h(T) \\
& \left.\leq g(\mathcal{G}(T)) h(\mathcal{H}(T))+\operatorname{th}(T)\left(g(T)+\frac{g(T)}{t+u}\right)-(t+1) g(T) h(T) \quad \text { (by (a) }\right) .
\end{aligned}
$$

Therefore, $g(\mathcal{A}) h(\mathcal{B}) \leq g(\mathcal{G}(T)) h(\mathcal{H}(T))$, and equality holds only if $u=0$.
Suppose $n=t+2$. This case requires a number of observations followed by the separate treatment of a few sub-cases.

Let $T_{1}=[t+1], T_{1}^{\prime}=T \cup\{t+2\}$, and $T_{2}=[t+2]$. For each $i \in\{t, t+1, t+2\}$ and each $\mathcal{F} \in\{\mathcal{A}, \mathcal{B}, \mathcal{G}, \mathcal{H}\}$, let $\mathcal{F}^{(i)}=\mathcal{F} \cap\binom{[t+2]}{i}$. Thus $\mathcal{A}=\mathcal{A}^{(t)} \cup \mathcal{A}^{(t+1)} \cup \mathcal{A}^{(t+2)}$ and $\mathcal{B}=\mathcal{B}^{(t)} \cup \mathcal{B}^{(t+1)} \cup \mathcal{B}^{(t+2)}$. Recall that $\mathcal{A}$ has no $t$-set, so $\mathcal{A}^{(t)}=\emptyset$. Since $\mathcal{A}^{(t+2)}, \mathcal{B}^{(t+2)} \subseteq$ $\binom{[t+2]}{t+2}=\{[t+2]\}$, we have $\mathcal{A}^{(t+2)} \subseteq \mathcal{G}(T)$ and $\mathcal{B}^{(t+2)} \subseteq \mathcal{H}(T)$. Let

$$
\mathcal{A}_{T}=\mathcal{A} \cap \mathcal{G}(T), \quad \mathcal{A}_{\bar{T}}=\mathcal{A} \backslash \mathcal{A}_{T}, \quad \mathcal{B}_{T}=\mathcal{B} \cap \mathcal{H}(T), \quad \mathcal{B}_{\bar{T}}=\mathcal{B} \backslash \mathcal{B}_{T}
$$

We have

$$
\begin{equation*}
\mathcal{A}_{T} \subseteq \mathcal{G}(T) \backslash\{T\}, \quad \mathcal{A}_{\bar{T}} \subseteq \mathcal{G}^{(t+1)} \backslash\left\{T_{1}, T_{1}^{\prime}\right\}, \quad \mathcal{B}_{\bar{T}} \subseteq \mathcal{H}^{(t)} \cup \mathcal{H}^{(t+1)} \backslash\left\{T_{1}, T_{1}^{\prime}\right\} \tag{2}
\end{equation*}
$$

Since $T \subsetneq T_{1} \subsetneq T_{2}$, we have $g\left(T_{1}\right) \leq \frac{g(T)}{t+u}, g\left(T_{2}\right) \leq \frac{g\left(T_{1}\right)}{t+u} \leq \frac{g(T)}{(t+u)^{2}}, h\left(T_{1}\right) \leq \frac{h(T)}{t+u}$, and $h\left(T_{2}\right) \leq \frac{h\left(T_{1}\right)}{t+u} \leq \frac{h(T)}{(t+u)^{2}}$. Clearly, for each $U \in \mathcal{G}^{(t)}$, there is a composition of leftcompressions that gives $T$ when applied to $U$, and hence $g(U) \leq g(T)$ by (b). Similarly, $h(V) \leq h(T)$ for each $V \in \mathcal{H}^{(t)}, g(U) \leq g\left(T_{1}\right)$ for each $U \in \mathcal{G}^{(t+1)}$, and $h(V) \leq h\left(T_{1}\right)$ for each $V \in \mathcal{H}^{(t+1)}$.

Suppose $\mathcal{A}^{(t+1)}=\emptyset$. Then $\mathcal{A}=\left\{T_{2}\right\}$. Since $T_{2} \in \mathcal{G}$ and $\mathcal{G}$ is hereditary, we have $T, T_{1}, T_{1}^{\prime}, T_{2} \in \mathcal{G}(T)$. Thus

$$
\begin{aligned}
g(\mathcal{G}(T)) & \geq g(T)+g\left(T_{1}\right)+g\left(T_{1}^{\prime}\right)+g\left(T_{2}\right) \\
& \geq(t+u)^{2} g\left(T_{2}\right)+2(t+u) g\left(T_{2}\right)+g\left(T_{2}\right) \\
& \geq((t+u)+1)^{2} g\left(T_{2}\right)=(t+u+1)^{2} g(\mathcal{A}),
\end{aligned}
$$

and hence $g(\mathcal{A}) \leq \frac{g(\mathcal{G}(T))}{(t+u+1)^{2}}$. Now

$$
\begin{aligned}
h(\mathcal{B}) & =h\left(\mathcal{B}_{T}\right)+h\left(\mathcal{B}_{\bar{T}}\right) \leq h(\mathcal{H}(T))+\left(\binom{t+2}{t}-1\right) h(T)+\left(\binom{t+2}{t+1}-2\right) h\left(T_{1}\right) \\
& \leq h(\mathcal{H}(T))+\left(\frac{(t+2)(t+1)}{2}-1\right) h(T)+t \frac{h(T)}{t}=h(\mathcal{H}(T))+\frac{t^{2}+3 t+2}{2} h(T) \\
& \leq h(\mathcal{H}(T))+\frac{t^{2}+3 t+2}{2} h(\mathcal{H}(T))=\frac{t^{2}+3 t+4}{2} h(\mathcal{H}(T)) .
\end{aligned}
$$

Thus

$$
g(\mathcal{A}) h(\mathcal{B}) \leq \frac{t^{2}+3 t+4}{2(t+u+1)^{2}} g(\mathcal{G}(T)) h(\mathcal{H}(T)) \leq \frac{t^{2}+3 t+4}{2(t+1)^{2}} g(\mathcal{G}(T)) h(\mathcal{H}(T)) .
$$

Hence $g(\mathcal{A}) h(\mathcal{B}) \leq g(\mathcal{G}(T)) h(\mathcal{H}(T))$, and equality holds only if $u=0$.
Suppose that $\mathcal{A}^{(t+1)}$ has at least 3 sets. Let $U_{1}, U_{2}$, and $U_{3}$ be 3 distinct sets in $\mathcal{A}^{(t+1)}$. Since $U_{1}, U_{2}, U_{3} \in\binom{[t+2]}{t+1}$, no $t$-set is a subset of each of $U_{1}, U_{2}$, and $U_{3}$. Thus no $t$-set $t$-intersects each of $U_{1}, U_{2}$, and $U_{3}$, and hence $\mathcal{B}^{(t)}=\emptyset$. We have
$g(\mathcal{A}) \leq g(\mathcal{G}(T))-g(T)+\left(\binom{t+2}{t+1}-2\right) g\left(T_{1}\right) \leq g(\mathcal{G}(T))-g(T)+t \frac{g(T)}{t+u} \leq g(\mathcal{G}(T))$.
Similarly, $h(\mathcal{B}) \leq h(\mathcal{H}(T))$. Thus $g(\mathcal{A}) h(\mathcal{B}) \leq g(\mathcal{G}(T)) h(\mathcal{H}(T))$, and equality holds only if $u=0$.

We still need to consider $1 \leq\left|\mathcal{A}^{(t+1)}\right| \leq 2$, for which we need more detailed observations. Let $\mathcal{C}_{0}=\mathcal{B}_{\bar{T}} \cap\binom{[t+2]}{t}, \mathcal{C}_{1}=\overline{\mathcal{B}}_{\bar{T}} \cap\binom{[t+2]}{t+1}, \mathcal{D}_{0}=\mathcal{H}(T) \cap\binom{[t+2]}{t}$, and $\mathcal{D}_{1}=\mathcal{H}(T) \cap\binom{[t+2]}{t+1}$. By (2) $), \mathcal{B}_{\bar{T}}=\mathcal{C}_{0} \cup \mathcal{C}_{1}$. If $\mathcal{H}^{(t+1)} \backslash\left\{T_{1}, T_{1}^{\prime}\right\}$ has a set $V$, then $t+2 \in V$, and hence there is a composition of left-compressions that gives $T_{1}^{\prime}$ when applied to $V$. Thus, if $\mathcal{H}^{(t+1)} \backslash\left\{T_{1}, T_{1}^{\prime}\right\}$ is non-empty, then $T_{1}, T_{1}^{\prime} \in \mathcal{H}(T)$ (as $\mathcal{H}$ is compressed, $T \subset T_{1}$, and $T \subset T_{1}^{\prime}$ ), and hence we have

$$
\begin{aligned}
h\left(\mathcal{C}_{1}\right) & \leq \sum_{V \in \mathcal{H}^{(t+1)} \backslash\left\{T_{1}, T_{1}^{\prime}\right\}} h(V) \quad(\text { by (2) }) \\
& \leq \sum_{V \in \mathcal{H}(t+1) \backslash\left\{T_{1}, T_{1}^{\prime}\right\}} h\left(T_{1}^{\prime}\right) \leq t h\left(T_{1}^{\prime}\right) \leq t \frac{h\left(T_{1}\right)+h\left(T_{1}^{\prime}\right)}{2}=\frac{t}{2}\left|\mathcal{D}_{1}\right| .
\end{aligned}
$$

If $\mathcal{H}^{(t+1)} \backslash\left\{T_{1}, T_{1}^{\prime}\right\}=\emptyset$, then $\mathcal{C}_{1}=\emptyset$, and hence we also have $h\left(\mathcal{C}_{1}\right) \leq \frac{t}{2} h\left(\mathcal{D}_{1}\right)$. With a slight abuse of notation, we set $g\left(T_{1}^{\prime}\right)=0$ if $T_{1}^{\prime} \notin \mathcal{G}$, and we set $g\left(T_{2}\right)=0$ if $T_{2} \notin \mathcal{G}$. Since $\mathcal{G}$ is hereditary, $T_{1}^{\prime} \in \mathcal{G}$ if $T_{2} \in \mathcal{G}$. Thus $g\left(T_{1}^{\prime}\right) \geq(t+u) g\left(T_{2}\right)$.

Suppose that $\mathcal{A}^{(t+1)}$ has exactly one set. Since $\mathcal{A}$ is compressed, $\mathcal{A}^{(t+1)}=\left\{T_{1}\right\}$. Thus $\mathcal{A} \subseteq\left\{T_{1}, T_{2}\right\}$, and hence $g(\mathcal{A}) \leq g\left(T_{1}\right)+g\left(T_{2}\right)=g(\mathcal{G}(T))-g(T)-g\left(T_{1}^{\prime}\right)$. The $t$-sets that $t$-intersect $T_{1}$ are those in $\binom{T_{1}}{t}$, so $\mathcal{B}^{(t)} \subseteq\binom{T_{1}}{t}$, and hence

$$
h\left(\mathcal{C}_{0}\right) \leq \sum_{V \in\left(\mathcal{H}^{(t)} \backslash\{T\}\right) \cap\binom{T_{1}}{t}} h(V) \leq\left(\binom{t+1}{t}-1\right) h(T)=\operatorname{th}\left(\mathcal{D}_{0}\right) .
$$

We have

$$
\begin{aligned}
g(\mathcal{A}) h(\mathcal{B}) & \leq\left(g(\mathcal{G}(T))-g(T)-g\left(T_{1}^{\prime}\right)\right)\left(h(\mathcal{H}(T))+h\left(\mathcal{B}_{\bar{T}}\right)\right) \\
& =g(\mathcal{G}(T)) h(\mathcal{H}(T))+g(\mathcal{G}(T)) h\left(\mathcal{B}_{\bar{T}}\right)-\left(g(T)+g\left(T_{1}^{\prime}\right)\right)\left(h(\mathcal{H}(T))+h\left(\mathcal{B}_{\bar{T}}\right)\right) \\
& =g(\mathcal{G}(T)) h(\mathcal{H}(T))+\left(g\left(T_{1}\right)+g\left(T_{2}\right)\right) h\left(\mathcal{B}_{\bar{T}}\right)-\left(g(T)+g\left(T_{1}^{\prime}\right)\right) h(\mathcal{H}(T)) \\
& \leq g(\mathcal{G}(T)) h(\mathcal{H}(T))+\left(\frac{g(T)}{t+u}+\frac{g\left(T_{1}^{\prime}\right)}{t+u}\right) h\left(\mathcal{B}_{\bar{T}}\right)-\left(g(T)+g\left(T_{1}^{\prime}\right)\right) h(\mathcal{H}(T)) \\
& =g(\mathcal{G}(T)) h(\mathcal{H}(T))+\left(g(T)+g\left(T_{1}^{\prime}\right)\right)\left(\frac{h\left(\mathcal{C}_{0}\right)+h\left(\mathcal{C}_{1}\right)}{t+u}-h(\mathcal{H}(T))\right) \\
& \leq g(\mathcal{G}(T)) h(\mathcal{H}(T))+\left(g(T)+g\left(T_{1}^{\prime}\right)\right)\left(\frac{t h\left(\mathcal{D}_{0}\right)+\frac{t}{2} h\left(\mathcal{D}_{1}\right)}{t+u}-\left(h\left(\mathcal{D}_{0}\right)+h\left(\mathcal{D}_{1}\right)\right)\right) .
\end{aligned}
$$

Thus $g(\mathcal{A}) h(\mathcal{B}) \leq g(\mathcal{G}(T)) h(\mathcal{H}(T))$, and equality holds only if $u=0$.
Suppose that $\mathcal{A}^{(t+1)}$ has exactly 2 sets. Since $\mathcal{A}$ is compressed, $\mathcal{A}^{(t+1)}=\left\{T_{1}, T_{1}^{\prime}\right\}$. The only $t$-set that $t$-intersects each of $T_{1}$ and $T_{1}^{\prime}$ is $T$, so $\mathcal{B}^{(t)} \subseteq\{T\}$. Thus $\mathcal{C}_{0}=\emptyset$, and hence $\mathcal{B}_{\bar{T}}=\mathcal{C}_{1}$. Since $\mathcal{D}_{1} \subseteq\left\{T_{1}, T_{1}^{\prime}\right\}, h\left(\mathcal{D}_{1}\right) \leq 2 \frac{h(T)}{t+u}=2 \frac{h\left(\mathcal{D}_{0}\right)}{t+u}$. Since $h(\mathcal{H}(T)) \geq$ $h\left(\mathcal{D}_{0}\right)+h\left(\mathcal{D}_{1}\right), h(\mathcal{H}(T)) \geq \frac{t+u}{2} h\left(\mathcal{D}_{1}\right)+h\left(\mathcal{D}_{1}\right)=\left(\frac{t+u}{2}+1\right) h\left(\mathcal{D}_{1}\right)$. We have

$$
\begin{aligned}
g(\mathcal{A}) h(\mathcal{B}) & \leq(g(\mathcal{G}(T))-g(T))\left(h(\mathcal{H}(T))+h\left(\mathcal{C}_{1}\right)\right) \\
& =g(\mathcal{G}(T)) h(\mathcal{H}(T))+g(\mathcal{G}(T)) h\left(\mathcal{C}_{1}\right)-g(T)\left(h(\mathcal{H}(T))+h\left(\mathcal{C}_{1}\right)\right) \\
& =g(\mathcal{G}(T)) h(\mathcal{H}(T))+\left(g\left(T_{1}\right)+g\left(T_{1}^{\prime}\right)+g\left(T_{2}\right)\right) h\left(\mathcal{C}_{1}\right)-g(T) h(\mathcal{H}(T)) \\
& \leq g(\mathcal{G}(T)) h(\mathcal{H}(T))+\left(\frac{2 g(T)}{t+u}+\frac{g(T)}{(t+u)^{2}}\right) \frac{t}{2} h\left(\mathcal{D}_{1}\right)-g(T)\left(\frac{t+u}{2}+1\right) h\left(\mathcal{D}_{1}\right) \\
& =g(\mathcal{G}(T)) h(\mathcal{H}(T))+g(T) h\left(\mathcal{D}_{1}\right)\left(\frac{t}{t+u}+\frac{t}{2(t+u)^{2}}-\frac{t+u}{2}-1\right) .
\end{aligned}
$$

Thus $g(\mathcal{A}) h(\mathcal{B}) \leq g(\mathcal{G}(T)) h(\mathcal{H}(T))$, and equality holds only if $u=0$.
Now consider $n \geq t+3$.
Define $\mathcal{H}_{0}=\{H \in \mathcal{H}: n \notin H\}$ and $\mathcal{H}_{1}=\{H \backslash\{n\}: n \in H \in \mathcal{H}\}$. Define $\mathcal{G}_{0}, \mathcal{G}_{1}, \mathcal{A}_{0}$, $\mathcal{A}_{1}, \mathcal{B}_{0}$, and $\mathcal{B}_{1}$ similarly. Since $\mathcal{A}, \mathcal{B}, \mathcal{G}$, and $\mathcal{H}$ are compressed, we clearly have that $\mathcal{A}_{0}, \mathcal{A}_{1}, \mathcal{B}_{0}, \mathcal{B}_{1}, \mathcal{G}_{0}, \mathcal{G}_{1}, \mathcal{H}_{0}$, and $\mathcal{H}_{1}$ are compressed. Since $\mathcal{G}$ and $\mathcal{H}$ are hereditary, we clearly have that $\mathcal{G}_{0}, \mathcal{G}_{1}, \mathcal{H}_{0}$, and $\mathcal{H}_{1}$ are hereditary, $\mathcal{G}_{1} \subseteq \mathcal{G}_{0}$, and $\mathcal{H}_{1} \subseteq \mathcal{H}_{0}$. Obviously, we have $\mathcal{A}_{0} \subseteq \mathcal{G}_{0} \subseteq 2^{[n-1]}, \mathcal{A}_{1} \subseteq \mathcal{G}_{1} \subseteq 2^{[n-1]}, \mathcal{B}_{0} \subseteq \mathcal{H}_{0} \subseteq 2^{[n-1]}$, and $\mathcal{B}_{1} \subseteq \mathcal{H}_{1} \subseteq 2^{[n-1]}$.

Let $h_{0}: \mathcal{H}_{0} \rightarrow \mathbb{R}^{+}$such that $h_{0}(H)=h(H)$ for each $H \in \mathcal{H}_{0}$. Let $h_{1}: \mathcal{H}_{1} \rightarrow \mathbb{R}^{+}$ such that $h_{1}(H)=h(H \cup\{n\})$ for each $H \in \mathcal{H}_{1}$ (note that $H \cup\{n\} \in \mathcal{H}$ by definition of $\mathcal{H}_{1}$ ). By (a) and (b), we have the following consequences. For any $A, B \in \mathcal{H}_{0}$ with $A \subsetneq B$ and $|A| \geq t$,

$$
\begin{equation*}
h_{0}(A)=h(A) \geq(t+u) h(B)=(t+u) h_{0}(B) . \tag{3}
\end{equation*}
$$

For any $C \in \mathcal{H}_{0}$ and any $i, j \in[n-1]$ with $i<j$,

$$
\begin{equation*}
h_{0}\left(\delta_{i, j}(C)\right)=h\left(\delta_{i, j}(C)\right) \geq h(C)=h_{0}(C) . \tag{4}
\end{equation*}
$$

For any $A, B \in \mathcal{H}_{1}$ with $A \subsetneq B$ and $|A| \geq t$,

$$
\begin{equation*}
h_{1}(A)=h(A \cup\{n\}) \geq(t+u) h(B \cup\{n\})=(t+u) h_{1}(B) . \tag{5}
\end{equation*}
$$

For any $C \in \mathcal{H}_{1}$ and any $i, j \in[n-1]$ with $i<j$,

$$
\begin{equation*}
h_{1}\left(\delta_{i, j}(C)\right)=h\left(\delta_{i, j}(C) \cup\{n\}\right)=h\left(\delta_{i, j}(C \cup\{n\})\right) \geq h(C \cup\{n\})=h_{1}(C) . \tag{6}
\end{equation*}
$$

Therefore, we have shown that properties (a) and (b) are inherited by $h_{0}$ and $h_{1}$.
Since $\mathcal{B}=\mathcal{B}_{0} \cup \mathcal{B}(\{n\}), \mathcal{B}_{0} \cap \mathcal{B}(\{n\})=\emptyset$, and $\mathcal{B}(\{n\})=\left\{B \cup\{n\}: B \in \mathcal{B}_{1}\right\}$, we have

$$
\begin{equation*}
h(\mathcal{B})=h\left(\mathcal{B}_{0}\right)+h(\mathcal{B}(\{n\}))=h_{0}\left(\mathcal{B}_{0}\right)+h_{1}\left(\mathcal{B}_{1}\right) . \tag{7}
\end{equation*}
$$

Along the same lines,

$$
\begin{align*}
h(\mathcal{H}(T)) & =h\left(\mathcal{H}_{0}(T)\right)+h(\{H \in \mathcal{H}: T \cup\{n\} \subseteq H\}) \\
& =h_{0}\left(\mathcal{H}_{0}(T)\right)+h\left(\left\{H \cup\{n\}: H \in \mathcal{H}_{1}(T)\right\}\right) \\
& =h_{0}\left(\mathcal{H}_{0}(T)\right)+h_{1}\left(\mathcal{H}_{1}(T)\right) . \tag{8}
\end{align*}
$$

Suppose $\mathcal{G}_{1}=\emptyset$. Clearly, $\mathcal{A}$ and $\mathcal{B}_{0}$ are cross- $t$-intersecting. Since $\mathcal{G}_{1}=\emptyset$, no set in $\mathcal{A}$ contains $n$, and hence $\mathcal{A}$ and $\mathcal{B}_{1}$ are cross- $t$-intersecting. Thus, by the induction hypothesis,

$$
\begin{equation*}
g(\mathcal{A}) h_{j}\left(\mathcal{B}_{j}\right) \leq g(\mathcal{G}(T)) h_{j}\left(\mathcal{H}_{j}(T)\right) \quad \text { for each } j \in\{0,1\} \tag{9}
\end{equation*}
$$

Together with (7) and (8), this gives us

$$
\begin{aligned}
g(\mathcal{A}) h(\mathcal{B}) & =g(\mathcal{A}) h_{0}\left(\mathcal{B}_{0}\right)+g(\mathcal{A}) h_{1}\left(\mathcal{B}_{1}\right) \\
& \leq g(\mathcal{G}(T)) h_{0}\left(\mathcal{H}_{0}(T)\right)+g(\mathcal{G}(T)) h_{1}\left(\mathcal{H}_{1}(T)\right) \\
& =g(\mathcal{G}(T)) h(\mathcal{H}(T))
\end{aligned}
$$

By (11), equality holds throughout, and hence $g(\mathcal{A}) h(\mathcal{B})=g(\mathcal{G}(T)) h(\mathcal{H}(T))$. Thus, in (9), we actually have equality. Suppose $u>0$. By the induction hypothesis, for each $j \in\{0,1\}$, we have $\mathcal{A}=\mathcal{G}\left(V_{j}\right)$ and $\mathcal{B}_{j}=\mathcal{H}_{j}\left(V_{j}\right)$ for some $V_{j} \in\binom{[n-1]}{t}$ such that $g\left(\mathcal{G}\left(V_{j}\right)\right)=g(\mathcal{G}(T))$ and $h_{j}\left(\mathcal{H}_{j}\left(V_{j}\right)\right)=h_{j}\left(\mathcal{H}_{j}(T)\right)$. Thus $g\left(\mathcal{G}\left(V_{0}\right)\right)>0$, and hence $\mathcal{G}\left(V_{0}\right) \neq \emptyset$. Thus, since $\mathcal{G}$ is hereditary and $\mathcal{A}=\mathcal{G}\left(V_{0}\right), V_{0} \in \mathcal{A}$. Since $\mathcal{A}$ and $\mathcal{B}$ are cross- $t$-intersecting, $\mathcal{B} \subseteq \mathcal{H}\left(V_{0}\right)$. Since $\mathcal{A}=\mathcal{G}\left(V_{0}\right)$, and since $\mathcal{G}\left(V_{0}\right)$ and $\mathcal{H}\left(V_{0}\right)$ are cross- $t$-intersecting, it follows by the choice of $\mathcal{A}$ and $\mathcal{B}$ that $\mathcal{B}=\mathcal{H}\left(V_{0}\right)$. By (1), $\mathcal{H}\left(V_{0}\right) \neq \emptyset$. Since $\mathcal{H}$ is hereditary, $V_{0} \in \mathcal{B}$. By Lemma 4.2, the result follows.

Now suppose that $\mathcal{G}_{1}$ is non-empty. If $\mathcal{H}_{1}=\emptyset$, then the result follows by an argument similar to that for the case $\mathcal{G}_{1}=\emptyset$ above. Thus we assume that $\mathcal{H}_{1}$ is non-empty. Since $\mathcal{G}_{1} \subseteq \mathcal{G}_{0}$ and $\mathcal{H}_{1} \subseteq \mathcal{H}_{0}, \mathcal{G}_{0}$ and $\mathcal{H}_{0}$ are non-empty too.

Similarly to $h_{0}$ and $h_{1}$, let $g_{0}: \mathcal{G}_{0} \rightarrow \mathbb{R}^{+}$such that $g_{0}(G)=g(G)$ for each $G \in \mathcal{G}_{0}$, and let $g_{1}: \mathcal{G}_{1} \rightarrow \mathbb{R}^{+}$such that $g_{1}(G)=g(G \cup\{n\})$ for each $G \in \mathcal{G}_{1}$ (note that $G \cup\{n\} \in \mathcal{G}$ by definition of $\mathcal{G}_{1}$ ). Then properties (a) and (b) are inherited by $g_{0}$ and $g_{1}$ in the same way they are inherited by $h_{0}$ and $h_{1}$, as shown above; that is, similarly to (3)-(6), we have the following. For any $A, B \in \mathcal{G}_{0}$ with $A \subsetneq B$ and $|A| \geq t$,

$$
\begin{equation*}
g_{0}(A) \geq(t+u) g_{0}(B) \tag{10}
\end{equation*}
$$

For any $C \in \mathcal{G}_{0}$ and any $i, j \in[n-1]$ with $i<j$,

$$
\begin{equation*}
g_{0}\left(\delta_{i, j}(C)\right) \geq g_{0}(C) \tag{11}
\end{equation*}
$$

For any $A, B \in \mathcal{G}_{1}$ with $A \subsetneq B$ and $|A| \geq t$,

$$
\begin{equation*}
g_{1}(A) \geq(t+u) g_{1}(B) \tag{12}
\end{equation*}
$$

For any $C \in \mathcal{G}_{1}$ and any $i, j \in[n-1]$ with $i<j$,

$$
\begin{equation*}
g_{1}\left(\delta_{i, j}(C)\right) \geq g_{1}(C) \tag{13}
\end{equation*}
$$

Similarly to (7) and (8), we have

$$
\begin{gather*}
g(\mathcal{A})=g_{0}\left(\mathcal{A}_{0}\right)+g_{1}\left(\mathcal{A}_{1}\right)  \tag{14}\\
g(\mathcal{G}(T))=g_{0}\left(\mathcal{G}_{0}(T)\right)+g_{1}\left(\mathcal{G}_{1}(T)\right) \tag{15}
\end{gather*}
$$

Clearly, $\mathcal{A}_{0}$ and $\mathcal{B}_{0}$ are cross- $t$-intersecting, $\mathcal{A}_{0}$ and $\mathcal{B}_{1}$ are cross- $t$-intersecting, and $\mathcal{A}_{1}$ and $\mathcal{B}_{0}$ are cross- $t$-intersecting.

Let us first assume that $\mathcal{A}_{1}$ and $\mathcal{B}_{1}$ are cross- $t$-intersecting too. Then, by the induction hypothesis,

$$
\begin{equation*}
g_{i}\left(\mathcal{A}_{i}\right) h_{j}\left(\mathcal{B}_{j}\right) \leq g_{i}\left(\mathcal{G}_{i}(T)\right) h_{j}\left(\mathcal{H}_{j}(T)\right) \quad \text { for any } i, j \in\{0,1\} \tag{16}
\end{equation*}
$$

Together with (7), (8), (14), and (15), this gives us

$$
\begin{aligned}
g(\mathcal{A}) h(\mathcal{B})= & g_{0}\left(\mathcal{A}_{0}\right) h_{0}\left(\mathcal{B}_{0}\right)+g_{0}\left(\mathcal{A}_{0}\right) h_{1}\left(\mathcal{B}_{1}\right)+g_{1}\left(\mathcal{A}_{1}\right) h_{0}\left(\mathcal{B}_{0}\right)+g_{1}\left(\mathcal{A}_{1}\right) h_{1}\left(\mathcal{B}_{1}\right) \\
\leq & g_{0}\left(\mathcal{G}_{0}(T)\right) h_{0}\left(\mathcal{H}_{0}(T)\right)+g_{0}\left(\mathcal{G}_{0}(T)\right) h_{1}\left(\mathcal{H}_{1}(T)\right)+ \\
& g_{1}\left(\mathcal{G}_{1}(T)\right) h_{0}\left(\mathcal{H}_{0}(T)\right)+g_{1}\left(\mathcal{G}_{1}(T)\right) h_{1}\left(\mathcal{H}_{1}(T)\right) \\
= & g(\mathcal{G}(T)) h(\mathcal{H}(T)) .
\end{aligned}
$$

By (1), equality holds throughout, and hence $g(\mathcal{A}) h(\mathcal{B})=g(\mathcal{G}(T)) h(\mathcal{H}(T))$. Thus, in (16), we actually have equality. Suppose $u>0$. By the induction hypothesis, we particularly have $\mathcal{A}_{0}=\mathcal{G}_{0}\left(V_{0}\right)$ and $\mathcal{B}_{0}=\mathcal{H}_{0}\left(V_{0}\right)$ for some $V_{0} \in\binom{[n-1]}{t}$ such that $g_{0}\left(\mathcal{G}_{0}\left(V_{0}\right)\right)=g_{0}\left(\mathcal{G}_{0}(T)\right)$ and $h_{0}\left(\mathcal{H}_{0}\left(V_{0}\right)\right)=h_{0}\left(\mathcal{H}_{0}(T)\right)$. Recall that $T \in \mathcal{G}$, so $T \in \mathcal{G}_{0}$, and hence $g_{0}\left(\mathcal{G}_{0}(T)\right)>0$. Thus $g_{0}\left(\mathcal{G}_{0}\left(V_{0}\right)\right)>0$, and hence $\mathcal{G}_{0}\left(V_{0}\right) \neq \emptyset$. Since $\mathcal{G}_{0}$ is hereditary, it follows that $V_{0} \in \mathcal{G}_{0}\left(V_{0}\right)$, and hence $V_{0} \in \mathcal{A}$. Similarly, $V_{0} \in \mathcal{B}$. By Lemma 4.2, the result follows.

We will now show that $\mathcal{A}_{1}$ and $\mathcal{B}_{1}$ are indeed cross- $t$-intersecting. Note that $\mathcal{A}_{1}$ and $\mathcal{B}_{1}$ are cross- $(t-1)$-intersecting.

Suppose that $\mathcal{A}_{1}$ and $\mathcal{B}_{1}$ are not cross- $t$-intersecting. Then there exists $A^{*} \in \mathcal{A}_{1}$ such that $\left|A^{*} \cap B^{*}\right|=t-1$ for some $B^{*} \in \mathcal{B}_{1}$. Let $r=\left|A^{*}\right|+1$ and $s=n-r+t$. Let

$$
\begin{aligned}
\mathcal{R} & =\left\{A \in \mathcal{A}_{1}:|A|=r-1,|A \cap B|=t-1 \text { for some } B \in \mathcal{B}_{1}\right\} \\
\mathcal{S} & =\left\{B \in \mathcal{B}_{1}:|B|=s-1,|A \cap B|=t-1 \text { for some } A \in \mathcal{A}_{1}\right\}
\end{aligned}
$$

We have $A^{*} \in \mathcal{R}$.

Consider any $R \in \mathcal{R}$ and $B \in \mathcal{B}_{1}$ such that $|R \cap B|<t$. Since $\mathcal{A}_{1}$ and $\mathcal{B}_{1}$ are cross- $(t-1)$-intersecting, $|R \cap B|=t-1$. We have

$$
\begin{aligned}
|B| & =|B \cap R|+|B \backslash R|=t-1+|B \backslash R| \\
& \leq t-1+|[n-1] \backslash R|=t-1+(n-1)-(r-1)=s-1 .
\end{aligned}
$$

Suppose $B \notin \mathcal{S}$. Then $|B|<s-1$. Thus we have

$$
|R \cup B|=|R|+|B|-|R \cap B| \leq r-1+s-2-t+1=n-2
$$

and hence $R \cup B \subsetneq[n-1]$. Let $c \in[n-1] \backslash(R \cup B)$. Since $B \in \mathcal{B}_{1}, B \cup\{n\} \in \mathcal{B}$. Let $C=\delta_{c, n}(B \cup\{n\})$. Since $c \notin B \cup\{n\}, C=B \cup\{c\}$. Since $\mathcal{B}$ is compressed, $C \in \mathcal{B}$. However, since $c \notin R \cup\{n\}$ and $|R \cap B|=t-1$, we have $|(R \cup\{n\}) \cap C|=t-1$, which is a contradiction as $\mathcal{A}$ and $\mathcal{B}$ are cross- $t$-intersecting, $R \cup\{n\} \in \mathcal{A}$, and $C \in \mathcal{B}$.

We have therefore shown that

$$
\begin{equation*}
\text { for each } B \in \mathcal{B}_{1} \text { such that }|R \cap B|<t \text { for some } R \in \mathcal{R}, B \in \mathcal{S} \text {. } \tag{17}
\end{equation*}
$$

By a similar argument,

$$
\begin{equation*}
\text { for each } A \in \mathcal{A}_{1} \text { such that }|A \cap S|<t \text { for some } S \in \mathcal{S}, A \in \mathcal{R} \text {. } \tag{18}
\end{equation*}
$$

For each $A \in \mathcal{A}_{1} \cup \mathcal{B}_{1}$, let $A^{\prime}=A \cup\{n\}$. Let $\mathcal{R}^{\prime}=\left\{R^{\prime}: R \in \mathcal{R}\right\}$ and $\mathcal{S}^{\prime}=\left\{S^{\prime}: S \in\right.$ $\mathcal{S}\}$. Since $\mathcal{R} \subseteq \mathcal{A}_{1}$ and $\mathcal{S} \subseteq \mathcal{B}_{1}, \mathcal{R}^{\prime} \subseteq \mathcal{A}(\{n\})$ and $\mathcal{S}^{\prime} \subseteq \mathcal{B}(\{n\})$. Let

$$
\mathcal{A}^{\prime}=\mathcal{A} \cup \mathcal{R}, \quad \mathcal{A}^{\prime \prime}=\mathcal{A} \backslash \mathcal{R}^{\prime}, \quad \mathcal{B}^{\prime}=\mathcal{B} \backslash \mathcal{S}^{\prime}, \quad \mathcal{B}^{\prime \prime}=\mathcal{B} \cup \mathcal{S}
$$

By (17), $\mathcal{A}^{\prime}$ and $\mathcal{B}^{\prime}$ are cross- $t$-intersecting. By (18), $\mathcal{A}^{\prime \prime}$ and $\mathcal{B}^{\prime \prime}$ are cross- $t$-intersecting. Since $\mathcal{G}$ and $\mathcal{H}$ are hereditary, and since $\mathcal{R}^{\prime} \subseteq \mathcal{A} \subseteq \mathcal{G}$ and $\mathcal{S}^{\prime} \subseteq \mathcal{B} \subseteq \mathcal{H}$, we have $\mathcal{R} \subseteq \mathcal{G}$ and $\mathcal{S} \subseteq \mathcal{H}$, and hence $\mathcal{A}^{\prime}, \mathcal{A}^{\prime \prime} \subseteq \mathcal{G}$ and $\mathcal{B}^{\prime}, \mathcal{B}^{\prime \prime} \subseteq \mathcal{H}$.

Let $x=g(\mathcal{A}), x_{1}=g\left(\mathcal{R}^{\prime}\right), y=h(\mathcal{B})$, and $y_{1}=h\left(\mathcal{S}^{\prime}\right)$. We use a double-counting argument to obtain $x \geq n x_{1} / r$ and $y \geq n y_{1} / s$. For any $R \in \mathcal{R}^{\prime}$ and any set $A$ such that $A=\delta_{i, n}(R)$ for some $i \in[n] \backslash R$, we write $A<R$. If $A<R \in \mathcal{R}^{\prime}$, then, since $\mathcal{A}$ is compressed and $n \in R \in \mathcal{A}$, we have $A \in \mathcal{A}_{0}$. For any $A \in \mathcal{A}_{0}$ and any $R \in \mathcal{R}^{\prime}$, let

$$
\chi(A, R)= \begin{cases}1 & \text { if } A<R \\ 0 & \text { otherwise }\end{cases}
$$

Then $\sum_{A \in \mathcal{A}_{0}} \chi(A, R)=n-r$ for each $R \in \mathcal{R}^{\prime}$. For each $A \in \mathcal{A}_{0}, \chi(A, R)=1$ only if $|A|=|R|$ and $R=(A \backslash\{i\}) \cup\{n\}$ for some $i \in A$. Thus $\sum_{R \in \mathcal{R}^{\prime}} \chi(A, R) \leq r$ for each $A \in \mathcal{A}_{0}$. We have

$$
\begin{aligned}
(n-r) x_{1} & =\sum_{R \in \mathcal{R}^{\prime}}(n-r) g(R)=\sum_{R \in \mathcal{R}^{\prime}} \sum_{A \in \mathcal{A}_{0}} \chi(A, R) g(R)=\sum_{A \in \mathcal{A}_{0}} \sum_{R \in \mathcal{R}^{\prime}} \chi(A, R) g(R) \\
& \leq \sum_{A \in \mathcal{A}_{0}} \sum_{R \in \mathcal{R}^{\prime}} \chi(A, R) g(A) \quad(\text { by }(\mathrm{b})) \\
& \leq \sum_{A \in \mathcal{A}_{0}} r g(A)=r g\left(\mathcal{A}_{0}\right)=r(x-g(\mathcal{A}(\{n\}))) \leq r\left(x-x_{1}\right),
\end{aligned}
$$

so $x \geq n x_{1} / r$. Similarly, $y \geq n y_{1} / s$.
Since $t-1=\left|A^{*} \cap B^{*}\right| \leq\left|A^{*}\right|=r-1, r \geq t$. By (17), $B^{*} \in \mathcal{S}$. Since $t-1=$ $\left|A^{*} \cap B^{*}\right| \leq\left|B^{*}\right|=s-1, s \geq t$.

Suppose $r=t$. Then $s=n$. Thus $\mathcal{S}^{\prime}=\{[n]\}$ and $\mathcal{S}=\{[n-1]\}$. Let $C^{*}=$ $[t-1] \cup\{n\}$. Since $A^{*} \in \mathcal{A}_{1}, \mathcal{A}_{1}$ is compressed, and $\left|A^{*}\right|=r-1=t-1$, we have $[t-1] \in \mathcal{A}_{1}$, and hence $C^{*} \in \mathcal{A}$.

Suppose that there exists $D^{*} \in \mathcal{B}$ such that $D^{*} \neq[n]$. Since $\mathcal{A}$ and $\mathcal{B}$ are cross- $t$ intersecting, we have $\left|C^{*} \cap D^{*}\right| \geq t$. Thus $C^{*} \subseteq D^{*}$ as $\left|C^{*}\right|=t$. Since $D^{*} \neq[n]$, there exists $c \in[n]$ such that $c \notin D^{*}$. Thus $c \notin C^{*}$. Since $\mathcal{A}$ is compressed, $\delta_{c, n}\left(C^{*}\right) \in \mathcal{A}$. However, $\left|\delta_{c, n}\left(C^{*}\right) \cap D^{*}\right|=\left|C^{*} \backslash\{n\}\right|=t-1$, which is a contradiction as $\mathcal{A}$ and $\mathcal{B}$ are cross- $t$-intersecting.

Therefore, $\mathcal{B}=\{[n]\}$. Since $n-1>t, h([n-1]) \geq(t+u) h([n]) \geq t h([n])$. We have

$$
h\left(\mathcal{B}^{\prime \prime}\right)=h([n])+h([n-1]) \geq h([n])+\operatorname{th}([n])=(t+1) h(\mathcal{B})=(t+1) y
$$

Since $x \geq n x_{1} / r=n x_{1} / t \geq(t+3) x_{1} / t, x_{1} \leq t x /(t+3)$. We have

$$
g\left(\mathcal{A}^{\prime \prime}\right)=x-x_{1} \geq x-\frac{t x}{t+3}=\frac{3 x}{t+3} .
$$

Thus we obtain

$$
g\left(\mathcal{A}^{\prime \prime}\right) h\left(\mathcal{B}^{\prime \prime}\right) \geq \frac{3(t+1) x y}{t+3}>x y=g(\mathcal{A}) h(\mathcal{B})
$$

contradicting the choice of $\mathcal{A}$ and $\mathcal{B}$.
Therefore, $r \geq t+1$. Similarly, $s \geq t+1$. Since $r-1 \geq t$ and each set in $\mathcal{R}$ is of size $r-1, g(\mathcal{R}) \geq(t+u) g\left(\mathcal{R}^{\prime}\right)$. Similarly, $h(\mathcal{S}) \geq(t+u) g\left(\mathcal{S}^{\prime}\right)$.

Consider any $R \in \mathcal{R}$. By definition of $\mathcal{R}$, there exists $B_{R} \in \mathcal{B}_{1}$ such that $\left|R \cap B_{R}\right|=$ $t-1$. Thus $\left|R \cap B_{R}{ }^{\prime}\right|=t-1$. Since $B_{R}{ }^{\prime} \in \mathcal{B}$, and since $\mathcal{A}$ and $\mathcal{B}$ are cross- $t$-intersecting, $R \notin \mathcal{A}$. Therefore, $\mathcal{A} \cap \mathcal{R}=\emptyset$. Similarly, $\mathcal{B} \cap \mathcal{S}=\emptyset$.

We have

$$
\begin{aligned}
& g\left(\mathcal{A}^{\prime}\right)=x+g(\mathcal{R}) \geq x+(t+u) g\left(\mathcal{R}^{\prime}\right)=x+(t+u) x_{1}, \\
& g\left(\mathcal{A}^{\prime \prime}\right)=x-x_{1} \\
& h\left(\mathcal{B}^{\prime}\right)=y-y_{1}, \\
& h\left(\mathcal{B}^{\prime \prime}\right)=y+h(\mathcal{S}) \geq y+(t+u) h\left(\mathcal{S}^{\prime}\right)=y+(t+u) y_{1} .
\end{aligned}
$$

By the choice of $\mathcal{A}$ and $\mathcal{B}$,

$$
g\left(\mathcal{A}^{\prime}\right) h\left(\mathcal{B}^{\prime}\right) \leq g(\mathcal{A}) h(\mathcal{B}) \quad \text { and } \quad g\left(\mathcal{A}^{\prime \prime}\right) h\left(\mathcal{B}^{\prime \prime}\right) \leq g(\mathcal{A}) h(\mathcal{B})
$$

Thus we have

$$
\begin{aligned}
& \left(x+(t+u) x_{1}\right)\left(y-y_{1}\right) \leq x y \quad \text { and } \quad\left(x-x_{1}\right)\left(y+(t+u) y_{1}\right) \leq x y \\
& \Rightarrow(t+u) x_{1} y \leq x y_{1}+(t+u) x_{1} y_{1} \quad \text { and } \quad(t+u) x y_{1} \leq x_{1} y+(t+u) x_{1} y_{1} \\
& \Rightarrow(t+u) x_{1} y+(t+u) x y_{1} \leq\left(x y_{1}+(t+u) x_{1} y_{1}\right)+\left(x_{1} y+(t+u) x_{1} y_{1}\right) \\
& \Rightarrow(t+u-1)\left(x_{1} y+x y_{1}\right) \leq 2(t+u) x_{1} y_{1} . \\
& \Rightarrow(t+u-1)\left(x_{1} \frac{n y_{1}}{s}+\frac{n x_{1}}{r} y_{1}\right) \leq 2(t+u) x_{1} y_{1} \\
& \Rightarrow(t+u-1)(r+s) x_{1} y_{1} n \leq 2(t+u) r s x_{1} y_{1} \\
& \Rightarrow(t+u-1)(n+t) n \leq 2(t+u) r(n-r+t) .
\end{aligned}
$$

Using differentiation, we find that the maximum value of the function $f(z)=z(n-z+t)$ occurs at $z=\frac{n+t}{2}$. Thus $r(n-r+t) \leq \frac{n+t}{2}\left(n-\frac{n+t}{2}+t\right)=(n+t)^{2} / 4$, and hence

$$
\begin{align*}
& (t+u-1)(n+t) n \leq 2(t+u)(n+t)^{2} / 4 \\
& \Rightarrow 2(t+u-1) n \leq(t+u)(n+t) \\
& \Rightarrow n \leq \frac{(t+u) t}{t+u-2} \tag{19}
\end{align*}
$$

Since $u>\frac{6-t}{3}, \frac{(t+u) t}{t+u-2}<t+3$. Thus we have $n<t+3$, which is a contradiction. Hence the result.

Remark 4.3 Note that the proof for the special case $n \leq t+2$ actually verifies the conjecture in Remark 1.4 for $n \leq t+2$. Also note that for $t \geq 3$, if we also settle the conjecture for $t+3 \leq n \leq t+6$, then we can take $u=0$ and proceed for $n \geq t+7$ in exactly the same way we did for $n \geq t+3$, because again we obtain a contradiction to (19); thus, as mentioned in Remark [1.4, this would settle the conjecture for $t \geq 3$.

## 5 Proof of Theorem 1.2

In this section, we use Theorem 1.3 to prove Theorem 1.2 ,
For a family $\mathcal{F}$ and an integer $r \geq 0$, we denote the families $\{F \in \mathcal{F}:|F|=r\}$ and $\{F \in \mathcal{F}:|F| \leq r\}$ by $\mathcal{F}^{(r)}$ and $\mathcal{F}^{(\leq r)}$, respectively.

We will need the following lemma only when dealing with the characterisation of the extremal structures in the proof of Theorem 1.2.

Lemma 5.1 Let $t, r, s, u, m$, and $n$ be as in Theorem 1.2. Let $i, j \in[\max \{m, n\}]$ with $i<j$. Let $\mathcal{G}=2^{[m]}$ and $\mathcal{H}=2^{[n]}$. Let $\mathcal{A} \subseteq \mathcal{G}^{(r)}$ and $\mathcal{B} \subseteq \mathcal{H}^{(s)}$ such that $\mathcal{A}$ and $\mathcal{B}$ are cross-t-intersecting. Suppose that $\Delta_{i, j}(\mathcal{A})=\mathcal{G}^{(r)}(T)$ and $\Delta_{i, j}(\mathcal{B})=\mathcal{H}^{(s)}(T)$ for some t-element subset $T$ of $[\min \{m, n\}]$. Then $\mathcal{A}=\mathcal{G}^{(r)}\left(T^{\prime}\right)$ and $\mathcal{B}=\mathcal{H}^{(s)}\left(T^{\prime}\right)$ for some $t$-element subset $T^{\prime}$ of $[\min \{m, n\}]$.

We prove the above lemma using the following special case of [11, Lemma 5.6].

Lemma 5.2 Let $t \geq 1, r \geq t+1, n \geq 2 r-t+2$, and $i, j \in[n]$. Let $\mathcal{H}=2^{[n]}$, and let $\mathcal{A}$ be a $t$-intersecting subfamily of $\mathcal{H}^{(r)}$. If $\Delta_{i, j}(\mathcal{A})$ is a largest $t$-star of $\mathcal{H}^{(r)}$, then $\mathcal{A}$ is a largest $t$-star of $\mathcal{H}^{(r)}$.

Proof of Lemma 5.1. We are given that $t \leq r \leq s$.
Suppose $r=t$. Then $\Delta_{i, j}(\mathcal{A})=\{T\}$. Thus $\mathcal{A}=\left\{T^{\prime}\right\}=\mathcal{H}^{(r)}\left(T^{\prime}\right)$ for some $T^{\prime} \in$ $\binom{[m]}{t}$. Since $\mathcal{A}$ and $\mathcal{B}$ are cross- $t$-intersecting, $T^{\prime} \subseteq B$ for all $B \in \mathcal{B}$. Thus $\mathcal{B} \subseteq \mathcal{H}^{(s)}\left(T^{\prime}\right)$. Since $\binom{n-t}{s-t}=\left|\mathcal{H}^{(s)}(T)\right|=\left|\Delta_{i, j}(\mathcal{B})\right|=|\mathcal{B}| \leq\left|\mathcal{H}^{(s)}\left(T^{\prime}\right)\right|=\binom{n-t}{s-t},|\mathcal{B}|=\binom{n-t}{s-t}$. Hence $\mathcal{B}=\mathcal{H}^{(s)}\left(T^{\prime}\right)$.

Now suppose $r \geq t+1$. Note that $T \backslash\{i\} \subseteq E$ for all $E \in \mathcal{A} \cup \mathcal{B}$.
Suppose that $\mathcal{A}$ is not $t$-intersecting. Then there exist $A_{1}, A_{2} \in \mathcal{A}$ such that $\mid A_{1} \cap$ $A_{2} \mid \leq t-1$, and hence $T \nsubseteq A_{l}$ for some $l \in\{1,2\}$; we may assume that $l=1$. Thus, since $\Delta_{i, j}(\mathcal{A})=\mathcal{H}^{(r)}(T)$, we have $A_{1} \notin \Delta_{i, j}(\mathcal{A}), A_{1} \neq \delta_{i, j}\left(A_{1}\right) \in \Delta_{i, j}(\mathcal{A}), \delta_{i, j}\left(A_{1}\right) \notin \mathcal{A}$ (because otherwise $\left.A_{1} \in \Delta_{i, j}(\mathcal{A})\right), i \in T, j \notin T, j \in A_{1}$, and $A_{1} \cap T=T \backslash\{i\}$. Since $T \backslash\{i\} \subseteq A_{1} \cap A_{2}$ and $\left|A_{1} \cap A_{2}\right| \leq t-1, A_{1} \cap A_{2}=T \backslash\{i\}$. Thus $j \notin A_{2}$, and hence $A_{2}=\delta_{i, j}\left(A_{2}\right)$. Since $\delta_{i, j}\left(A_{2}\right) \in \Delta_{i, j}(\mathcal{A})=\mathcal{H}^{(r)}(T), T \subseteq A_{2}$. Let $X=[n] \backslash\left(A_{1} \cup A_{2}\right)$. We have

$$
\begin{aligned}
|X| & =n-\left|A_{1} \cup A_{2}\right|=n-\left(\left|A_{1}\right|+\left|A_{2}\right|-\left|A_{1} \cap A_{2}\right|\right)=n-2 r+t-1 \\
& \geq(t+u+2)(s-t)+r-1-2(r-t)-(t+1) \geq(t+u)(s-t)-1 \\
& >\left(2+\frac{2 t}{3}\right)(s-t)-1=\left(1+\frac{2 t}{3}\right)(s-t)+s-(t+1) .
\end{aligned}
$$

Since $t+1 \leq r \leq s$, we have $|X|>s-t$, and hence $\binom{X}{s-t} \neq \emptyset$. Let $C \in\binom{X}{s-t}$ and $D=C \cup T$. Then $D \in \mathcal{H}^{(s)}(T)$ and $D \cap A_{1}=T \backslash\{i\}$, meaning that $D \in \Delta_{i, j}(\mathcal{B})$ and $\left|D \cap A_{1}\right|=t-1$. Since $\mathcal{A}$ and $\mathcal{B}$ are cross-t-intersecting, we obtain $D \notin \mathcal{B}$ and $(D \backslash\{i\}) \cup\{j\} \in \mathcal{B}$, which is a contradiction since $\left|((D \backslash\{i\}) \cup\{j\}) \cap A_{2}\right|=|T \backslash\{i\}|=t-1$.

Therefore, $\mathcal{A}$ is $t$-intersecting. Similarly, $\mathcal{B}$ is $t$-intersecting. Now $\mathcal{H}^{(r)}(T)$ is a largest $t$-star of $\mathcal{H}^{(r)}$, and $\mathcal{H}^{(s)}(T)$ is a largest $t$-star of $\mathcal{H}^{(s)}$. Since $t+1 \leq r \leq s$ and
$\max \{m, n\} \geq(t+u+2)(s-t)+r-1=(t+u)(s-t)+(2 s-t+2)+r-(t+1)-2$,
we have

$$
\max \{m, n\}-(2 s-t+2)>\left(2+\frac{2 t}{3}\right)(s-t)-2 \geq \frac{2 t}{3}(s-t)>0
$$

and hence $\max \{m, n\}>2 s-t+2$. By Lemma 5.2, $\mathcal{A}=\mathcal{H}^{(r)}\left(T^{\prime}\right)$ for some $T^{\prime} \in\binom{[m]}{t}$, and $\mathcal{B}=\mathcal{H}^{(s)}\left(T^{*}\right)$ for some $T^{*} \in\binom{[n]}{t}$.

Suppose $T^{\prime} \neq T^{*}$. Let $z \in T^{*} \backslash T^{\prime}$. Since $m>2 s-t+2>r$, we can choose $A^{\prime} \in \mathcal{H}^{(r)}\left(T^{\prime}\right)$ such that $z \notin A^{\prime}$. Since $n>2 s-t+2 \geq r+s-t+2>r+s-t$ and $z \in T^{*} \backslash A^{\prime}$, we can choose $B^{*} \in \mathcal{H}^{(s)}\left(T^{*}\right)$ such that $\left|A^{\prime} \cap B^{*}\right| \leq t-1$; however, this is a contradiction since $\mathcal{A}=\mathcal{H}^{(r)}\left(T^{\prime}\right), \mathcal{B}=\mathcal{H}^{(s)}\left(T^{*}\right)$, and $\mathcal{A}$ and $\mathcal{B}$ are cross- $t$-intersecting. Therefore, $T^{\prime}=T^{*}$.

Proof of Theorem 1.2. If $\mathcal{A}=\emptyset$ or $\mathcal{B}=\emptyset$, then $|\mathcal{A}||\mathcal{B}|=0$. Thus we assume that $\mathcal{A} \neq \emptyset$ and $\mathcal{B} \neq \emptyset$. Let $l=\max \{m, n\}$, so $\mathcal{A}, \mathcal{B} \subseteq 2^{[l]}$.

As explained in Section 3, we apply left-compressions to $\mathcal{A}$ and $\mathcal{B}$ simultaneously until we obtain two compressed cross- $t$-intersecting families $\mathcal{A}^{*}$ and $\mathcal{B}^{*}$, respectively. We have $\mathcal{A}^{*} \subseteq\binom{[m]}{r}, \mathcal{B}^{*} \subseteq\binom{[n]}{s},\left|\mathcal{A}^{*}\right|=|\mathcal{A}|$, and $\left|\mathcal{B}^{*}\right|=|\mathcal{B}|$. In view of Lemma 5.1] we may therefore assume that $\mathcal{A}$ and $\mathcal{B}$ are compressed. Thus, by Lemma 3.1(ii),

$$
\begin{equation*}
|A \cap B \cap[r+s-t]| \geq t \text { for any } A \in \mathcal{A} \text { and any } B \in \mathcal{B} . \tag{20}
\end{equation*}
$$

Let $p=r+s-t$. Let $\mathcal{G}=\binom{[p]}{\leq r}$ and $\mathcal{H}=\binom{[p]}{\leq s}$. Let $g: \mathcal{G} \rightarrow \mathbb{N}$ such that $g(G)=\binom{m-p}{r-|G|}$ for each $G \in \mathcal{G}$. Let $h: \mathcal{H} \rightarrow \mathbb{N}$ such that $h(H)=\binom{n-p}{s-|H|}$ for each $H \in \mathcal{H}$.

For every $F, G \in \mathcal{G}$ with $F \subsetneq G$ and $t \leq|F|=|G|-1$, we have

$$
\begin{aligned}
\frac{g(F)-(t+u) g(G)}{\binom{m-p}{r-|F|}} & =1-\frac{(t+u)\binom{m-p}{r-|F|-1}}{\binom{m-p}{r-|F|}}=1-\frac{(t+u)(r-|F|)}{m-p-(r-|F|)+1} \\
& =\frac{m-p-(t+u+1)(r-|F|)+1}{m-p-(r-|F|)+1} \\
& \geq \frac{m-p-(t+u+1)(r-t)+1}{m-p-(r-|F|)+1} \\
& =\frac{m-(t+u+2)(r-t)-s+1}{m-p-(r-|F|)+1} \\
& \geq \frac{(t+u+2)(s-t)+r-1-((t+u+2)(r-t)+s-1)}{m-p-(r-|F|)+1} \geq 0
\end{aligned}
$$

and hence $g(F) \geq(t+u) g(G)$. It follows that $g(F) \geq(t+u) g(G)$ for every $F, G \in \mathcal{G}$ with $F \subsetneq G$ and $|F| \geq t$. Similarly, $h(F) \geq(t+u) g(H)$ for every $F, H \in \mathcal{H}$ with $F \subsetneq H$ and $|F| \geq t$.

We have $g\left(\delta_{i, j}(G)\right)=g(G)$ for every $G \in \mathcal{G}$ and every $i, j \in[p]$. Similarly, $h\left(\delta_{i, j}(H)\right)=h(H)$ for every $H \in \mathcal{H}$ and every $i, j \in[p]$.

Let $\mathcal{C}=\{A \cap[p]: A \in \mathcal{A}\}$ and $\mathcal{D}=\{B \cap[p]: B \in \mathcal{B}\}$. Then $\mathcal{C} \subseteq \mathcal{G}, \mathcal{D} \subseteq \mathcal{H}$, and, by (20), $\mathcal{C}$ and $\mathcal{D}$ are cross- $t$-intersecting. Let $T=[t]$. By Theorem [1.3,

$$
\begin{equation*}
g(\mathcal{C}) h(\mathcal{D}) \leq g(\mathcal{G}(T)) h(\mathcal{H}(T)) \tag{21}
\end{equation*}
$$

and if $u>0$, then equality holds only if $\mathcal{C}=\mathcal{G}\left(T^{\prime}\right)$ and $\mathcal{D}=\mathcal{H}\left(T^{\prime}\right)$ for some $T^{\prime} \in\binom{[p]}{t}$.
We have

$$
\begin{align*}
& |\mathcal{A}|=\left|\bigcup_{i=0}^{r}\left\{A \in \mathcal{A}: A \cap[p] \in \mathcal{C}^{(i)}\right\}\right| \leq \sum_{i=0}^{r}\left|\mathcal{C}^{(i)}\right|\binom{m-p}{r-i}=g(\mathcal{C})  \tag{22}\\
& |\mathcal{B}|=\left|\bigcup_{j=0}^{s}\left\{B \in \mathcal{B}: B \cap[p] \in \mathcal{D}^{(j)}\right\}\right| \leq \sum_{j=0}^{s}\left|\mathcal{D}^{(j)}\right|\binom{n-p}{s-j}=h(\mathcal{D}), \tag{23}
\end{align*}
$$

and hence, by (21),

$$
\begin{equation*}
|\mathcal{A}||\mathcal{B}| \leq g(\mathcal{G}(T)) h(\mathcal{H}(T)) \tag{24}
\end{equation*}
$$

Now

$$
\begin{aligned}
g(\mathcal{G}(T)) & =\sum_{i=t}^{r}\left|\mathcal{G}^{(i)}(T)\right|\binom{m-p}{r-i}=\left|\bigcup_{i=t}^{r}\left\{A \in\binom{[m]}{r}: T \subseteq A,|A \cap[p]|=i\right\}\right| \\
& =\left|\left\{A \in\binom{[m]}{r}: T \subseteq A\right\}\right|=\binom{m-t}{r-t}
\end{aligned}
$$

and, similarly, $h(\mathcal{H}(T))=\binom{n-t}{s-t}$. Together with (24), this gives us

$$
|\mathcal{A}||\mathcal{B}| \leq\binom{ m-t}{r-t}\binom{n-t}{s-t}
$$

as required.
Suppose $|\mathcal{A}||\mathcal{B}|=\binom{m-t}{r-t}\binom{n-t}{s-t}$ and $u>0$. Then equality holds throughout in each of (21)-(24), and hence $\mathcal{C}=\mathcal{G}\left(T^{\prime}\right)$ and $\mathcal{D}=\mathcal{H}\left(T^{\prime}\right)$ for some $T^{\prime} \in\binom{[p]}{t}$. It follows that $\mathcal{A} \subseteq\left\{A \in\binom{[m]}{r}: T^{\prime} \subseteq A\right\}$ and $\mathcal{B} \subseteq\left\{B \in\binom{[n]}{s}: T^{\prime} \subseteq B\right\}$. Since $|\mathcal{A}||\mathcal{B}|=\binom{m-t}{r-t}\binom{n-t}{s-t}$, both inclusion relations are actually equalities, $T^{\prime} \subseteq[m]$, and $T^{\prime} \subseteq[n]$.

## 6 Proof of Theorem 1.5

In this section, we use Theorem 1.3 to prove Theorem 1.5,
We start by defining a compression operation for labeled sets. For any $x, y \in \mathbb{N}$, let

$$
\gamma_{x, y}(A)= \begin{cases}(A \backslash\{(x, y)\}) \cup\{(x, 1)\} & \text { if }(x, y) \in A \\ A & \text { otherwise }\end{cases}
$$

for any labeled set $A$, and let

$$
\Gamma_{x, y}(\mathcal{A})=\left\{\gamma_{x, y}(A): A \in \mathcal{A}\right\} \cup\left\{A \in \mathcal{A}: \gamma_{x, y}(A) \in \mathcal{A}\right\}
$$

for any family $\mathcal{A}$ of labeled sets.
Note that $\left|\Gamma_{x, y}(\mathcal{A})\right|=|\mathcal{A}|$ and that if $\mathcal{A} \subseteq \mathcal{S}_{\mathbf{c}, r}$, then $\Gamma_{x, y}(\mathcal{A}) \subseteq \mathcal{S}_{\mathbf{c}, r}$. It is well known that if $\mathcal{A}$ and $\mathcal{B}$ are cross- $t$-intersecting families of labeled sets, then so are $\Gamma_{x, y}(\mathcal{A})$ and $\Gamma_{x, y}(\mathcal{B})$. We present a result that gives more than this.

For any IP sequence $\mathbf{c}=\left(c_{1}, \ldots, c_{n}\right)$ and any $r \in[n]$, let $\mathcal{S}_{\mathbf{c}, \leq r}$ denote the union $\bigcup_{i=1}^{r} \mathcal{S}_{\mathbf{c}, i}$.

Lemma 6.1 Let $\mathbf{c}=\left(c_{1}, \ldots, c_{m}\right)$ and $\mathbf{d}=\left(d_{1}, \ldots, d_{n}\right)$ be IP sequences. Let $x, y \in \mathbb{N}$, $y \geq 2$. Let $l=\max \{m, n\}$ and $h=\max \left\{c_{m}, d_{n}\right\}$. Let $V \subseteq[l] \times[2, h]$. Let $\mathcal{A} \subseteq \mathcal{S}_{\mathbf{c}, \leq m}$ and $\mathcal{B} \subseteq \mathcal{S}_{\mathbf{d}, \leq n}$ such that $|(A \cap B) \backslash V| \geq t$ for every $A \in \mathcal{A}$ and every $B \in \mathcal{B}$. Then $|(C \cap D) \backslash(V \cup\{(x, y)\})| \geq t$ for every $C \in \Gamma_{x, y}(\mathcal{A})$ and every $D \in \Gamma_{x, y}(\mathcal{B})$.

Proof. Suppose $C \in \Gamma_{x, y}(\mathcal{A})$ and $D \in \Gamma_{x, y}(\mathcal{B})$. We first show that $|(C \cap D) \backslash V| \geq t$. Let $C^{\prime}=(C \backslash\{(x, 1)\}) \cup\{(x, y)\}$ and $D^{\prime}=(D \backslash\{(x, 1)\}) \cup\{(x, y)\}$. If $C \in \mathcal{A}$ and $D \in \mathcal{B}$, then $|(C \cap D) \backslash V| \geq t$. If $C \notin \mathcal{A}$ and $D \notin \mathcal{B}$, then $(x, 1) \in C \cap D, C^{\prime} \in \mathcal{A}$,
$D^{\prime} \in \mathcal{B}$, and hence, since $(x, 1) \notin V,|(C \cap D) \backslash V| \geq\left|\left(C^{\prime} \cap D^{\prime}\right) \backslash V\right| \geq t$. Suppose $C \notin \mathcal{A}$ and $D \in \mathcal{B}$. Then $(x, 1) \in C$ and $C^{\prime} \in \mathcal{A}$. If $(x, y) \notin D$, then, since $C^{\prime} \in \mathcal{A}$ and $D \in \mathcal{B}, t \leq\left|\left(C^{\prime} \cap D\right) \backslash V\right| \leq|(C \cap D) \backslash V|$. If $(x, y) \in D$, then $\gamma_{x, y}(D) \in \mathcal{B}$ (because otherwise $\left.D \notin \Gamma_{x, y}(\mathcal{B})\right)$, and hence, since $C^{\prime} \in \mathcal{A}, t \leq\left|\left(C^{\prime} \cap \gamma_{x, y}(D)\right) \backslash V\right|=|(C \cap D) \backslash V|$. Similarly, if $C \in \mathcal{A}$ and $D \notin \mathcal{B}$, then $|(C \cap D) \backslash V| \geq t$.

Now suppose $(C \cap D) \backslash(V \cup\{(x, y)\})<t$. Since $|(C \cap D) \backslash V| \geq t,(x, y) \in C \cap D$. Thus $C, \gamma_{x, y}(C) \in \mathcal{A}, D, \gamma_{x, y}(D) \in \mathcal{B}$, and $\left|\left(C \cap \gamma_{x, y}(D)\right) \backslash V\right|=|(C \cap D) \backslash(V \cup\{(x, y)\})|<t$, a contradiction.

Corollary 6.2 Let $\mathbf{c}=\left(c_{1}, \ldots, c_{m}\right), \mathbf{d}=\left(d_{1}, \ldots, d_{n}\right)$, l, and $h$ be as in Lemma 6.1. Let $\mathcal{A} \subseteq \mathcal{S}_{\mathbf{c}, \leq m}$ and $\mathcal{B} \subseteq \mathcal{S}_{\mathbf{d}, \leq n}$ such that $\mathcal{A}$ and $\mathcal{B}$ are cross-t-intersecting. Let

$$
\begin{aligned}
\mathcal{A}^{*} & =\Gamma_{l, h} \circ \cdots \circ \Gamma_{l, 2} \circ \cdots \circ \Gamma_{2, h} \circ \cdots \circ \Gamma_{2,2} \circ \Gamma_{1, h} \circ \cdots \circ \Gamma_{1,2}(\mathcal{A}), \\
\mathcal{B}^{*} & =\Gamma_{l, h} \circ \cdots \circ \Gamma_{l, 2} \circ \cdots \circ \Gamma_{2, h} \circ \cdots \circ \Gamma_{2,2} \circ \Gamma_{1, h} \circ \cdots \circ \Gamma_{1,2}(\mathcal{B}) .
\end{aligned}
$$

Then $|A \cap B \cap([l] \times[1])| \geq t$ for every $A \in \mathcal{A}^{*}$ and every $B \in \mathcal{B}^{*}$.
Proof. Let $Z=[l] \times[2, h]$. By repeated application of Lemma 6.1, $|(A \cap B) \backslash Z| \geq t$ for every $A \in \mathcal{A}^{*}$ and every $B \in \mathcal{B}^{*}$. The result follows since $(A \cap B) \backslash Z=A \cap B \cap$ $([l] \times[1])$.

The next lemma is needed for the characterisation of the extremal structures in Theorem 1.5

Lemma 6.3 Let $\mathbf{c}=\left(c_{1}, \ldots, c_{m}\right), \mathbf{d}=\left(d_{1}, \ldots, d_{n}\right), l$, and $h$ be as in Lemma 6.1. Suppose $c_{1} \geq 3$ and $d_{1} \geq 3$. Let $r \in[m], s \in[n]$, and $t \in[\min \{r, s\}]$. Let $\mathcal{A} \subseteq \mathcal{S}_{\mathbf{c}, r}$ and $\mathcal{B} \subseteq \mathcal{S}_{\mathbf{d}, s}$ such that $\mathcal{A}$ and $\mathcal{B}$ are cross-t-intersecting. Suppose $\Gamma_{x, y}(\mathcal{A})=\mathcal{S}_{\mathbf{c}, r}(T)$ and $\Gamma_{x, y}(\mathcal{B})=\mathcal{S}_{\mathbf{d}, s}(T)$ for some $(x, y) \in[l] \times[h]$ and some labeled set $T \in\binom{[l] \times[h]}{t}$. Then $\mathcal{A}=\mathcal{S}_{\mathbf{c}, r}\left(T^{\prime}\right)$ and $\mathcal{B}=\mathcal{S}_{\mathbf{d}, s}\left(T^{\prime}\right)$ for some labeled set $T^{\prime} \in\binom{[l] \times[h]}{t}$.

Proof. The result is immediate if $\mathcal{A}=\Gamma_{x, y}(\mathcal{A})$ and $\mathcal{B}=\Gamma_{x, y}(\mathcal{B})$. Suppose $\mathcal{A} \neq$ $\Gamma_{x, y}(\mathcal{A})$ or $\mathcal{B} \neq \Gamma_{x, y}(\mathcal{B})$. We may assume that $\mathcal{A} \neq \Gamma_{x, y}(\mathcal{A})$. Thus there exists $A_{1} \in$ $\mathcal{A} \backslash \Gamma_{x, y}(\mathcal{A})$ such that $\gamma_{x, y}\left(A_{1}\right) \in \Gamma_{x, y}(\mathcal{A}) \backslash \mathcal{A}$. Then $(x, 1) \neq(x, y) \in A_{1}$ and $\gamma_{x, y}\left(A_{1}\right)=$ $\left(A_{1} \backslash\{(x, y)\}\right) \cup\{(x, 1)\}$.

Suppose $(x, 1) \notin T$. Together with $\gamma_{x, y}\left(A_{1}\right) \in \Gamma_{x, y}(\mathcal{A})=\mathcal{S}_{\mathbf{c}, r}(T)$, this gives us $T \subseteq A_{1}$, which contradicts $A_{1} \notin \Gamma_{x, y}(\mathcal{A})$.

Therefore, $(x, 1) \in T$. Let $\left(a_{1}, b_{1}\right), \ldots,\left(a_{t}, b_{t}\right)$ be the elements of $T$, where $\left(a_{t}, b_{t}\right)=$ $(x, 1)$. Let $T^{\prime}=(T \backslash\{(x, 1)\}) \cup\{(x, y)\}$. Since $\gamma_{x, y}\left(A_{1}\right) \in \mathcal{S}_{\mathbf{c}, r}(T)$, we have $T^{\prime} \subseteq A_{1}$, and hence $\mathcal{S}_{\mathbf{c}, r}\left(T^{\prime}\right) \neq \emptyset$. Note that $\left|\mathcal{S}_{\mathbf{c}, r}\left(T^{\prime}\right)\right|=\left|\mathcal{S}_{\mathbf{c}, r}(T)\right|$.

Let $A^{*} \in \mathcal{S}_{\mathbf{c}, r}\left(T^{\prime}\right)$. If $s>t$, then let $x_{1}, \ldots, x_{s-t}$ be distinct elements of $[n] \backslash\left\{a_{1}, \ldots, a_{t}\right\}$. For each $i \in[n]$, let $D_{i}=\{i\} \times\left[d_{i}\right]$. We are given that $3 \leq d_{1} \leq \cdots \leq d_{n}$. By definition of a labeled set, for each $i \in[n]$, we have $\left|A \cap D_{i}\right| \leq 1$ for all $A \in \mathcal{S}_{\mathbf{c}, r}$. Thus $\left|D_{i} \backslash\left(A_{1} \cup A^{*}\right)\right| \geq d_{i}-2 \geq 1$ for each $i \in[n]$. If $s>t$, then let $\left(x_{i}, y_{i}\right) \in D_{x_{i}} \backslash\left(A_{1} \cup A^{*}\right)$ for each $i \in[s-t]$, and let $B^{*}=T^{\prime} \cup\left\{\left(x_{1}, y_{1}\right), \ldots,\left(x_{s-t}, y_{s-t}\right)\right\}$. If $s=t$, then let $B^{*}=T^{\prime}$. Thus $B^{*} \in \mathcal{S}_{\mathbf{d}, s}\left(T^{\prime}\right)$. Since $\Gamma_{x, y}(\mathcal{B})=\mathcal{S}_{\mathbf{d}, s}(T)$, we have $B^{*} \in \mathcal{B}$ or $\gamma_{x, y}\left(B^{*}\right) \in \mathcal{B}$.

However, $\left|\gamma_{x, y}\left(B^{*}\right) \cap A_{1}\right|=\left|T \cap A_{1}\right|=|T \backslash\{(x, 1)\}|=t-1$, so $B^{*} \in \mathcal{B}$. Since $\Gamma_{x, y}(\mathcal{A})=\mathcal{S}_{\mathbf{c}, r}(T)$, we have $A^{*} \in \mathcal{A}$ or $\gamma_{x, y}\left(A^{*}\right) \in \mathcal{A}$. However, $\left|\gamma_{x, y}\left(A^{*}\right) \cap B^{*}\right|=t-1$, so $A^{*} \in \mathcal{A}$.

We have therefore shown that $\mathcal{S}_{\mathbf{c}, r}\left(T^{\prime}\right) \subseteq \mathcal{A}$. Since $|\mathcal{A}|=\left|\Gamma_{x, y}(\mathcal{A})\right|=\left|\mathcal{S}_{\mathbf{c}, r}(T)\right|=$ $\left|\mathcal{S}_{\mathbf{c}, r}\left(T^{\prime}\right)\right|$, we actually have $\mathcal{A}=\mathcal{S}_{\mathbf{c}, r}\left(T^{\prime}\right)$. Clearly, for each $L \in \mathcal{S}_{\mathbf{d}, s}$ with $T^{\prime} \nsubseteq L$, there exists $L^{\prime} \in \mathcal{S}_{\mathbf{c}, r}\left(T^{\prime}\right)$ such that $\left|L \cap L^{\prime}\right|=\left|L \cap T^{\prime}\right|<\left|T^{\prime}\right|=t$. Thus, since $\mathcal{A}=\mathcal{S}_{\mathbf{c}, r}\left(T^{\prime}\right)$, each set in $\mathcal{B}$ contains $T^{\prime}$. Hence $\mathcal{B} \subseteq \mathcal{S}_{\mathbf{d}, s}\left(T^{\prime}\right)$. Since $|\mathcal{B}|=\left|\Gamma_{x, y}(\mathcal{B})\right|=\left|\mathcal{S}_{\mathbf{d}, s}(T)\right|=$ $\left|\mathcal{S}_{\mathbf{d}, s}\left(T^{\prime}\right)\right|$, we actually have $\mathcal{B}=\mathcal{S}_{\mathbf{d}, s}\left(T^{\prime}\right)$.

The next lemma allows us to translate the setting in Theorem 1.5 to one given by Theorem 1.3

Lemma 6.4 Let $\mathbf{c}$ be an IP sequence $\left(c_{1}, \ldots, c_{n}\right)$. Let $r \in[n]$. Let $w:\binom{[n]}{\leq r} \rightarrow \mathbb{N}$ such that for each $A \in\binom{[n]}{\leq r}$,

$$
w(A)=\left|\left\{L \in \mathcal{S}_{\mathbf{c}, r}: L \cap([n] \times[1])=A \times[1]\right\}\right|
$$

Then:
(i) $w(A) \geq\left(c_{1}-1\right) w(B)$ for every $A, B \in\binom{[n]}{\leq r}$ with $A \subsetneq B$.
(ii) $w\left(\delta_{i, j}(A)\right) \geq w(A)$ for every $A \in\binom{[n]}{\leq r}$ and every $i, j \in[n]$ with $i<j$.

Proof. (i) Let $A, B \in\binom{[n]}{\leq r}$ with $A \subsetneq B$. Let $B^{\prime}=B \backslash A$. Thus $\left|B^{\prime}\right| \geq 1$. For each $L \in \mathcal{S}_{\mathbf{c}, r}$, let $\sigma(L)=\left\{x \in[n]:(x, y) \in L\right.$ for some $\left.y \in\left[c_{i}\right]\right\}$. We have

$$
\begin{aligned}
w(A) & \geq\left|\left\{L \in \mathcal{S}_{\mathbf{c}, r}: L \cap([n] \times[1])=A \times[1], B^{\prime} \subseteq \sigma(L)\right\}\right| \\
& =\sum_{E \in\binom{\left[n \backslash \backslash\left(A \cup B^{\prime}\right)\right.}{r|A|| |\left|B^{\prime}\right|}} \prod_{b \in B^{\prime}}\left(c_{b}-1\right) \prod_{e \in E}\left(c_{e}-1\right) \\
& =\prod_{b \in B^{\prime}}\left(c_{b}-1\right)\left(\sum_{E \in\binom{[n \backslash \mid B}{r-|B|}} \prod_{e \in E}\left(c_{e}-1\right)\right) \\
& =w(B) \prod_{b \in B^{\prime}}\left(c_{b}-1\right) \geq\left(c_{1}-1\right)^{\left|B^{\prime}\right|} w(B) \geq\left(c_{1}-1\right) w(B) .
\end{aligned}
$$

(ii) Let $A \in\binom{[n]}{<r}$, and let $i, j \in[n]$ with $i<j$. Suppose $\delta_{i, j}(A) \neq A$. Then $j \in A$, $i \notin A$, and $\delta_{i, j}(A)=(A \backslash\{j\}) \cup\{i\}$. Let $B=A \backslash\{j\}$. Let

$$
\begin{gathered}
\mathcal{E}_{0}=\binom{[n] \backslash(B \cup\{i, j\})}{r-|A|}, \\
\mathcal{E}_{1}=\left\{E \in\binom{[n] \backslash(B \cup\{i\})}{r-|A|}: j \in E\right\}, \\
\mathcal{E}_{2}=\left\{E \in\binom{[n] \backslash(B \cup\{j\})}{r-|A|}: i \in E\right\} .
\end{gathered}
$$

We have

$$
\begin{aligned}
& w(B \cup\{i\})=\sum_{E \in\left(\begin{array}{c}
{[n] \backslash(B \cup\{i\})} \\
r-|A| \\
)
\end{array}\right.} \prod_{e \in E}\left(c_{e}-1\right) \\
& =\sum_{D \in \mathcal{E}_{0}} \prod_{d \in D}\left(c_{d}-1\right)+\sum_{F \in \mathcal{E}_{1}} \prod_{f \in F}\left(c_{f}-1\right) \\
& \geq \sum_{D \in \mathcal{E}_{0}} \prod_{d \in D}\left(c_{d}-1\right)+\sum_{F \in \mathcal{E}_{1}} \prod_{f \in F}\left(c_{f}-1\right) \frac{c_{i}-1}{c_{j}-1} \quad\left(\text { since } c_{i} \leq c_{j}\right) \\
& =\sum_{D \in \mathcal{E}_{0}} \prod_{d \in D}\left(c_{d}-1\right)+\sum_{F \in \mathcal{E}_{2}} \prod_{f \in F}\left(c_{f}-1\right) \\
& =\sum_{\substack{E \in\left(\begin{array}{c}
[n] \backslash(B \cup\{j\}) \\
r| | A \mid\\
)
\end{array}\right.}} \prod_{e \in E}\left(c_{e}-1\right)=w(B \cup\{j\}),
\end{aligned}
$$

and hence $w\left(\delta_{i, j}(A)\right) \geq w(A)$.
Proof of Theorem 1.5. Let $\mathcal{G}=\binom{[m]}{\leq r}$. Let $v: \mathcal{G} \rightarrow \mathbb{N}$ such that for each $G \in \mathcal{G}$,

$$
v(G)=\left|\left\{L \in \mathcal{S}_{\mathbf{c}, r}: L \cap([m] \times[1])=G \times[1]\right\}\right|
$$

Let $\mathcal{H}=\binom{[n]}{\leq s}$. Let $w: \mathcal{H} \rightarrow \mathbb{N}$ such that for each $H \in \mathcal{H}$,

$$
w(H)=\left|\left\{L \in \mathcal{S}_{\mathbf{d}, s}: L \cap([n] \times[1])=H \times[1]\right\}\right| .
$$

Let $l=\max \{m, n\}$ and $h=\max \left\{c_{m}, d_{n}\right\}$. Let

$$
\begin{aligned}
\mathcal{A}^{*} & =\Gamma_{l, h} \circ \cdots \circ \Gamma_{l, 2} \circ \cdots \circ \Gamma_{2, h} \circ \cdots \circ \Gamma_{2,2} \circ \Gamma_{1, h} \circ \cdots \circ \Gamma_{1,2}(\mathcal{A}), \\
\mathcal{B}^{*} & =\Gamma_{l, h} \circ \cdots \circ \Gamma_{l, 2} \circ \cdots \circ \Gamma_{2, h} \circ \cdots \circ \Gamma_{2,2} \circ \Gamma_{1, h} \circ \cdots \circ \Gamma_{1,2}(\mathcal{B}) .
\end{aligned}
$$

Now let

$$
\begin{aligned}
\mathcal{C} & =\left\{G \in \mathcal{G}: E \cap([m] \times[1])=G \times[1] \text { for some } E \in \mathcal{A}^{*}\right\}, \\
\mathcal{D} & =\left\{H \in \mathcal{H}: F \cap([n] \times[1])=H \times[1] \text { for some } F \in \mathcal{B}^{*}\right\} .
\end{aligned}
$$

Then $\mathcal{C} \subseteq \mathcal{G} \subseteq 2^{[l]}, \mathcal{D} \subseteq \mathcal{H} \subseteq 2^{[l]}$, and, by Corollary $6.2, \mathcal{C}$ and $\mathcal{D}$ are cross-t-intersecting. We have

$$
\begin{align*}
& \mathcal{A}^{*} \subseteq \bigcup_{C \in \mathcal{C}}\left\{L \in \mathcal{S}_{\mathbf{c}, r}: L \cap([m] \times[1])=C \times[1]\right\}  \tag{25}\\
& \mathcal{B}^{*} \subseteq \bigcup_{D \in \mathcal{D}}\left\{L \in \mathcal{S}_{\mathbf{d}, s}: L \cap([n] \times[1])=D \times[1]\right\} \tag{26}
\end{align*}
$$

Thus

$$
\begin{align*}
\left|\mathcal{A}^{*}\right| & \leq \sum_{C \in \mathcal{C}} v(C)=v(\mathcal{C})  \tag{27}\\
\left|\mathcal{B}^{*}\right| & \leq \sum_{D \in \mathcal{D}} w(D)=w(\mathcal{D}) . \tag{28}
\end{align*}
$$

Since $|\mathcal{A}|=\left|\mathcal{A}^{*}\right|$ and $|\mathcal{B}|=\left|\mathcal{B}^{*}\right|$, we therefore have

$$
\begin{align*}
& |\mathcal{A}| \leq v(\mathcal{C})  \tag{29}\\
& |\mathcal{B}| \leq w(\mathcal{D}) \tag{30}
\end{align*}
$$

Let $T_{0}=[t]$. Let $\mathcal{I}=\mathcal{G}\left(T_{0}\right), \mathcal{J}=\mathcal{H}\left(T_{0}\right), \mathcal{X}=\mathcal{S}_{\mathbf{c}, r}\left(T_{0} \times[1]\right)$, and $\mathcal{Y}=\mathcal{S}_{\mathbf{d}, s}\left(T_{0} \times[1]\right)$. By Lemma 6.4 and Theorem 1.3 ,

$$
\begin{equation*}
v(\mathcal{C}) w(\mathcal{D}) \leq v(\mathcal{I}) w(\mathcal{J}) \tag{31}
\end{equation*}
$$

Now

$$
\begin{aligned}
v(\mathcal{I}) & =\left(\sum_{I \in \mathcal{I}} v(I)\right)=\left(\sum_{I \in \mathcal{I}}\left|\left\{L \in \mathcal{S}_{\mathbf{c}, r}: L \cap([m] \times[1])=I \times[1]\right\}\right|\right) \\
& =\left|\bigcup_{I \in \mathcal{I}}\left\{L \in \mathcal{S}_{\mathbf{c}, r}: L \cap([m] \times[1])=I \times[1]\right\}\right|=|\mathcal{X}|
\end{aligned}
$$

and, similarly, $w(\mathcal{J})=|\mathcal{Y}|$. Together with (29) $-(31)$, this gives us $|\mathcal{A}||\mathcal{B}| \leq|\mathcal{X}||\mathcal{Y}|$, which establishes the first part of the theorem.

We now prove the second part of the theorem. The sufficiency condition is trivial, so we prove the necessary condition.

Suppose $|\mathcal{A}||\mathcal{B}|=|\mathcal{X}||\mathcal{Y}|$ and $u>0$. Then all the inequalities in (27) - (31) are equalities. Having equality throughout in each of (27) and (28) implies that equality holds in each of (25) and (261). By Theorem (1.3), equality in (31) gives us that $\mathcal{C}=\mathcal{G}\left(T_{1}\right)$ and $\mathcal{D}=\mathcal{H}\left(T_{1}\right)$ for some $T_{1} \in\binom{[l]}{t}$. Together with equality in each of (25) and (26), this gives us that $\mathcal{A}^{*}=\mathcal{S}_{\mathbf{c}, r}\left(T_{2}\right)$ and $\mathcal{B}^{*}=\mathcal{S}_{\mathrm{d}, s}\left(T_{2}\right)$, where $T_{2}=T_{1} \times[1]$. By Lemma 6.3, $\mathcal{A}=\mathcal{S}_{\mathbf{c}, r}\left(T_{3}\right)$ and $\mathcal{B}=\mathcal{S}_{\mathbf{d}, s}\left(T_{3}\right)$ for some $T_{3} \in\binom{[l] \times[h]}{t}$. Since $|\mathcal{A}||\mathcal{B}|=|\mathcal{X}||\mathcal{Y}|>0$, we clearly have $T_{3} \in \mathcal{S}_{\mathbf{c}, t} \cap \mathcal{S}_{\mathbf{d}, t}$.

## 7 Proof of Theorem 1.6

In this section, we use Theorem 1.3 to prove Theorem 1.6 .
As in Section 5, for any family $\mathcal{F}, \mathcal{F}^{(r)}$ denotes $\{F \in \mathcal{F}:|F|=r\}$. For any $n, r \in \mathbb{N}$ and any family $\mathcal{A}$, let $M_{n, r, \mathcal{A}}$ denote the set $\left\{A \in M_{n, r}: \mathrm{S}_{A} \in \mathcal{A}\right\}$.

Lemma 7.1 If $n, r \in \mathbb{N}, i, j \in[n]$, and $\mathcal{A} \subseteq 2^{[n]}$, then $\left|M_{n, r, \Delta_{i, j}(\mathcal{A})}\right|=\left|M_{n, r, \mathcal{A}}\right|$.
Proof. Let $\mathcal{B}=\Delta_{i, j}(\mathcal{A})$. Clearly, $\left|\mathcal{B}^{(p)}\right|=\left|\mathcal{A}^{(p)}\right|$ for each $p \in[n]$. We have

$$
\begin{aligned}
\left|M_{n, r, \mathcal{B}}\right| & =\sum_{B \in \mathcal{B}}\left|M_{n, r,\{B\}}\right|=\sum_{p=1}^{n} \sum_{B \in \mathcal{B}^{(p)}}\left|M_{n, r,\{B\}}\right|=\sum_{p=1}^{n}\left|\mathcal{B}^{(p)}\right|\left|M_{n, r,\{[p]\}}\right| \\
& =\sum_{p=1}^{n}\left|\mathcal{A}^{(p)}\right|\left|M_{n, r,\{[p]\}}\right|=\sum_{p=1}^{n} \sum_{A \in \mathcal{A}^{(p)}}\left|M_{n, r,\{A\}}\right|=\sum_{A \in \mathcal{A}}\left|M_{n, r,\{A\}}\right|=\left|M_{n, r, \mathcal{A}}\right|,
\end{aligned}
$$

as required.

Proof of Theorem 1.6. Let $\mathcal{C}=\left\{\mathrm{S}_{A}: A \in \mathcal{A}\right\}$ and $\mathcal{D}=\left\{\mathrm{S}_{B}: B \in \mathcal{B}\right\}$. Clearly, $\mathcal{A} \subseteq M_{m, r, \mathcal{C}}, \mathcal{B} \subseteq M_{n, s, \mathcal{D}}$, and, since $\mathcal{A}$ and $\mathcal{B}$ are cross- $t$-intersecting, $M_{m, r, \mathcal{C}}$ and $M_{n, s, \mathcal{D}}$ are cross- $t$-intersecting. Thus we assume that

$$
\begin{equation*}
\mathcal{A}=M_{m, r, \mathcal{C}} \quad \text { and } \quad \mathcal{B}=M_{n, s, \mathcal{D}} \tag{32}
\end{equation*}
$$

As explained in Section 3, we apply left-compressions to $\mathcal{C}$ and $\mathcal{D}$ simultaneously until we obtain two compressed cross- $t$-intersecting families $\mathcal{C}^{*}$ and $\mathcal{D}^{*}$, respectively. Since $\mathcal{C} \subseteq\binom{[m]}{\leq r}$ and $\mathcal{D} \subseteq\binom{[n]}{\leq s}$, we have $\mathcal{C}^{*} \subseteq\binom{[m]}{\leq r}$ and $\mathcal{D}^{*} \subseteq\binom{[n]}{\leq s}$. By Lemma 3.1(ii),

$$
\begin{equation*}
|C \cap D \cap[r+s-t]| \geq t \text { for any } C \in \mathcal{C}^{*} \text { and any } D \in \mathcal{D}^{*} \tag{33}
\end{equation*}
$$

Let $p=r+s-t$. Let $\mathcal{G}=\binom{[p]}{\leq r}$ and $\mathcal{H}=\binom{[p]}{\leq s}$. Let $g: \mathcal{G} \rightarrow \mathbb{N}$ such that $g(G)=\binom{m+r-p-1}{r-|G|}$ for each $G \in \mathcal{G}$. Let $h: \mathcal{H} \rightarrow \mathbb{N}$ such that $h(H)=\binom{n+s-p-1}{s-|H|}$ for each $H \in \mathcal{H}^{r}$.

For every $F, G \in \mathcal{G}$ with $F \subsetneq G$ and $t \leq|F|=|G|-1$, we have

$$
\begin{aligned}
\frac{g(F)-(t+u) g(G)}{\binom{m+r-p-1}{r-|F|}} & =1-\frac{(t+u)\binom{m+r-p-1}{r-|F|-1}}{\binom{m+r-p-1}{r-|F|}}=1-\frac{(t+u)(r-|F|)}{m-p+|F|} \\
& =\frac{m-p+|F|-(t+u)(r-|F|)}{m-p+|F|} \\
& \geq \frac{m-p+t-(t+u)(r-t)}{m-p+|F|}=\frac{m-(t+u+1)(r-t)-s+t}{m-p+|F|} \\
& \geq \frac{(t+u+1)(s-t)+r-t-((t+u+1)(r-t)+s-t)}{m-p+|F|} \geq 0,
\end{aligned}
$$

and hence $g(F) \geq(t+u) g(G)$. It follows that $g(F) \geq(t+u) g(G)$ for every $F, G \in \mathcal{G}$ with $F \subsetneq G$ and $|F| \geq t$. Similarly, $h(F) \geq(t+u) g(H)$ for every $F, H \in \mathcal{H}$ with $F \subsetneq H$ and $|F| \geq t$.

We have $g\left(\delta_{i, j}(G)\right)=g(G)$ for every $G \in \mathcal{G}$ and every $i, j \in[p]$. Similarly, $h\left(\delta_{i, j}(H)\right)=h(H)$ for every $H \in \mathcal{H}$ and every $i, j \in[p]$.

Let $\mathcal{E}=\left\{C \cap[p]: C \in \mathcal{C}^{*}\right\}$ and $\mathcal{F}=\left\{D \cap[p]: D \in \mathcal{D}^{*}\right\}$. Then $\mathcal{E} \subseteq \mathcal{G}, \mathcal{F} \subseteq \mathcal{H}$, and, by (331), $\mathcal{E}$ and $\mathcal{F}$ are cross- $t$-intersecting. Let $T=[t]$. By Theorem [1.3,

$$
\begin{equation*}
g(\mathcal{E}) h(\mathcal{F}) \leq g(\mathcal{G}(T)) h(\mathcal{H}(T)), \tag{34}
\end{equation*}
$$

and if $u>0$, then equality holds only if $\mathcal{E}=\mathcal{G}\left(T^{\prime}\right)$ and $\mathcal{F}=\mathcal{H}\left(T^{\prime}\right)$ for some $T^{\prime} \in\binom{[p]}{t}$.
By (32) and Lemma 7.1,

$$
\begin{align*}
|\mathcal{A}| & =\left|M_{m, r, \mathcal{C}^{*}}\right| \leq \mid\left\{A \in M_{m, r}: \mathrm{S}_{A} \cap[p]=E \text { for some } E \in \mathcal{E}\right\} \mid \\
& =\sum_{E \in \mathcal{E}} \mid\left\{A \in M_{m, r}: \mathrm{S}_{A} \cap[p]=E\right\} \\
& =\sum_{E \in \mathcal{E}}\left|\left\{\left(a_{1}, \ldots, a_{r-|E|}\right): a_{1} \leq \cdots \leq a_{r-|E|}, a_{1}, \ldots, a_{r-|E|} \in E \cup[p+1, m]\right\}\right| \\
& =\sum_{E \in \mathcal{E}}\left|M_{|E|+m-p, r-|E|}\right|=\sum_{E \in \mathcal{E}}\binom{m+r-p-1}{r-|E|}=g(\mathcal{E}) . \tag{35}
\end{align*}
$$

Similarly,

$$
\begin{equation*}
|\mathcal{B}| \leq h(\mathcal{F}) . \tag{36}
\end{equation*}
$$

By (34)-(36),

$$
\begin{equation*}
|\mathcal{A}||\mathcal{B}| \leq g(\mathcal{G}(T)) h(\mathcal{H}(T)) . \tag{37}
\end{equation*}
$$

Now, similarly to (35),

$$
\begin{aligned}
g(\mathcal{G}(T)) & =\mid\left\{A \in M_{n, r}: \mathrm{S}_{A} \cap[p]=E \text { for some } E \in \mathcal{G}(T)\right\} \mid \\
& =\left|\left\{A \in M_{n, r}: T \subseteq \mathrm{~S}_{A}\right\}\right|=\binom{m+r-t-1}{r-t} .
\end{aligned}
$$

Similarly, $h(\mathcal{H}(T))=\binom{n+s-t-1}{s-t}$. By (37), it follows that

$$
|\mathcal{A}||\mathcal{B}| \leq\binom{ m+r-t-1}{r-t}\binom{n+s-t-1}{s-t}
$$

as required.
Suppose $|\mathcal{A}||\mathcal{B}|=\binom{m+r-t-1}{r-t}\binom{n+s-t-1}{s-t}$ and $u>0$. Then equality holds throughout in each of (34)-(37), and hence $\mathcal{E}=\mathcal{G}\left(T^{\prime}\right)$ and $\mathcal{F}=\mathcal{H}\left(T^{\prime}\right)$ for some $T^{\prime} \in\binom{[p]}{t}$. Having equality throughout in (35) implies that $M_{m, r, \mathcal{C}^{*}}=\left\{A \in M_{m, r}: \mathrm{S}_{A} \cap[p]=\right.$ $E$ for some $E \in \mathcal{E}\}=\left\{A \in M_{m, r}: T^{\prime} \subseteq \mathrm{S}_{A}\right\}$. Thus $T^{\prime} \in \mathcal{C}^{*}$, and hence there exists $T_{1} \in\binom{[m]}{t}$ such that $T_{1} \in \mathcal{C}$. Similarly, there exists $T_{2} \in\binom{[n]}{t}$ such that $T_{2} \in \mathcal{D}$. Since $\mathcal{C}$ and $\mathcal{D}$ are cross- $t$-intersecting, we have $T_{1}=T_{2}, \mathcal{C} \subseteq\left\{C \in\binom{[m]}{\leq r}: T_{1} \subseteq C\right\}$, and $\mathcal{D} \subseteq\left\{D \in\binom{[n]}{\leq s}: T_{1} \subseteq D\right\}$. Consequently, $\mathcal{A} \subseteq\left\{A \in M_{m, r}: T_{1} \subseteq S_{A}\right\}$ and $\mathcal{B} \subseteq$ $\left\{B \in M_{n, s}: T_{1} \subseteq S_{B}\right\}$. Since $|\mathcal{A}||\mathcal{B}|=\binom{m+r-t-1}{r-t}\binom{n+s-t-1}{s-t}$, both inclusion relations are actually equalities.

## 8 The remaining cases

Each of Theorems 1.2, 1.5, and 1.6 solves the particular cross- $t$-intersection problem under consideration for all cases where the ground sets are not smaller than a certain value dependent on $r, s$, and $t$. Solving any of these problems completely appears to be very difficult and would take this area of study to a significantly deeper level. We conjecture that the complete solutions are (38)-(40) below.

For any $n \in \mathbb{N}$ and any $r, t, i, j \in\{0\} \cup[n]$ with $1 \leq t \leq r$ and $t+i+j \leq n$, let $\mathcal{M}_{n, r, t, i, j}=\left\{A \in\binom{[n]}{r}:|A \cap[t+i+j]| \geq t+i\right\}$. In [22], Frankl conjectured that the size of a largest $t$-intersecting subfamily of $\binom{[n]}{r}$ is $\max \left\{\left|\mathcal{M}_{m, r, t, i, i}\right|: i, j \in\{0\} \cup \mathbb{N}, t+2 i \leq\right.$ $n\}$, and this was verified in [1]. Hirschorn suggested an analogous conjecture 29, Conjecture 4] for cross- $t$-intersecting families $\mathcal{A}$ and $\mathcal{B}$ with $\mathcal{A} \subseteq\binom{[n]}{r}$ and $\mathcal{B} \subseteq\binom{[n]}{s}$. Generalising Hirschorn's conjecture, we conjecture that if $m, n \in \mathbb{N}, r \in[m], s \in[n]$, $t \in[\min \{r, s\}], \mathcal{A} \subseteq\binom{[m]}{r}, \mathcal{B} \subseteq\binom{[n]}{s}$, and $\mathcal{A}$ and $\mathcal{B}$ are cross- $t$-intersecting, then

$$
\begin{equation*}
|\mathcal{A}||\mathcal{B}| \leq \max \left\{\left|\mathcal{M}_{m, r, t, i, j}\right|\left|\mathcal{M}_{n, s, t, j, i}\right|: i, j \in\{0\} \cup \mathbb{N}, t+i+j \leq \min \{m, n\}\right\} . \tag{38}
\end{equation*}
$$

For any IP sequence $\mathbf{c}=\left(c_{1}, \ldots, c_{n}\right)$ and any $r, t, i, j \in\{0\} \cup[n]$ with $1 \leq t \leq r$ and $t+i+j \leq n$, let $\mathcal{S}_{\mathbf{c}, r, t, i, j}=\left\{A \in \mathcal{S}_{\mathbf{c}, r}:|A \cap([t+i+j] \times[1])| \geq t+i\right\}$. We conjecture that if $\mathbf{c}=\left(c_{1}, \ldots, c_{m}\right)$ and $\mathbf{d}=\left(d_{1}, \ldots, d_{n}\right)$ are IP sequences, $c_{1} \geq 2, d_{1} \geq 2, r \in[m]$, $s \in[n], t \in[\min \{r, s\}], \mathcal{A} \subseteq \mathcal{S}_{\mathbf{c}, r}, \mathcal{B} \subseteq \mathcal{S}_{\mathbf{d}, s}$, and $\mathcal{A}$ and $\mathcal{B}$ are cross- $t$-intersecting, then

$$
\begin{equation*}
|\mathcal{A}||\mathcal{B}| \leq \max \left\{\left|\mathcal{S}_{\mathbf{c}, r, t, i, j}\right|\left|\mathcal{S}_{\mathbf{d}, s, t, j, i}\right|: i, j \in\{0\} \cup \mathbb{N}, t+i+j \leq \min \{m, n\}\right\} \tag{39}
\end{equation*}
$$

This generalises [40, Conjecture 3], which is a conjecture for the case $r=s=m=n$.
For any $n \in \mathbb{N}$ and any $r, t, i, j \in\{0\} \cup[n]$ with $1 \leq t \leq r$ and $t+i+j \leq n$, let $M_{n, r, t, i, j}=\left\{A \in M_{n, r}:\left|\mathrm{S}_{A} \cap[t+i+j]\right| \geq t+i\right\}$. We conjecture that if $m, n, r, s \in \mathbb{N}$, $t \in[\min \{r, s\}], \mathcal{A} \subseteq M_{m, r}, \mathcal{B} \subseteq M_{n, s}$, and $\mathcal{A}$ and $\mathcal{B}$ are cross- $t$-intersecting, then

$$
\begin{equation*}
|\mathcal{A}||\mathcal{B}| \leq \max \left\{\left|M_{m, r, t, i, j}\right|\left|M_{n, s, t, j, i}\right|: i, j \in\{0\} \cup \mathbb{N}, t+i+j \leq \min \{m, n\}\right\} . \tag{40}
\end{equation*}
$$

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