# TO NUMERICAL MODELING WITH STRONG ORDERS 1.0, 1.5, AND 2.0 OF CONVERGENCE FOR MULTIDIMENSIONAL DYNAMICAL SYSTEMS WITH RANDOM DISTURBANCES 

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#### Abstract

The article is devoted to explicit one-step numerical methods with strong orders $1.0,1.5$, and 2.0 of convergence for Ito stochastic differential equations with multidimensional and non-commutative noise. For numerical modeling of iterated Ito stochastic integrals with multiplicities 1 to 4 we use the method of multiple Fourier-Legendre series converging in the sense of norm in Hilbert space $L_{2}\left([t, T]^{k}\right), k=1,2,3,4$. The article is addressed to engineers who use numerical modeling in stochastic control and for solving the nonlinear filtering problem.


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## 1. Introduction

The Ito stochastic differential equations (SDEs) are known to be adequate mathematical models of the dynamical systems of various physical nature subjected to random perturbations [2]-[5]. On the assumption of strong convergence criterion [2], the need for numerical integration of Ito SDEs arises

[^0]at solving the different mathematical problems. Among them we mention the following problems: stochastic optimal control (also with incomplete data) [2], 6], signal filtering in random noise in various formulations [2], 6], estimating the parameters of stochastic systems [2], 3]. It is common knowledge that one of the promising approaches to the numerical integration of Ito SDEs is the approach based on the stochastic analogues of the Taylor formula, the so-called Taylor-Ito and Taylor-Stratonovich expansions [2], [3], 7]-[12]. This approach makes use of finite discretization of the time variable and implies numerical modeling of the solution of Ito SDE at the discrete time instants using the stochastic analogues of the Taylor formula obtained by iterative application of the Ito formula.

Let $(\Omega, \mathrm{F}, \mathrm{P})$ be a complete probability space, let $\left\{\mathrm{F}_{t}, t \in[0, T]\right\}$ be a nondecreasing right-continous family of $\sigma$-algebras of F , and let $\mathbf{f}_{t}$ be a standard $m$-dimensional Wiener stochastic process, which is $\mathrm{F}_{t}$-measurable for any $t \in[0, T]$. We assume that the components $\mathbf{f}_{t}^{(i)}(i=1, \ldots, m)$ of this process are independent. Consider an Ito SDE in the integral form

$$
\begin{equation*}
\mathbf{x}_{t}=\mathbf{x}_{0}+\int_{0}^{t} \mathbf{a}\left(\mathbf{x}_{\tau}, \tau\right) d \tau+\int_{0}^{t} B\left(\mathbf{x}_{\tau}, \tau\right) d \mathbf{f}_{\tau}, \quad \mathbf{x}_{0}=\mathbf{x}(0, \omega), \quad \omega \in \Omega \tag{1}
\end{equation*}
$$

Here $\mathbf{x}_{t}$ is some $n$-dimensional stochastic process satisfying to the Ito SDE (1). The nonrandom functions a : $\mathbb{R}^{n} \times[0, T] \rightarrow \mathbb{R}^{n}, B: \mathbb{R}^{n} \times[0, T] \rightarrow \mathbb{R}^{n \times m}$ guarantee the existence and uniqueness up to stochastic equivalence of a solution of the Ito SDE (1) [1]. The second integral on the right-hand side of (11) is interpreted as an Ito stochastic integral. Let $\mathbf{x}_{0}$ be an $n$-dimensional random variable, which is $\mathrm{F}_{0}$-measurable and $\mathrm{M}\left\{\left|\mathbf{x}_{0}\right|^{2}\right\}<\infty\left(\mathrm{M}\right.$ denotes a mathematical expectation). We assume that $\mathbf{x}_{0}$ and $\mathbf{f}_{t}-\mathbf{f}_{0}$ are independent when $t>0$.

The most important feature of stochastic analogues of the Taylor formula [2], [3, [7]-12] for solutions of the Ito SDE (11) consists in the presence of iterated Ito and Stratonovich stochastic integrals. These stochastic integrals are complicated functionals from the components of the multidimensional Wiener process. In one of the most general forms of notation of the present paper, the aforementioned iterated Ito and Stratonovich stochastic integrals are given, respectively, by

$$
\begin{align*}
J\left[\psi^{(k)}\right]_{T, t} & =\int_{t}^{T} \psi_{k}\left(t_{k}\right) \ldots \int_{t}^{t_{2}} \psi_{1}\left(t_{1}\right) d \mathbf{w}_{t_{1}}^{\left(i_{1}\right)} \ldots d \mathbf{w}_{t_{k}}^{\left(i_{k}\right)}  \tag{2}\\
J^{*}\left[\psi^{(k)}\right]_{T, t} & =\int_{t}^{* T} \psi_{k}\left(t_{k}\right) \ldots \int_{t}^{* t_{2}} \psi_{1}\left(t_{1}\right) d \mathbf{w}_{t_{1}}^{\left(i_{1}\right)} \ldots d \mathbf{w}_{t_{k}}^{\left(i_{k}\right)} \tag{3}
\end{align*}
$$

where every $\psi_{l}(\tau)(l=1, \ldots, k)$ is a nonrandom function on $[t, T], \mathbf{w}_{\tau}^{(i)}=\mathbf{f}_{\tau}^{(i)}$ for $i=1, \ldots, m$ and $\mathbf{w}_{\tau}^{(0)}=\tau$,

$$
\int \text { and } \int^{*}
$$

denote Ito and Stratonovich stochastic integrals, respectively, $i_{1}, \ldots, i_{k}=0,1, \ldots, m$ (in this paper, we use the definition of the Stratonovich stochastic integral from [2]).

Consequently, the systems of stochastic integrals like (2), (3) play an important part in solving the problem of numerical integration of the Ito SDEs (1). In terms of the mean-square convergence criterion, the problem of efficient joint numerical modeling of the totalities of stochastic integrals of
the kind (2), (3) (the case of a multidimensional Wiener process) is not only important, but also sufficiently complex in both the theoretical and computational terms. We note that the aforementioned problem does not arise at using the Euler method for the Ito SDEs (1) [2], 7]. However, despite its simplicity, the Euler method under the standard conditions [2, [7] for coefficients of the Ito SDE (1) has the mean-square convergence order 0.5 [2], [7] and its accuracy is insufficient to solve a number of practical problems. This fact motivates one to construct numerical methods for the Ito SDEs (1) having higher orders of strong convergence.

It may seem at the first glance that the stochastic integrals from the families (21), (3) can be approximated by the multiple integral sums. However, this leads to partitioning of the interval of integration $[t, T]$ of the iterated stochastic integrals. The mentioned interval is already a small value because it represents a step of integration in the numerical methods for Ito SDEs. As the numerical experiments show [13], the above partitioning gives rise to an unacceptably high computing costs.

A number of publications are devoted to methods of numerical modeling of families of stochastic integrals like (2), (3), which do not use partitioning of the aforementioned interval of integration $[t, T]$ and converge in the mean-square sense. It was suggested in [7] to use converging in the meansquare sense trigonometric Fourier expansions of the Wiener processes, which underlie the iterated stochastic integral. By this method, the mean-square approximations of the simplest integrals like (2) of multiplicities 1 and $2\left(k=2 ; \psi_{1}(s), \psi_{2}(s) \equiv 1 ; i_{1}, i_{2}=0,1, \ldots, m\right)$ were obtained in [7]. These approximations were used in [7] to construct a numerical method for the Ito SDE (11), which under certain conditions [7] has the order 1.0 of the mean-square convergence and is known as the Milstein method.

A more general method of the mean-square approximation of the stochastic integrals like (3), which based on the generalized iterated Fourier series was proposed in [14, [15]. It enables one to use the complete orthonormal systems of Legendre polynomials and trigonometric functions in the space $L_{2}([t, T])$. In virtue of its characteristics, the method from [7] admits the application of only trigonometric basis functions.

In [2], 3, [16] ,17] an attempt was made to extend the method from [7] to the stochastic integrals like (3) for $k=3 ; \psi_{1}(s), \ldots, \psi_{3}(s) \equiv 1 ; i_{1}, \ldots, i_{3}=0,1, \ldots, m$.

We note that the methods [2, 3], [16], 17] $(k=3)$ and 14] $(k \geq 3)$ lead to iterated application of the operation of limit transition. As a result, these methods allow us to represent the integrals (3) as iterated series of products of standard Gaussian random variables (the operation of passing to the limit is carried out iteratively). This fact is essential and imposes some constraints related with the method of summation of the aforementioned series [2], [3], [14, [16], [17] if we consider the stochastic integrals like (2), (3) of multiplicities 3 and higher (we mean here at least triple integration over the Wiener processes). Additionally, the aforementioned methods in virtue of their features prevent precise calculation of the mean-square error of approximation with the exception of the simplest iterated stochastic integrals of multiplicity 2 . This means that at the stage of realization of the numerical methods for Ito SDEs, possibly, one will need to allow for the redundant terms of the expansions of iterated stochastic integrals, which increases the computing costs and reduces efficiency of the numerical methods.

We notice [2], 13] that to construct numerical methods for the Ito SDE (11) having orders 1.5 and 2.0 of strong convergence one has to approximate (proceeding from the mean-square convergence criterion) the stochastic integrals not only of multiplicities 1 and 2 , but also 3 and 4 from the families (2), (3). Some publications [2], 7], (8] contain the aforementioned numerical schemes with orders 1.5 and 2.0 of strong convergence but without the contained in them efficient procedures of the mean-square approximation of iterated stochastic integrals for the case of a multidimensional Wiener process, which corresponds to $i_{1}, \ldots, i_{4}=1, \ldots, m$ in (22), (3). Part of publications (see, for example, [2] , [8]) contain representations of the stochastic integrals of multiplicities 3 and 4 like (22), (3)) only for the simplest case $\psi_{1}(s), \ldots, \psi_{4}(s) \equiv 1, i_{1}=\ldots=i_{4}$ (representations based on the Hermit polynomials). Some publications [8] use other simplifying assumptions about the Ito SDE (1). For example, assumptions are made about additivity of the stochastic perturbation or its smallness,
which corresponds, respectively, to $B(\mathbf{x}, t) \equiv C(t)$ or $B(\mathbf{x}, t) \equiv \varepsilon D(\mathbf{x}, t)$. Here, $\varepsilon>0$ is a fixed small number and $C:[0, T] \rightarrow \mathbb{R}^{n \times m}, D: \mathbb{R}^{n} \times[0, T] \rightarrow \mathbb{R}^{n \times m}$.

In the case at hand, the problem of efficient joint numerical modeling of the iterated stochastic integrals from the families (22), (31) becomes simpler due to the absence of some terms in the expressions of the numerical methods or the possibility of disregarding some of the aforementioned terms. Also, one may encounter approximation method [18] for iterated stochastic integrals of multiplicity 3 from the familiy (2) for $\psi_{1}(s), \psi_{2}(s), \psi_{3}(s) \equiv 1\left(i_{1}, i_{2}, i_{3}=1, \ldots, m\right)$ based on partitioning of the interval of integration $[t, T]$ of the iterated stochastic integrals and using multiple integral sums whose disadvantages were mentioned above.

The present paper is devoted to the development of efficient procedures for joint numerical modeling of the iterated stochastic integrals from the families (2), (3) in accordance with the mean-square criterion of convergence. At that we do not use any essential simplifying assumptions, that is, the Wiener process involved in the Ito SDE (11) is assumed to be the multidimensional one which corresponds to the condition $i_{1}, \ldots, i_{k}=0,1, \ldots, m$ in (2), (3). In addition, it is assumed that the stochastic perturbation is nonadditive (the simplifying assumptions about the function $B: \mathbb{R}^{n} \times$ $[0, T] \rightarrow \mathbb{R}^{n \times m}$ involved in (1) are not introduced). Additionally, the functions $\psi_{1}(s), \ldots, \psi_{k}(s)$ in (2), (3) are, generally speaking, assumed to be different. Moreover, the assumption of commutativity [2], 3] of the stochastic perturbation is also not introduced.

More precisely, in this paper we consider the method of the mean-square approximation of iterated Ito stochastic integrals from the family (2), which is based on the generalized multiple (not iterated) Fourier series converging in the sense of norm in Hilbert space $L_{2}\left([t, T]^{k}\right)(k \in \mathbb{N})$ [13] (2006), [19][55]. Multiple series (the operation of limit transition is implemented only once) are more convenient for approximation than the iterated ones (iterated application of the operation of limit transition), since partial sums of multiple series converge for any possible case of convergence to infinity of their upper limits of summation (let us denote them as $p_{1}, \ldots, p_{k}$ ). For example, when $p_{1}=\ldots=p_{k}=$ $p \rightarrow \infty$. For iterated series, the condition $p_{1}=\ldots=p_{k}=p \rightarrow \infty$ obviously does not guarantee the convergence of this series. However, in [2] (Sect. 5.8, pp. 202-204), 3] (pp. 82-84), 16] (pp. 438-439), [17] (pp. 263-264) the authors use (without rigorous proof) the condition $p_{1}=p_{2}=p_{3}=p \rightarrow \infty$ within the frames of the approach based on the Karhunen-Loeve expansion of the Brownian bridge process [7] together with the Wong-Zakai approximation [56]-58. See discussions in [32] (Sect. 2.18, 6.2), [33], 34] (Sect. 2.6.2, 6.2) for details.

## 2. Numerical Schemes With the Orders 1.0 , 1.5, and 2.0 of Strong Convergence

Consider the partition $\left\{\tau_{j}\right\}_{j=0}^{N}$ of the segment $[0, T]$ with the partition rank $\Delta_{N}$ such that

$$
0=\tau_{0}<\tau_{1}<\ldots<\tau_{N}=T
$$

Denote by $\mathbf{y}_{\tau_{j}} \stackrel{\text { def }}{=} \mathbf{y}_{j} ; j=0,1, \ldots, N$ the discrete approximation of the process $\mathbf{x}_{t}, t \in[0, T]$ (solution of the Ito SDE (1) corresponding to the maximal step of discretization $\Delta_{N}$.

Definition 1 [2]. We will say that the discrete approximation (numerical method) $\mathbf{y}_{j} ; j=0,1, \ldots, N$ corresponding to the maximal step of discretization $\Delta_{N}$ converges strongly with the order $\gamma>0$ at the time instant $T$ to the process $\mathbf{x}_{t}, t \in[0, T]$ if there exist a constant $C>0$ independent of $\Delta_{N}$ and a number $\delta>0$ such that

$$
\begin{equation*}
\mathrm{M}\left\{\left|\mathbf{x}_{T}-\mathbf{y}_{T}\right|\right\} \leq C\left(\Delta_{N}\right)^{\gamma} \tag{4}
\end{equation*}
$$

for all $\Delta_{N} \in(0, \delta)$.

We note that the authors of some publications [7, [8] prefer to consider the mean-square convergence instead of the strong convergence.

Definition 2 7, [8. We will say that the numerical method $\mathbf{y}_{j} ; j=0,1, \ldots, N$ converges in the mean-square sense with the order $\gamma>0$ to the process $\mathbf{x}_{t}, t \in[0, T]$ if there exist a constant $C>0$ independent of $\Delta_{N}, j$ and a number $\delta>0$ such that

$$
\left(\mathrm{M}\left\{\left|\mathbf{x}_{j}-\mathbf{y}_{j}\right|^{2}\right\}\right)^{1 / 2} \leq C\left(\Delta_{N}\right)^{\gamma}
$$

for all $\Delta_{N} \in(0, \delta)$.
Here, $\mathbf{x}_{\tau_{j}} \stackrel{\text { def }}{=} \mathbf{x}_{j} ; j=0,1, \ldots, N$.
We notice that sometimes the condition (4) in Definition 1 is replaced by the condition (2)

$$
\mathrm{M}\left\{\left|\mathbf{x}_{j}-\mathbf{y}_{j}\right|\right\} \leq C\left(\Delta_{N}\right)^{\gamma} \quad(j=0,1, \ldots, N)
$$

At that, the constant $C$ is independent of $\Delta_{N}$ and $j$.
Strong convergence follows, obviously, from the mean-square convergence in virtue of the Lyapunov inequality. In what follows, we rely on Definition 1 of strong convergence.

Consider the following explicit one-step numerical method

$$
\begin{gather*}
\mathbf{y}_{p+1}=\mathbf{y}_{p}+\sum_{i=1}^{m} B_{i} \hat{I}_{(0) \tau_{p+1}, \tau_{p}}^{(i)}+\Delta \mathbf{a}+\sum_{i, j=1}^{m} G_{j} B_{i} \hat{I}_{(00) \tau_{p+1}, \tau_{p}}^{(j i)}+ \\
+\sum_{i=1}^{m}\left(G_{i} \mathbf{a}\left(\Delta \hat{I}_{(0) \tau_{p+1}, \tau_{p}}^{(i)}+\hat{I}_{(1) \tau_{p+1}, \tau_{p}}^{(i)}\right)-L B_{i} \hat{I}_{(1) \tau_{p+1}, \tau_{p}}^{(i)}\right)+ \\
\quad+\sum_{i, j, l=1}^{m} G_{l} G_{j} B_{i} \hat{I}_{(000) \tau_{p+1}, \tau_{p}}^{(l i)}+\frac{\Delta^{2}}{2} L \mathbf{a} \tag{5}
\end{gather*}
$$

corresponding to the constant discretization step $\Delta=T / N\left(\tau_{p}=p \Delta ; p=0,1, \ldots, N ; N>1\right)$, where $\hat{I}_{\left(l_{1} \ldots l_{k}\right) s, t}^{\left(i_{1} \ldots i_{k}\right)}$ denotes approximation of the iterated Ito stochastic integral

$$
\begin{equation*}
I_{\left(l_{1} \ldots l_{k}\right) s, t}^{\left(i_{1} \ldots i_{k}\right)}=\int_{t}^{s}\left(t-\tau_{k}\right)^{l_{k}} \ldots \int_{t}^{\tau_{2}}\left(t-\tau_{1}\right)^{l_{1}} d \mathbf{f}_{\tau_{1}}^{\left(i_{1}\right)} \ldots d \mathbf{\tau}_{\tau_{k}}^{\left(i_{k}\right)} \tag{6}
\end{equation*}
$$

and

$$
\begin{gathered}
L=\frac{\partial}{\partial t}+\sum_{i=1}^{n} \mathbf{a}_{i}(\mathbf{x}, t) \frac{\partial}{\partial \mathbf{x}_{i}}+\frac{1}{2} \sum_{j=1}^{m} \sum_{l, i=1}^{n} B_{l j}(\mathbf{x}, t) B_{i j}(\mathbf{x}, t) \frac{\partial^{2}}{\partial \mathbf{x}_{l} \partial \mathbf{x}_{i}}, \\
G_{i}=\sum_{j=1}^{n} B_{j i}(\mathbf{x}, t) \frac{\partial}{\partial \mathbf{x}_{j}} \quad(i=1, \ldots, m),
\end{gathered}
$$

$l_{1}, \ldots, l_{k}=0,1,2 \ldots ; i_{1}, \ldots, i_{k}=1, \ldots, m ; k=1,2, \ldots ; B_{i}$ and $B_{i j}$ are, respectively, the $i$ th column and $i j$ th element of the matrix function $B ; \mathbf{a}_{i}$ and $\mathbf{x}_{i}$ are, respectively, the $i$ th components of the vector function $\mathbf{a}$ and column $\mathbf{x}$; the columns

$$
B_{i}, \quad \mathbf{a}, \quad G_{j} B_{i}, \quad G_{i} \mathbf{a}, \quad L B_{i}, \quad G_{l} G_{j} B_{i}, \quad L \mathbf{a}
$$

are calculated at the point $\left(\mathbf{y}_{p}, p\right)$.
The numerical scheme (5) can be found, for example, in a somewhat different form in [2], [7], 8]. The difference here lies in that the author of this work used in (5) the relation

$$
\begin{equation*}
\Delta I_{(0) \tau_{p+1}, \tau_{p}}^{(i)}+I_{(1) \tau_{p+1}, \tau_{p}}^{(i)}=\int_{\tau_{p}}^{\tau_{p+1}} \int_{\tau_{p}}^{\tau} d \mathbf{f}_{s}^{(i)} d \tau \tag{7}
\end{equation*}
$$

which follows with probability 1 from the Ito formula and enables one to reduce by one the number of iterated Ito stochastic integrals to be approximated. This is due to the fact that the Ito stochastic integral on the right-hand side of (7) is expressed as a linear combination of the Ito stochastic integrals

$$
I_{(0) \tau_{p+1}, \tau_{p}}^{(i)} \quad \text { and } \quad I_{(1) \tau_{p+1}, \tau_{p}}^{(i)}
$$

whose approximations are already included in the right-hand side of (5).
It is common knowledge that under certain conditions [2] the discrete approximation (numerical method) (5) has the order 1.5 of strong convergence. Among the aforementioned conditions we note only the condition for approximations of the iterated Ito stochastic integrals involved in (5)

$$
\begin{equation*}
\mathrm{M}\left\{\left(I_{\left(l_{1} \ldots l_{k}\right) \tau_{p+1}, \tau_{p}}^{\left(i_{1} \ldots i_{k}\right)}-\hat{I}_{\left(l_{1} \ldots l_{k}\right) \tau_{p+1}, \tau_{p}}^{\left(i_{1} \ldots i_{k}\right)}\right)^{2}\right\} \leq C \Delta^{r} \tag{8}
\end{equation*}
$$

where $r=4$ and the constant $C$ is independent of $\Delta$, because the present paper deals mostly with the approximation of the aforementioned stochastic integrals.

Conditions somewhat different from [2] are given in [8]. Under them the numerical method (5) has the order 1.5 of the mean-square convergence.

Note that the Milstein method [7] (method with the order 1.0 of strong convergence) corresponds to the first line in (5).

Consider the explicit one-step numerical method with the order 2.0 of strong convergence given by

$$
\begin{gather*}
\mathbf{y}_{p+1}=\mathbf{y}_{p}+\sum_{i=1}^{m} B_{i} \hat{I}_{(0) \tau_{p+1}, \tau_{p}}^{(i)}+\Delta \mathbf{a}+\sum_{i, j=1}^{m} G_{j} B_{i} \hat{I}_{(00) \tau_{p+1}, \tau_{p}}^{(j i)}+ \\
+\sum_{i=1}^{m}\left(G_{i} \mathbf{a}\left(\Delta \hat{I}_{(0) \tau_{p+1}, \tau_{p}}^{(i)}+\hat{I}_{(1) \tau_{p+1}, \tau_{p}}^{(i)}\right)-L B_{i} \hat{I}_{(1) \tau_{p+1}, \tau_{p}}^{(i)}\right)+ \\
+\sum_{i, j, l=1}^{m} G_{l} G_{j} B_{i} \hat{I}_{(000) \tau_{p+1}, \tau_{p}}^{(l j i)}+\frac{\Delta^{2}}{2} L \mathbf{a}+ \\
+\sum_{i, j=1}^{m}\left(G_{0}^{(j)} L B_{i}\left(\hat{I}_{(10) \tau_{p+1}, \tau_{p}}^{(j i)}-\hat{I}_{(01) \tau_{p+1}, \tau_{p}}^{(j i)}\right)-L G_{j} B_{i} \hat{I}_{(10) \tau_{p+1}, \tau_{p}}^{(j i)}+\right. \\
\left.+G_{j} G_{i} \mathbf{a}\left(\hat{I}_{(01) \tau_{p+1}, \tau_{p}}^{(j i)}+\Delta \hat{I}_{(00) \tau_{p+1}, \tau_{p}}^{(j i)}\right)\right)+ \\
+\sum_{i, j, l, r=1}^{m} G_{r} G_{l} G_{j} B_{i} \hat{I}_{(0000) \tau_{p+1}, \tau_{p}}^{(r l j i)} \tag{9}
\end{gather*}
$$

where notation corresponds to (5).
The numerical scheme (9) can be found in another representation in [2], 8]. In this case the distinctions are due to the fact that along with (7) the author used in (9) the equalities

$$
\begin{align*}
I_{(01) \tau_{p+1}, \tau_{p}}^{(j i)}+\Delta I_{(00) \tau_{p+1}, \tau_{p}}^{(j i)} & =\int_{\tau_{p}}^{\tau_{p+1}} \int_{\tau_{p}}^{\theta} \int_{\tau_{p}}^{\tau} d \mathbf{f}_{s}^{(j)} d \mathbf{f}_{\tau}^{(i)} d \theta  \tag{10}\\
I_{(10) \tau_{p+1}, \tau_{p}}^{(j i)}-I_{(01) \tau_{p+1}, \tau_{p}}^{(j i)} & =\int_{\tau_{p}}^{\tau_{p+1}} \int_{\tau_{p}}^{\theta} \int_{\tau_{p}}^{\tau} d \mathbf{f}_{s}^{(j)} d \tau d \mathbf{f}_{\theta}^{(i)}
\end{align*}
$$

which follow with probability 1 from the Ito formula and enable one to reduce by one more unit the number of iterated Ito stochastic integrals to be approximated. This is due to the fact that the Ito stochastic integrals on the right-hand sides of (10) and (11) are expressed as linear combinations of the Ito stochastic integrals

$$
I_{(01) \tau_{p+1}, \tau_{p}}^{(j i)}, \quad I_{(10) \tau_{p+1}, \tau_{p}}^{(j i)}, \quad I_{(00) \tau_{p+1}, \tau_{p}}^{(j i)}
$$

whose approximations are already included in the right-hand side of (9).
We notice that under certain conditions [2] the numerical method (9) has the order 2.0 of strong convergence. Among the aforementioned conditions we mark only the condition (8) for $r=5$ intended for approximations of the iterated Ito stochastic integrals included in (9).

Some modifications of the numerical methods (5) and (9) were constructed in 2], 8. Among which there are finite-difference methods of the Runge-Kutta type as well as the implicit and twostep methods (also see [13], [19]-[22, , 30]-34]). In all aforementioned methods, however, a need arises for efficient joint mean-square approximation of the iterated Ito stochastic integrals of multiplicities 1 to 4. The collection of these integrals is the same as in the numerical methods (5) and (9).

## 3. Expansion of Iterated Ito Stochastic Integrals of Multiplicity $k(k \in \mathbb{N})$ Based on Generalized Multiple Fourier Series

An efficient mean-square approximation method for the iterated Ito stochastic integrals like (2) was proposed and developed by the author of this article in [13], [19]-[55] (see Theorems 1, 2 below). This method based on the generalized multiple Fourier series converging in the mean-square sense in the space $L_{2}\left([t, T]^{k}\right), k \in \mathbb{N}$. At that the method [13, [19]-55 allows to use different complete orthonormal systems of functions in the space $L_{2}\left([t, T]^{k}\right), k \in \mathbb{N}$. In this article, we use the system of Legendre polynomials, which has a series of advantages over the system of trigonometric functions in the framework of the considered problem [43, 44. Moreover, in this method the passage to the limit is carried out only once, which leads to a correct choice of the lengths of sequences of the standard Gaussian random variables required to approximate the iterated Ito stochastic integrals.

Suppose that every $\psi_{l}(\tau)(l=1, \ldots, k)$ is a nonrandom function from the space $L_{2}([t, T])$. Define the following function on the hypercube $[t, T]^{k}$

$$
K\left(t_{1}, \ldots, t_{k}\right)=\left\{\begin{array}{ll}
\psi_{1}\left(t_{1}\right) \ldots \psi_{k}\left(t_{k}\right) & \text { for } t_{1}<\ldots<t_{k}  \tag{12}\\
0 & \text { otherwise }
\end{array}, \quad t_{1}, \ldots, t_{k} \in[t, T], \quad k \geq 2\right.
$$

and $K\left(t_{1}\right) \equiv \psi_{1}\left(t_{1}\right)$ for $t_{1} \in[t, T]$.
Suppose that $\left\{\phi_{j}(x)\right\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_{2}([t, T])$. The function $K\left(t_{1}, \ldots, t_{k}\right)$ belongs to the space $L_{2}\left([t, T]^{k}\right)$. At this situation it is well known that the generalized multiple Fourier series of $K\left(t_{1}, \ldots, t_{k}\right) \in L_{2}\left([t, T]^{k}\right)$ is converging to $K\left(t_{1}, \ldots, t_{k}\right)$ in the hypercube $[t, T]^{k}$ in the mean-square sense, i.e.

$$
\lim _{p_{1}, \ldots, p_{k} \rightarrow \infty}\left\|K\left(t_{1}, \ldots, t_{k}\right)-\sum_{j_{1}=0}^{p_{1}} \ldots \sum_{j_{k}=0}^{p_{k}} C_{j_{k} \ldots j_{1}} \prod_{l=1}^{k} \phi_{j_{l}}\left(t_{l}\right)\right\|_{L_{2}\left([t, T]^{k}\right)}=0
$$

where

$$
\begin{equation*}
C_{j_{k} \ldots j_{1}}=\int_{[t, T]^{k}} K\left(t_{1}, \ldots, t_{k}\right) \prod_{l=1}^{k} \phi_{j_{l}}\left(t_{l}\right) d t_{1} \ldots d t_{k} \tag{13}
\end{equation*}
$$

is the Fourier coefficient,

$$
\|f\|_{L_{2}\left([t, T]^{k}\right)}=\left(\int_{t t, T]^{k}} f^{2}\left(t_{1}, \ldots, t_{k}\right) d t_{1} \ldots d t_{k}\right)^{1 / 2}
$$

and the Parceval equality

$$
\begin{equation*}
\int_{[t, T]^{k}} K^{2}\left(t_{1}, \ldots, t_{k}\right) d t_{1} \ldots d t_{k}=\lim _{p_{1}, \ldots, p_{k} \rightarrow \infty} \sum_{j_{1}=0}^{p_{1}} \ldots \sum_{j_{k}=0}^{p_{k}} C_{j_{k} \ldots j_{1}}^{2} \tag{14}
\end{equation*}
$$

takes place.
Consider the partition $\left\{\tau_{j}\right\}_{j=0}^{N}$ of $[t, T]$ such that

$$
\begin{equation*}
t=\tau_{0}<\ldots<\tau_{N}=T, \quad \Delta_{N}=\max _{0 \leq j \leq N-1} \Delta \tau_{j} \rightarrow 0 \quad \text { if } N \rightarrow \infty, \quad \Delta \tau_{j}=\tau_{j+1}-\tau_{j} \tag{15}
\end{equation*}
$$

Theorem 1 [13] (2006), [19]-[55]. Suppose that every $\psi_{l}(\tau)(l=1, \ldots, k)$ is a continuous nonrandom function on $[t, T]$ and $\left\{\phi_{j}(x)\right\}_{j=0}^{\infty}$ is a complete orthonormal system of continuous functions in the space $L_{2}([t, T])$. Then

$$
\begin{align*}
& J\left[\psi^{(k)}\right]_{T, t}={\underset{p_{1}, \ldots, p_{k} \rightarrow \infty}{\text { l.i.m. }} \sum_{j_{1}=0}^{p_{1}} \ldots \sum_{j_{k}=0}^{p_{k}} C_{j_{k} \ldots j_{1}}\left(\prod_{l=1}^{k} \zeta_{j_{l}}^{\left(i_{l}\right)}-\right.}^{\left.-\operatorname{li.i.m.~}_{N \rightarrow \infty} \sum_{\left(l_{1}, \ldots, l_{k}\right) \in \mathrm{G}_{k}} \phi_{j_{1}}\left(\tau_{l_{1}}\right) \Delta \mathbf{w}_{\tau_{l_{1}}}^{\left(i_{1}\right)} \ldots \phi_{j_{k}}\left(\tau_{l_{k}}\right) \Delta \mathbf{w}_{\tau_{l_{k}}}^{\left(i_{k}\right)}\right),}
\end{align*}
$$

where $J\left[\psi^{(k)}\right]_{T, t}$ is defined by (2),

$$
\begin{gathered}
\mathrm{G}_{k}=\mathrm{H}_{k} \backslash \mathrm{~L}_{k}, \quad \mathrm{H}_{k}=\left\{\left(l_{1}, \ldots, l_{k}\right): l_{1}, \ldots, l_{k}=0,1, \ldots, N-1\right\}, \\
\mathrm{L}_{k}=\left\{\left(l_{1}, \ldots, l_{k}\right): l_{1}, \ldots, l_{k}=0,1, \ldots, N-1 ; l_{g} \neq l_{r}(g \neq r) ; g, r=1, \ldots, k\right\},
\end{gathered}
$$

l.i.m. is a limit in the mean-square sense, $i_{1}, \ldots, i_{k}=0,1, \ldots, m$,

$$
\begin{equation*}
\zeta_{j}^{(i)}=\int_{t}^{T} \phi_{j}(s) d \mathbf{w}_{s}^{(i)} \tag{17}
\end{equation*}
$$

are independent standard Gaussian random variables for various i or $j($ if $i \neq 0), C_{j_{k} \ldots j_{1}}$ is the Fourier coefficient (13), $\Delta \mathbf{w}_{\tau_{j}}^{(i)}=\mathbf{w}_{\tau_{j+1}}^{(i)}-\mathbf{w}_{\tau_{j}}^{(i)}(i=0,1, \ldots, m),\left\{\tau_{j}\right\}_{j=0}^{N}$ is a partition of the interval $[t, T]$, which satisfies the condition (15).

It was shown in [20]-27, 30]-34] that Theorem 1 is valid for convergence in the mean of degree $2 n(n \in \mathbb{N})$. Moreover, the convergence with probability 1 in Theorem 1 is proved in [32]- 34, [63]. In addition, the complete orthonormal systems of Haar and Rademacher-Walsh functions in $L_{2}([t, T])$ also can be applied in Theorem 1 [13, [19-27, [30]-34. The modification of Theorem 1 for complete orthonormal with weigth $r(x) \geq 0$ systems of functions in the space $L_{2}([t, T])$ can be found in [31], [32]-34, [40. Application of Theorem 1 and Theorem 2 (see below) to the approximation of iterated stochastic integrals with respect to the infinite-dimensional $Q$-Wiener process contains in 32-34 (Chapter 7), 46, [55, 64, 66.

In order to evaluate the significance of Theorem 1 for practice we will demonstrate its transformed particular cases for $k=1, \ldots, 5$ [13 (2006), [19]-55] (the cases $k=6,7$ and $k>7(k \in \mathbb{N})$ can also be found in these papers)

$$
\begin{align*}
& J\left[\psi^{(1)}\right]_{T, t}=\underset{p_{1} \rightarrow \infty}{\operatorname{li.m} . \mathrm{m}} \sum_{j_{1}=0}^{p_{1}} C_{j_{1}} \zeta_{j_{1}}^{\left(i_{1}\right)},  \tag{18}\\
& J\left[\psi^{(2)}\right]_{T, t}=\underset{p_{1}, p_{2} \rightarrow \infty}{\text { l.i.m. }} \sum_{j_{1}=0}^{p_{1}} \sum_{j_{2}=0}^{p_{2}} C_{j_{2} j_{1}}\left(\zeta_{j_{1}}^{\left(i_{1}\right)} \zeta_{j_{2}}^{\left(i_{2}\right)}-\mathbf{1}_{\left\{i_{1}=i_{2} \neq 0\right\}} \mathbf{1}_{\left\{j_{1}=j_{2}\right\}}\right),  \tag{19}\\
& J\left[\psi^{(3)}\right]_{T, t}=\underset{p_{1}, p_{2}, p_{3} \rightarrow \infty}{\text { l.i.m. }} \sum_{j_{1}=0}^{p_{1}} \sum_{j_{2}=0}^{p_{2}} \sum_{j_{3}=0}^{p_{3}} C_{j_{3} j_{2} j_{1}}\left(\zeta_{j_{1}}^{\left(i_{1}\right)} \zeta_{j_{2}}^{\left(i_{2}\right)} \zeta_{j_{3}}^{\left(i_{3}\right)}-\right. \\
& \left.-\mathbf{1}_{\left\{i_{1}=i_{2} \neq 0\right\}} \mathbf{1}_{\left\{j_{1}=j_{2}\right\}} \zeta_{j_{3}}^{\left(i_{3}\right)}-\mathbf{1}_{\left\{i_{2}=i_{3} \neq 0\right\}} \mathbf{1}_{\left\{j_{2}=j_{3}\right\}} \zeta_{j_{1}}^{\left(i_{1}\right)}-\mathbf{1}_{\left\{i_{1}=i_{3} \neq 0\right\}} \mathbf{1}_{\left\{j_{1}=j_{3}\right\}} \zeta_{j_{2}}^{\left(i_{2}\right)}\right),  \tag{20}\\
& J\left[\psi^{(4)}\right]_{T, t}=\underset{p_{1}, \ldots, p_{4} \rightarrow \infty}{\text { l.i.m. }} \sum_{j_{1}=0}^{p_{1}} \ldots \sum_{j_{4}=0}^{p_{4}} C_{j_{4} \ldots j_{1}}\left(\prod_{l=1}^{4} \zeta_{j_{l}}^{\left(i_{l}\right)}-\right. \\
& -\mathbf{1}_{\left\{i_{1}=i_{2} \neq 0\right\}} \mathbf{1}_{\left\{j_{1}=j_{2}\right\}} \zeta_{j_{3}}^{\left(i_{3}\right)} \zeta_{j_{4}}^{\left(i_{4}\right)}-\mathbf{1}_{\left\{i_{1}=i_{3} \neq 0\right\}} \mathbf{1}_{\left\{j_{1}=j_{3}\right\}} \zeta_{j_{2}}^{\left(i_{2}\right)} \zeta_{j_{4}}^{\left(i_{4}\right)}- \\
& -\mathbf{1}_{\left\{i_{1}=i_{4} \neq 0\right\}} \mathbf{1}_{\left\{j_{1}=j_{4}\right\}} \zeta_{j_{2}}^{\left(i_{2}\right)} \zeta_{j_{3}}^{\left(i_{3}\right)}-\mathbf{1}_{\left\{i_{2}=i_{3} \neq 0\right\}} \mathbf{1}_{\left\{j_{2}=j_{3}\right\}} \zeta_{j_{1}}^{\left(i_{1}\right)} \zeta_{j_{4}}^{\left(i_{4}\right)}-
\end{align*}
$$

$$
\begin{align*}
& -\mathbf{1}_{\left\{i_{2}=i_{4} \neq 0\right\}} \mathbf{1}_{\left\{j_{2}=j_{4}\right\}} \zeta_{j_{1}}^{\left(i_{1}\right)} \zeta_{j_{3}}^{\left(i_{3}\right)}-\mathbf{1}_{\left\{i_{3}=i_{4} \neq 0\right\}} \mathbf{1}_{\left\{j_{3}=j_{4}\right\}} \zeta_{j_{1}}^{\left(i_{1}\right)} \zeta_{j_{2}}^{\left(i_{2}\right)}+ \\
& +\mathbf{1}_{\left\{i_{1}=i_{2} \neq 0\right\}} \mathbf{1}_{\left\{j_{1}=j_{2}\right\}} \mathbf{1}_{\left\{i_{3}=i_{4} \neq 0\right\}} \mathbf{1}_{\left\{j_{3}=j_{4}\right\}}+\mathbf{1}_{\left\{i_{1}=i_{3} \neq 0\right\}} \mathbf{1}_{\left\{j_{1}=j_{3}\right\}} \mathbf{1}_{\left\{i_{2}=i_{4} \neq 0\right\}} \mathbf{1}_{\left\{j_{2}=j_{4}\right\}}+ \\
& \left.+\mathbf{1}_{\left\{i_{1}=i_{4} \neq 0\right\}} \mathbf{1}_{\left\{j_{1}=j_{4}\right\}} \mathbf{1}_{\left\{i_{2}=i_{3} \neq 0\right\}} \mathbf{1}_{\left\{j_{2}=j_{3}\right\}}\right),  \tag{21}\\
& J\left[\psi^{(5)}\right]_{T, t}={\underset{p}{1, \ldots, p_{5} \rightarrow \infty}}_{\text {l.i.m. }} \sum_{j_{1}=0}^{p_{1}} \ldots \sum_{j_{5}=0}^{p_{5}} C_{j_{5} \ldots j_{1}}\left(\prod_{l=1}^{5} \zeta_{j_{l}}^{\left(i_{l}\right)}-\right. \\
& -\mathbf{1}_{\left\{i_{1}=i_{2} \neq 0\right\}} \mathbf{1}_{\left\{j_{1}=j_{2}\right\}} \zeta_{j_{3}}^{\left(i_{3}\right)} \zeta_{j_{4}}^{\left(i_{4}\right)} \zeta_{j_{5}}^{\left(i_{5}\right)}-\mathbf{1}_{\left\{i_{1}=i_{3} \neq 0\right\}} \mathbf{1}_{\left\{j_{1}=j_{3}\right\}} \zeta_{j_{2}}^{\left(i_{2}\right)} \zeta_{j_{4}}^{\left(i_{4}\right)} \zeta_{j_{5}}^{\left(i_{5}\right)}- \\
& -\mathbf{1}_{\left\{i_{1}=i_{4} \neq 0\right\}} \mathbf{1}_{\left\{j_{1}=j_{4}\right\}} \zeta_{j_{2}}^{\left(i_{2}\right)} \zeta_{j_{3}}^{\left(i_{3}\right)} \zeta_{j_{5}}^{\left(i_{5}\right)}-\mathbf{1}_{\left\{i_{1}=i_{5} \neq 0\right\}} \mathbf{1}_{\left\{j_{1}=j_{5}\right\}} \zeta_{j_{2}}^{\left(i_{2}\right)} \zeta_{j_{3}}^{\left(i_{3}\right)} \zeta_{j_{4}}^{\left(i_{4}\right)}- \\
& -\mathbf{1}_{\left\{i_{2}=i_{3} \neq 0\right\}} \mathbf{1}_{\left\{j_{2}=j_{3}\right\}} \zeta_{j_{1}}^{\left(i_{1}\right)} \zeta_{j_{4}}^{\left(i_{4}\right)} \zeta_{j_{5}}^{\left(i_{5}\right)}-\mathbf{1}_{\left\{i_{2}=i_{4} \neq 0\right\}} \mathbf{1}_{\left\{j_{2}=j_{4}\right\}} \zeta_{j_{1}}^{\left(i_{1}\right)} \zeta_{j_{3}}^{\left(i_{3}\right)} \zeta_{j_{5}}^{\left(i_{5}\right)}- \\
& -\mathbf{1}_{\left\{i_{2}=i_{5} \neq 0\right\}} \mathbf{1}_{\left\{j_{2}=j_{5}\right\}} \zeta_{j_{1}}^{\left(i_{1}\right)} \zeta_{j_{3}}^{\left(i_{3}\right)} \zeta_{j_{4}}^{\left(i_{4}\right)}-\mathbf{1}_{\left\{i_{3}=i_{4} \neq 0\right\}} \mathbf{1}_{\left\{j_{3}=j_{4}\right\}} \zeta_{j_{1}}^{\left(i_{1}\right)} \zeta_{j_{2}}^{\left(i_{2}\right)} \zeta_{j_{5}}^{\left(i_{5}\right)}- \\
& -\mathbf{1}_{\left\{i_{3}=i_{5} \neq 0\right\}} \mathbf{1}_{\left\{j_{3}=j_{5}\right\}} \zeta_{j_{1}}^{\left(i_{1}\right)} \zeta_{j_{2}}^{\left(i_{2}\right)} \zeta_{j_{4}}^{\left(i_{4}\right)}-\mathbf{1}_{\left\{i_{4}=i_{5} \neq 0\right\}} \mathbf{1}_{\left\{j_{4}=j_{5}\right\}} \zeta_{j_{1}}^{\left(i_{1}\right)} \zeta_{j_{2}}^{\left(i_{2}\right)} \zeta_{j_{3}}^{\left(i_{3}\right)}+ \\
& +\mathbf{1}_{\left\{i_{1}=i_{2} \neq 0\right\}} \mathbf{1}_{\left\{j_{1}=j_{2}\right\}} \mathbf{1}_{\left\{i_{3}=i_{4} \neq 0\right\}} \mathbf{1}_{\left\{j_{3}=j_{4}\right\}} \zeta_{j_{5}}^{\left(i_{5}\right)}+\mathbf{1}_{\left\{i_{1}=i_{2} \neq 0\right\}} \mathbf{1}_{\left\{j_{1}=j_{2}\right\}} \mathbf{1}_{\left\{i_{3}=i_{5} \neq 0\right\}} \mathbf{1}_{\left\{j_{3}=j_{5}\right\}} \zeta_{j_{4}}^{\left(i_{4}\right)}+ \\
& +\mathbf{1}_{\left\{i_{1}=i_{2} \neq 0\right\}} \mathbf{1}_{\left\{j_{1}=j_{2}\right\}} \mathbf{1}_{\left\{i_{4}=i_{5} \neq 0\right\}} \mathbf{1}_{\left\{j_{4}=j_{5}\right\}} \zeta_{j_{3}}^{\left(i_{3}\right)}+\mathbf{1}_{\left\{i_{1}=i_{3} \neq 0\right\}} \mathbf{1}_{\left\{j_{1}=j_{3}\right\}} \mathbf{1}_{\left\{i_{2}=i_{4} \neq 0\right\}} \mathbf{1}_{\left\{j_{2}=j_{4}\right\}} \zeta_{j_{5}}^{\left(i_{5}\right)}+ \\
& +\mathbf{1}_{\left\{i_{1}=i_{3} \neq 0\right\}} \mathbf{1}_{\left\{j_{1}=j_{3}\right\}} \mathbf{1}_{\left\{i_{2}=i_{5} \neq 0\right\}} \mathbf{1}_{\left\{j_{2}=j_{5}\right\}} \zeta_{j_{4}}^{\left(i_{4}\right)}+\mathbf{1}_{\left\{i_{1}=i_{3} \neq 0\right\}} \mathbf{1}_{\left\{j_{1}=j_{3}\right\}} \mathbf{1}_{\left\{i_{4}=i_{5} \neq 0\right\}} \mathbf{1}_{\left\{j_{4}=j_{5}\right\}} \zeta_{j_{2}}^{\left(i_{2}\right)}+ \\
& +\mathbf{1}_{\left\{i_{1}=i_{4} \neq 0\right\}} \mathbf{1}_{\left\{j_{1}=j_{4}\right\}} \mathbf{1}_{\left\{i_{2}=i_{3} \neq 0\right\}} \mathbf{1}_{\left\{j_{2}=j_{3}\right\}} \zeta_{j_{5}}^{\left(i_{5}\right)}+\mathbf{1}_{\left\{i_{1}=i_{4} \neq 0\right\}} \mathbf{1}_{\left\{j_{1}=j_{4}\right\}} \mathbf{1}_{\left\{i_{2}=i_{5} \neq 0\right\}} \mathbf{1}_{\left\{j_{2}=j_{5}\right\}} \zeta_{j_{3}}^{\left(i_{3}\right)}+ \\
& +\mathbf{1}_{\left\{i_{1}=i_{4} \neq 0\right\}} \mathbf{1}_{\left\{j_{1}=j_{4}\right\}} \mathbf{1}_{\left\{i_{3}=i_{5} \neq 0\right\}} \mathbf{1}_{\left\{j_{3}=j_{5}\right\}} \zeta_{j_{2}}^{\left(i_{2}\right)}+\mathbf{1}_{\left\{i_{1}=i_{5} \neq 0\right\}} \mathbf{1}_{\left\{j_{1}=j_{5}\right\}} \mathbf{1}_{\left\{i_{2}=i_{3} \neq 0\right\}} \mathbf{1}_{\left\{j_{2}=j_{3}\right\}} \zeta_{j_{4}}^{\left(i_{4}\right)}+ \\
& +\mathbf{1}_{\left\{i_{1}=i_{5} \neq 0\right\}} \mathbf{1}_{\left\{j_{1}=j_{5}\right\}} \mathbf{1}_{\left\{i_{2}=i_{4} \neq 0\right\}} \mathbf{1}_{\left\{j_{2}=j_{4}\right\}} \zeta_{j_{3}}^{\left(i_{3}\right)}+\mathbf{1}_{\left\{i_{1}=i_{5} \neq 0\right\}} \mathbf{1}_{\left\{j_{1}=j_{5}\right\}} \mathbf{1}_{\left\{i_{3}=i_{4} \neq 0\right\}} \mathbf{1}_{\left\{j_{3}=j_{4}\right\}} \zeta_{j_{2}}^{\left(i_{2}\right)}+ \\
& +\mathbf{1}_{\left\{i_{2}=i_{3} \neq 0\right\}} \mathbf{1}_{\left\{j_{2}=j_{3}\right\}} \mathbf{1}_{\left\{i_{4}=i_{5} \neq 0\right\}} \mathbf{1}_{\left\{j_{4}=j_{5}\right\}} \zeta_{j_{1}}^{\left(i_{1}\right)}+\mathbf{1}_{\left\{i_{2}=i_{4} \neq 0\right\}} \mathbf{1}_{\left\{j_{2}=j_{4}\right\}} \mathbf{1}_{\left\{i_{3}=i_{5} \neq 0\right\}} \mathbf{1}_{\left\{j_{3}=j_{5}\right\}} \zeta_{j_{1}}^{\left(i_{1}\right)}+ \\
& \left.+\mathbf{1}_{\left\{i_{2}=i_{5} \neq 0\right\}} \mathbf{1}_{\left\{j_{2}=j_{5}\right\}} \mathbf{1}_{\left\{i_{3}=i_{4} \neq 0\right\}} \mathbf{1}_{\left\{j_{3}=j_{4}\right\}} \zeta_{j_{1}}^{\left(i_{1}\right)}\right), \tag{22}
\end{align*}
$$

where $\mathbf{1}_{A}$ is the indicator of the set $A$.
For further consideration, let us consider the generalization of formulas (18)-(22) for the case of an arbitrary multiplicity $k(k \in \mathbb{N})$ of the iterated Ito stochastic integral $J\left[\psi^{(k)}\right]_{T, t}$ defined by (22). In order to do this, let us introduce some notations. Consider the unordered set $\{1,2, \ldots, k\}$ and separate it into two parts: the first part consists of $r$ unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k-2 r$ numbers. So, we have

$$
\begin{equation*}
(\{\underbrace{\left\{g_{1}, g_{2}\right\}, \ldots,\left\{g_{2 r-1}, g_{2 r}\right\}}_{\text {part } 1}\},\{\underbrace{q_{1}, \ldots, q_{k-2 r}}_{\text {part } 2}\}) \tag{23}
\end{equation*}
$$

where

$$
\left\{g_{1}, g_{2}, \ldots, g_{2 r-1}, g_{2 r}, q_{1}, \ldots, q_{k-2 r}\right\}=\{1,2, \ldots, k\}
$$

braces mean an unordered set, and parentheses mean an ordered set.
We will say that (23) is a partition and consider the sum with respect to all possible partitions

$$
\begin{equation*}
\sum_{\substack{\left(\left\{\left\{g_{1}, g_{2}\right\}, \ldots,\left\{g_{2 r-1}, g_{2 r}\right\}\right\},\left\{q_{1}, \ldots, q_{k-2 r}\right\}\right) \\\left\{g_{1}, g_{2}, \ldots, g_{2 r-1}, g_{2 r}, q_{1}, \ldots, q_{k-2 r}\right\}=\{1,2, \ldots, k\}}} a_{g_{1} g_{2}, \ldots, g_{2 r-1} g_{2 r}, q_{1} \ldots q_{k-2 r}} \tag{24}
\end{equation*}
$$

Below there are several examples of sums in the form (24)

$$
\begin{aligned}
& \sum_{\substack{\left(\left\{g_{1}, g_{2}\right\}\right) \\
\left\{g_{1}, g_{2}\right\}=\{1,2\}}} a_{g_{1} g_{2}}=a_{12}, \\
& \sum_{\substack{\left(\left\{\left\{g_{1}, g_{2}\right\},\left\{g_{3}, g_{4}\right\}\right\}\right) \\
\left\{g_{1}, g_{2}, g_{3}, g_{4}\right\}=\{1,2,3,4\}}} a_{g_{1} g_{2} g_{3} g_{4}}=a_{1234}+a_{1324}+a_{2314}, \\
& \sum_{\substack{\left(\left\{g_{1}, g_{2}\right\},\left\{q_{1}, q_{2}\right\}\right) \\
\left\{g_{1}, g_{2}, q_{1}, q_{2}\right\}=\{1,2,3,4\}}} a_{g_{1} g_{2}, q_{1} q_{2}}= \\
& =a_{12,34}+a_{13,24}+a_{14,23}+a_{23,14}+a_{24,13}+a_{34,12}, \\
& \sum_{\substack{\left(\left\{g_{1}, g_{2}\right\},\left\{q_{1}, q_{2}, q_{3}\right\}\right) \\
\left\{g_{1}, g_{2}, q_{1}, q_{2}, q_{3}\right\}=\{1,2,3,4,5\}}} a_{g_{1} g_{2}, q_{1} q_{2} q_{3}}= \\
& =a_{12,345}+a_{13,245}+a_{14,235}+a_{15,234}+a_{23,145}+a_{24,135}+ \\
& +a_{25,134}+a_{34,125}+a_{35,124}+a_{45,123}, \\
& \sum_{\substack{\left(\left\{\left\{g_{1}, g_{2}\right\},\left\{g_{3}, g_{4}\right\}\right\},\left\{q_{1}\right\}\right) \\
\left\{g_{1}, g_{2}, g_{3}, g_{4}, q_{1}\right\}=\{1,2,3,4,5\}}} a_{g_{1} g_{2}, g_{3} g_{4}, q_{1}}= \\
& =a_{12,34,5}+a_{13,24,5}+a_{14,23,5}+a_{12,35,4}+a_{13,25,4}+a_{15,23,4}+ \\
& +a_{12,54,3}+a_{15,24,3}+a_{14,25,3}+a_{15,34,2}+a_{13,54,2}+a_{14,53,2}+ \\
& +a_{52,34,1}+a_{53,24,1}+a_{54,23,1} .
\end{aligned}
$$

Now we can write (16) as

$$
\begin{gather*}
J\left[\psi^{(k)}\right]_{T, t}=\sum_{p_{1}, \ldots, p_{k} \rightarrow \infty}^{\text {l.i.m. }} \sum_{j_{1}=0}^{p_{1}} \ldots \sum_{j_{k}=0}^{p_{k}} C_{j_{k} \ldots j_{1}}\left(\prod_{l=1}^{k} \zeta_{j_{l}}^{\left(i_{l}\right)}+\sum_{r=1}^{[k / 2]}(-1)^{r} \times\right. \\
\left.\left.\left.\times \sum_{\substack{\left(\left\{\left\{g_{1}, g_{2}\right\}, \ldots,\left\{g_{2 r-1}, g_{2 r}\right\}\right\},\left\{q_{1}, \ldots, q_{k-2 r}\right\}\right) \\
\left\{g_{1}, g_{2}, \ldots, g_{2 r-1}, g_{2 r}, q_{1}, \ldots, q_{k-2 r}\right\}=\{1,2, \ldots, k\}}} \prod_{s=1}^{r} \mathbf{1}_{\left\{i_{g_{2 s-1}}\right.}=i_{g_{2 s}} \neq 0\right\} \mathbf{1}_{\left\{j_{g_{2 s-1}}\right.}=j_{g_{2 s}}\right\} \prod_{l=1}^{k-2 r} \zeta_{j_{q_{l}}}^{\left(i_{q_{l}}\right)}\right) \tag{25}
\end{gather*}
$$

where $[x]$ is an integer part of a real number $x$; another notations are the same as in Theorem 1.
In particular, from (25) for $k=5$ we obtain

$$
\begin{gathered}
J\left[\psi^{(5)}\right]_{T, t}=\sum_{p_{1}, \ldots, p_{5} \rightarrow \infty}^{\text {li.m. }} \sum_{j_{1}=0}^{p_{1}} \ldots \sum_{j_{5}=0}^{p_{5}} C_{j_{5} \ldots j_{1}}\left(\prod_{l=1}^{5} \zeta_{j_{l}}^{\left(i_{l}\right)}-\right. \\
-\sum_{\substack{\left(\left\{g_{1}, g_{2}\right\},\left\{q_{1}, q_{2}, q_{3}\right\}\right) \\
\left\{g_{1}, g_{2}, q_{1}, q_{2}, q_{3}\right\}=\{1,2,3,4,5\}}} \mathbf{1}_{\left\{i_{g_{1}}=i_{g_{2}} \neq 0\right\}} \mathbf{1}_{\left\{j_{g_{1}}=j_{g_{2}}\right\}} \prod_{l=1}^{3} \zeta_{j_{q_{l}}}^{\left(i_{q_{l}}\right)}+ \\
\left.+\sum_{\substack{\left(\left\{\left\{g_{1}, g_{2}\right\},\left\{g_{3}, g_{4}\right\}\right\},\left\{q_{1}\right\}\right) \\
\left\{g_{1}, g_{2}, g_{3}, g_{4}, q_{1}\right\}=\{1,2,3,4,5\}}} \mathbf{1}_{\left\{i_{g_{1}}=i_{g_{2}} \neq 0\right\}} \mathbf{1}_{\left\{j_{g_{1}}=j_{\left.g_{2}\right\}}\right\}} \mathbf{1}_{\left\{i_{g_{3}}=i_{g_{4}} \neq 0\right\}} \mathbf{1}_{\left\{j_{g_{3}}=j_{g_{4}}\right\}} \zeta_{j_{q_{1}}}^{\left(i_{q_{1}}\right)}\right)
\end{gathered}
$$

The last equality obviously agrees with (22).
Let us consider the generalization of Theorem 1 for the case of an arbitrary complete orthonormal systems of functions in the space $L_{2}([t, T])$ and $\psi_{1}(\tau), \ldots, \psi_{k}(\tau) \in L_{2}([t, T])$.

Theorem 232 (Sect. 1.11), 39 (Sect. 15). Suppose that $\psi_{1}(\tau), \ldots, \psi_{k}(\tau) \in L_{2}([t, T])$ and $\left\{\phi_{j}(x)\right\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_{2}([t, T])$. Then the following expansion

$$
\begin{align*}
& \quad J\left[\psi^{(k)}\right]_{T, t}=\sum_{p_{1}, \ldots, p_{k} \rightarrow \infty}^{\text {li.m. }} \sum_{j_{1}=0}^{p_{1}} \ldots \sum_{j_{k}=0}^{p_{k}} C_{j_{k} \ldots j_{1}}\left(\prod_{l=1}^{k} \zeta_{j_{l}}^{\left(i_{l}\right)}+\sum_{r=1}^{[k / 2]}(-1)^{r} \times\right. \\
& \left.\times \sum_{\substack{\left(\left\{\left\{g_{1}, g_{2}\right\}, \ldots,\left\{g_{2 r-1}, g_{2 r}\right\}\right\},\left\{q_{1}, \ldots, q_{k-2 r}\right\}\right) \\
\left\{g_{1}, g_{2}, \ldots, g_{2 r-1}, g_{2 r}, q_{1}, \ldots, q_{k-2 r}\right\}=\{1,2, \ldots, k\}}} \prod_{s=1}^{r} \mathbf{1}_{\left\{i_{g_{2 s-1}}=i_{g_{2 s}} \neq 0\right\}} \mathbf{1}_{\left\{j_{g_{2 s-1}}=j_{g_{2 s}}\right\}} \prod_{l=1}^{k-2 r} \zeta_{j_{q_{l}}}^{\left(i_{q_{l}}\right)}\right) \tag{26}
\end{align*}
$$

converging in the mean-square sense is valid, where $[x]$ is an integer part of a real number $x$; another notations are the same as in Theorem 1.

It should be noted that an analogue of Theorem 2 was considered in 68. Note that we use another notations [32] (Sect. 1.11), 39] (Sect. 15) in comparison with 68. Moreover, the proof of an analogue of Theorem 2 from [68] is somewhat different from the proof given in 32] (Sect. 1.11), [39] (Sect. 15).

## 4. Expansion of Iterated Stratonovich Stochastic Integrals of Multiplicities 1 to 6

As it turned out [23]-27], 30]-34], 47, [71, [72] Theorems 1, 2 can be adapted for the iterated Stratonovich stochastic integrals (3). At that the expansions of the integrals (3) turn out to be much simpler than the expansions of the iterated Ito stochastic integrals (2). Let us first present some old results as the following theorem.

Theorem 3 [23]-[27, [30]-34], 47]. Assume that the following conditions are fulfilled:

1. $\left\{\phi_{j}(x)\right\}_{j=0}^{\infty}$ is the complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_{2}([t, T])$.
2. The function $\psi_{2}(\tau)$ is continuously differentiable at the interval $[t, T]$, and the functions $\psi_{1}(\tau)$, $\psi_{3}(\tau)$ are twice continuously differentiable at the interval $[t, T]$ (in (27) and (29)).

Then, the iterated Stratonovich stochastic integrals (3) of multiplicities 2-4 are expanded into the mean-square converging multiple series

$$
\begin{align*}
& J^{*}\left[\psi^{(2)}\right]_{T, t}=\underset{q_{1}, q_{2} \rightarrow \infty}{\operatorname{l.i.m.}} \sum_{j_{1}=0}^{q_{1}} \sum_{j_{2}=0}^{q_{2}} C_{j_{2} j_{1}} \zeta_{j_{1}}^{\left(i_{1}\right)} \zeta_{j_{2}}^{\left(i_{2}\right)},  \tag{27}\\
& J^{*}\left[\psi^{(3)}\right]_{T, t}=\underset{q_{1}, q_{2}, q_{3} \rightarrow \infty}{\operatorname{li.m.}} \sum_{j_{1}=0}^{q_{1}} \sum_{j_{2}=0}^{q_{2}} \sum_{j_{3}=0}^{q_{3}} C_{j_{3} j_{2} j_{1}} \zeta_{j_{1}}^{\left(i_{1}\right)} \zeta_{j_{2}}^{\left(i_{2}\right)} \zeta_{j_{3}}^{\left(i_{3}\right)},  \tag{28}\\
& J^{*}\left[\psi^{(3)}\right]_{T, t}=\underset{q \rightarrow \infty}{\operatorname{li.m.}} \sum_{j_{1}, j_{2}, j_{3}=0}^{q} C_{j_{3} j_{2} j_{1}} \zeta_{j_{1}}^{\left(i_{1}\right)} \zeta_{j_{2}}^{\left(i_{2}\right)} \zeta_{j_{3}}^{\left(i_{3}\right)},  \tag{29}\\
& J^{*}\left[\psi^{(4)}\right]_{T, t}=\underset{q \rightarrow \infty}{\operatorname{li.m.m}} \sum_{j_{1}, j_{2}, j_{3}, j_{4}=0}^{q} C_{j_{4} j_{3} j_{2} j_{1}} \zeta_{j_{1}}^{\left(i_{1}\right)} \zeta_{j_{2}}^{\left(i_{2}\right)} \zeta_{j_{3}}^{\left(i_{3}\right)} \zeta_{j_{4}}^{\left(i_{4}\right)}, \tag{30}
\end{align*}
$$

where we assume that $i_{1}, i_{2}, i_{3}=1, \ldots, m$ in (27)-(29) and $i_{1}, \ldots, i_{4}=0,1, \ldots, m$ in (30). Additionally, we assume in (28) and (30) that $\psi_{1}(\tau), \ldots, \psi_{4}(\tau) \equiv 1$. Another notations are the same as in Theorems 1, 2.

Recently, a new approach to the expansion and mean-square approximation of iterated Stratonovich stochastic integrals has been obtained 32] (Sect. 2.10-2.16), 37] (Sect. 5-11), 47 (Sect. 13-19), 49] (Sect. 7-13), 71] (Sect. 4-9), 72]. Let us formulate four theorems that were obtained using this approach.

Theorem 4 [32, [37], 47], [49], [71]. Suppose that $\left\{\phi_{j}(x)\right\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_{2}([t, T])$. Furthermore, let $\psi_{1}(\tau), \psi_{2}(\tau)$, $\psi_{3}(\tau)$ are continuously differentiable nonrandom functions on $[t, T]$. Then, for the iterated Stratonovich stochastic integral of third multiplicity

$$
J^{*}\left[\psi^{(3)}\right]_{T, t}=\int_{t}^{* T} \psi_{3}\left(t_{3}\right) \int_{t}^{* t_{3}} \psi_{2}\left(t_{2}\right) \int_{t}^{* t_{2}} \psi_{1}\left(t_{1}\right) d \mathbf{w}_{t_{1}}^{\left(i_{1}\right)} d \mathbf{w}_{t_{2}}^{\left(i_{2}\right)} d \mathbf{w}_{t_{3}}^{\left(i_{3}\right)} \quad\left(i_{1}, i_{2}, i_{3}=0,1, \ldots, m\right)
$$

the following relations

$$
\begin{gather*}
J^{*}\left[\psi^{(3)}\right]_{T, t}=\operatorname{li.i.m.~}_{p \rightarrow \infty} \sum_{j_{1}, j_{2}, j_{3}=0}^{p} C_{j_{3} j_{2} j_{1}} \zeta_{j_{1}}^{\left(i_{1}\right)} \zeta_{j_{2}}^{\left(i_{2}\right)} \zeta_{j_{3}}^{\left(i_{3}\right)},  \tag{31}\\
\mathrm{M}\left\{\left(J^{*}\left[\psi^{(3)}\right]_{T, t}-\sum_{j_{1}, j_{2}, j_{3}=0}^{p} C_{j_{3} j_{2} j_{1}} \zeta_{j_{1}}^{\left(i_{1}\right)} \zeta_{j_{2}}^{\left(i_{2}\right)} \zeta_{j_{3}}^{\left(i_{3}\right)}\right)^{2}\right\} \leq \frac{C}{p}
\end{gather*}
$$

are fulfilled, where $i_{1}, i_{2}, i_{3}=0,1, \ldots, m$ in (31) and $i_{1}, i_{2}, i_{3}=1, \ldots, m$ in (32), constant $C$ is independent of $p$,

$$
C_{j_{3} j_{2} j_{1}}=\int_{t}^{T} \psi_{3}\left(t_{3}\right) \phi_{j_{3}}\left(t_{3}\right) \int_{t}^{t_{3}} \psi_{2}\left(t_{2}\right) \phi_{j_{2}}\left(t_{2}\right) \int_{t}^{t_{2}} \psi_{1}\left(t_{1}\right) \phi_{j_{1}}\left(t_{1}\right) d t_{1} d t_{2} d t_{3}
$$

and

$$
\zeta_{j}^{(i)}=\int_{t}^{T} \phi_{j}(\tau) d \mathbf{f}_{\tau}^{(i)}
$$

are independent standard Gaussian random variables for various $i$ or $j$ (in the case when $i \neq 0$ ); another notations are the same as in Theorems 1, 2.

Theorem 5 [32], [37], [47], 49, [71. Let $\left\{\phi_{j}(x)\right\}_{j=0}^{\infty}$ be a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_{2}([t, T])$. Furthermore, let $\psi_{1}(\tau), \ldots, \psi_{4}(\tau)$ be continuously differentiable nonrandom functions on $[t, T]$. Then, for the iterated Stratonovich stochastic integral of fourth multiplicity

$$
\begin{equation*}
J^{*}\left[\psi^{(4)}\right]_{T, t}=\int_{t}^{* T} \psi_{4}\left(t_{4}\right) \int_{t}^{* t_{4}} \psi_{3}\left(t_{3}\right) \int_{t}^{*^{t_{3}}} \psi_{2}\left(t_{2}\right) \int_{t}^{* t_{2}} \psi_{1}\left(t_{1}\right) d \mathbf{w}_{t_{1}}^{\left(i_{1}\right)} d \mathbf{w}_{t_{2}}^{\left(i_{2}\right)} d \mathbf{w}_{t_{3}}^{\left(i_{3}\right)} d \mathbf{w}_{t_{4}}^{\left(i_{4}\right)} \tag{33}
\end{equation*}
$$

the following relations

$$
\begin{align*}
& J^{*}\left[\psi^{(4)}\right]_{T, t}=\underset{p \rightarrow \infty}{\operatorname{li.m} .} \sum_{j_{1}, j_{2}, j_{3}, j_{4}=0}^{p} C_{j_{4} j_{3} j_{2} j_{1}} \zeta_{j_{1}}^{\left(i_{1}\right)} \zeta_{j_{2}}^{\left(i_{2}\right)} \zeta_{j_{3}}^{\left(i_{3}\right)} \zeta_{j_{4}}^{\left(i_{4}\right)},  \tag{34}\\
& \mathrm{M}\left\{\left(J^{*}\left[\psi^{(4)}\right]_{T, t}-\sum_{j_{1}, j_{2}, j_{3}, j_{4}=0}^{p} C_{j_{4} j_{3} j_{2} j_{1}} \zeta_{j_{1}}^{\left(i_{1}\right)} \zeta_{j_{2}}^{\left(i_{2}\right)} \zeta_{j_{3}}^{\left(i_{3}\right)} \zeta_{j_{4}}^{\left(i_{4}\right)}\right)^{2}\right\} \leq \frac{C}{p^{1-\varepsilon}}
\end{align*}
$$

are fulfilled, where $i_{1}, \ldots, i_{4}=0,1, \ldots, m$ in (33), (34) and $i_{1}, \ldots, i_{4}=1, \ldots, m$ in (35), constant $C$ does not depend on $p, \varepsilon$ is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space $L_{2}([t, T])$ and $\varepsilon=0$ for the case of complete orthonormal system of trigonometric functions in the space $L_{2}([t, T])$,

$$
\begin{gathered}
C_{j_{4} j_{3} j_{2} j_{1}}= \\
=\int_{t}^{T} \psi_{4}\left(t_{4}\right) \phi_{j_{4}}\left(t_{4}\right) \int_{t}^{t_{4}} \psi_{3}\left(t_{3}\right) \phi_{j_{3}}\left(t_{3}\right) \int_{t}^{t_{3}} \psi_{2}\left(t_{2}\right) \phi_{j_{2}}\left(t_{2}\right) \int_{t}^{t_{2}} \psi_{1}\left(t_{1}\right) \phi_{j_{1}}\left(t_{1}\right) d t_{1} d t_{2} d t_{3} d t_{4}
\end{gathered}
$$

another notations are the same as in Theorem 4.
Theorem 6 [32, [37, [47, 49, [71. Assume that $\left\{\phi_{j}(x)\right\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_{2}([t, T])$ and $\psi_{1}(\tau), \ldots, \psi_{5}(\tau)$ are continuously differentiable nonrandom functions on $[t, T]$. Then, for the iterated Stratonovich stochastic integral of fifth multiplicity

$$
\begin{equation*}
J^{*}\left[\psi^{(5)}\right]_{T, t}=\int_{t}^{* T} \psi_{5}\left(t_{5}\right) \ldots \int_{t}^{* t_{2}} \psi_{1}\left(t_{1}\right) d \mathbf{w}_{t_{1}}^{\left(i_{1}\right)} \ldots d \mathbf{w}_{t_{5}}^{\left(i_{5}\right)} \tag{36}
\end{equation*}
$$

the following relations

$$
\begin{gather*}
J^{*}\left[\psi^{(5)}\right]_{T, t}=\operatorname{li.i.m.}_{p \rightarrow \infty} \sum_{j_{1}, \ldots, j_{5}=0}^{p} C_{j_{5} \ldots j_{1}} \zeta_{j_{1}}^{\left(i_{1}\right)} \ldots \zeta_{j_{5}}^{\left(i_{5}\right)},  \tag{37}\\
\mathrm{M}\left\{\left(J^{*}\left[\psi^{(5)}\right]_{T, t}-\sum_{j_{1}, \ldots, j_{5}=0}^{p} C_{j_{5} \ldots j_{1}} \zeta_{j_{1}}^{\left(i_{1}\right)} \ldots \zeta_{j_{5}}^{\left(i_{5}\right)}\right)^{2}\right\} \leq \frac{C}{p^{1-\varepsilon}}
\end{gather*}
$$

are fulfilled, where $i_{1}, \ldots, i_{5}=0,1, \ldots, m$ in (36), (37) and $i_{1}, \ldots, i_{5}=1, \ldots, m$ in (38), constant $C$ is independent of $p, \varepsilon$ is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space $L_{2}([t, T])$ and $\varepsilon=0$ for the case of complete orthonormal system of trigonometric functions in the space $L_{2}([t, T])$,

$$
C_{j_{5} \ldots j_{1}}=\int_{t}^{T} \psi_{5}\left(t_{5}\right) \phi_{j_{5}}\left(t_{5}\right) \ldots \int_{t}^{t_{2}} \psi_{1}\left(t_{1}\right) \phi_{j_{1}}\left(t_{1}\right) d t_{1} \ldots d t_{5}
$$

another notations are the same as in Theorems 4, 5.
Theorem 7 [32], 37], 47], 49], [72]. Suppose that $\left\{\phi_{j}(x)\right\}_{j=0}^{\infty}$ is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space $L_{2}([t, T])$. Then, for the iterated Stratonovich stochastic integral of sixth multiplicity

$$
\begin{equation*}
J_{T, t}^{*\left(i_{1} \ldots i_{6}\right)}=\int_{t}^{* T} \ldots \int_{t}^{* t_{2}} d \mathbf{w}_{t_{1}}^{\left(i_{1}\right)} \ldots d \mathbf{w}_{t_{6}}^{\left(i_{6}\right)} \tag{39}
\end{equation*}
$$

the following expansion

$$
J_{T, t}^{*\left(i_{1} \ldots i_{6}\right)}=\operatorname{limim.~}_{p \rightarrow \infty} \sum_{j_{1}, \ldots, j_{6}=0}^{p} C_{j_{6} \ldots j_{1}} \zeta_{j_{1}}^{\left(i_{1}\right)} \ldots \zeta_{j_{6}}^{\left(i_{6}\right)}
$$

that converges in the mean-square sense is valid, where $i_{1}, \ldots, i_{6}=0,1, \ldots, m$,

$$
C_{j_{6} \ldots j_{1}}=\int_{t}^{T} \phi_{j_{6}}\left(t_{6}\right) \ldots \int_{t}^{t_{2}} \phi_{j_{1}}\left(t_{1}\right) d t_{1} \ldots d t_{6}
$$

another notations are the same as in Theorems 4-6.
5. Legendre Polynomial-Based Approximation of the Iterated Ito and Stratonovich Stochastic Integrals Used in the Applications

We notice that the collection of iterated Ito stochastic integrals used in the numerical methods (5), (9) is given by

$$
\begin{equation*}
I_{(0) T, t}^{\left(i_{1}\right)}, \quad I_{(1) T, t}^{\left(i_{1}\right)}, \quad I_{(00) T, t}^{\left(i_{1} i_{2}\right)}, \quad I_{(000) T, t}^{\left(i_{1} i_{2} i_{3}\right)}, \quad I_{(01) T, t}^{\left(i_{1} i_{2}\right)}, \quad I_{(10) T, t}^{\left(i_{1} i_{2}\right)}, \quad I_{(0000) T, t}^{\left(i_{1} i_{2} i_{3} i_{4}\right)} \tag{40}
\end{equation*}
$$

where $i_{1}, \ldots, i_{4}=1, \ldots, m$.
The functions $K\left(t_{1}, \ldots, t_{k}\right)$ like (12) for the collection (40) are given, respectively, by

$$
\begin{gathered}
K_{0}\left(t_{1}\right) \equiv 1, \quad K_{1}\left(t_{1}\right)=t-t_{1}, \quad K_{00}\left(t_{1}, t_{2}\right)=\mathbf{1}_{\left\{t_{1}<t_{2}\right\}} \\
K_{000}\left(t_{1}, t_{2}, t_{3}\right)=\mathbf{1}_{\left\{t_{1}<t_{2}<t_{3}\right\}}, \quad K_{01}\left(t_{1}, t_{2}\right)=\left(t-t_{2}\right) \mathbf{1}_{\left\{t_{1}<t_{2}\right\}} \\
K_{10}\left(t_{1}, t_{2}\right)=\left(t-t_{1}\right) \mathbf{1}_{\left\{t_{1}<t_{2}\right\}}, \quad K_{0000}\left(t_{1}, \ldots, t_{4}\right)=\mathbf{1}_{\left\{t_{1}<t_{2}<t_{3}<t_{4}\right\}},
\end{gathered}
$$

where $t_{1}, \ldots, t_{4} \in[t, T]$ and $\mathbf{1}_{A}$ is the indicator of the set $A$.
For a finite-degree polynomial, the simplest (having a finite number of terms) expansion into Fourier series by the complete orthonormal system of functions in the space $L_{2}([t, T])$ is the FourierLegendre series expansion. The polynomial functions are included in the functions $K_{1}\left(t_{1}\right), K_{01}\left(t_{1}, t_{2}\right)$, $K_{10}\left(t_{1}, t_{2}\right)$ as their components. Therefore, it is logical to expect that the simplest expansions of these functions into multiple Fourier series are their Fourier-Legendre expansions.

The following example illustrates rather well the noticed feature.
Consider the approximation $I_{(1) T, t}^{\left(i_{1}\right) q}$ of the stochastic integral $I_{(1) T, t}^{\left(i_{1}\right)}$ based on the expansion of the Brownian bridge process into the trigonometric Fourier series with random coefficients [7]

$$
\begin{equation*}
I_{(1) T, t}^{\left(i_{1}\right) q}=-\frac{(T-t)^{3 / 2}}{2}\left(\zeta_{0}^{\left(i_{1}\right)}-\frac{\sqrt{2}}{\pi}\left(\sum_{r=1}^{q} \frac{1}{r} \zeta_{2 r-1}^{\left(i_{1}\right)}+\sqrt{\alpha_{q}} \xi_{q}^{\left(i_{1}\right)}\right)\right) \tag{41}
\end{equation*}
$$

where

$$
\xi_{q}^{\left(i_{1}\right)}=\frac{1}{\sqrt{\alpha_{q}}} \sum_{r=q+1}^{\infty} \frac{1}{r} \zeta_{2 r-1}^{\left(i_{1}\right)}, \quad \alpha_{q}=\frac{\pi^{2}}{6}-\sum_{r=1}^{q} \frac{1}{r^{2}},
$$

where $\zeta_{0}^{\left(i_{1}\right)}, \zeta_{2 r-1}^{\left(i_{1}\right)}, \xi_{q}^{\left(i_{1}\right)} ; r=1, \ldots, q ; i_{1}=1, \ldots, m$ are independent standard Gaussian random variables.

On the other hand, it is possible to obtain the following equality

$$
\begin{equation*}
I_{(1) T, t}^{\left(i_{1}\right)}=-\frac{(T-t)^{3 / 2}}{2}\left(\zeta_{0}^{\left(i_{1}\right)}+\frac{1}{\sqrt{3}} \zeta_{1}^{\left(i_{1}\right)}\right) \tag{42}
\end{equation*}
$$

which is valid with probability 1 and based on the expansion of the function $t-t_{1}$ into the FourierLegendre series at the interval $[t, T]$ (this expansion has just two terms).

The above example demonstrates the advantage of the Legendre polynomials over the trigonometric functions in the context of the issue under consideration. More detailed comparison can be found in [32]-34, 43], 44].

We notice that, as was established in [13, [19]-27], 30]-34], in the Fourier method (Theorem 1) it is also possible to use the Haar and Rademacher-Walsh functions (also see Theorem 2). However, in [13], [19]-27, [30]-34] it was shown that the expansions of the iterated Ito stochastic integrals (2) of multiplicities 1 and 2 obtained with the use of Theorem 1 and systems of Haar and RademacherWalsh functions are overcomplicated as compared with their analogues obtained on the basis of the Legendre polynomials. In this connection, practical application of such expansions is hindered.

Consider approximations of the remaining stochastic integrals from the family (40) obtained using Theorems 1, 2 and complete orthonormal system of Legendre polynomials in the space $L_{2}([t, T])$. First, we consider approximations of stochastic integrals of multiplicities 1 and 2

$$
\begin{gather*}
I_{(0) T, t}^{\left(i_{1}\right)}=\sqrt{T-t} \zeta_{0}^{\left(i_{1}\right)},  \tag{43}\\
I_{(00) T, t}^{\left(i_{1} i_{2}\right) q}=I_{(00) T, t}^{*\left(i_{1} i_{2}\right) q}-\frac{1}{2} \mathbf{1}_{\left\{i_{1}=i_{2}\right\}}(T-t), \\
I_{(00) T, t}^{*\left(i_{1} i_{2}\right) q}=\frac{T-t}{2}\left(\zeta_{0}^{\left(i_{1}\right)} \zeta_{0}^{\left(i_{2}\right)}+\sum_{i=1}^{q} \frac{1}{\sqrt{4 i^{2}-1}}\left(\zeta_{i-1}^{\left(i_{1}\right)} \zeta_{i}^{\left(i_{2}\right)}-\zeta_{i}^{\left(i_{1}\right)} \zeta_{i-1}^{\left(i_{2}\right)}\right)\right),  \tag{45}\\
I_{(10) T, t}^{\left(i_{1} i_{2}\right) q}=I_{(10) T, t}^{*\left(i_{1} i_{2}\right) q}+\frac{1}{4} \mathbf{1}_{\left\{i_{1}=i_{2}\right\}}(T-t)^{2}, \quad I_{(01) T, t}^{\left(i_{1} i_{2}\right) q}=I_{(01) T, t}^{*\left(i_{1} i_{2}\right) q}+\frac{1}{4} \mathbf{1}_{\left\{i_{1}=i_{2}\right\}}(T-t)^{2}, \\
\left.+\sum_{i=0}^{q}\left(\frac{(i+2) \zeta_{i}^{\left(i_{1}\right)} \zeta_{i+2}^{\left(i_{2}\right)}-(i+1) \zeta_{i+2}^{\left(i_{1}\right)} \zeta_{i}^{\left(i_{2}\right)}}{\sqrt{(2 i+1)(2 i+5)}(2 i+3)}-\frac{\zeta_{i}^{\left(i_{1}\right)} \zeta_{i}^{\left(i_{2}\right)}}{(2 i-1)(2 i+3)}\right)\right), \\
I_{(01) T, t}^{*\left(i_{2}\right) q}=-\frac{T-t}{2} I_{(00) T, t}^{*\left(i_{1} i_{2}\right) q}-\frac{(T-t)^{2}}{4}\left(\frac{1}{\sqrt{3}} \zeta_{0}^{\left(i_{1}\right)} \zeta_{1}^{\left(i_{2}\right)}+\right. \\
I_{(10) T, t}^{*\left(i_{1} i_{2}\right) q}=-\frac{T-t}{2} I_{(00) T, t}^{*\left(i_{1} i_{2}\right) q}-\frac{(T-t)^{2}}{4}\left(\frac{1}{\sqrt{3}} \zeta_{0}^{\left(i_{2}\right)} \zeta_{1}^{\left(i_{1}\right)}+\right. \\
\left.+\sum_{i=0}^{q}\left(\frac{(i+1) \zeta_{i+2}^{\left(i_{2}\right)} \zeta_{i}^{\left(i_{1}\right)}-(i+2) \zeta_{i}^{\left(i_{2}\right)} \zeta_{i+2}^{\left(i_{1}\right)}}{\sqrt{(2 i+1)(2 i+5)}(2 i+3)}+\frac{\zeta_{i}^{\left(i_{1}\right)} \zeta_{i}^{\left(i_{2}\right)}}{(2 i-1)(2 i+3)}\right)\right)
\end{gather*}
$$

where here and below

$$
I_{\left(l_{1} \ldots l_{k}\right) s, t}^{*\left(i_{1} \ldots i_{k}\right) q} \quad \text { and } \quad I_{\left(l_{1} \ldots l_{k}\right) s, t}^{\left(i_{1} \ldots i_{k}\right) q}
$$

are the approximations of the iterated Stratonovich and Ito stochastic integrals like

$$
\begin{equation*}
I_{\left(l_{1} \ldots l_{k}\right) s, t}^{*\left(i_{1} \ldots i_{k}\right)}=\int_{t}^{*^{s}}\left(t-\tau_{k}\right)^{l_{k}} \ldots \int_{t}^{* \tau_{2}}\left(t-\tau_{1}\right)^{l_{1}} d \mathbf{f}_{\tau_{1}}^{\left(i_{1}\right)} \ldots d \mathbf{f}_{\tau_{k}}^{\left(i_{k}\right)} \tag{48}
\end{equation*}
$$

and, correspondingly, like (6) ; $\zeta_{j}^{(i)}$ are independent standard Gaussian random variables for various $i$ or $j ; j=0,1, \ldots, p+2 ; i=1, \ldots, m$.

Calculate the mean-square errors of approximations (44)-(47). A precise formula for pairwise different $i_{1}, \ldots, i_{k}=1, \ldots, m$ was established in [13, 31]-34, 38]

$$
\begin{equation*}
\mathrm{M}\left\{\left(J\left[\psi^{(k)}\right]_{T, t}-J\left[\psi^{(k)}\right]_{T, t}^{q}\right)^{2}\right\}=\int_{[t, T]^{k}} K^{2}\left(t_{1}, \ldots, t_{k}\right) d t_{1} \ldots d t_{k}-\sum_{j_{1}, \ldots, j_{k}=0}^{q} C_{j_{k} \ldots j_{1}}^{2} \tag{49}
\end{equation*}
$$

where in virtue of the Parseval equality (14) the right-hand side of (49) tends to zero for $q \rightarrow \infty$; $J\left[\psi^{(k)}\right]_{T, t}$ has the form (2), and $J\left[\psi^{(k)}\right]_{T, t}^{q}$ is the approximation of $J\left[\psi^{(k)}\right]_{T, t}$ defined as the prelimit expression in (26) for $p_{1}=\ldots=p_{k}=q$ (also see the prelimit expressions in (18)-(22)); the sense of the rest notations is the same as in Theorems 1, 2.

The following formula [13, 31]-34, [38] takes place

$$
\begin{gather*}
\mathrm{M}\left\{\left(J\left[\psi^{(2)}\right]_{T, t}-J\left[\psi^{(2)}\right]_{T, t}^{q}\right)^{2}\right\}= \\
=\int_{[t, T]^{2}} K^{2}\left(t_{1}, t_{2}\right) d t_{1} d t_{2}-\sum_{j_{1}, j_{2}=0}^{q} C_{j_{2} j_{1}}^{2}-\sum_{j_{1}, j_{2}=0}^{q} C_{j_{1} j_{2}} C_{j_{2} j_{1}} \quad\left(i_{1}=i_{2}\right), \tag{50}
\end{gather*}
$$

where notations are the same as in (49).
The value $\mathrm{M}\left\{\left(J\left[\psi^{(k)}\right]_{T, t}-J\left[\psi^{(k)}\right]_{T, t}^{q}\right)^{2}\right\}$ can be calculated exactly.
Theorem 8 32] (Sect. 1.12), 38 (Sect. 6). Suppose that $\left\{\phi_{j}(x)\right\}_{j=0}^{\infty}$ is an arbitrary complete orthonormal system of functions in the space $L_{2}([t, T])$ and $\psi_{1}(\tau), \ldots, \psi_{k}(\tau) \in L_{2}([t, T]), i_{1}, \ldots, i_{k}=$ $1, \ldots, m$. Then

$$
\begin{gather*}
\mathrm{M}\left\{\left(J\left[\psi^{(k)}\right]_{T, t}-J\left[\psi^{(k)}\right]_{T, t}^{q}\right)^{2}\right\}=\int_{[t, T]^{k}} K^{2}\left(t_{1}, \ldots, t_{k}\right) d t_{1} \ldots d t_{k}- \\
-\sum_{j_{1}, \ldots, j_{k}=0}^{q} C_{j_{k} \ldots j_{1}} \mathrm{M}\left\{J\left[\psi^{(k)}\right]_{T, t} \sum_{\left(j_{1}, \ldots, j_{k}\right)} \int_{t}^{T} \phi_{j_{k}}\left(t_{k}\right) \ldots \int_{t}^{t_{2}} \phi_{j_{1}}\left(t_{1}\right) d \mathbf{f}_{t_{1}}^{\left(i_{1}\right)} \ldots d \mathbf{f}_{t_{k}}^{\left(i_{k}\right)}\right\}, \tag{51}
\end{gather*}
$$

where $i_{1}, \ldots, i_{k}=1, \ldots, m ;$ the expression

$$
\sum_{\left(j_{1}, \ldots, j_{k}\right)}
$$

means the sum with respect to all possible permutations $\left(j_{1}, \ldots, j_{k}\right)$. At the same time if $j_{r}$ swapped with $j_{q}$ in the permutation $\left(j_{1}, \ldots, j_{k}\right)$, then $i_{r}$ swapped with $i_{q}$ in the permutation $\left(i_{1}, \ldots, i_{k}\right)$; another notations are the same as in Theorems 1, 2.

Using (49) and (50), we get

$$
\begin{gather*}
\mathrm{M}\left\{\left(I_{(00) T, t}^{\left(i_{1} i_{2}\right)}-I_{(00) T, t}^{\left(i_{1} i_{2}\right) q}\right)^{2}\right\}=\frac{(T-t)^{2}}{2}\left(\frac{1}{2}-\sum_{i=1}^{q} \frac{1}{4 i^{2}-1}\right)\left(i_{1} \neq i_{2}\right),  \tag{52}\\
\mathrm{M}\left\{\left(I_{(10) T, t}^{\left(i_{1} i_{2}\right)}-I_{(10) T, t}^{\left(i_{1} i_{2}\right) q}\right)^{2}\right\}=\mathrm{M}\left\{\left(I_{(01) T, t}^{\left(i_{1} i_{2}\right)}-I_{(01) T, t}^{\left(i_{1} i_{2}\right) q}\right)^{2}\right\}=\frac{(T-t)^{4}}{16} \times \\
\times\left(\frac{5}{9}-2 \sum_{i=2}^{q} \frac{1}{4 i^{2}-1}-\sum_{i=1}^{q} \frac{1}{(2 i-1)^{2}(2 i+3)^{2}}-\sum_{i=0}^{q} \frac{(i+2)^{2}+(i+1)^{2}}{(2 i+1)(2 i+5)(2 i+3)^{2}}\right) \tag{53}
\end{gather*}
$$

for $i_{1} \neq i_{2}$ and

$$
\begin{gather*}
\mathrm{M}\left\{\left(I_{(10) T, t}^{\left(i_{1} i_{1}\right)}-I_{(10) T, t}^{\left(i_{1} i_{1}\right) q}\right)^{2}\right\}=\mathrm{M}\left\{\left(I_{(01) T, t}^{\left(i_{1} i_{1}\right)}-I_{(01) T, t}^{\left(i_{1} i_{1}\right) q}\right)^{2}\right\}= \\
=\frac{(T-t)^{4}}{16}\left(\frac{1}{9}-\sum_{i=0}^{q} \frac{1}{(2 i+1)(2 i+5)(2 i+3)^{2}}-2 \sum_{i=1}^{q} \frac{1}{(2 i-1)^{2}(2 i+3)^{2}}\right) . \tag{54}
\end{gather*}
$$

Let us consider the numerical modeling of the iterated Ito stochastic integral of multiplicity 3 $I_{(000) T, t}^{\left(i_{1} i_{2} i_{3}\right)}$. Using Theorems 1, 2 for the case $k=3$ (see (20)), we obtain

$$
\begin{align*}
I_{(000) T, t}^{\left(i_{1} i_{2} i_{3}\right) q} & =\sum_{j_{1}, j_{2}, j_{3}=0}^{q} C_{j_{3} j_{2} j_{1}}\left(\zeta_{j_{1}}^{\left(i_{1}\right)} \zeta_{j_{2}}^{\left(i_{2}\right)} \zeta_{j_{3}}^{\left(i_{3}\right)}-\mathbf{1}_{\left\{i_{1}=i_{2}\right\}} \mathbf{1}_{\left\{j_{1}=j_{2}\right\}} \zeta_{j_{3}}^{\left(i_{3}\right)}-\right. \\
& \left.-\mathbf{1}_{\left\{i_{2}=i_{3}\right\}} \mathbf{1}_{\left\{j_{2}=j_{3}\right\}} \zeta_{j_{1}}^{\left(i_{1}\right)}-\mathbf{1}_{\left\{i_{1}=i_{3}\right\}} \mathbf{1}_{\left\{j_{1}=j_{3}\right\}} \zeta_{j_{2}}^{\left(i_{2}\right)}\right), \tag{55}
\end{align*}
$$

where $i_{1}, i_{2}, i_{3}=1, \ldots, m$ and

$$
\begin{gather*}
C_{j_{3} j_{2} j_{1}}=\int_{t}^{T} \phi_{j_{3}}(z) \int_{t}^{z} \phi_{j_{2}}(y) \int_{t}^{y} \phi_{j_{1}}(x) d x d y d z= \\
=\frac{\sqrt{\left(2 j_{1}+1\right)\left(2 j_{2}+1\right)\left(2 j_{3}+1\right)}}{8}(T-t)^{3 / 2} \bar{C}_{j_{3} j_{2} j_{1}}  \tag{56}\\
\bar{C}_{j_{3} j_{2} j_{1}}=\int_{-1}^{1} P_{j_{3}}(z) \int_{-1}^{z} P_{j_{2}}(y) \int_{-1}^{y} P_{j_{1}}(x) d x d y d z \tag{57}
\end{gather*}
$$

where $P_{i}(x)(i=0,1,2, \ldots)$ is the Legendre polynomial.

For the case $i_{1}=i_{2}=i_{3}$, one can use the well known equality which follows from the Ito formula and is valid with probability 1 [2]

$$
\begin{equation*}
I_{(000) T, t}^{\left(i_{1} i_{1} i_{1}\right)}=\frac{1}{6}(T-t)^{3 / 2}\left(\left(\zeta_{0}^{\left(i_{1}\right)}\right)^{3}-3 \zeta_{0}^{\left(i_{1}\right)}\right) \tag{58}
\end{equation*}
$$

The procedure of numerical modeling of the iterated Ito stochastic integral $I_{(000) T, t}^{\left(i_{1} i_{2} i_{3}\right)}$ may follow (55)-(58). The Fourier-Legendre coefficients $\bar{C}_{j_{3} j_{2} j_{1}}$ of the form (57) being precisely calculable for the given number $q$ by PYTHON, DERIVE or MAPLE. The mean-square error of approximation is checked by (49) for $k=3$ as well as by the formulas established in [31]-34], [38]

$$
\begin{align*}
& \mathrm{M}\left\{\left(J\left[\psi^{(3)}\right]_{T, t}-J\left[\psi^{(3)}\right]_{T, t}^{q}\right)^{2}\right\}=\int_{[t, T]^{3}} K^{2}\left(t_{1}, t_{2}, t_{3}\right) d t_{1} d t_{2} d t_{3}- \\
&  \tag{59}\\
& -\sum_{j_{3}, j_{2}, j_{1}=0}^{q} C_{j_{3} j_{2} j_{1}}^{2}-\sum_{j_{3}, j_{2}, j_{1}=0}^{q} C_{j_{3} j_{1} j_{2}} C_{j_{3} j_{2} j_{1}} \quad\left(i_{1}=i_{2} \neq i_{3}\right),
\end{align*}
$$

$$
\mathrm{M}\left\{\left(J\left[\psi^{(3)}\right]_{T, t}-J\left[\psi^{(3)}\right]_{T, t}^{q}\right)^{2}\right\}=\int_{[t, T]^{3}} K^{2}\left(t_{1}, t_{2}, t_{3}\right) d t_{1} d t_{2} d t_{3}-
$$

$$
\begin{equation*}
-\sum_{j_{3}, j_{2}, j_{1}=0}^{q} C_{j_{3} j_{2} j_{1}}^{2}-\sum_{j_{3}, j_{2}, j_{1}=0}^{q} C_{j_{2} j_{3} j_{1}} C_{j_{3} j_{2} j_{1}} \quad\left(i_{1} \neq i_{2}=i_{3}\right) \tag{60}
\end{equation*}
$$

$$
\begin{align*}
& \mathrm{M}\left\{\left(J\left[\psi^{(3)}\right]_{T, t}-J\left[\psi^{(3)}\right]_{T, t}^{q}\right)^{2}\right\}=\int_{[t, T]^{3}} K^{2}\left(t_{1}, t_{2}, t_{3}\right) d t_{1} d t_{2} d t_{3}- \\
&  \tag{61}\\
& -\sum_{j_{3}, j_{2}, j_{1}=0}^{q} C_{j_{3} j_{2} j_{1}}^{2}-\sum_{j_{3}, j_{2}, j_{1}=0}^{q} C_{j_{3} j_{2} j_{1}} C_{j_{1} j_{2} j_{3}} \quad\left(i_{1}=i_{3} \neq i_{2}\right) .
\end{align*}
$$

The following estimate [31-34, 38 can also be applied for the case $k=3$

$$
\begin{gather*}
\mathrm{M}\left\{\left(J\left[\psi^{(k)}\right]_{T, t}-J\left[\psi^{(k)}\right]_{T, t}^{q}\right)^{2}\right\} \leq \\
\leq k!\left(\int_{t, T]^{k}} K^{2}\left(t_{1}, \ldots, t_{k}\right) d t_{1} \ldots d t_{k}-\sum_{j_{1}, \ldots, j_{k}=0}^{q} C_{j_{k} \ldots j_{1}}^{2}\right) \tag{62}
\end{gather*}
$$

where $i_{1}, \ldots, i_{k}=1, \ldots, m$ and $0<T-t<\infty$ or $i_{1}, \ldots, i_{k}=0,1, \ldots, m$ and $0<T-t<1$.
In particular, for the pairwise different $i_{1}, i_{2}, i_{3}=1, \ldots, m$ and $q=6$ we get from (49)

$$
\begin{equation*}
\mathrm{M}\left\{\left(I_{(000) T, t}^{\left(i_{1} i_{2} i_{3}\right)}-I_{(000) T, t}^{\left(i_{1} i_{2} i_{3}\right) 6}\right)^{2}\right\} \approx 0.01956(T-t)^{3} \tag{63}
\end{equation*}
$$

Taking into consideration that $T-t$ is the integration step of numerical methods for the Ito SDE (11) and $T-t$ is a sufficiently small number, we get that already for $q=6$ the mean-square error of approximation of the stochastic integral $I_{000}^{\left(i_{1} i_{2} i_{3}\right)}$ is sufficiently small as well (see (63)).

Consider now the iterated Ito stochastic integral $I_{(0000) T, t}^{\left(i_{1} i_{2} i_{3} i_{4}\right)}$ of multiplicity 4. Using Theorems 1, 2, we get the representation

$$
\begin{gather*}
I_{(0000) T, t}^{\left(i_{1} i_{2} i_{3} i_{4}\right) q}=\sum_{j_{1}, j_{2}, j_{3}, j_{4}=0}^{q} C_{j_{4} j_{3} j_{2} j_{1}}\left(\zeta_{j_{1}}^{\left(i_{1}\right)} \zeta_{j_{2}}^{\left(i_{2}\right)} \zeta_{j_{3}}^{\left(i_{3}\right)} \zeta_{j_{4}}^{\left(i_{4}\right)}-\right. \\
-\mathbf{1}_{\left\{i_{1}=i_{2}\right\}} \mathbf{1}_{\left\{j_{1}=j_{2}\right\}} \zeta_{j_{3}}^{\left(i_{3}\right)} \zeta_{j_{4}}^{\left(i_{4}\right)}-\mathbf{1}_{\left\{i_{1}=i_{3}\right\}} \mathbf{1}_{\left\{j_{1}=j_{3}\right\}} \zeta_{j_{2}}^{\left(i_{2}\right)} \zeta_{j_{4}}^{\left(i_{4}\right)}- \\
-\mathbf{1}_{\left\{i_{1}=i_{4}\right\}} \mathbf{1}_{\left\{j_{1}=j_{4}\right\}} \zeta_{j_{2}}^{\left(i_{2}\right)} \zeta_{j_{3}}^{\left(i_{3}\right)}-\mathbf{1}_{\left\{i_{2}=i_{3}\right\}} \mathbf{1}_{\left\{j_{2}=j_{3}\right\}} \zeta_{j_{1}}^{\left(i_{1}\right)} \zeta_{j_{4}}^{\left(i_{4}\right)}- \\
-\mathbf{1}_{\left\{i_{2}=i_{4}\right\}} \mathbf{1}_{\left\{j_{2}=j_{4}\right\}} \zeta_{\left.j_{1}\right)}^{\left(i_{1}\right)} \zeta_{j_{3}}^{\left(i_{3}\right)}-\mathbf{1}_{\left\{i_{3}=i_{4}\right\}} \mathbf{1}_{\left\{j_{3}=j_{4}\right\}} \zeta_{j_{1}}^{\left(i_{1}\right)} \zeta_{j_{2}}^{\left(i_{2}\right)}+ \\
+\mathbf{1}_{\left\{i_{1}=i_{2}\right\}} \mathbf{1}_{\left\{j_{1}=j_{2}\right\}} \mathbf{1}_{\left\{i_{3}=i_{4}\right\}} \mathbf{1}_{\left\{j_{3}=j_{4}\right\}}+\mathbf{1}_{\left\{i_{1}=i_{3}\right\}} \mathbf{1}_{\left\{j_{1}=j_{3}\right\}} \mathbf{1}_{\left\{i_{2}=i_{4}\right\}} \mathbf{1}_{\left\{j_{2}=j_{4}\right\}}+ \\
\left.+\mathbf{1}_{\left\{i_{1}=i_{4}\right\}} \mathbf{1}_{\left\{j_{1}=j_{4}\right\}} \mathbf{1}_{\left\{i_{2}=i_{3}\right\}} \mathbf{1}_{\left\{j_{2}=j_{3}\right\}}\right) \tag{64}
\end{gather*}
$$

where $i_{1}, i_{2}, i_{3}, i_{4}=1, \ldots, m$ and

$$
\begin{gathered}
C_{j_{4} j_{3} j_{2} j_{1}}=\int_{t}^{T} \phi_{j_{4}}(u) \int_{t}^{u} \phi_{j_{3}}(z) \int_{t}^{z} \phi_{j_{2}}(y) \int_{t}^{y} \phi_{j_{1}}(x) d x d y d z d u= \\
=\frac{\sqrt{\left(2 j_{1}+1\right)\left(2 j_{2}+1\right)\left(2 j_{3}+1\right)\left(2 j_{4}+1\right)}}{16} \Delta^{2} \bar{C}_{j_{4} j_{3} j_{2} j_{1}} \\
\bar{C}_{j_{4} j_{3} j_{2} j_{1}}=\int_{-1}^{1} P_{j_{4}}(u) \int_{-1}^{u} P_{j_{3}}(z) \int_{-1}^{z} P_{j_{2}}(y) \int_{-1}^{y} P_{j_{1}}(x) d x d y d z d u
\end{gathered}
$$

where $P_{i}(x)(i=0,1,2, \ldots)$ is the Legendre polynomial.
For precise calculation of the Fourier-Legendre coefficients $C_{j_{4} j_{3} j_{2} j_{1}}$ we can use the previous recommendations and check the mean-square error of approximation of the iterated Ito stochastic integral $I_{(0000) T, t}^{\left(i_{1} i_{2} i_{3} i_{4}\right)}$, for example, using the estimate (62) for $k=4$.

In particular, for pairwise different $i_{1}, \ldots, i_{4}=1, \ldots, m$ we get from (49) with regard for smallness of $T-t$ already for $q=2$ a sufficiently good accuracy of the mean-square approximation

$$
\begin{equation*}
\mathrm{M}\left\{\left(I_{(0000) T, t}^{\left(i_{1} i_{2} i_{3} i_{4}\right)}-I_{(0000) T, t}^{\left(i_{1} i_{2} i_{3} i_{4}\right) 2}\right)^{2}\right\} \approx 0.0236084(T-t)^{4} \tag{65}
\end{equation*}
$$

We notice that at deriving (63) and (65) the coefficients $\bar{C}_{j_{3} j_{2} j_{1}}$ and $\bar{C}_{j_{4} j_{3} j_{2} j_{1}}$ were precisely calculated using the DERIVE package.

Note that the formulas (27)-(30) are simpler than (19)-(21). However, calculation of the meansquare approximation error for the iterated Stratonovich stochastic integrals (3) turned out more complex than for the iterated Ito stochastic integrals (22) 32]-34, 42, [52].

## 6. Algorithms of Numerical Modeling With the Orders 1.5 and 2.0 of Strong Convergence

We formulate in algorithmic form the above formulas and recommendations for the numerical method of the order 1.5 of strong convergence. We assume that the necessary Fourier-Legendre coefficients $\bar{C}_{j_{3} j_{2} j_{1}}, \bar{C}_{j_{4} j_{3} j_{2} j_{1}}$ are already calculated. In particular, several tables of the precisely calculated Fourier-Legendre coefficients $\bar{C}_{j_{3} j_{2} j_{1}}, \bar{C}_{j_{4} j_{3} j_{2} j_{1}}$ were presented in [13, [31]-34]. These coefficients were calculated by DERIVE. It should be noted that in [61, 62] the database with 270,000 precisely calculated Fourier-Legendre coefficients is presented. In 61] 62] we used the PYTHON programming language.

## Algorithm. 1.

Step 1. Given are the initial parameters of the problem such as the interval of integration $[0, T]$, step of integration $\Delta$ (for example, constant $\Delta=T / N, N \geq 1$, although a variable step of integration is admissible), initial condition $\mathbf{y}_{0}$, and constant $C$ involved in the condition (8).

Step 2. Assume that $p=0$.
Step 3. Selection of the minimal natural numbers $q$ and $q_{1}\left(q \ll q_{1}\right)$ ensuring the necessary accuracy of approximation of the stochastic integrals

$$
I_{(00) \tau_{p+1}, \tau_{p}}^{\left(i_{1} i_{2}\right)}, \quad I_{(000) \tau_{p+1}, \tau_{k}}^{\left(i_{1} i_{2} i_{3}\right)} \quad\left(\tau_{p}=p \Delta\right)
$$

and satisfying the conditions

$$
\begin{gather*}
\mathrm{M}\left\{\left(I_{(00) \tau_{p+1}, \tau_{p}}^{\left(i_{1} i_{2}\right)}-I_{(00) \tau_{p+1}, \tau_{p}}^{\left(i_{1} i_{2}\right) q_{1}}\right)^{2}\right\}=\frac{\Delta^{2}}{2}\left(\frac{1}{2}-\sum_{i=1}^{q_{1}} \frac{1}{4 i^{2}-1}\right) \leq C \Delta^{4}  \tag{66}\\
\mathrm{M}\left\{\left(I_{(000) \tau_{p+1}, \tau_{p}}^{\left(i_{1} i_{2} i_{3}\right)}-I_{(000) \tau_{p+1}, \tau_{p}}^{\left(i_{1} i_{2} i_{3}\right) q}\right)^{2}\right\} \leq 6\left(\frac{\Delta^{3}}{6}-\sum_{j_{3}, j_{2}, j_{1}=0}^{q} C_{j_{3} j_{2} j_{1}}^{2}\right) \leq C \Delta^{4} . \tag{67}
\end{gather*}
$$

Remark 1. If it is required to check the mean-square approximation error of the iterated Ito stochastic integral $I_{(000) \tau_{p+1}, \tau_{p}}^{\left(i_{1} i_{2} i_{3}\right.}$ using the precise formulas (49), (59)-(61), rather than the estimate (67) (see (62)), then instead of the condition (67) one has to take the following conditions

$$
\begin{gathered}
E_{p, q, \Delta}^{\left(i_{1} i_{2} i_{3}\right)}=\frac{\Delta^{3}}{6}-\sum_{j_{3}, j_{2}, j_{1}=0}^{q} C_{j_{3} j_{2} j_{1}}^{2} \leq C \Delta^{4} \quad\left(i_{1} \neq i_{2}, \quad i_{1} \neq i_{3}, \quad i_{2} \neq i_{3}\right), \\
E_{p, q, \Delta}^{\left(i_{1} i_{2} i_{3}\right)}=\frac{\Delta^{3}}{6}-\sum_{j_{3}, j_{2}, j_{1}=0}^{q} C_{j_{3} j_{2} j_{1}}^{2}-\sum_{j_{3}, j_{2}, j_{1}=0}^{q} C_{j_{2} j_{3} j_{1}} C_{j_{3} j_{2} j_{1}} \leq C \Delta^{4} \quad\left(i_{1} \neq i_{2}=i_{3}\right),
\end{gathered}
$$

$$
\begin{aligned}
& E_{p, q, \Delta}^{\left(i_{1} i_{2} i_{3}\right)}=\frac{\Delta^{3}}{6}-\sum_{j_{3}, j_{2}, j_{1}=0}^{q} C_{j_{3} j_{2} j_{1}}^{2}-\sum_{j_{3}, j_{2}, j_{1}=0}^{q} C_{j_{3} j_{2} j_{1}} C_{j_{1} j_{2} j_{3}} \leq C \Delta^{4} \quad\left(i_{1}=i_{3} \neq i_{2}\right), \\
& E_{p, q, \Delta}^{\left(i_{1} i_{2} i_{3}\right)}=\frac{\Delta^{3}}{6}-\sum_{j_{3}, j_{2}, j_{1}=0}^{q} C_{j_{3} j_{2} j_{1}}^{2}-\sum_{j_{3}, j_{2}, j_{1}=0}^{q} C_{j_{3} j_{1} j_{2}} C_{j_{3} j_{2} j_{1}} \leq C \Delta^{4} \quad\left(i_{1}=i_{2} \neq i_{3}\right),
\end{aligned}
$$

where

$$
\mathrm{M}\left\{\left(I_{(000) \tau_{p+1}, \tau_{p}}^{\left(i_{1} i_{2} i_{3}\right)}-I_{(000) \tau_{p+1}, \tau_{p}}^{\left(i_{1} i_{2} i_{3}\right) q}\right)^{2}\right\} \stackrel{\text { def }}{=} E_{p, q, \Delta}^{\left(i_{1} i_{2} i_{3}\right)}
$$

Step 4. Modeling of the sequence of independent standard Gaussian random variables $\zeta_{l}^{(i)}(l=$ $\left.0,1, \ldots, q_{1} ; i=1, \ldots, m\right)$.

Step 5. Modeling of the iterated Ito stochastic integrals

$$
I_{(0) \tau_{p+1}, \tau_{p}}^{\left(i_{1}\right)}, \quad I_{(1) \tau_{p+1}, \tau_{p}}^{\left(i_{1}\right)}, \quad I_{(00) \tau_{p+1}, \tau_{p}}^{\left(i_{1} i_{2}\right)}, \quad I_{(000) \tau_{p+1}, \tau_{p}}^{\left(i_{1} i_{2} i_{3}\right)}
$$

using the formulas

$$
\begin{gathered}
I_{(0) \tau_{k+1}, \tau_{k}}^{\left(i_{1}\right)}=\sqrt{T-t} \zeta_{0}^{\left(i_{1}\right)}, \\
I_{(1) \tau_{p+1}, \tau_{p}}^{\left(i_{1}\right)}=-\frac{(T-t)^{3 / 2}}{2}\left(\zeta_{0}^{\left(i_{1}\right)}+\frac{1}{\sqrt{3}} \zeta_{1}^{\left(i_{1}\right)}\right) \\
I_{(00) \tau_{p+1}, \tau_{p}}^{\left(i_{1} i_{2}\right) q_{1}}=\frac{T-t}{2}\left(\zeta_{0}^{\left(i_{1}\right)} \zeta_{0}^{\left(i_{2}\right)}+\sum_{i=1}^{q_{1}} \frac{1}{\sqrt{4 i^{2}-1}}\left(\zeta_{i-1}^{\left(i_{1}\right)} \zeta_{i}^{\left(i_{2}\right)}-\zeta_{i}^{\left(i_{1}\right)} \zeta_{i-1}^{\left(i_{2}\right)}\right)-\mathbf{1}_{\left\{i_{1}=i_{2}\right\}}\right), \\
I_{(000) \tau_{p+1}, \tau_{p}}^{\left(i_{1} i_{2} i_{3}\right) q}=\sum_{j_{1}, j_{2}, j_{3}=0}^{q} C_{j_{3} j_{2} j_{1}}\left(\zeta_{j_{1}}^{\left(i_{1}\right)} \zeta_{j_{2}}^{\left(i_{2}\right)} \zeta_{j_{3}}^{\left(i_{3}\right)}-\mathbf{1}_{\left\{i_{1}=i_{2}\right\}} \mathbf{1}_{\left\{j_{1}=j_{2}\right\}} \zeta_{j_{3}}^{\left(i_{3}\right)}-\right. \\
\left.-\mathbf{1}_{\left\{i_{2}=i_{3}\right\}} \mathbf{1}_{\left\{j_{2}=j_{3}\right\}} \zeta_{j_{1}}^{\left(i_{1}\right)}-\mathbf{1}_{\left\{i_{1}=i_{3}\right\}} \mathbf{1}_{\left\{j_{1}=j_{3}\right\}} \zeta_{j_{2}}^{\left(i_{2}\right)}\right)
\end{gathered}
$$

where $i_{1}, i_{2}, i_{3}=1, \ldots, m$.
Remark 2. In the case of $i_{1}=i_{2}=i_{3}$, it is advisable to model the stochastic integral $I_{(000) \tau_{p+1}, \tau_{p}}^{\left(i_{1} i_{2} i_{3}\right)}$ using the formula (58), where one has to assume that $T-t=\Delta$.

Step 6. Calculate $\mathbf{y}_{p+1}$ from (5).
Step 7. If $p<N-1$, then assume that $p=p+1$ and go to Step 4 ; otherwise, go to Step 8 .
Step 8. End.

We briefly note how to modify the algorithm to enable numerical modeling with the order 2.0 of strong convergence.

At Step 3 one has to take the following three iterated Ito stochastic integrals

$$
I_{(10) \tau_{p+1}, \tau_{p}}^{\left(i_{1} i_{2}\right)}, \quad I_{(01) \tau_{p+1}, \tau_{p}}^{\left(i_{1} i_{2}\right)}, \quad I_{(0000) \tau_{p+1}, \tau_{p}}^{\left(i_{1} i_{2} i_{3} i_{4}\right)}
$$

whose approximations obey (45)-(47), (64) and add to the considered stochastic integrals. Moreover, we replace $C \Delta^{4}$ by $C \Delta^{5}$ in (66), (67). At that, one can use the estimate (62) for $k=4$ and the formulas (53), (54) to check the accuracy of modeling of the aforementioned integrals. As the result, we get the following conditions

$$
\begin{gathered}
\mathrm{M}\left\{\left(I_{(10) \tau_{p+1}, \tau_{p}}^{\left(i_{1} i_{2}\right)}-I_{(10) \tau_{p+1}, \tau_{p}}^{\left(i_{1} i_{2}\right) q_{2}}\right)^{2}\right\}=\mathrm{M}\left\{\left(I_{(01) \tau_{p+1}, \tau_{p}}^{\left(i_{1} i_{2}\right)}-I_{(01) \tau_{p+1}, \tau_{p}}^{\left(i_{1} i_{2}\right) q_{2}}\right)^{2}\right\}=\frac{\Delta^{4}}{16} \times \\
\times\left(\frac{5}{9}-2 \sum_{i=2}^{q_{2}} \frac{1}{4 i^{2}-1}-\sum_{i=1}^{q_{2}} \frac{1}{(2 i-1)^{2}(2 i+3)^{2}}-\sum_{i=0}^{q_{2}} \frac{(i+2)^{2}+(i+1)^{2}}{(2 i+1)(2 i+5)(2 i+3)^{2}}\right) \leq C \Delta^{5}
\end{gathered}
$$

for $i_{1} \neq i_{2}$ and

$$
\begin{aligned}
& \mathrm{M}\left\{\left(I_{(10) \tau_{p+1}, \tau_{p}}^{\left(i_{1} i_{1}\right)}-I_{(10) \tau_{p+1}, \tau_{p}}^{\left(i_{1} i_{1}\right) q_{3}}\right)^{2}\right\}=\mathrm{M}\left\{\left(I_{(01) \tau_{p+1}, \tau_{p}}^{\left(i_{1} i_{1}\right)}-I_{(01) \tau_{p+1}, \tau_{p}}^{\left(i_{1} i_{1}\right) q_{3}}\right)^{2}\right\}= \\
= & \frac{\Delta^{4}}{16}\left(\frac{1}{9}-\sum_{i=0}^{q_{3}} \frac{1}{(2 i+1)(2 i+5)(2 i+3)^{2}}-2 \sum_{i=1}^{q_{3}} \frac{1}{(2 i-1)^{2}(2 i+3)^{2}}\right) \leq C \Delta^{5}
\end{aligned}
$$

for $i_{1}=i_{2} ;$

$$
\begin{equation*}
\mathrm{M}\left\{\left(I_{(0000) \tau_{p+1}, \tau_{p}}^{\left(i_{1} i_{2} i_{3} i_{4}\right)}-I_{(0000) \tau_{p+1}, \tau_{p}}^{\left(i_{1} i_{2} i_{3} i_{4}\right) q_{4}}\right)^{2}\right\} \leq 24\left(\frac{\Delta^{4}}{24}-\sum_{j_{1}, j_{2}, j_{3}, j_{4}=0}^{q_{4}} C_{j_{4} j_{3} j_{2} j_{1}}^{2}\right) \leq C \Delta^{5} \tag{68}
\end{equation*}
$$

where $i_{1}, i_{2}, i_{3}, i_{4}=1, \ldots, m ; q_{2}, q_{3}, q_{4}<q<q_{1}$.
Carry out Step 5 with allowance of the stochastic integrals

$$
I_{(10) \tau_{p+1}, \tau_{p}}^{\left(i_{1} i_{2}\right)}, I_{(01) \tau_{p+1}, \tau_{p}}^{\left(i_{1} i_{2}\right)}, I_{(0000) \tau_{p+1}, \tau_{p}}^{\left(i_{1} i_{2} i_{3} i_{4}\right)}
$$

and calculate $\mathbf{y}_{p+1}$ at Step 6 according to (9).
It should be noted that instead of the estimate (68) we can use the precise relations for the value

$$
\mathrm{M}\left\{\left(I_{(0000) \tau_{p+1}, \tau_{p}}^{\left(i_{1} i_{2} i_{3} i_{4}\right)}-I_{(0000) \tau_{p+1}, \tau_{p}}^{\left(i_{1} i_{2} i_{3} i_{4}\right) q_{4}}\right)^{2}\right\}
$$

which were obtained in [31-[34, 38] for all possible combinations of $i_{1}, i_{2}, i_{3}, i_{4}=1, \ldots, m$. Note that the optimization of the mentioned procedure is considered in 69.

## 7. Conclusions

The present paper provided efficient procedures for the mean-square approximation of iterated Ito and Stratonovich stochastic integrals of multiplicities 1 to 4 based on multiple Fourier-Legendre series. These results can be used for implementation of the numerical methods with the orders 1.0, 1.5 , and 2.0 of strong convergence for Ito stochastic differential equations with multidimensional noncommutative noise. The results of the article can be applied for numerical solution of the problems of optimal stochastic control and signal filtering in random noise in different formulations. The development of the approaches from this work can be found in [13], [19]-[55], [61]-67], [69], [70].

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