

SPECTRAL APPROACH TO LINEAR PROGRAMMING BOUNDS ON CODES

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ABSTRACT. We give new proofs of asymptotic upper bounds of coding theory obtained within the frame of Delsarte's linear programming method. The proofs rely on the analysis of eigenvectors of some finite-dimensional operators related to orthogonal polynomials. The examples of the method considered in the paper include binary codes, binary constant-weight codes, spherical codes, and codes in the projective spaces.

1. Introduction. Let X be a compact metric space with distance function d . A code C is a finite subset of X . Define the minimum distance of C as $d(C) = \min_{x,y \in C, x \neq y} d(x, y)$. A variety of metric spaces that arise from different applications include the binary Hamming space, the binary Johnson space, the sphere in \mathbb{R}^n , real and complex projective spaces, Grassmann manifolds, etc. Estimating the maximum size of the code with a given value of d is one of the main problems of coding theory. Let M be the cardinality of C . A powerful technique to bound M above as a function of $d(C)$ that is applicable in a wide class of metric spaces including all of the aforementioned examples is Delsarte's linear programming method [2]. The first such examples to be considered were the binary Hamming space $H_n = \{0, 1\}^n$ and the Johnson space $J^{n,w} \subset H_n$ which is formed by all the vectors of H_n of Hamming weight w , with the distance given by the Hamming metric. The best currently known asymptotic estimates of the size of binary codes and binary constant weight codes were obtained in McEliece, Rodemich, Rumsey, Welch [11] and are called the MRRW bounds. Shortly thereafter, Kabatiansky and Levenshtein [7] established an analogous bound for codes on the unit sphere in \mathbb{R}^n with Euclidean metric and some related spaces. This paper also introduced a general approach to bounding the code size in distance-transitive metric spaces based on harmonic analysis of their isometry group. This approach was furthered in papers [8, 10] which also explored the limits of Delsarte's method.

In this paper we suggest a new proof method for linear programming upper bounds of coding theory. Our approach, which relies on the analysis of eigenvectors of some finite-dimensional operators related to orthogonal polynomials arguably makes some steps of the proofs conceptually more transparent than those previously known. We also consider some of the main examples mentioned above. The linear-algebraic ideas that we follow were introduced in a recent paper by Bachoc [1] in which a similar approach has been taken to establish an asymptotic bound for codes in the real Grassmann manifold.

2. A bound on the code size. We assume that X is a distance-transitive space which means that its isometry group G acts doubly transitively on ordered pairs of points at a given distance. In this case the zonal spherical kernels $K_i(\mathbf{x}, \mathbf{y})$ associated with irreducible regular representations of G depend only on the distance between \mathbf{x} and \mathbf{y} . In all the examples mentioned above, except for the Grassmann manifold, $K_i(\mathbf{x}, \mathbf{y})$ can be expressed as a univariate polynomial $p_i(x)$ of degree i , where $x = \tau(d)$ is some function of the distance $d(\mathbf{x}, \mathbf{y})$.

Let D be the (finite or infinite) set of the possible values of the distance in X . We will assume that $\tau(d(\mathbf{x}, \mathbf{y}))$ is a monotone function that sends D to a segment $[a, b]$. For instance, for the Hamming space, $D = \{0, 1, \dots, n\}$ and τ can be taken the identity function. For the sphere $S^{n-1}(\mathbb{R})$, $D = [0, 2]$. In this

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case it is convenient to take $\tau(d) = 1 - d^2/2$ to be the scalar product $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_i x_i y_i$. The invariant measure on G induces a measure $d\mu$ on $[a, b]$. For instance, for $X = \{0, 1\}^n$, the measure $d\mu$ corresponds to the binomial probability distribution on $\{0, 1, \dots, n\}$, so $\int_D d\mu = 1$. We will assume that the last condition holds in general and normalize μ when this is not the case.

The kernels $K_i(\mathbf{x}, \mathbf{y})$, $i = 0, 1, \dots$, are positive semidefinite which means that $\sum_{\mathbf{x}, \mathbf{y} \in C} K_i(\mathbf{x}, \mathbf{y}) \geq 0$ for any finite set $C \subset X$. This property together with the fact that $K_i(\mathbf{x}, \mathbf{y})$ can be expressed as a polynomial of one variable gives rise to the following set of inequalities

$$(1) \quad \sum_{\mathbf{x}, \mathbf{y} \in C} p_i(\tau(d(\mathbf{x}, \mathbf{y}))) \geq 0, \quad i = 0, 1, \dots$$

called the Delsarte inequalities in coding theory.

The function τ can be chosen in such a way that the polynomials p_i , $i = 0, 1, \dots$, are orthogonal on $[a, b]$ with respect to the scalar product $\langle f, g \rangle = \int f g d\mu$. Below we denote by V the space $L_2(d\mu)$ of square-integrable functions on $[a, b]$.

We will assume that the polynomials p_i are orthonormal, i.e., $\|p_i\|^2 = \langle p_i, p_i \rangle = 1$. Note that this implies that $p_0 \equiv 1$. Another assumption used below is that the product $p_i p_j$ for all $i, j \geq 0$ expands into the basis $\{p_i\}$ with nonnegative coefficients, i.e.,

$$(2) \quad p_i p_j = \sum_k q_{i,j}^k p_k \quad (q_{i,j}^k \geq 0).$$

This property is again implied by the fact that the zonal spherical kernels are positive semidefinite, see [7].

Since the polynomials $\{p_i\}$ are orthogonal, they satisfy a three-term recurrence [12] of the form

$$(3) \quad x p_k = \alpha_k p_{k+1} + \beta_k p_k + \gamma_k p_{k-1} \quad (k = 0, 1, \dots; p_{-1} = 0).$$

Let $P_1 = \varepsilon p_1$, where $\varepsilon > 0$ is some constant. We will write this recurrence in the form

$$(4) \quad P_1 p_k = a_k p_{k+1} + b_k p_k + c_k p_{k-1},$$

which follows from (3) upon noticing that P_1 is a linear function. By (2), the coefficients a_k, b_k, c_k are nonnegative.

Let $C \subset X$ be a code of size M and distance d . Denote by $\Delta(C) = \{\tau(d(\mathbf{x}, \mathbf{y})), \mathbf{x}, \mathbf{y} \in C, \mathbf{x} \neq \mathbf{y}\}$ the set of values that the function τ takes on the distances between distinct code points. Let $\tau_0 = \tau(0)$.

The main theorem of the linear-programming method asserts the following.

Theorem 1. [2, 7] *Let $C \subset X$ be a code of size M . Let $F(t) = \sum_{i=0}^m F_i p_i(x)$ be a polynomial that satisfies*

(i) $F_0 > 0, F_i \geq 0, i = 1, 2, \dots, m$;

(ii) $F(x) \leq 0$ for $x \in \Delta(C)$.

Then $M \leq F(\tau_0)/F_0$.

The proof is obvious because on the one hand, by assumption (ii)

$$\sum_{\mathbf{x}, \mathbf{y} \in C} F(\tau(d(\mathbf{x}, \mathbf{y}))) \leq M F(\tau_0);$$

on the other hand, because of (1), assumption (i) and the fact that $p_0 = 1$,

$$\sum_{\mathbf{x}, \mathbf{y} \in C} F(\tau(d(\mathbf{x}, \mathbf{y}))) = \sum_i F_i \sum_{\mathbf{x}, \mathbf{y}} p_i(\tau(d(\mathbf{x}, \mathbf{y}))) \geq F_0 M^2.$$

This theorem is equivalent to a duality theorem for a linear programming problem whose variables are the coefficients of the distance distribution of the code C and whose constraints are given by the Delsarte inequalities. For this reason, estimates obtained from this theorem are called the linear programming bounds. Our objective in this section is to present a new method of obtaining bounds on M based on this theorem.

We shall use a generic notation $A_k(c_i, b_i, a_i)$ for a tridiagonal matrix of the form

$$A_k = \begin{bmatrix} b_0 & a_0 & 0 & 0 & \dots & 0 \\ c_1 & b_1 & a_1 & 0 & \dots & 0 \\ & c_2 & b_2 & a_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & a_{k-1} \\ 0 & 0 & \dots & \dots & c_k & b_k \end{bmatrix}.$$

The largest eigenvalue of a square symmetric matrix M will be denoted by $\lambda_{\max}(M)$.

Throughout the paper we use bold letters to denote operators acting on V and regular letters to denote their matrices in the basis $\{p_i\}$. Let V_k be the space of polynomials of degree $\leq k$ considered as a subspace of V . Let \mathbf{E}_k be the orthogonal projection from V to V_k . Consider the operator

$$\mathbf{S}_k = \mathbf{E}_k \circ P_1 : V_k \rightarrow V_k,$$

i.e., multiplication by P_1 followed by projection on V_k . The argument that follows relies on the fact that this operator is self-adjoint (with respect to the bilinear form $\langle \cdot, \cdot \rangle$). Indeed, both multiplication by a function and the orthogonal projection are self-adjoint operators. Therefore, the matrix $S_k = A_k(c_i, b_i, a_i)$ is symmetric. In other words,

$$a_i = \langle P_1 p_i, p_{i+1} \rangle = \langle p_i, P_1 p_{i+1} \rangle = c_{i+1}.$$

A $p \times p$ matrix $A \geq 0$ (i.e., a matrix with nonnegative entries) is called irreducible if for any partition of the set of indices $\{1, 2, \dots, p\}$ into two disjoint subsets I and J , $|I| + |J| = p$, the matrix $(a_{i,j})_{i \in I, j \in J}$ is nonzero (in other words, a directed graph G with vertices $\{1, 2, \dots, p\}$ and edges (i, j) whenever $A_{ij} > 0$ is strongly connected). For instance, the matrix S_k is nonnegative and irreducible.

In the next lemma we collect the properties of irreducible matrices used below.

Lemma 1. *Let $A \geq 0$ be a $p \times p$ irreducible symmetric matrix.*

(a) *Its largest eigenvalue $\lambda_{\max}(A)$ is positive and has multiplicity one. There exists a vector $y > 0$ such that $Ay = \lambda_{\max}(A)y$.*

(b) $\lambda_{\max}(A) \leq \max_{1 \leq i \leq p} \sum_j A_{ij}$.

(c) *For any $y \neq 0$, $\lambda_{\max}(A) \geq \frac{(Ay, y)}{(y, y)}$.*

(d) *If $0 \leq B \leq A$ for some matrix B , or if B is a principal minor of A , then $|\lambda_{\max}(B)| \leq \lambda_{\max}(A)$.*

Here claims (a),(b),(d) form a part of the Perron–Frobenius theory (see, e.g., [5]), and claim (c) is obvious and holds true for any symmetric matrix.

The suggested method for deriving upper bounds is based on the following theorem.

Theorem 2. *Let $C \subset X$ be an (M, d) code and let $\rho_k = a_k p_{k+1}(\tau_0)/p_k(\tau_0)$. Then*

$$M \leq \frac{4\rho_k p_k^2(\tau_0)}{P_1(\tau_0) - \lambda_{\max}(S_k)}$$

for all k such that $\lambda_{\max}(S_{k-1}) \geq P_1(x)$ for all $x \in \Delta(C)$.

Proof: Let $g = \sum_{i=1}^k g_i p_i \in V_k$. Fix some $\rho > 0$ (its value to be chosen later). Consider the operator $\mathbf{T}_k : V_k \rightarrow V_k$ defined by

$$(5) \quad \mathbf{T}_k g = \mathbf{S}_k g - \rho g_k p_k,$$

and let θ_k be its largest eigenvalue. Recall that T_k is the matrix of this operator in the basis $\{p_i\}$. (T_k is the same as S_k except that $(T_k)_{k+1, k+1} = (S_k)_{k+1, k+1} - \rho$.) We may “shift” the matrix T_k by a multiple of the identity matrix I to make all of its elements nonnegative. For instance, we may consider $T_k + \rho_k I \geq 0$. Therefore, by Lemma 1(d) we have

$$\lambda_{\max}(S_{k-1} + \rho I) < \theta_k + \rho < \lambda_{\max}(S_k + \rho I),$$

whence we get

$$(6) \quad \lambda_{\max}(S_{k-1}) < \theta_k < \lambda_{\max}(S_k).$$

Moreover, the eigenvalue θ_k is of multiplicity one. Denote by $f = (f_0, f_1, \dots, f_k) \in V_k$ the eigenvector that corresponds to it. By (5) we have

$$P_1 f = \theta_k f + \rho f_k p_k + f_k a_k p_{k+1},$$

so

$$f = \frac{\rho p_k + a_k p_{k+1}}{P_1 - \theta_k} f_k.$$

Consider the polynomial $F = (\rho p_k + a_k p_{k+1})f$. By Lemma 1(a), f can be chosen to have positive coordinates. Therefore by (2), the coefficients of the expansion of F into the basis $\{p_i\}$ are nonnegative. Next, if $\lambda_{\max}(S_{k-1}) \geq P_1(x)$ for $x \in \Delta(C)$, then by (6) we have $F(x) \leq 0$ for $x \in \Delta(C)$, i.e., $F(x)$ satisfies condition (ii) of Theorem 1. Since multiplication by f is a self-adjoint operator, we compute

$$F_0 = \langle (\rho p_k + a_k p_{k+1})f, 1 \rangle = \langle \rho p_k + a_k p_{k+1}, f \rangle = \rho f_k > 0$$

and

$$F(\tau_0) = \frac{(\rho p_k(\tau_0) + a_k p_{k+1}(\tau_0))^2}{P_1(\tau_0) - \theta_k} f_k < \frac{(\rho p_k(\tau_0) + a_k p_{k+1}(\tau_0))^2}{P_1(\tau_0) - \lambda_{\max}(S_k)} f_k$$

provided that $\lambda_{\max}(S_k) < n$. Thus,

$$\frac{F(\tau_0)}{F_0} < \frac{(\rho p_k(\tau_0) + a_k p_{k+1}(\tau_0))^2}{\rho(P_1(\tau_0) - \lambda_{\max}(S_k))}.$$

The value of ρ minimizing the left-hand side is $\rho = \rho_k$. The claimed estimate is obtained by using the polynomial $F = (\rho_k p_k + a_k p_{k+1})f$ in Theorem 1. ■

Remark 1. Note that by Lemma 1(d), the $\{\lambda_{\max}(S_k)\}$ form a monotone increasing sequence. Therefore, the last condition of the theorem holds for all k greater than some value k_0 .

Next let us estimate the largest eigenvalue of S_k .

Lemma 2. *Let $a_{i+1} > a_i, b_{i+1} > b_i, i = 0, 1, \dots$. Then for all $s = 1, \dots, k+1$,*

$$\frac{1}{s}(2(s-1)a_{k-s+1} + s b_{k-s+1}) \leq \lambda_{\max}(S_k) \leq a_{k-1} + \max(a_{k-1} + b_{k-1}, b_k).$$

Proof: By Lemma 1(b)

$$\lambda_{\max}(S_k) \leq \max(a_{k-2} + b_{k-1} + a_{k-1}, a_{k-1} + b_k),$$

hence the upper bound. On the other hand, take $y = (0^{k-s+1} 1^s)^t$ where t denotes transposition. Then by part (c) of the same lemma,

$$\lambda_{\max}(S_k) \geq \frac{1}{s} \left(2 \sum_{p=1}^{s-1} a_{k-p} + \sum_{p=0}^{s-1} b_{k-p} \right).$$

Since we assumed that the coefficients a_i, b_i are monotone increasing on i this implies the lower bound. ■

Remark 2. In effect, Lemma 2 provides an estimate of the extremal zero of p_{k+1} . Indeed, consider the operator $\mathbf{X}_k = \mathbf{E}_k \circ x : V_k \rightarrow V_k$. It is self-adjoint, so its matrix in the basis $\{p_i\}$ is tridiagonal symmetric and is given by $X_k = A_k(\gamma_i, \beta_i, \alpha_i)$, where the elements $\alpha_i, \beta_i, \gamma_i$ are the coefficients in the three-term recurrence (3).

It is well known (e.g., [6]) that the spectrum of X_k coincides with the set of zeros of p_{k+1} . [A proof goes as follows: let $p_{k+1}(\lambda) = 0$. Consider the action of \mathbf{X}_k on the polynomial $f = p_{k+1}/(\lambda - x) \in V_k$:

$$\lambda f - \mathbf{X}_k f = \lambda f - \mathbf{E}_k(xf) = \mathbf{E}_k((\lambda - x)f) = \mathbf{E}_k p_{k+1} = 0.$$

Conversely, if $f \in V_k$, $f \neq 0$, and $0 = \lambda f - \mathbf{X}_k f = \mathbf{E}_k((\lambda - x)f)$, this implies that $(\lambda - x)f$ is a constant multiple of p_{k+1} . Therefore, $p_{k+1}(x)$ is proportional¹ to $\det(xI_{k+1} - X_k)$.] Then the largest zero x_{k+1}^+ of p_{k+1} can be found as $x_{k+1}^+ = \lambda_{\max}(X_k) = \max_{\|y\|=1}(X_k y, y)$, or more concretely as

$$x_{k+1}^+ = \max_{\|y\|=1} \left\{ \sum_{i=0}^k \beta_i y_i^2 + 2 \sum_{i=0}^{k-1} \alpha_i y_i y_{i+1} \right\}.$$

This formula was first published in [10, p.580] with a different proof.

We note that the relation between the extremal zero of p_{k+1} and the largest eigenvalue $\lambda_{\max}(X_k)$ makes the task of finding the zero computationally much easier than the direct approach because of the existence of very efficient iterative algorithms for the symmetric eigenvalue problem. This property is helpful for computing linear programming bounds on codes such as the bounds considered in the next section and other similar results for codes of moderate or even large length (on the order of several thousands).

3. Examples. In this section we consider a few examples of interest to coding theory.

3.1. Binary codes. Let $X = \{0, 1\}^n$ be the binary Hamming space. It is known [2, 7] that the polynomials p_i are given by the (normalized) Krawtchouk polynomials $\{\tilde{K}_k(x), k = 0, 1, \dots, n\}$. We have $\mu(i) = 2^{-n} \binom{n}{i}$, so the bilinear form can be written as $\langle f, g \rangle = \sum_{i=0}^n \mu(i) f(i) g(i)$. Let C be a binary code of length n , size M and minimum Hamming distance $d = d(C)$. We choose $\tau(k) = k$ to obtain $\Delta(C) \subset \{d, d+1, \dots, n\}$. This inclusion may be proper depending on the code C , but we will ignore this and assume that $\Delta(C) = \{d, d+1, \dots, n\}$ since this assumption can only relax the linear programming bound on M .

The polynomials \tilde{K}_k satisfy a three-term recurrence relation [12]

$$(7) \quad 2x\tilde{K}_k(x) = -\sqrt{(n-k)(k+1)}\tilde{K}_{k+1}(x) + n\tilde{K}_k(x) - \sqrt{(n-k+1)k}\tilde{K}_{k-1}(x),$$

$$\tilde{K}_0 = 1, \tilde{K}_i(x)\tilde{K}_j(x) = \sum_k q_{i,j}^k \tilde{K}_k(x) \text{ with } q_{i,j}^k \geq 0, \text{ and}$$

$$(8) \quad \tilde{K}_k(0) = \sqrt{\binom{n}{k}}.$$

Choose in (4) $P_1 = \sqrt{n}p_1 = n - 2x$. From (7) we then obtain $S_k = A_k(a_{i-1}, 0, a_i)$, where $a_i = \sqrt{(i+1)(n-i)}$, $i = 0, 1, \dots$, or more explicitly,

$$S_k = \begin{bmatrix} 0 & \sqrt{n} & 0 & \dots & \dots & 0 \\ \sqrt{n} & 0 & \sqrt{2(n-1)} & \dots & \dots & 0 \\ 0 & \sqrt{2(n-1)} & 0 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & 0 & \sqrt{(k-1)(n-k+2)} & 0 \\ \dots & \dots & \dots & \sqrt{(k-1)(n-k+2)} & 0 & \sqrt{k(n-k+1)} \\ 0 & 0 & \dots & 0 & \sqrt{k(n-k+1)} & 0 \end{bmatrix}.$$

The monotonicity assumption of Lemma 2 clearly holds because $a_k > a_{k-1}$ as long as $k < n/2$. Therefore for the largest eigenvalue of S_k we obtain the following estimate:

$$\frac{2(s-1)}{s} \sqrt{(k-s+2)(n-k+s-1)} \leq \lambda_{\max}(S_k) \leq 2\sqrt{k(n-k+1)}.$$

Letting $n \rightarrow \infty, s \rightarrow \infty, s = o(n)$, we obtain the exact asymptotic behavior of the main term:

$$(9) \quad \lim_{n \rightarrow \infty, k/n \rightarrow \tau} \frac{\lambda_{\max}(S_k)}{n} = 2\sqrt{\tau(1-\tau)}.$$

¹The coefficient equals $\alpha_0 \alpha_1 \dots \alpha_{k-1}$ and can be found recursively from (3) and the equality $p_0 \equiv 1$.

Since $\tau_0 = 0$ and $\rho_k = n - k$, the bound of Theorem 2 takes the form

$$(10) \quad M \leq \frac{4(n-k)}{n - \lambda_{\max}(S_k)} \binom{n}{k}$$

for all k such that $\lambda_{\max}(S_{k-1}) \geq P_1(d) = n - 2d$. This estimate together with (9) leads to the following asymptotic result (the asymptotic MRRW bound for binary codes [11]):

$$\frac{1}{n} \log M \leq h(1/2 - \sqrt{\delta(1-\delta)})(1 + o(1)).$$

Here $h(x) = x \log_2 x - (1-x) \log_2 (1-x)$ is the binary entropy function. Indeed, let $\lim \frac{d}{n} = \delta$ and assume that $\delta \leq 1/2$. We need to choose k so that $\frac{\lambda_{k-1}}{n} \geq (1-2\delta)(1+o(1))$ as $n \rightarrow \infty$. In the limit, this amounts to taking τ that satisfies $2\sqrt{\tau(1-\tau)} \geq 1-2\delta$, or $\tau \geq 1/2 - \sqrt{\delta(1-\delta)}$. The result now follows by the Stirling approximation.

Remark 3. Specializing Remark 2 to the case at hand, we observe from (7) that

$$X_k = 1/2(nI_{k+1} - S_k) = 1/2A_k(-\sqrt{i(n-i+1)}, n, -\sqrt{(i+1)(n-i)}).$$

Therefore we obtain the following expression for the largest root of \tilde{K}_{k+1} :

$$x_{k+1}^+ = \frac{n}{2} + \max_{\|y\|=1} \sum_{i=0}^{k-1} y_i y_{i+1} \sqrt{(i+1)(n-i)}.$$

This result is originally due to [9]. Although more accurate estimates of the extremal zeros are available in the literature [10, 4], our Lemma 2 suffices to compute the correct value of the main term.

Remark 4. The bound (10) is close to the previously known estimates obtained within the frame of Delsarte's method. In particular, Levenshtein [8, 10] constructed a sequence of polynomials that are optimal in the Delsarte problem (with some qualifiers). His results imply that the above estimate does not improve the known bounds on M . The result of [11] is also of the form similar to (10).

Remarks 2–4, modified appropriately, apply also to the other examples in this section.

3.2. Constant-weight codes. Now let $X \subset J^{n,w}$ the binary Johnson space, i.e., the set of vectors in $\{0, 1\}^n$ of Hamming weight w . We take d to be the Hamming metric so that $D = \{0, 2, \dots, 2w\}$ and put $\tau(d) = d/2$. The relevant family of orthogonal polynomials is given by the Hahn polynomials $H_k(x)$ [2]. They are orthogonal on $\tau(D) = \{0, 1, \dots, w\}$ with respect to the weight $\mu_J(i) = \frac{\binom{w}{i} \binom{n-w}{i}}{\binom{n}{i}}$ according to $\int H_k H_m d\mu_J = \frac{n-2k+1}{n-k+1} \binom{n}{k} \delta_{km}$ and satisfy a three-term recurrence

$$(11) \quad \begin{aligned} (k+1)(w-k)(n-w-k)(n-2k+2)(n-2k+3)H_{k+1}(x) = \\ (n-2k-1)(n-2k+3)[(n+2)w(n-w) - nk(n-k+1) - (n-2k)(n-2k+2)x]H_k(x) \\ - (n-2k-1)(n-2k)(w-k+1)(n-w-k+1)(n-k+2)H_{k-1}(x). \end{aligned}$$

Note that $\sum_{i=1}^w \mu_J(i) = 1$. Let us normalize H_k by setting $\tilde{H}_k = \left(\frac{n-2k+1}{n-k+1} \binom{n}{k}\right)^{-1/2} H_k$. As above, we have

$$\tilde{H}_i(x) \tilde{H}_j(x) = \sum_{k=0}^w q_{i,j}^k \tilde{H}_k(x) \quad (q_{i,j}^k \geq 0)$$

and

$$\tilde{H}_k(0) = \sqrt{\frac{n-2k+1}{n-k+1} \binom{n}{k}}.$$

Let us take

$$P_1(x) = (n-1)^{-1/2} \tilde{H}_1(x) = 1 - \frac{nx}{w(n-w)}.$$

Let us write out the matrix of the operator $\mathbf{S}_k = \mathbf{E}_k \circ P_1$ in the orthonormal basis. We have $S_k = A_k(a_{i-1}, b_i, a_i)$, where the matrix elements can be computed from (11). We obtain

$$a_i = \frac{n(w-i)(n-w-i)}{w(n-w)(n-2i)} \sqrt{\frac{(i+1)(n-i+1)}{(n-2i+1)(n-2i-1)}},$$

$$b_i = \frac{(n-2w)^2 i(n-i+1)}{w(n-w)(n-2i)(n-2i+2)}, \quad i \geq 0.$$

Let $C \subset J^{n,w}$ be a code of size M and distance $2d$. Let us apply Theorem 2 to bounding M as a function of d . We have $\tau_0 = 0, \tilde{H}_0 = 1$,

$$\rho_k = a_k \frac{\tilde{H}_{k+1}(0)}{\tilde{H}_k(0)} = \frac{n(w-k)(n-w-k)(n-k+1)}{w(n-w)(n-2k)(n-2k+1)},$$

and $\Delta(C) = \{0, 1, \dots, d\}$. Thus, we obtain the following estimate.

Theorem 3.

$$M \leq \frac{4n(w-k)(n-w-k)}{(1 - \lambda_{\max}(S_k))w(n-w)(n-2k)} \binom{n}{k}$$

for all k such that $\lambda_{\max}(S_{k-1}) \geq 1 - \frac{nd}{w(n-w)}$.

Let us find the minimum k that satisfies the required condition. First we use Lemma 2 to compute the asymptotic behavior of $\lambda_{\max}(S_k)$.

Lemma 3.

$$\lim_{\substack{n \rightarrow \infty \\ w/n \rightarrow \omega, k/n \rightarrow \tau}} \lambda_{\max}(S_k) = \frac{2\omega(1-\omega) + \sqrt{\tau(1-\tau)}}{\omega(1-\omega)(1 + 2\sqrt{\tau(1-\tau)})} \sqrt{\tau(1-\tau)}.$$

Proof: Note that for the upper bound in Lemma 2 it suffices to prove that the value $a_i + b_i + a_{i-1}$ grows on i . Letting $\alpha = \frac{i}{n}$, we compute

$$\begin{aligned} a_{i-1} + b_i + a_i &= \frac{2(\omega - \alpha)(1 - \omega - \alpha)\sqrt{\alpha(1-\alpha)} + (1 - 2\omega)^2 \alpha(1 - \alpha)}{\omega(1 - \omega)(1 - 2\alpha)^2} (1 + o(1)). \\ &= \frac{2\omega(1 - \omega)\sqrt{\alpha(1-\alpha)} + \alpha(1 - \alpha)}{\omega(1 - \omega)(1 + 2\sqrt{\alpha(1-\alpha)})} (1 + o(1)). \end{aligned}$$

The main term on the right-hand side of the last expression is a growing function of α . Indeed, $\sqrt{\alpha(1-\alpha)}$ grows on α for $\alpha < 1/2$, so we only need to check that the function $t(2\omega(1-\omega) + t)/(1 + 2t)$ increases on t for $0 \leq t \leq 1/2$ which is straightforward. Thus we put $i = k - 1$ and obtain for $\lambda_{\max}(S_k)$ an upper bound of the form claimed. Lemma 2 also implies a matching lower bound. Namely, from its proof,

$$\lambda_{\max}(S_k) \geq \frac{1}{s} \left(2 \sum_{p=1}^{s-1} a_{k-p} + \sum_{p=0}^{s-1} b_{k-p} \right) \quad (s = 1, \dots, k+1).$$

For large values of the parameters, we can write

$$\lambda_{\max}(S_k) \geq (a_{k-s} + b_{k-s+1} + a_{k-s+1})(1 + o(1)).$$

The proof is completed by letting $s \rightarrow \infty, s = o(n)$. ■

Let us use this lemma in Theorem 3. Assume that $n \rightarrow \infty, d = \delta n$. The condition on k in this theorem will be fulfilled for any $k = \tau/n$ that satisfies

$$\frac{2\omega(1-\omega) + \sqrt{\tau(1-\tau)}}{\omega(1-\omega)(1 + 2\sqrt{\tau(1-\tau)})} \sqrt{\tau(1-\tau)} > 1 - \frac{\delta}{\omega(1-\omega)}$$

or

$$\delta > \frac{(\omega - \tau)(1 - \omega - \tau)}{1 + 2\sqrt{\tau(1 - \tau)}}.$$

We conclude that Theorem 3 implies the following estimate for an $(n, M, 2\delta n)$ code $C \subset J^{n,w}$ (the asymptotic MRRW bound for constant weight codes [11]):

$$\frac{1}{n} \log M \leq h(\tau)(1 + o(1)),$$

where $\delta = (\omega - \tau)(1 - \omega - \tau)/(1 + 2\sqrt{\tau(1 - \tau)})$.

3.3. Spherical codes. Consider codes on the unit sphere S^{n-1} in \mathbb{R}^n . The polynomials p_i in this case belong to the family of Gegenbauer polynomials $C_k(x)$ [12, pp.80ff]. We have

$$\int_{-1}^1 C_i(x)C_j(x)(1-x^2)^{\frac{n-3}{2}} dx = \frac{\binom{n+i-3}{i}}{n+2i-2} \omega_n \delta_{i,j},$$

where $\omega_n = \frac{\pi \Gamma(n-2)}{2^{n-2} \Gamma^2(\frac{n-2}{2})}$, and in particular for $i = j = 0$, $\int_{-1}^1 (1-x^2)^{\frac{n-3}{2}} dx = \omega_n/(n-2)$. We also have $C_k(1) = \binom{n+k-3}{k}$.

Normalizing the measure, we obtain $d\mu(x) = \frac{n-2}{\omega_n} (1-x^2)^{(n-3)/2} dx$. The normalized Gegenbauer polynomials are then given by

$$\tilde{C}_k = \sqrt{\frac{n+2k-2}{(n-2)\binom{n+k-3}{k}}} C_k.$$

The polynomials \tilde{C}_k satisfy a three-term recurrence of the form

$$x\tilde{C}_k(x) = a_k \tilde{C}_{k+1}(x) + a_{k-1} \tilde{C}_{k-1}(x),$$

where $a_i = \sqrt{\frac{(n+i-2)(i+1)}{(n+2i)(n+2i-2)}}$, $i = 0, \dots$, and $\tilde{C}_{-1} = 0, \tilde{C}_0 = 1$. Further, $\tilde{C}_i \tilde{C}_j = \sum_k q_{i,j}^k \tilde{C}_k$ where $q_{i,j}^k \geq 0$ and

$$\tilde{C}_k(1) = \sqrt{\frac{n+2k-2}{n-2} \binom{n+k-3}{k}}.$$

Let $C(n, M, t)$ denote a code in which the angle between any two distinct vectors $\mathbf{x}_i, \mathbf{x}_j$ satisfies $\cos(\widehat{\mathbf{x}_i, \mathbf{x}_j}) \leq t$. As remarked above, we take $\tau(d) = 1 - d^2/2$. We have $D = [0, 2], \tau(D) = [-1, 1], \Delta(C) \subset [-1, t], \tau_0 = 1$. Choose $P_1(x) = n^{-1/2} \tilde{C}_1(x) = x$, then the matrix S_k has the form $A_k(a_{i-1}, 0, a_i)$, so

$$\rho_k = a_k \frac{\tilde{C}_{k+1}(1)}{\tilde{C}_k(1)} = \frac{n+k-2}{n+2k-2}.$$

From Theorem 2 we obtain

Theorem 4.

$$(12) \quad M \leq \frac{4}{1 - \lambda_{\max}(S_k)} \binom{n+k-2}{k}$$

for all k such that $\lambda_{\max}(S_{k-1}) \geq t$.

This coincides with the original bound of [7].

Lemma 4. For any $s = 2, \dots, k$

$$\frac{2(s-1)}{s} \sqrt{\frac{(n+k-s-1)(k-s+2)}{(n+2k-2s+2)(n+2k-2s)}} \leq \lambda_{\max}(S_k) \leq 2 \sqrt{\frac{(n+k-3)k}{(n+2k-2)(n+2k-4)}}.$$

In particular,

$$\lim_{n \rightarrow \infty, \frac{k}{n} \rightarrow \rho} \frac{\lambda_{\max}(S_k)}{n} = 2 \frac{\sqrt{\rho(1+\rho)}}{1+2\rho}.$$

Proof: We only need to check that $a_i \geq a_{i+1}$. For $n \geq 5$,

$$a_i^2 - a_{i-1}^2 = \frac{(n-2)(n-4)}{(n+2i)(n+2i-2)(n+2i-4)} > 0,$$

so a_i is an increasing function of i . The inequalities in the claim now follow directly from Lemma 2. Letting $s \rightarrow \infty$, $s = o(n)$ and taking the limit gives the asymptotic behavior of $\lambda_{\max}(S_k)$. ■

Theorem 4 and Lemma 4 together enable us to recover the asymptotic bound of [7]. Namely, using the Stirling approximation we obtain

$$\frac{1}{n} \log M \leq ((1+\rho) \log(1+\rho) - \rho \log \rho)(1+o(1))$$

under the condition $t \leq \lambda_{\max}(X_{k-1})$ which in the limit of $n \rightarrow \infty$, $\frac{k}{n} \rightarrow \rho$ translates into $\rho \geq \frac{1-\sqrt{1-t^2}}{2\sqrt{1-t^2}}$.

3.4. Codes in projective spaces. A class of spaces related to the real sphere is given by the projective spaces $\mathbb{P}L^{n-1}$ where $L = \mathbb{R}$ or \mathbb{C} of \mathbb{H} . The zonal spherical functions in these spaces are given by the Jacobi polynomials $P_k^{\alpha,\beta}(x)$ [12], where $\alpha = \sigma(n-1) - 1$, $\beta = \sigma - 1$, and $\sigma = 1/2, 1, 2$, respectively.

The polynomials $P_k^{\alpha,\beta}(x)$ satisfy

$$\int_{-1}^1 P_i^{\alpha,\beta}(x) P_j^{\alpha,\beta}(x) (1-x)^\alpha (1+x)^\beta dx = \frac{2^{\alpha+\beta+1} (k+\alpha)! (k+\beta)!}{(2k+\alpha+\beta+1) k! (k+\alpha+\beta)!} \delta_{i,j},$$

$$P_k(1) = \binom{k+\alpha}{\alpha},$$

where by definition $x! = \Gamma(x+1)$. The coefficients of three-term recurrence (3) have the form

$$\alpha_k = \frac{2(k+1)(k+\alpha+\beta+1)}{(2k+\alpha+\beta+1)(2k+\alpha+\beta+2)}, \quad \beta_k = \frac{\beta^2 - \alpha^2}{(2k+\alpha+\beta)(2k+\alpha+\beta+2)},$$

$$\gamma_k = \frac{2(k+\alpha)(k+\beta)}{(2k+\alpha+\beta)(2k+\alpha+\beta+1)}.$$

Define the bilinear form on V by $\langle f, g \rangle = \int_{-1}^1 f g d\mu$, where

$$d\mu(x) = \frac{(\alpha+\beta+1) \binom{\alpha+\beta}{\alpha}}{2^{\alpha+\beta+1}} (1-x)^\alpha (1+x)^\beta dx.$$

Then the squared norm of P_k is equal to

$$\|P_k^{\alpha,\beta}\|^2 = \frac{(\alpha+\beta+1)(\alpha+\beta)!(k+\alpha)!(k+\beta)!}{(2k+\alpha+\beta+1)\alpha!\beta!k!(k+\alpha+\beta)!}.$$

Denote by $\tilde{P}_k = P_k^{\alpha,\beta} / \|P_k^{\alpha,\beta}\|$ the normalized Jacobi polynomials.

We will take in (4)

$$P_1(x) = P_1^{\alpha,\beta}(x) = \frac{1}{2}((\alpha+\beta+2)x + \alpha - \beta),$$

then the coefficients of the recurrence are found to be

$$a_k = \frac{\alpha + \beta + 2}{2k + \alpha + \beta + 2} \sqrt{\frac{(k+\alpha+1)(k+\beta+1)(k+1)(k+\alpha+\beta+1)}{(2k+\alpha+\beta+3)(2k+\alpha+\beta+1)}},$$

$$b_k = \frac{2(\alpha - \beta)k(k+\alpha+\beta+1)}{(2k+\alpha+\beta)(2k+\alpha+\beta+2)},$$

and $c_k = a_{k-1}$.

Let $C \subset X$ be a code of size M in which $|\langle \mathbf{x}_i, \mathbf{x}_j \rangle| \leq t$ for any two distinct vectors $\mathbf{x}_i, \mathbf{x}_j$. We have $D = [0, \sqrt{2}]$, so choosing $\tau(d) = 2(1 - d^2/2)^2 - 1$ we obtain $\tau(D) = [-1, 1]$, $\Delta(C) \subset [-1, 2t^2 - 1]$. We compute

$$\tilde{P}_k^2(1) = \frac{(2k + \alpha + \beta + 1)}{\alpha + \beta + 1} \frac{\binom{k+\alpha}{\alpha} \binom{k+\alpha+\beta}{k}}{\binom{k+\beta}{\beta}}.$$

$$\rho_k = a_k \frac{\tilde{P}_{k+1}(1)}{\tilde{P}_k(1)} = \frac{(\alpha + \beta + 2)(k + \alpha + 1)(k + \alpha + \beta + 1)}{(2k + \alpha + \beta + 1)(2k + \alpha + \beta + 2)},$$

Using these expressions in Theorem 2 we obtain

Theorem 5.

$$M \leq \frac{4(\alpha + \beta + 2)(k + \alpha + 1)}{(2k + \alpha + \beta + 2)(1 - \lambda_{\max}(S_k))} \frac{\binom{k+\alpha}{\alpha} \binom{k+\alpha+\beta+1}{k}}{\binom{k+\beta}{\beta}}.$$

Let us use Lemma 2 to derive the asymptotic behavior of $\lambda_{\max}(S_k)$ as $k \rightarrow \infty, \alpha = ak, \beta = bk, a > 0, b \geq 0$. We obtain

$$\frac{\lambda_{\max}(S_k)}{k} \rightarrow \frac{2((a+b)\sqrt{(a+1)(b+1)(a+b+1)} + (a-b)(a+b+1))}{(a+b+2)^2}.$$

The condition for Theorem 2 to be applicable is

$$(13) \quad \lambda_{\max}(S_k) > P_1(2t^2 - 1) = (\alpha + \beta + 2)t^2 - \beta - 1.$$

For instance, let us derive a bound for the case $X = \mathbb{P}\mathbb{R}^{n-1}$. Letting $k = sn/2, \alpha = (n-3)/2, \beta = -1/2$, we obtain $a = 1/s, b = 0$,

$$\frac{\lambda_{\max}(S_k)}{k} \rightarrow \frac{4(1+s)}{(1+2s)^2}.$$

Therefore, for large values of the parameters condition (13) becomes

$$\frac{4(1+s)}{(1+2s)^2} = \frac{t^2}{s},$$

or $s = 1/2((1/\sqrt{1-t^2}) - 1)$. From Theorem 2 we obtain the asymptotic bound of [7] on the code size:

$$\frac{1}{n} \log M \leq (1+s) \log(1+s) - s \log s.$$

In a similar way we can recover the asymptotic bounds of [7] in the other cases mentioned.

The method presented is a linear-algebraic alternative to the analytic methods of [11, 7, 10]. It is equivalent to them in the sense that it gives the same asymptotic results, although for finite parameters the bounds derived by these two approaches generally do not coincide.

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