Problems of Information Transmission, vol. 43, no. 2, pp. 3-24, 2007.

M. V. Burnashev

## CODE SPECTRUM AND RELIABILITY FUNCTION: GAUSSIAN CHANNEL ${ }^{1}$

A new approach for upper bounding the channel reliability function using the code spectrum is described. It allows to treat both low and high rate cases in a unified way. In particular, the earlier known upper bounds are improved, and a new derivation of the sphere-packing bound is presented.

## § 1. Introduction and main results

We consider the discrete time channel with independent additive Gaussian noise, i.e. if $\boldsymbol{x}=\left(x_{1}, \ldots, x_{n}\right)$ is the input codeword then the received block $\boldsymbol{y}=\left(y_{1}, \ldots, y_{n}\right)$ is

$$
y_{i}=x_{i}+\xi_{i}, \quad i=1, \ldots, n,
$$

where $\left(\xi_{1}, \ldots, \xi_{n}\right)$ are independent Gaussian r.v.'s with $\mathbf{E} \xi_{i}=0, \mathbf{E} \xi_{i}^{2}=1$.
For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{n}$ denote $(\boldsymbol{x}, \boldsymbol{y})=\sum_{i=1}^{n} x_{i} y_{i},\|\boldsymbol{x}\|^{2}=(\boldsymbol{x}, \boldsymbol{x}), d(\boldsymbol{x}, \boldsymbol{y})=\|\boldsymbol{x}-\boldsymbol{y}\|^{2}$ and $S^{n-1}(b)=\left\{\boldsymbol{x} \in R^{n}:\|\boldsymbol{x}\|=b\right\}$. We assume that all codewords $\boldsymbol{x}$ satisfy the condition $\|\boldsymbol{x}\|^{2}=A n$, where $A>0$ is a given constant. A subset $\mathcal{C}=\left\{\boldsymbol{x}_{1}, \ldots, \boldsymbol{x}_{M}\right\} \subset S^{n-1}(\sqrt{A n})$, $M=e^{R n}$, is called a ( $R, A, n$ )-code of rate $R$ and length $n$. The minimum distance of the code $\mathcal{C}$ is $d(\mathcal{C})=\min \left\{d\left(\boldsymbol{x}_{i}, \boldsymbol{x}_{j}\right): i \neq j\right\}$.

The channel reliability function [1, 2] is defined as

$$
E(R, A)=\limsup _{n \rightarrow \infty} \frac{1}{n} \ln \frac{1}{P_{\mathrm{e}}(R, A, n)},
$$

where $P_{\mathrm{e}}(R, A, n)$ is the minimal possible decoding error probability for a $(R, A, n)$-code.
After the fundamental results of the paper [1], further improvements of various bounds for $E(R, A)$ have been obtained in [2-9]. In particular, on the exact form of the function $E(R, A)$ it was known only that [1]

$$
\begin{equation*}
E(0, A)=\frac{A}{4}, \quad E(R, A)=E_{\mathrm{sp}}(R, A), \quad R_{\mathrm{crit}}(A) \leq R \leq C(A) \tag{1}
\end{equation*}
$$

[^0]where
\[

$$
\begin{gather*}
C=C(A)=\frac{1}{2} \ln (1+A), \quad R_{\text {crit }}(A)=\frac{1}{2} \ln \frac{2+A+\sqrt{A^{2}+4}}{4},  \tag{2}\\
E_{\mathrm{sp}}(R, A)=\frac{A}{2}-\frac{\sqrt{A\left(1-e^{-2 R}\right)} g(R, A)}{2}-\ln g(R, A)+R,  \tag{3}\\
g(R, A)=\frac{1}{2}\left(\sqrt{A\left(1-e^{-2 R}\right)}+\sqrt{A\left(1-e^{-2 R}\right)+4}\right) .
\end{gather*}
$$
\]

Moreover, recently [8] the exact form of $E(R, A)$ for a new region $\bar{R}_{1}(A) \leq R \leq R_{\text {crit }}(A)$ was claimed under some restriction on $A$. Similar to the case of the binary symmetric channel (BSC), that assertion follows from a useful observation that the tangent (it has the slope $(-1))$ to the function $E_{\mathrm{sp}}(R, A)$ at the point $R=R_{\text {crit }}(A)$ touches the previously known upper bound for $E(R, A)$ [5-7]. Since those results from [5-7] were proved under some restrictions on $A$, those restrictions were remaining in [8] as well. Since there are some inaccuracies in the formulation of that result in [8] we do not expose corresponding formulas from [8] (moreover, they have a different from ours form).

From theorem 1 and the formula (9) (see below) the exact form of $E(R, A)$ follows for the region $\bar{R}_{1}(A) \leq R \leq R_{\text {crit }}(A)$ for any $A>0$. Moreover, if $A>A_{0} \approx 2.288$ (see (14)) then from theorem 2 below the exact form of $E(R, A)$ follows for a wider region $\bar{R}_{3}(A) \leq R \leq R_{\text {crit }}(A)$, where $\bar{R}_{3}(A)<\bar{R}_{1}(A)$ and $\bar{R}_{3}(A) \approx R_{\text {crit }}(A)-0.06866, A \geq A_{0}$.

For $0<R<\bar{R}_{1}(A), 0<A \leq A_{0}$, or $0<R<\bar{R}_{3}(A), A>A_{0}$, still only lower and upper bounds for $E(R, A)$ are known [1-9], and in this paper the most accurate of the upper bounds is improved.

We begin by explaining what constituted the difficulty in upper bounding the function $E(R, A)$ in the earlier papers [5-9]. Note that when testing only two codewords $\boldsymbol{x}_{i}, \boldsymbol{x}_{j}$ with large distance $\left\|\boldsymbol{x}_{i}-\boldsymbol{x}_{j}\right\|^{2}=d$ we have the decoding error probability $P_{\mathrm{e}} \sim e^{-d / 8}$. Let $B_{\rho n}$ be the average number of each codeword $\boldsymbol{x}_{i}$ neighbors on the approximate distance $2 A(1-\rho) n$. It was shown in [5] that for a $(R, A, n)$-code there exists $\rho$ such that $B_{\rho n} \gtrsim 2^{b(\rho) n}$, where the function $b(\rho)>0$ is described below, and $2 A(1-\rho) n$ does not exceed the best upper bound (linear programming) for the minimal code distance $d(\mathcal{C})$. Therefore, if each codeword $\boldsymbol{x}_{i}$ has approximately $B_{\rho n}$ neighbors on the distance $2 A(1-\rho) n$, then it is natural to expect that $P_{\mathrm{e}} \gtrsim B_{\rho n} e^{-A(1-\rho) n / 4}$ for large $n$ (and not very small $\rho$ ), i.e. a variant of an additive lower bound for the probability of the union of events holds.

The first variant of such additive bound was obtained in 5 under rather severe constraints on $R$ and $A$. Those results of [5] have been strengthened in [6, 7], using the method of [10-12]. However there were still certain constraints on $R$ and $A$. It should be noted that the investigation of $E(R, A)$ for the Gaussian channel is similar to the investigation of $E(R, A)$ for the BSC. The difference is only that due to the discrete structure of a binary alphabet some expressions become simpler. For the BSC the method of [6] was recently [14, 15] further developed. Although the approach of [14, 15] is still based on [6], some additional arguments allowed the approach to be essentially strengthened and simplified.

It should also be noted that until the papers [14, 15], all papers mentioned made use of various variants of the second order Bonferroni inequalities.

The main aim of this paper is to prove an additive bound without any constraints on $R$ or $A$. For that purpose the method of [14, 15] is applied. It is also worth noting that Bonferroni inequalities are not used. This approach allows us to treat both low and high rate $R$ cases in a unified way. As an example, in $\S 2$ a new derivation of the sphere-packing bound is presented.

Introduce some notations. For a code $\mathcal{C}=\left\{\boldsymbol{x}_{1}, \ldots, \boldsymbol{x}_{M}\right\} \subset S^{n-1}(\sqrt{A n})$ denote

$$
\begin{equation*}
\rho_{i j}=\frac{\left(\boldsymbol{x}_{i}, \boldsymbol{x}_{j}\right)}{A n}, \quad d_{i j}=\left\|\boldsymbol{x}_{i}-\boldsymbol{x}_{j}\right\|^{2}=2 A n\left(1-\rho_{i j}\right) . \tag{4}
\end{equation*}
$$

Below it will be convenient to use the parametric representation of the transmission rate $R=R(t)$ via the monotonic increasing function

$$
\begin{equation*}
R(t)=(1+t) \ln (1+t)-t \ln t, \quad t \geq 0 \tag{5}
\end{equation*}
$$

Consequently, for a rate $R \geq 0$ introduce $t_{R} \geq 0$ as the unique root of the equation

$$
\begin{equation*}
R=R\left(t_{R}\right)=\left(1+t_{R}\right) \ln \left(1+t_{R}\right)-t_{R} \ln t_{R} . \tag{6}
\end{equation*}
$$

Introduce also the functions

$$
\begin{equation*}
\tau(t)=\frac{2 \sqrt{t(1+t)}}{1+2 t}, \quad \tau_{R}=\tau\left(t_{R}\right) \tag{7}
\end{equation*}
$$

We shall need the values

$$
\begin{gather*}
\bar{t}_{1}(A)=\frac{\sqrt{2+\sqrt{4+A^{2}}}-2}{4}, \quad \bar{\tau}_{1}(A)=\tau\left(\bar{t}_{1}(A)\right)=\frac{A}{2+\sqrt{4+A^{2}}}  \tag{8}\\
\bar{R}_{1}(A)= \\
R\left(\bar{t}_{1}(A)\right)
\end{gather*}
$$

where the functions $\tau(t), R(t)$ are defined in (7) and (6). Sometimes below we shall omit the argument $A$ in $\bar{t}_{1}(A), \bar{\tau}_{1}(A), \bar{R}_{1}(A)$.

One of the main results of the paper is
T h e o r em 1. For any $A>0$ the following relations hold:

$$
E(R, A)=\left\{\begin{array}{l}
E_{\mathrm{sp}}\left(R_{\text {crit }}, A\right)+R_{\text {crit }}-R, \quad \bar{R}_{1} \leq R \leq R_{\text {crit }}  \tag{9}\\
E_{\mathrm{sp}}(R, A), \quad R_{\text {crit }} \leq R \leq C
\end{array}\right.
$$

and

$$
\begin{equation*}
E(R, A) \leq \frac{A\left(1-\tau_{R}\right)}{4}+\ln \left(1+2 t_{R}\right)-R, \quad 0 \leq R \leq \bar{R}_{1} \tag{10}
\end{equation*}
$$

where $R_{\text {crit }}(A), \bar{R}_{1}(A), \tau_{R}$ and $t_{R}$ are defined in (2), (8), (77) and (6), respectively.
Remark 1. We have $\bar{R}_{1}(A)<R_{\text {crit }}(A), A>0$. Moreover, $\max _{A}\left\{R_{\text {crit }}(A)-\bar{R}_{1}(A)\right\} \approx$ 0.06866, and it is attained for $A=A_{0} \approx 2.288$.

Remark 2. Note that (see the formulas (9) and (10) for $R=\bar{R}_{1}$ )

$$
\begin{equation*}
E_{\text {sp }}\left(R_{\text {crit }}, A\right)+R_{\text {crit }}=\frac{A\left(1-\bar{\tau}_{1}\right)}{4}+\ln \left(1+2 \bar{t}_{1}\right) . \tag{11}
\end{equation*}
$$

Validity of (11) can be checked using the formulas (6), (7) and the relations

$$
\begin{align*}
1+2 \bar{t}_{1} & =\sqrt{\frac{A}{4 \bar{\tau}_{1}}}, \quad R_{\text {crit }}=\frac{1}{2} \ln \frac{1}{1-\bar{\tau}_{1}},  \tag{12}\\
A\left(1-e^{-2 R_{\text {crit }}}\right) & =A \bar{\tau}_{1}=\frac{A}{\bar{\tau}_{1}}-4, \quad g\left(R_{\text {crit }}\right)=\frac{\left(1+\bar{\tau}_{1}\right) \sqrt{A}}{2 \sqrt{\bar{\tau}_{1}}} .
\end{align*}
$$

If $A>A_{0} \approx 2.288$ (see (14)) then the upper bound (10) can be slightly improved, and, moreover, the validity region of the first of formulas (9) can be enlarged to $\bar{R}_{3} \leq R \leq R_{\text {crit }}$, where $\bar{R}_{3}(A)<\bar{R}_{1}(A)$ (see (14)). To explain the possibility of such an improvement consider the problem of upper bounding the minimal code distance $\delta(R, n)$ of a spherical code. The best upper bound for $\delta(R, n)$ was obtained in [4] using the linear programming bound. It was also noticed in [4, p. 20] that for $R>0.234$ a better upper bound for $\delta(R, n)$ is obtained if the linear programming bound is applied not directly to the original spherical code, but to its subcode on a spherical cap. That observation was recently used in [9] when estimating the code spectrum and the function $E(R, A)$. Using the approach of 6] an upper bound for $E(R, A)$ was obtained in [9]. But it is rather difficult to use that upper bound since it is expressed as an optimization problem over four parameters. In fact, it is possible to get a more accurate and rather simple bound that constitutes theorem 2 below.

Introduce the function

$$
\begin{equation*}
D(t)=\ln \frac{1+t}{t}-\frac{1}{2 \sqrt{t(1+t)}}-\frac{1}{1+2 t}, \quad t>0 \tag{13}
\end{equation*}
$$

and denote $\bar{t}_{2} \approx 0.061176$ the unique root of the equation $D(t)=0$. The equivalent equation (with a sign misprint) appeared earlier in [4, p. 20]. Denote also

$$
\begin{gather*}
\bar{R}_{2}=R\left(\bar{t}_{2}\right) \approx 0.2339, \quad \bar{\tau}_{2}=\tau\left(\bar{t}_{2}\right) \approx 0.4540 \\
\bar{R}_{3}(A)=R_{\text {crit }}(A)+\bar{R}_{2}+\frac{1}{2} \ln \left(1-\bar{\tau}_{2}\right) \approx R_{\text {crit }}(A)-0.0687  \tag{14}\\
A_{0}=\min \left\{A: \bar{R}_{1}(A) \geq \bar{R}_{2}\right\} \approx 2.288
\end{gather*}
$$

The next result strengthens theorem 1 when $A>A_{0}$.
Theorem 2. If $A>A_{0} \approx 2.288$ then the following relations hold:

$$
E(R, A)=\left\{\begin{array}{l}
E_{\mathrm{sp}}\left(R_{\text {crit }}, A\right)+R_{\text {crit }}-R, \quad \bar{R}_{3} \leq R \leq R_{\text {crit }}  \tag{15}\\
E_{\mathrm{sp}}(R, A), \quad R_{\text {crit }} \leq R \leq C
\end{array}\right.
$$

and

$$
E(R, A) \leq\left\{\begin{array}{l}
\frac{1}{4} A\left(1-\tau_{R}\right)+\ln \left(1+2 t_{R}\right)-R, \quad 0<R \leq \bar{R}_{2}  \tag{16}\\
\frac{1}{4} A a e^{-2 R}-\frac{1}{2} \ln \left(2-a e^{-2 R}\right)-\frac{1}{2} \ln a, \quad \bar{R}_{2} \leq R \leq \bar{R}_{3}(A)
\end{array}\right.
$$

where $a=\left(1-\bar{\tau}_{2}\right) e^{2 \bar{R}_{2}} \approx 0.8717$.
For a comparison purpose we present also the best known lower bound for the function $E(R, A)[1 ; 3$, Theorem 7.4.4]

$$
E(R, A) \geq\left\{\begin{array}{l}
A\left(1-\sqrt{1-e^{-2 R}}\right) / 4, \quad 0 \leq R \leq R_{\text {low }}  \tag{17}\\
E_{\text {sp }}\left(R_{\text {crit }}, A\right)+R_{\text {crit }}-R, \quad R_{\text {low }} \leq R \leq R_{\text {crit }} \\
E_{\text {sp }}(R, A), \quad R_{\text {crit }} \leq R \leq C(A)
\end{array}\right.
$$

where

$$
\begin{equation*}
R_{\mathrm{low}}(A)=\frac{1}{2} \ln \frac{2+\sqrt{A^{2}+4}}{4} . \tag{18}
\end{equation*}
$$

Combining analytical and numerical methods it can be shown that for $A>A_{0}$ we have

$$
\begin{equation*}
R_{\text {low }}(A)<\bar{R}_{2}<\bar{R}_{3}(A)<\bar{R}_{1}(A)<R_{\text {crit }}(A) . \tag{19}
\end{equation*}
$$

On the figure the plots of upper (15), (16) and lower (17) bounds for $E(R, A)$ with $A=4$ are presented.

The paper is organized as follows. In $\S 2$ the main analytical tool (proposition 1) is presented and, as an example, the sphere-packing upper bound is derived. In $\S 3$ proposition 1 and the code spectrum are combined in propositions $2-3$. In $\S 4$ (using results of $\S 3$ and the known bound for the code spectrum - theorem 3) theorem 1 is proved. In $\S 5$ theorem 2 is proved. Proofs of some auxiliary results are presented in Appendix.

## § 2. New approach and sphere-packing exponent

For the conditional output probability distribution density $p(\boldsymbol{y} \mid \boldsymbol{x})$ of the input codeword $\boldsymbol{x}$ the formula holds

$$
\ln p(\boldsymbol{y} \mid \boldsymbol{x})=-\frac{1}{2} d(\boldsymbol{y}, \boldsymbol{x})-\frac{n}{2} \ln (2 \pi), \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{n}
$$

(in a similar formula in [6] there is a misprint - the minus sign is missing). To describe our approach, we fix a small $\delta=o(1), n \rightarrow \infty$, and $s>0$ and for an output $\boldsymbol{y}$ define the set:

$$
\begin{equation*}
\boldsymbol{X}_{s}(\boldsymbol{y})=\left\{\boldsymbol{x}_{i} \in \mathcal{C}:\left|d\left(\boldsymbol{y}, \boldsymbol{x}_{i}\right)-s n\right| \leq \delta n\right\}, \quad \boldsymbol{y} \in \mathbb{R}^{n} \tag{20}
\end{equation*}
$$

All codewords $\left\{\boldsymbol{x}_{i}\right\}$ are assumed equiprobable. For a chosen decoding method denote $P\left(e \mid \boldsymbol{y}, \boldsymbol{x}_{i}\right)$ the conditional decoding error probability provided that $\boldsymbol{x}_{i}$ was transmitted and $\boldsymbol{y}$ was received. Denote $p_{\mathrm{e}}(\boldsymbol{y})$ the probability distribution density to get the output $\boldsymbol{y}$ and to make a decoding error. Then

$$
\begin{aligned}
& p_{\mathrm{e}}(\boldsymbol{y})=M^{-1} \sum_{i=1}^{M} p\left(\boldsymbol{y} \mid \boldsymbol{x}_{i}\right) P\left(e \mid \boldsymbol{y}, \boldsymbol{x}_{i}\right) \geq M^{-1} \sum_{\boldsymbol{x}_{i} \in \boldsymbol{X}_{s}(\boldsymbol{y})} p\left(\boldsymbol{y} \mid \boldsymbol{x}_{i}\right) P\left(e \mid \boldsymbol{y}, \boldsymbol{x}_{i}\right)= \\
& =M^{-1}(2 \pi)^{-n / 2} \sum_{\boldsymbol{x}_{i} \in \boldsymbol{X}_{s}(\boldsymbol{y})} e^{-d\left(\boldsymbol{y}, \boldsymbol{x}_{i}\right) / 2} P\left(e \mid \boldsymbol{y}, \boldsymbol{x}_{i}\right) \geq \\
& \geq M^{-1}\left(2 \pi e^{s+\delta}\right)^{-n / 2} \sum_{\boldsymbol{x}_{i} \in \boldsymbol{X}_{s}(\boldsymbol{y})} P\left(e \mid \boldsymbol{y}, \boldsymbol{x}_{i}\right) \geq M^{-1}\left(2 \pi e^{s+\delta}\right)^{-n / 2}\left[\left|\boldsymbol{X}_{s}(\boldsymbol{y})\right|-1\right]_{+},
\end{aligned}
$$

where $[z]_{+}=\max \{0, z\}$ and $|A|$ - the cardinality of the set $A$. For the decoding error probability $P_{\mathrm{e}}$ we get

$$
P_{\mathrm{e}}=\int_{\boldsymbol{y} \in \mathbb{R}^{n}} p_{\mathrm{e}}(\boldsymbol{y}) d \boldsymbol{y} \geq M^{-1}\left(2 \pi e^{s+\delta}\right)^{-n / 2} \int_{\boldsymbol{y}:\left|\boldsymbol{X}_{s}(\boldsymbol{y})\right| \geq 2}\left[\left|\boldsymbol{X}_{s}(\boldsymbol{y})\right|-1\right] d \boldsymbol{y} .
$$

Since $(a-1) \geq a / 2, a \geq 2$, we have

$$
\begin{equation*}
P_{\mathrm{e}} \geq(2 M)^{-1}\left(2 \pi e^{s+\delta}\right)^{-n / 2} \int_{\boldsymbol{y}:\left|\boldsymbol{X}_{s}(\boldsymbol{y})\right| \geq 2}\left|\boldsymbol{X}_{s}(\boldsymbol{y})\right| d \boldsymbol{y} \tag{21}
\end{equation*}
$$

where $\boldsymbol{X}_{s}(\boldsymbol{y})$ is defined in (20). To develop further the right-hand side of (21) we fix some $r>0$ and for each $\boldsymbol{x}_{i}$ introduce the set

$$
\begin{align*}
& \boldsymbol{Z}_{s, r}(i)=\left\{\boldsymbol{y}:\left|\|\boldsymbol{y}\|^{2}-r n\right| \leq \delta n,\left|d\left(\boldsymbol{y}, \boldsymbol{x}_{i}\right)-s n\right| \leq \delta n,\left|\boldsymbol{X}_{s}(\boldsymbol{y})\right| \geq 2\right\}= \\
& =\left\{\boldsymbol{y}: \begin{array}{c}
\left|\|\boldsymbol{y}\|^{2}-r n\right| \leq \delta n,\left|d\left(\boldsymbol{y}, \boldsymbol{x}_{i}\right)-s n\right| \leq \delta n \text { and } \\
\text { there exists } \boldsymbol{x}_{j} \neq \boldsymbol{x}_{i} \text { with }\left|d\left(\boldsymbol{x}_{j}, \boldsymbol{y}\right)-s n\right| \leq \delta n
\end{array}\right\} . \tag{22}
\end{align*}
$$

For a measurable set $A \subseteq \mathbb{R}^{n}$ denote by $m(A)$ its Lebesque measure. Then

$$
\int_{\boldsymbol{y}:\left|\boldsymbol{X}_{s}(\boldsymbol{y})\right| \geq 2}\left|\boldsymbol{X}_{s}(\boldsymbol{y})\right| d \boldsymbol{y} \geq \sum_{i=1}^{M} m\left(\boldsymbol{Z}_{s, r}(i)\right)
$$

and from (21) we get
Proposition 1. With any $\delta>0$ for the decoding error probability $P_{\mathrm{e}}$ the lower bound holds

$$
\begin{equation*}
P_{\mathrm{e}} \geq \frac{1}{2 M} \max _{s, r}\left\{\left(2 \pi e^{s+\delta}\right)^{-n / 2} \sum_{i=1}^{M} m\left(\boldsymbol{Z}_{s, r}(i)\right)\right\} \tag{23}
\end{equation*}
$$

where $\boldsymbol{Z}_{s, r}(i)$ is defined in (22).
Example: sphere-packing upper bound. We show first how to get the spherepacking upper bound $E(R, A) \leq E_{\mathrm{sp}}(R, A)$ from (23) (cf. [1;3, Chapter 7.4]). To simplify formulas we write below $a \approx b$ if $|a-b| \leq \delta$, where $\delta=o(1), n \rightarrow \infty$. Note that

$$
\begin{gathered}
\boldsymbol{Z}_{s, r}(i)=\boldsymbol{Z}_{s, r}^{(1)}(i) \backslash \boldsymbol{Z}_{s, r}^{(2)}(i), \quad \boldsymbol{Z}_{s, r}^{(1)}(i)=\left\{\boldsymbol{y}:\|\boldsymbol{y}\|^{2} / n \approx r, d\left(\boldsymbol{y}, \boldsymbol{x}_{i}\right) / n \approx s\right\}, \\
\boldsymbol{Z}_{s, r}^{(2)}(i)=\left\{\boldsymbol{y}:\|\boldsymbol{y}\|^{2} / n \approx r, d\left(\boldsymbol{y}, \boldsymbol{x}_{i}\right) / n \approx s,\left|\boldsymbol{X}_{s}(\boldsymbol{y})\right|=1\right\}= \\
=\left\{\boldsymbol{y}:\|\boldsymbol{y}\|^{2} / n \approx r, d\left(\boldsymbol{y}, \boldsymbol{x}_{i}\right) / n \approx s \text { and there is no } \boldsymbol{x}_{j} \neq \boldsymbol{x}_{i} \text { with } d\left(\boldsymbol{x}_{j}, \boldsymbol{y}\right) / n \approx s\right\} .
\end{gathered}
$$

Then we have

$$
\begin{gathered}
\bigcup_{i=1}^{M} \boldsymbol{Z}_{s, r}^{(2)}(i)=\boldsymbol{Y}_{s}=\left\{\boldsymbol{y}:\|\boldsymbol{y}\|^{2} / n \approx r,\left|\boldsymbol{X}_{s}(\boldsymbol{y})\right|=1\right\}= \\
=\left\{\boldsymbol{y}:\|\boldsymbol{y}\|^{2} / n \approx r \text { and there exists exactly one } \boldsymbol{x}_{i} \text { with } d\left(\boldsymbol{y}, \boldsymbol{x}_{i}\right) / n \approx s\right\} \\
\boldsymbol{Y}_{s} \subseteq \boldsymbol{Y}(r)=\left\{\boldsymbol{y}:\|\boldsymbol{y}\|^{2} / n \approx r\right\}
\end{gathered}
$$

and the lower bound (23) takes the form

$$
P_{\mathrm{e}} \geq(2 M)^{-1}\left(2 \pi e^{s+\delta}\right)^{-n / 2}\left[M m\left(\left|\boldsymbol{Z}_{s, r}^{(1)}(1)\right|\right)-m(\boldsymbol{Y}(r))\right]_{+} .
$$

The surface area of a $n$-dimensional sphere of radius $a$ is $S_{n}(a)=n \pi^{n / 2} a^{n-1} / \Gamma(n / 2+1) \sim$ $\left(2 \pi e a^{2} / n\right)^{n / 2}$. Then from a standard geometry we get

$$
\begin{gathered}
m\left(\left|\boldsymbol{Z}_{s, r}^{(1)}(1)\right|\right) \sim\left(2 \pi e r_{1}\right)^{n / 2}, \quad m(\boldsymbol{Y}(r)) \sim(2 \pi e r)^{n / 2} \\
r_{1}=s-\frac{(r-A-s)^{2}}{4 A}=r-\frac{(r+A-s)^{2}}{4 A}
\end{gathered}
$$

Therefore the lower bound (23) takes the form

$$
\begin{equation*}
P_{\mathrm{e}} \gtrsim M^{-1}\left(e^{s+\delta-1}\right)^{-n / 2}\left[M r_{1}^{n / 2}-r^{n / 2}\right]_{+} . \tag{24}
\end{equation*}
$$

We want to maximize the right-hand side of (24) over $s$, $r$. Since we are interested only in exponents in $n$, we may assume that $M r_{1}^{n / 2}=r^{n / 2}$, i.e. $e^{2 R} r_{1}=r$. Then we should maximize the function $f(s, r)=\ln r-s$ provided

$$
s-\frac{(r-A-s)^{2}}{4 A}-r e^{-2 R}=0
$$

As usual, considering the function

$$
g(s, r)=\ln r-s+\lambda\left[s-\frac{(r-A-s)^{2}}{4 A}-r e^{-2 R}\right]
$$

and solving the equations $g_{s}^{\prime}=g_{r}^{\prime}=0$, we get

$$
r=\frac{1}{1-\lambda\left(1-e^{-2 R}\right)}, \quad s=r+A-\frac{2 A}{\lambda}
$$

where $\lambda$ satisfies the equation

$$
\left(1-e^{-2 R}\right) \lambda^{2}+A\left(1-e^{-2 R}\right) \lambda-A=0 .
$$

Therefore

$$
\lambda=\frac{\sqrt{A}}{g_{1} \sqrt{1-e^{-2 R}}},
$$

where $g_{1}=g_{1}(R, A)$ is defined in (3). Note that

$$
\begin{gathered}
g^{2}-1=g \sqrt{A\left(1-e^{-2 R}\right)}, \quad 1-\lambda\left(1-e^{-2 R}\right)=\frac{1}{g^{2}}, \\
\ln r-s=2 \ln g-1-A+g \sqrt{A\left(1-e^{-2 R}\right)} .
\end{gathered}
$$

Taking into account that $e^{2 R} r_{1}=r$, we get from (24) and (3)

$$
\begin{gathered}
\frac{1}{n} \ln \frac{1}{P_{\mathrm{e}}} \leq \frac{s-1}{2}-\ln r_{1}=\frac{s-1}{2}+R-\frac{1}{2} \ln r= \\
=\frac{A-\sqrt{A\left(1-e^{-2 R}\right)} g(R, A)}{2}-\ln g(R, A)+R=E_{\mathrm{sp}}(R, A),
\end{gathered}
$$

which gives the sphere-packing upper bound $E(R, A) \leq E_{\mathrm{sp}}(R, A)$.

## § 3. Lower bound (23) and code spectrum

For a code $\mathcal{C} \subset S^{n-1}(\sqrt{A n})$ introduce the code spectrum function

$$
\begin{equation*}
B(s, t)=\frac{1}{|\mathcal{C}|}\left|\left\{\boldsymbol{u}, \boldsymbol{v} \in \mathcal{C}: s \leq \frac{(\boldsymbol{u}, \boldsymbol{v})}{A n}<t\right\}\right| \tag{25}
\end{equation*}
$$

and denote

$$
b(\rho, \varepsilon)=\frac{1}{n} \ln B(\rho-\varepsilon, \rho+\varepsilon), \quad 0<\varepsilon<\rho .
$$

To simplify notation we write below $a \approx b$ if $|a-b| \leq \delta$, where $\delta=1 / \sqrt{A n}$. For some $r>0$ we consider only the set of outputs

$$
\begin{equation*}
\boldsymbol{Y}(r)=\left\{\boldsymbol{y}:\|\boldsymbol{y}\|^{2} / n \approx r\right\} \subseteq \mathbb{R}^{n} \tag{26}
\end{equation*}
$$

To investigate the function $E(R, A), R<R_{\text {crit }}$, we use a variant of the lower bound (23)

$$
\begin{equation*}
P_{\mathrm{e}} \geq(2 M)^{-1} \max _{s, r>0} \max _{\rho}\left\{\left(2 \pi e^{s+\delta}\right)^{-n / 2} \sum_{i=1}^{M} m\left(\boldsymbol{Z}_{s, r}(\rho, i)\right)\right\} \tag{27}
\end{equation*}
$$

where

$$
\boldsymbol{Z}_{s, r}(\rho, i)=\left\{\boldsymbol{y} \in \boldsymbol{Y}(r): \begin{array}{c}
\text { there exists } \boldsymbol{x}_{j} \text { with } \rho_{i j} \approx \rho \text { and }  \tag{28}\\
d\left(\boldsymbol{x}_{i}, \boldsymbol{y}\right) / n \approx d\left(\boldsymbol{x}_{j}, \boldsymbol{y}\right) / n \approx s
\end{array}\right\}
$$

and $\rho_{i j}$ is defined in (4). We develop the lower bound (27), relating it to the code spectrum (25), i.e. to the distribution of the pairwise inner products $\left\{\rho_{i j}\right\}$.

For codewords $\boldsymbol{x}_{i}, \boldsymbol{x}_{j}$ with $\rho_{i j} \approx \rho$ introduce the set

$$
\begin{equation*}
\boldsymbol{Z}_{s, r}(\rho, i, j)=\left\{\boldsymbol{y} \in \boldsymbol{Y}(r): d\left(\boldsymbol{x}_{i}, \boldsymbol{y}\right) / n \approx d\left(\boldsymbol{x}_{j}, \boldsymbol{y}\right) / n \approx s\right\} \tag{29}
\end{equation*}
$$

Then for any $i$ from (28) and (29) we have

$$
\begin{equation*}
\boldsymbol{Z}_{s, r}(\rho, i)=\bigcup_{j: \rho_{i j} \approx \rho} \boldsymbol{Z}_{s, r}(\rho, i, j) . \tag{30}
\end{equation*}
$$

Denoting

$$
\begin{equation*}
Z(s, r, \rho)=m\left(\boldsymbol{Z}_{s, r}(\rho, i, j)\right) \tag{31}
\end{equation*}
$$

(since the measure of that set does not depend on indices $(i, j)$ ), we have (see Appendix)

$$
\begin{equation*}
\frac{1}{n} \ln Z(s, r, \rho)=\frac{1}{2} \ln [2 \pi e z(s, r, \rho)]+o(1), \quad n \rightarrow \infty \tag{32}
\end{equation*}
$$

where

$$
\begin{equation*}
z(s, r, \rho)=r-\frac{(A+r-s)^{2}}{2 A(1+\rho)} \tag{33}
\end{equation*}
$$

Note that due to (30), for the sum in the right-hand side of (27) for any $\rho$ we have

$$
\begin{gather*}
\sum_{i=1}^{M} m\left(\boldsymbol{Z}_{s, r}(\rho, i)\right) \leq \sum_{(i, j): \rho_{i j} \approx \rho} m\left(\boldsymbol{Z}_{s, r}(\rho, i, j)\right)=Z(s, r, \rho)\left|\left\{(i, j): \rho_{i j} \approx \rho\right\}\right|=  \tag{34}\\
=\exp \left\{\frac{n}{2} \ln [2 \pi e z(s, r, \rho)]+[R+b(\rho)] n+o(n)\right\}
\end{gather*}
$$

since for $b(\rho)=b(\rho, \delta)$ the following formula holds (see (25))

$$
\left|\left\{(i, j): \rho_{i j} \approx \rho\right\}\right|=e^{R n} B(\rho-\delta, \rho+\delta)=e^{(R+b(\rho)) n}
$$

Suppose that for some $\rho=\rho_{0}$ in the relation (34) the following asymptotic equality holds:

$$
\begin{equation*}
\frac{1}{n} \ln \left[\sum_{i=1}^{M} m\left(\boldsymbol{Z}_{s, r}\left(\rho_{0}, i\right)\right)\right]=\frac{1}{2} \ln \left[2 \pi e z\left(s, r, \rho_{0}\right)\right]+R+b\left(\rho_{0}\right)+o(1), \quad n \rightarrow \infty \tag{35}
\end{equation*}
$$

Using the functions $s=s(\rho), r=r(\rho)$ (they are chosen below), from (27), (35) and (33) for such $\rho_{0}$ we get

$$
\begin{equation*}
\frac{1}{n} \ln \frac{1}{P_{\mathrm{e}}} \leq \frac{s-1}{2}-\frac{1}{2} \ln \left[r-\frac{(A+r-s)^{2}}{2 A\left(1+\rho_{0}\right)}\right]-b\left(\rho_{0}\right)+o(1) . \tag{36}
\end{equation*}
$$

We set below

$$
\begin{equation*}
s(\rho)=\frac{A(1-\rho)}{2}+1, \quad r(\rho)=\frac{A(1+\rho)}{2}+1 \tag{37}
\end{equation*}
$$

Such choice of $s(\rho), r(\rho)$ minimizes (over $s, r$ ) the right-hand side of (36). Optimality of such $s, r$ can also be deduced from the formulas (72) (see Appendix).

For such $s(\rho), r(\rho)$ we have $r-(A+r-s)^{2} /[2 A(1+\rho)]=1$, and then (36) takes the simple form

$$
\begin{equation*}
\frac{1}{n} \ln \frac{1}{P_{\mathrm{e}}} \leq \frac{A\left(1-\rho_{0}\right)}{4}-b\left(\rho_{0}\right)+o(1) . \tag{38}
\end{equation*}
$$

Note that $b(\rho) \geq 0$ if there exists a pair $\left(\boldsymbol{x}_{i}, \boldsymbol{x}_{j}\right)$ with $\rho_{i j} \approx \rho$, and $b(\rho)=-\infty$ if there is no any pair with $\rho_{i j} \approx \rho$.

We formulate the result obtained as follows.
Proposition 2. If for some $\rho_{0}$ the condition (35) is fulfilled, then the inequality (38) for the decoding error probability $P_{\mathrm{e}}$ holds.

We show that as such $\rho_{0}$ we may choose the value $\rho_{0}$, minimizing the right-hand side of (38). In other words, define $\rho_{0}$ as follows

$$
\begin{equation*}
A \rho_{0}+4 b\left(\rho_{0}\right)=\max _{|\rho| \leq 1}\{A \rho+4 b(\rho)\} \tag{39}
\end{equation*}
$$

Remark 3. If there are several such $\rho_{0}$, we may use any of them. It is not important that we do not know the function $b(\rho)$. We may use as $b(\rho)$ any lower bound for it (see proofs of theorems 1 and 2).

Proposit i o n 3. For $\rho_{0}$ from (39) the condition (35) holds and therefore the inequality (38) is valid.

Proof. It is convenient to "quantize" the range of possible values of the normalized inner products $\rho_{i j}$. For that purpose we partition the whole range $[-1 ; 1]$ of values $\rho_{i j}$ on subintervals of the length $\delta=1 / \sqrt{A n}$. There will be $n_{1}=2 / \delta$ of such subintervals. We may assume that $\rho_{i j}$ takes values from the set $\left\{-1=\rho_{1}<\ldots<\rho_{n_{1}}=1\right\}$.

We call $\left(\boldsymbol{x}_{i}, \boldsymbol{x}_{j}\right)$ a $\rho$-pair if $\left(\boldsymbol{x}_{i}, \boldsymbol{x}_{j}\right) /(A n) \approx \rho$. Then $M e^{n b(\rho)}$ is the total number of $\rho$-pairs. We use $s=s\left(\rho_{0}\right), r=r\left(\rho_{0}\right)$ from (37) and consider only outputs $\boldsymbol{y} \in \boldsymbol{Y}(r)=$ $\boldsymbol{Y}\left(r\left(\rho_{0}\right)\right)$. We say that such a point $\boldsymbol{y}$ is $\rho$-covered if there exists a $\rho$-pair $\left(\boldsymbol{x}_{i}, \boldsymbol{x}_{j}\right)$ such that $d\left(\boldsymbol{x}_{i}, \boldsymbol{y}\right) / n \approx d\left(\boldsymbol{x}_{j}, \boldsymbol{y}\right) / n \approx s$. Then the total (taking into account the covering multiplicities) Lebesque measure of all $\rho$-covered points $\boldsymbol{y}$ equals $M e^{n b(\rho)} Z(s, r, \rho)$.
Introduce the set $\boldsymbol{Y}\left(\rho_{0}, \rho\right)$ of all $\rho$-covered points $\boldsymbol{y}$

$$
\boldsymbol{Y}\left(\rho_{0}, \rho\right)=\{\boldsymbol{y} \in \boldsymbol{Y}(r): \boldsymbol{y} \text { is } \rho \text {-covered }\} .
$$

We consider the set $\boldsymbol{Y}\left(\rho_{0}, \rho\right)$ and perform its "cleaning", excluding from it all points $\boldsymbol{y}$ that are also $\rho$-covered for any $\rho$ such that $\left|\rho-\rho_{0}\right| \geq 4 \delta$, i.e. we consider the set

$$
\begin{gather*}
\boldsymbol{Y}^{\prime}\left(\rho_{0}, \rho_{0}\right)=\boldsymbol{Y}\left(\rho_{0}, \rho_{0}\right) \backslash \bigcup_{\left|\rho-\rho_{0}\right| \geq 4 \delta} \boldsymbol{Y}\left(\rho_{0}, \rho\right)=  \tag{40}\\
=\left\{\boldsymbol{y} \in \boldsymbol{Y}(r): \begin{array}{c}
\boldsymbol{y} \text { is } \rho_{0} \text {-covered and is not } \rho \text {-covered } \\
\text { for any } \rho \text { such that }\left|\rho-\rho_{0}\right| \geq 4 \delta
\end{array}\right\} .
\end{gather*}
$$

Each point $\boldsymbol{y} \in \boldsymbol{Y}^{\prime}\left(\rho_{0}, \rho_{0}\right)$ can be $\rho$-covered only if $\left|\rho-\rho_{0}\right|<4 \delta$. We show that both sets $\boldsymbol{Y}\left(\rho_{0}, \rho_{0}\right)$ and $\boldsymbol{Y}^{\prime}\left(\rho_{0}, \rho_{0}\right)$ have essentially the same Lebesque measures. Note that a $\rho$-pair $\left(\boldsymbol{x}_{i}, \boldsymbol{x}_{j}\right) \rho$-covers the set $\boldsymbol{Z}_{s, r}(\rho, i, j)$ from (29) with the Lebesque measure $Z(s, r, \rho)$. We compare the values $\sum_{\left|\rho-\rho_{0}\right| \geq 4 \delta} e^{n b(\rho)} Z(s, r, \rho)$ and $e^{n b\left(\rho_{0}\right)} Z\left(s, r, \rho_{0}\right)$ (see (40)). For that purpose we consider the function

$$
\begin{equation*}
g(\rho)=\frac{1}{n} \ln \frac{e^{n b(\rho)} Z(s, r, \rho)}{e^{n b\left(\rho_{0}\right)} Z\left(s, r, \rho_{0}\right)}=b(\rho)-b\left(\rho_{0}\right)+\frac{1}{2} \ln \frac{z(s, r, \rho)}{z\left(s, r, \rho_{0}\right)}+o(1), \tag{41}
\end{equation*}
$$

where $z(s, r, \rho)$ is defined in (33). From (33) we also have

$$
z(s, r, \rho)=1+\frac{A\left(1+\rho_{0}\right)\left(\rho-\rho_{0}\right)}{2(1+\rho)} .
$$

Since $b(\rho) \leq b\left(\rho_{0}\right)-A\left(\rho-\rho_{0}\right) / 4$ (see (39)), for the function $g(\rho)$ from (41) we get

$$
\begin{equation*}
g(\rho) \leq \frac{1}{2} \ln \left[1+\frac{A\left(1+\rho_{0}\right)\left(\rho-\rho_{0}\right)}{2(1+\rho)}\right]-\frac{A\left(\rho-\rho_{0}\right)}{4} \leq-\frac{A\left(\rho-\rho_{0}\right)^{2}}{4(1+\rho)} . \tag{42}
\end{equation*}
$$

Since $\rho-\rho_{0}=i \delta,|i| \geq 4$, after simple calculations we have

$$
\frac{\sum_{\left|\rho-\rho_{0}\right| \geq 4 \delta} e^{n b(\rho)} Z(s, r, \rho)}{e^{n b\left(\rho_{0}\right)} Z\left(s, r, \rho_{0}\right)}=\sum_{\left|\rho-\rho_{0}\right| \geq 4 \delta} e^{n g(\rho)} \leq 2 \sum_{i \geq 4} \exp \left\{-\frac{A n \delta^{2} i^{2}}{8}\right\}=2 \sum_{i \geq 4} e^{-i^{2} / 8}<\frac{1}{2}
$$

Therefore we get

$$
e^{n b\left(\rho_{0}\right)} Z\left(s, r, \rho_{0}\right)-\sum_{\left|\rho-\rho_{0}\right| \geq 4 \delta} e^{n b(\rho)} Z(s, r, \rho)>\frac{1}{2} e^{n b\left(\rho_{0}\right)} Z\left(s, r, \rho_{0}\right)
$$

Then the total (taking into account the covering multiplicities) Lebesque measure of all $\rho$-covered points $\boldsymbol{y} \in \boldsymbol{Y}^{\prime}\left(\rho_{0}, \rho_{0}\right)$ exceeds $M e^{n b\left(\rho_{0}\right)} Z\left(s, r, \rho_{0}\right) / 2$. Remind that any point $\boldsymbol{y} \in$ $\boldsymbol{Y}^{\prime}\left(\rho_{0}, \rho_{0}\right)$ can be $\rho$-covered only if $\left|\rho-\rho_{0}\right|<4 \delta$.

For each point $\boldsymbol{y} \in \boldsymbol{Y}^{\prime}\left(\rho_{0}, \rho_{0}\right)$ consider the set $\boldsymbol{X}_{s}(\boldsymbol{y})$ defined in (20), i.e. the set of all codewords $\left\{\boldsymbol{x}_{i}\right\}$ such that $d\left(\boldsymbol{x}_{i}, \boldsymbol{y}\right) / n \approx s$. The codewords from $\boldsymbol{X}_{s}(\boldsymbol{y})$ satisfy also the condition $\left|\left(\boldsymbol{x}_{i}, \boldsymbol{x}_{j}\right) /(A n)-\rho_{0}\right|<4 \delta$, i.e. the set $\left\{\boldsymbol{x}_{i}\right\}$ constitutes almost a simplex. It is rather clear that the number $\left|\boldsymbol{X}_{s}(\boldsymbol{y})\right|$ of such codewords is not exponential on $n$, i.e.

$$
\begin{equation*}
\max _{\boldsymbol{y} \in \boldsymbol{Y}^{\prime}\left(\rho_{0}, \rho_{0}\right)}\left\{\frac{1}{n} \ln \left|\boldsymbol{X}_{s}(\boldsymbol{y})\right|\right\}=o(1), \quad n \rightarrow \infty \tag{43}
\end{equation*}
$$

Formally the validity of (43) follows from lemma 2 (see below).
Note that if $A_{1}, \ldots, A_{N} \subset \mathbb{R}^{n}$ are a measurable sets, and any point $a \in \bigcup_{i} A_{i}$ is covered by the sets $\left\{A_{i}\right\}$ not more than $K$ times, then

$$
\begin{equation*}
m\left(\bigcup_{i=1}^{N} A_{i}\right) \geq \frac{1}{K} \sum_{i=1}^{N} m\left(A_{i}\right) \tag{44}
\end{equation*}
$$

For $\boldsymbol{y} \in \boldsymbol{Y}^{\prime}\left(\rho_{0}, \rho_{0}\right)$ denote

$$
\begin{gather*}
\boldsymbol{X}_{i}(\boldsymbol{y})=\left\{\boldsymbol{x}_{j}: d\left(\boldsymbol{x}_{i}, \boldsymbol{y}\right) / n \approx d\left(\boldsymbol{x}_{j}, \boldsymbol{y}\right) / n \approx s, \rho_{i j} \approx \rho_{0}\right\}, \\
X_{\max }=\max _{i, \boldsymbol{y} \in \boldsymbol{Y}^{\prime}\left(\rho_{0}, \rho_{0}\right)}\left|\boldsymbol{X}_{i}(\boldsymbol{y})\right| . \tag{45}
\end{gather*}
$$

Due to (43) we have

$$
\begin{equation*}
\frac{1}{n} \ln X_{\max }=o(1), \quad n \rightarrow \infty . \tag{46}
\end{equation*}
$$

Since any point $\boldsymbol{y} \in \boldsymbol{Y}^{\prime}\left(\rho_{0}, \rho_{0}\right)$ can be $\rho$-covered not more than $X_{\text {max }}$ times and $\boldsymbol{Y}^{\prime}\left(\rho_{0}, \rho_{0}\right) \subseteq$ $\boldsymbol{Y}\left(\rho_{0}, \rho_{0}\right)$, then from (43)-(46) we get

$$
\begin{gather*}
\frac{1}{n} \ln \left[\sum_{i=1}^{M} m\left(\boldsymbol{Z}_{s, r}\left(\rho_{0}, i\right)\right)\right] \geq \frac{1}{n} \ln m\left(\boldsymbol{Y}^{\prime}\left(\rho_{0}, \rho_{0}\right)\right) \geq \\
\quad \geq \frac{1}{n} \ln \left(M e^{n b\left(\rho_{0}\right)} Z\left(s, r, \rho_{0}\right)\right)+o(1)=  \tag{47}\\
=\frac{1}{2} \ln \left[2 \pi e z\left(s, r, \rho_{0}\right)\right]+R+b\left(\rho_{0}\right)+o(1), \quad n \rightarrow \infty .
\end{gather*}
$$

Therefore due to the inequalities (34) and (47), the condition (35) is fulfilled, and then the relation (38) holds.

To complete the proof of proposition 2 it remains to establish the formula (43). We prove it first for a simpler (but a more natural) case $\rho^{*} \leq \bar{\tau}_{1}$, and then consider the general case.

C a s e $\rho_{0} \leq \bar{\tau}_{1}$. In that case the relation (43)) follows from simple lemma (see proof in Appendix).

Lemma 1. Let $\boldsymbol{y} \in \mathbb{R}^{n}$ with $\|\boldsymbol{y}\|^{2}=r n$. Let $\mathcal{C}=\left\{\boldsymbol{x}_{1}, \ldots, \boldsymbol{x}_{M}\right\} \subset S^{n-1}(\sqrt{A n})$ be a code with $\left\|\boldsymbol{x}_{i}-\boldsymbol{y}\right\|^{2}=s n, i=1, \ldots, M$, and $\max _{i \neq j}\left(\boldsymbol{x}_{i}, \boldsymbol{x}_{j}\right) \leq$ An $\rho$. If

$$
\begin{equation*}
A+r-s \geq 2 \sqrt{A r \rho} \tag{48}
\end{equation*}
$$

then $M \leq 2 n$.
For $s(\rho), r(\rho)$ from (37) the condition (48) holds, if

$$
\begin{equation*}
\rho \leq \frac{A}{2+\sqrt{4+A^{2}}}=\bar{\tau}_{1}(A) . \tag{49}
\end{equation*}
$$

From lemma 1 and (49) the relation (43) follows.
General case. Although a code with $\rho_{0}>\bar{\tau}_{1}$ can hardly decrease the decoding error probability $P_{\mathrm{e}}$, its investigation needs a bit more efforts. The relation (43) follows from lemma (see proof in Appendix).

Lemma2. Let for a code $\mathcal{C}=\left\{\boldsymbol{x}_{1}, \ldots, \boldsymbol{x}_{M}\right\} \subset S^{n-1}(\sqrt{A n})$ and some $\rho<1$ it holds that

$$
\max _{i \neq j}\left|\left(\boldsymbol{x}_{i}, \boldsymbol{x}_{j}\right)-A \rho n\right|=o(n), \quad n \rightarrow \infty
$$

Then $\ln M=o(n), n \rightarrow \infty$.
It completes the proof of proposition 3.
Using proposition 3 and two lower bounds for $b(\rho)$ we shall prove theorems 1 and 2 .

## § 4. Proof of theorem 1

First we investigate the function $E(R, A)$ for $0<R \leq \bar{R}_{1}(A)$ and prove the upper bound (10). Then for $\bar{R}_{1}(A)<R<R_{\text {crit }}(A)$, using the "straight-line bound" [2, we will prove the formula (9). To apply proposition 3 we use the known bound for the code spectrum. The next result is a slight refinement of [5, Theorem 9] (see also [6, Theorem 1]).

T heorem 3. Let $\mathcal{C} \subset S^{n-1}(\sqrt{A n})$ be a code with $|\mathcal{C}|=e^{R n}, R>0$. Then for any $\varepsilon=\varepsilon(n)>0$ there exists $\rho$ such that $\rho \geq \tau_{R}$ and

$$
\begin{gather*}
b(\rho)=\frac{1}{n} \ln B(\rho-\varepsilon, \rho+\varepsilon) \geq R-J\left(t_{R}, \rho\right)+\frac{\ln \varepsilon}{n}+o(1), \quad n \rightarrow \infty \\
J(t, \rho)=(1+2 t) \ln [2 t \rho+q(t, \rho)]-\ln q(t, \rho)-t \ln [4 t(1+t)]  \tag{50}\\
q(t, \rho)=\rho+\sqrt{(1+2 t)^{2} \rho^{2}-4 t(1+t)}
\end{gather*}
$$

where $t_{R}, \tau_{R}$ are defined in (4) and (7), and o(1) does not depend on $\varepsilon$.
Note that

$$
\begin{gather*}
J_{\rho}^{\prime}(t, \rho)=\frac{4 t(1+t)}{\rho+\sqrt{(1+2 t)^{2} \rho^{2}-4 t(1+t)}}, \\
J_{\rho \rho}^{\prime \prime}(t, \rho)=-\frac{4 t(1+t)}{\left[\rho+\sqrt{(1+2 t)^{2} \rho^{2}-4 t(1+t)}\right]^{2}}\left[1+\frac{(1+2 t)^{2} \rho}{\sqrt{(1+2 t)^{2} \rho^{2}-4 t(1+t)}}\right]  \tag{51}\\
J_{t}^{\prime}(t, \rho)=2 \ln [2 t \rho+q(t, \rho)]-\ln [4 t(1+t)] \\
{[R(t)-J(t, \rho)]_{t}^{\prime}=2 \ln \frac{2(1+t)}{2 t \rho+q}>0, \quad J\left(t_{R}, \tau_{R}\right)=\ln \left(1+2 t_{R}\right), \quad J\left(t_{R}, 1\right)=R .}
\end{gather*}
$$

Proposition 4. For the function $E(R, A)$ the upper bound (10) holds.
Proof. Due to theorem 2 there exists $\rho \geq \tau_{R}$ such that the inequality (50) holds. Denote $\rho^{*}$ the largest of such $\rho$. Since $b\left(\rho_{0}\right) \geq b\left(\rho^{*}\right)-A\left(\rho_{0}-\rho^{*}\right) / 4$ (cm. (39)), from (38) and (50) we get

$$
\begin{gather*}
\frac{1}{n} \ln \frac{1}{P_{\mathrm{e}}} \leq \frac{A\left(1-\rho_{0}\right)}{4}-b\left(\rho_{0}\right)+o(1) \leq \frac{A\left(1-\rho^{*}\right)}{4}-b\left(\rho^{*}\right)+o(1) \leq  \tag{52}\\
\leq \frac{A\left(1-\rho^{*}\right)}{4}+J\left(t_{R}, \rho^{*}\right)-R+o(1)
\end{gather*}
$$

Note that if $\tau_{R} \leq \bar{\tau}_{1}$ (i.e. if $R \leq \bar{R}_{1}(A)$ ) then (see Appendix)

$$
\begin{equation*}
\left[J\left(t_{R}, \rho\right)-A \rho / 4\right]_{\rho}^{\prime} \leq 0, \quad \rho \geq \tau_{R} \tag{53}
\end{equation*}
$$

and therefore the function $J\left(t_{R}, \rho\right)-A \rho / 4$ monotone decreases on $\rho \geq \tau_{R}$. Since $\rho^{*} \geq \tau_{R}$ then for $\tau_{R} \leq \bar{\tau}_{1}$ we can continue (52) as follows

$$
\begin{align*}
& \frac{1}{n} \ln \frac{1}{P_{\mathrm{e}}} \leq \frac{A\left(1-\tau_{R}\right)}{4}+J\left(t_{R}, \tau_{R}\right)-R+o(1)=  \tag{54}\\
= & \frac{A\left(1-\tau_{R}\right)}{4}+\ln \left(1+2 t_{R}\right)-R, \quad 0<R \leq \bar{R}_{1},
\end{align*}
$$

which is the desired upper bound (10).
To prove the relation (9) note that the best upper bound for $E(R, A)$ is a combination of the upper bound (10) and the sphere-packing bound via the "straight-line bound" [2], which gives

$$
E(R, A) \leq \frac{A\left(1-\bar{\tau}_{1}\right)}{4}+\ln \left(1+2 \bar{t}_{1}\right)-R, \quad \bar{R}_{1} \leq R \leq R_{\text {crit }}
$$

On the other hand, the random coding bound [1, 3] gives

$$
E(R, A) \geq E_{\text {sp }}\left(R_{\text {crit }}, A\right)+R_{\text {crit }}-R, \quad R \leq R_{\text {crit }}
$$

where $E_{\mathrm{sp}}(R, A)$ is defined in (3). Together with the formula (11) it completes the proof of theorem 1.

## § 5. Proof of theorem 2

As was already mentioned in $\S 1$, for $R>0.234$ the upper bounds for the minimal code distance [4, p. 20] of a spherical code and its spectrum [9] can be improved, if the linear programming bound is not directly applied to the original spherical code, but to its subcodes on spherical caps. The same approach allows to improve the upper bound for $E(R, A)$ as well. For that purpose we will need a bound for a code spectrum better than (50). The bound obtained below (theorem 4), probably, is equivalent to the similar bound in [9, Theorem 3] (expressed in a different terms), but its derivation is simpler and a more accurate.

Since we are interested only in angles between codewords $\boldsymbol{x}_{i}, \boldsymbol{x}_{j}$, for the formulas simplification we may set $A n=1$, and consider a code $\mathcal{C} \subset S^{n-1}(1)=S^{n-1}$. Let $T_{\theta}^{n}(\boldsymbol{z})$ be the spherical cap with half-angle $0 \leq \theta \leq \pi / 2$ and center $\boldsymbol{z} \in S^{n-1}$, i.e.

$$
T_{\theta}^{n}(\boldsymbol{z})=\left\{\boldsymbol{x} \in S^{n-1}:(\boldsymbol{x}, \boldsymbol{z}) \geq \cos \theta\right\}
$$

It will be convenient to consider subcodes of $\mathcal{C}$ not on spherical caps $T_{\theta}^{n}(\boldsymbol{z})$, but on related with them thin ring-shaped surfaces $D_{\theta}^{n}(\boldsymbol{z})$. We set further $\delta=1 / n^{2}$, and denote $D_{\theta}^{n}(\boldsymbol{z})$ as

$$
\begin{equation*}
D_{\theta}^{n}(\boldsymbol{z})=T_{\theta}^{n}(\boldsymbol{z}) \backslash T_{\theta-\delta}^{n}(\boldsymbol{z})=\left\{\boldsymbol{x} \in S^{n-1}: \cos \theta \leq(\boldsymbol{x}, \boldsymbol{z}) \leq \cos (\theta-\delta)\right\} . \tag{55}
\end{equation*}
$$

Denote $D_{n}(\theta)$ the surface area of $D_{\theta}^{n}(\boldsymbol{z})$. Then [1, formula (21)]

$$
D_{n}(\theta)=\frac{(n-1) \pi^{(n-1) / 2}}{\Gamma((n+1) / 2)} \int_{\theta-\delta}^{\theta} \sin ^{n-2} u d u, \quad \delta \leq \theta \leq \pi / 2
$$

It is not difficult to show that

$$
1-\frac{1}{2 n \sin \theta} \leq \frac{D_{n}(\theta) \Gamma((n+1) / 2) n^{2}}{\pi^{(n-1) / 2}(n-1) \sin ^{n-2} \theta} \leq 1
$$

Since the surface area $\left|S^{n-1}\right|$ of the sphere $S^{n-1}$ equals $n \pi^{n / 2} / \Gamma(n / 2+1)$, we have uniformly over $1 / n \leq \theta \leq \pi / 2$

$$
\frac{1}{n} \ln \frac{D_{n}(\theta)}{\left|S^{n-1}\right|}=\ln \sin \theta+o(1), \quad n \rightarrow \infty
$$

For the code $\mathcal{C} \subset S^{n-1}$ and $\theta$ such that max $\left\{\arcsin e^{-R}, 1 / n\right\} \leq \theta \leq \pi / 2$, and $\boldsymbol{z} \in S^{n-1}$ we consider the subcode $\mathcal{C}(\theta, \boldsymbol{z})=\mathcal{C} \cap D_{\theta}^{n}(\boldsymbol{z})$ with $|\mathcal{C}(\theta, \boldsymbol{z})|=e^{n r(\boldsymbol{z})}$ codewords. Then

$$
\frac{1}{m\left(S^{n-1}\right)} \int_{\boldsymbol{z} \in S^{n-1}}|\mathcal{C}(\theta, \boldsymbol{z})| d \boldsymbol{z}=\frac{|\mathcal{C}| D_{n}(\theta)}{\left|S^{n-1}\right|}=\exp \{(R+\ln \sin \theta) n+o(n)\}
$$

i.e. in average (over $\boldsymbol{z} \in S^{n-1}$ ) a subcode $\mathcal{C}(\theta, \boldsymbol{z})$ has the rate $r=R+\ln \sin \theta+o(1)$. All its $|\mathcal{C}(\theta, \boldsymbol{z})|$ codewords are located in the ball $B^{n}\left(\sin \theta, \boldsymbol{z}^{\prime}\right)$ of radius $\sin \theta$ and centered at $\boldsymbol{z}^{\prime}=\boldsymbol{z} \cos \theta$. Moreover, they are located in a thin (of thickness $\sim \delta$ ) torus orthogonal to $\boldsymbol{z}$. If $\boldsymbol{x} \in D_{\theta}^{n}(\boldsymbol{z})$, then we denote $\boldsymbol{x}^{\prime}=\boldsymbol{x}-\boldsymbol{z}^{\prime}$ the corresponding vector from $B^{n}\left(\sin \theta, \boldsymbol{z}^{\prime}\right)$. The original angle $\varphi$ between two vectors $\boldsymbol{x}, \boldsymbol{y} \in D_{\theta}^{n}(\boldsymbol{z})$ becomes the angle $\varphi^{\prime}+O(\delta)$ between the vectors $\boldsymbol{x}^{\prime}, \boldsymbol{y}^{\prime} \in B^{n}\left(\sin \theta, \boldsymbol{z}^{\prime}\right)$, where $\sin \left(\varphi^{\prime} / 2\right)=\sin (\varphi / 2) / \sin \theta$. The original value $\rho=\cos \varphi$ becomes the value $\rho^{\prime}+O(\delta)$, where $\rho^{\prime}=\cos \varphi^{\prime}$ is defined by the formula

$$
\begin{equation*}
1-\rho=\left(1-\rho^{\prime}\right) \sin ^{2} \theta \tag{56}
\end{equation*}
$$

since

$$
\rho^{\prime}=\cos \left(2 \arcsin \left(\frac{\sin (\varphi / 2)}{\sin \theta}\right)\right)=1-\frac{2 \sin ^{2}(\varphi / 2)}{\sin ^{2} \theta}=1-(1-\rho) e^{2(R-r)} .
$$

The angle $\varphi^{\prime}$ and the value $\rho^{\prime}$ correspond to the case when the vectors $\boldsymbol{x}^{\prime}, \boldsymbol{y}^{\prime}$ are orthogonal to $\boldsymbol{z}$. The code $\mathcal{C}(\theta, \boldsymbol{z})$ is then transferred to the code $\mathcal{C}^{\prime}(\boldsymbol{z})=\mathcal{C}^{\prime}(\theta, \boldsymbol{z}) \subset B^{n}\left(\sin \theta, \boldsymbol{z}^{\prime}\right)$.

To evaluate the average number $e^{n b_{\mathcal{C}}(\rho)}$ of $\rho$-neighbors in the code $\mathcal{C}$, we consider any pair $\boldsymbol{x}_{i}, \boldsymbol{x}_{j}$ with $\left(\boldsymbol{x}_{i}, \boldsymbol{x}_{j}\right)=\rho$ and introduce the sets

$$
\begin{gathered}
\boldsymbol{Z}(\boldsymbol{x}, a)=\left\{\boldsymbol{z} \in S^{n-1}:(\boldsymbol{x}, \boldsymbol{z}) \geq a\right\} \\
\boldsymbol{Z}(\boldsymbol{x}, \boldsymbol{y}, a)=\left\{\boldsymbol{z} \in S^{n-1}:(\boldsymbol{x}, \boldsymbol{z}) \geq a \text { and }(\boldsymbol{y}, \boldsymbol{z}) \geq a\right\}
\end{gathered}
$$

Denote by $\Omega_{n}(\theta)$ the surface area of the spherical cap $T_{\theta}^{n}(\boldsymbol{z})$. For $0 \leq \theta<\pi / 2$ we have

$$
\Omega_{n}(\theta)=\frac{\pi^{(n-1) / 2} \sin ^{n-1} \theta}{\Gamma((n+1) / 2) \cos \theta}(1+o(1)), \quad n \rightarrow \infty
$$

Then for the Lebesque measure $m(a)$ of the set $\boldsymbol{Z}(\boldsymbol{x}, a)$ we have

$$
m(a)=m(\boldsymbol{Z}(\boldsymbol{x}, a))=\Omega_{n}(\arccos a)
$$

We evaluate the Lebesque measure $m(\rho, a)$ of the set $\boldsymbol{Z}(\boldsymbol{x}, \boldsymbol{y}, a)$ provided $(\boldsymbol{x}, \boldsymbol{y})=\rho$. Note that if $\boldsymbol{x}, \boldsymbol{y} \in S^{n-1}$ and $(\boldsymbol{x}, \boldsymbol{y})=\rho$, then $\|\boldsymbol{x}+\boldsymbol{y}\|^{2}=2(1+\rho)$. Therefore $\boldsymbol{v}=$ $(\boldsymbol{x}+\boldsymbol{y}) / \sqrt{2(1+\rho)} \in S^{n-1}$, and then

$$
\begin{gathered}
\boldsymbol{Z}(\boldsymbol{x}, \boldsymbol{y}, a) \subseteq\left\{\boldsymbol{z} \in S^{n-1}:(\boldsymbol{x}+\boldsymbol{y}, \boldsymbol{z}) \geq 2 a\right\}= \\
=\left\{\boldsymbol{z} \in S^{n-1}:(\boldsymbol{v}, \boldsymbol{z}) \geq a \sqrt{2 /(1+\rho)}\right\}=\boldsymbol{Z}(\boldsymbol{v}, a \sqrt{2 /(1+\rho)}) .
\end{gathered}
$$

Therefore we get

$$
m(\rho, a)=m(\boldsymbol{Z}(\boldsymbol{x}, \boldsymbol{y}, a)) \leq m(\boldsymbol{Z}(\boldsymbol{v}, a \sqrt{2 /(1+\rho)}))=\Omega_{n}(\arccos (a \sqrt{2 /(1+\rho)}))
$$

That upper bound for $m(\rho, a)$ is logarithmically (as $n \rightarrow \infty$ ) exact. In particular, if $a=$ $\cos \theta$ and $(\boldsymbol{x}, \boldsymbol{y})=\rho$, then

$$
\begin{aligned}
\frac{1}{n} \ln \frac{m(\cos \theta)}{m(\rho, \cos \theta)} & \geq \ln \sin \theta-\ln \sin (\arccos (\sqrt{2 /(1+\rho)} \cos \theta))= \\
= & \ln \sin \theta-\ln \sqrt{1-2 \cos ^{2} \theta /(1+\rho)}
\end{aligned}
$$

We use below the values $\rho^{\prime}=\rho^{\prime}(\rho, \theta)$ from and (56) and $\varepsilon^{\prime}=\varepsilon / \sin ^{2} \theta$. Then denoting $B_{\mathcal{C}}(\rho)=B_{\mathcal{C}}(\rho-\varepsilon, \rho+\varepsilon), B_{\mathcal{C}^{\prime}(\boldsymbol{z})}\left(\rho^{\prime}\right)=B_{\mathcal{C}^{\prime}(\boldsymbol{z})}\left(\rho^{\prime}-\varepsilon^{\prime}, \rho^{\prime}+\varepsilon^{\prime}\right)$, for any $\rho, \varepsilon$ we have

$$
\begin{equation*}
B_{\mathcal{C}}(\rho)|\mathcal{C}|=\frac{1}{m(\rho, \cos \theta)} \int_{\boldsymbol{z} \in S^{n-1}} B_{\mathcal{C}^{\prime}(\boldsymbol{z})}\left(\rho^{\prime}\right)\left|\mathcal{C}^{\prime}(\boldsymbol{z})\right| d \boldsymbol{z} \tag{57}
\end{equation*}
$$

Indeed, the value $B_{\mathcal{C}}(\rho)|\mathcal{C}|$ is the total number of pairs $\boldsymbol{x}_{i}, \boldsymbol{x}_{j} \in \mathcal{C}$ with $\left|\left(\boldsymbol{x}_{i}, \boldsymbol{x}_{j}\right)-\rho\right| \leq \varepsilon$, and $B_{\mathcal{C}^{\prime}(\boldsymbol{z})}\left(\rho^{\prime}\right)\left|\mathcal{C}^{\prime}(\boldsymbol{z})\right|$ is the total number of similar pairs $\boldsymbol{x}_{i}^{\prime}, \boldsymbol{x}_{j}^{\prime} \in \mathcal{C}^{\prime}(\boldsymbol{z})$ with $\mid\left(\boldsymbol{x}_{i}^{\prime}, \boldsymbol{x}_{j}^{\prime}\right) /\left(\left\|\boldsymbol{x}_{i}^{\prime}\right\|\right.$. $\left.\left\|\boldsymbol{x}_{j}^{\prime}\right\|\right)-\rho^{\prime} \mid \leq \varepsilon^{\prime}$. Moreover, each pair $\boldsymbol{x}_{i}^{\prime}, \boldsymbol{x}_{j}^{\prime} \in \mathcal{C}^{\prime}(\boldsymbol{z})$ gives the contribution $m(\rho, \cos \theta)$ to the integral, from which the formula (57) follows. From (57) for any set $\mathcal{A} \subseteq S^{n-1}$ we have

$$
\begin{equation*}
e^{n b_{\mathcal{C}}(\rho)} \geq \frac{1}{m(\rho, \cos \theta)|\mathcal{C}|} \int_{\boldsymbol{z} \in \mathcal{A}} e^{n b_{\mathcal{C}^{\prime}}(\boldsymbol{z})\left(\rho^{\prime}\right)}\left|\mathcal{C}^{\prime}(\boldsymbol{z})\right| d \boldsymbol{z} \tag{58}
\end{equation*}
$$

and also

$$
|\mathcal{C}|=\frac{1}{m(\cos \theta)} \int_{\boldsymbol{z} \in S^{n-1}}\left|\mathcal{C}^{\prime}(\boldsymbol{z})\right| d \boldsymbol{z} \geq \frac{1}{m(\cos \theta)} \int_{\boldsymbol{z} \in \mathcal{A}}\left|\mathcal{C}^{\prime}(\boldsymbol{z})\right| d \boldsymbol{z}
$$

The code $\mathcal{C}^{\prime}(\boldsymbol{z})$ has the rate $r(\boldsymbol{z})=\left(\ln \left|\mathcal{C}^{\prime}(\boldsymbol{z})\right|\right) / n$. Then there exists $r_{0}$ such that

$$
\begin{gather*}
|\mathcal{C}|=\frac{e^{o(n)}}{m(\cos \theta)} \max _{t}\left\{e^{t n} m\left(\boldsymbol{z} \in S^{n-1}:|r(\boldsymbol{z})-t| \leq \varepsilon\right)\right\}=\frac{e^{r_{0} n+o(n)} m\left(S_{0}\right)}{m(\cos \theta)}  \tag{59}\\
S_{0}=\left\{\boldsymbol{z} \in S^{n-1}:\left|r(\boldsymbol{z})-r_{0}\right| \leq \varepsilon\right\}
\end{gather*}
$$

Since $m\left(S_{0}\right) \leq m\left(S^{n-1}\right)$ then

$$
\begin{equation*}
r_{0} \geq \frac{1}{n} \ln \frac{|\mathcal{C}| m(\cos \theta)}{m\left(S^{n-1}\right)}=R+\ln \sin \theta+o(1) \tag{60}
\end{equation*}
$$

We set $\mathcal{A}=S_{0}$ and $\varepsilon=o(1), n \rightarrow \infty$. Then using the Jensen inequality, from (58) and (59) we have

$$
\begin{aligned}
& e^{n b_{\mathcal{C}}(\rho)} \geq \frac{1}{m(\rho, \cos \theta)|\mathcal{C}|} \int_{\boldsymbol{z} \in S_{0}} e^{n b_{\mathcal{C}^{\prime}}(\boldsymbol{z})\left(\rho^{\prime}\right)}\left|\mathcal{C}^{\prime}(\boldsymbol{z})\right| d \boldsymbol{z} \geq \\
& \quad \geq \frac{m(\cos \theta) e^{o(n)}}{m(\rho, \cos \theta) m\left(S_{0}\right)} \int_{\boldsymbol{z} \in S_{0}} e^{n b_{\mathcal{C}^{\prime}}(\boldsymbol{z})\left(\rho^{\prime}\right)} d \boldsymbol{z} \geq \\
& \geq \frac{m(\cos \theta) e^{o(n)}}{m(\rho, \cos \theta)} \exp \left\{\frac{n}{m\left(S_{0}\right)} \int_{\boldsymbol{z} \in S_{0}} b_{\mathcal{C}^{\prime}(\boldsymbol{z})}\left(\rho^{\prime}\right) d \boldsymbol{z}\right\},
\end{aligned}
$$

from which we get

$$
\begin{equation*}
b_{\mathcal{C}}(\rho) \geq \frac{1}{n} \ln \frac{m(\cos \theta)}{m(\rho, \cos \theta)}+\frac{1}{m\left(S_{0}\right)} \int_{\boldsymbol{z} \in S_{0}} b_{\mathcal{C}^{\prime}(\boldsymbol{z})}\left(\rho^{\prime}\right) d \boldsymbol{z}+o(1) . \tag{61}
\end{equation*}
$$

Due to theorem 3 for each code $\mathcal{C}^{\prime}(\boldsymbol{z}), \boldsymbol{z} \in S_{0}$, there exists $\rho^{\prime \prime}=\rho^{\prime \prime}(\boldsymbol{z})$ such that $\rho^{\prime \prime} \geq \tau_{r_{0}}$ and

$$
b_{\mathcal{C}^{\prime}(\boldsymbol{z})}\left(\rho^{\prime \prime}\right) \geq r_{0}-J\left(t_{r_{0}}, \rho^{\prime \prime}\right)+o(1) .
$$

Therefore there exists $\rho^{\prime} \geq \tau_{r_{0}}$ and the corresponding $\rho=\rho\left(\rho^{\prime}\right)$ from (56) such that from the inequality (61) we get

$$
\begin{gather*}
b_{\mathcal{C}}(\rho) \geq \frac{1}{n} \ln \frac{m(\cos \theta)}{m(\rho, \cos \theta)}+r_{0}-J\left(t_{r_{0}}, \rho^{\prime}\right)+o(1) \geq \\
=\frac{1}{n} \ln \frac{m(\cos \theta)}{m(\rho, \cos \theta)}+R+\ln \sin \theta-J\left(t_{\left.R+\ln \sin \theta, \rho^{\prime}\right)+o(1) \geq}^{\geq R+} 2 \ln \sin \theta-J\left(t_{R+\ln \sin \theta}, \rho^{\prime}\right)-\ln \sqrt{1-2 \cos ^{2} \theta /(1+\rho)}+o(1)=\right.  \tag{62}\\
= \\
R+\ln \sin \theta-J\left(t_{R+\ln \sin \theta}, \rho^{\prime}\right)+\frac{1}{2} \ln \frac{(1+\rho)}{\left(1+\rho^{\prime}\right)}+o(1),
\end{gather*}
$$

where we used the formula (60) and monotonicity of the function $r-J\left(t_{r}, \rho\right)$ on $r$ (see (51)), and $\rho^{\prime}=\rho^{\prime}(\rho, \theta)$ is defined in (56). After the variable change $\sin \theta=e^{r-R}$ from (62) we get

Theorem 4. Let $\mathcal{C} \subset S^{n-1}(1)$ be a code with $|\mathcal{C}|=e^{R n}, R>0$. Then for any $r \leq R$ there exists $\rho^{\prime}$ such that $\rho^{\prime} \geq \tau_{r}$ and for $\rho=1-\left(1-\rho^{\prime}\right) e^{2(r-R)}$ the following inequality holds

$$
\begin{equation*}
b_{\mathcal{C}}(\rho) \geq r-J\left(t_{r}, \rho^{\prime}\right)+\frac{1}{2} \ln \frac{(1+\rho)}{\left(1+\rho^{\prime}\right)}+o(1) \tag{63}
\end{equation*}
$$

Using the relation (63) in the inequality (38) we prove theorem 2 . We have

$$
\begin{gather*}
\frac{1}{n} \ln \frac{1}{P_{\mathrm{e}}} \leq \min _{r \leq R} \max _{\rho^{\prime} \geq \tau_{r}}\left\{\frac{A(1-\rho)}{4}-b(\rho)\right\}+o(1) \leq \\
\leq \min _{r \leq R} \max _{\rho^{\prime} \geq \tau_{r}}\left\{\frac{A\left(1-\rho^{\prime}\right) e^{2(r-R)}}{4}-r+J\left(t_{r}, \rho^{\prime}\right)+\frac{1}{2} \ln \frac{1+\rho^{\prime}}{1+\rho}\right\}=\min _{r \leq R} \max _{\rho \geq \tau_{r}} f(r, \rho) \tag{64}
\end{gather*}
$$

where

$$
f(r, \rho)=\frac{A(1-\rho) e^{2(r-R)}}{4}+R-2 r+J\left(t_{r}, \rho\right)+\frac{1}{2} \ln \frac{1+\rho}{2 e^{2(R-r)}+\rho-1} .
$$

With $t=t_{r}$ and $\left(1-\tau_{r}\right) e^{2(r-R)}=2 z$ we have

$$
\begin{gathered}
f_{\rho}^{\prime}=-\frac{A e^{2(r-R)}}{4}-\frac{1}{2\left(2 e^{2(R-r)}+\rho-1\right)}+\frac{4 t(1+t)}{\rho+\sqrt{(1+2 t)^{2} \rho^{2}-4 t(1+t)}}+\frac{1}{2(1+\rho)} \\
\left.f_{\rho}^{\prime}\right|_{\rho=\tau_{r}}=-\frac{A e^{2(r-R)}}{4}-\frac{1}{2\left(2 e^{2(R-r)}+\tau_{r}-1\right)}+\frac{1}{2\left(1-\tau_{r}\right)}= \\
=\frac{A z^{2}-(A+2) z+1}{2(1-z)\left(1-\tau_{r}\right)}, \quad f_{\rho \rho}^{\prime \prime}<0 .
\end{gathered}
$$

Since $f_{\rho \rho}^{\prime \prime}<0$ then $\rho=\tau_{r}$ is optimal if $\left.f_{\rho}^{\prime}\right|_{\rho=\tau_{r}} \leq 0$. Since $r \leq R$ then $z \leq 1$. Therefore $\left.f_{\rho}^{\prime}\right|_{\rho=\tau_{r}} \leq 0$ if the following inequalities are fulfilled:

$$
\begin{equation*}
\frac{2}{A+2+\sqrt{A^{2}+4}} \leq z \leq \frac{A+2+\sqrt{A^{2}+4}}{2 A} . \tag{65}
\end{equation*}
$$

The right one of the inequalities (65) is always satisfied. The left one of the inequalities (65)) is equivalent to the inequality

$$
\begin{equation*}
f_{2}(r)=2 r+\ln \left(1-\tau_{r}\right) \geq 2 R-2 R_{\text {crit }}(A) \tag{66}
\end{equation*}
$$

The next simple technical lemma concerns the function $f_{2}(r)$ in the left-hand side of (66).
$\mathrm{L} \mathrm{e} \mathrm{m} \mathrm{m} \mathrm{a} \mathrm{3} .\mathrm{The} \mathrm{function} f_{2}(r)$ from (66) monotone decreases on $0 \leq r<\bar{R}_{2}$, and monotone increases on $r>\bar{R}_{2}$, where $\bar{R}_{2}$ is defined in (14). Moreover, the formula holds

$$
\begin{equation*}
\ln \left(1-\bar{\tau}_{1}(A)\right)=-2 R_{\text {crit }}(A), \quad A>0 \tag{67}
\end{equation*}
$$

Since the function $E(R, A), R \geq \bar{R}_{1}(A)$, is known exactly (see theorem 1), we consider only the case $R<\bar{R}_{1}(A)$. Then two cases are possible: $R \leq \min \left\{\bar{R}_{1}(A), \bar{R}_{2}\right\}$ and $\bar{R}_{2}<$ $R<\bar{R}_{1}(A)$.

C a s e $R \leq \min \left\{\bar{R}_{1}(A), \bar{R}_{2}\right\}$. For $R \leq \bar{R}_{2}$ minimum (over $r \leq R$ ) in the left-hand side of (66) is attained when $r=R$, and then due to (67) the inequality (66) reduces to the condition $\tau_{R} \leq \bar{\tau}_{1}(A)$, i.e. to $R \leq \bar{R}_{1}(A)$. Therefore if $r \leq R \leq \min \left\{\bar{R}_{1}(A), \bar{R}_{2}\right\}$ then the inequalities (66) and (65) are fulfilled, and then $\rho=\tau_{r}$ is optimal in the right-hand side of (64). Since $J\left(t_{r}, \tau_{r}\right)=\ln \left(1+2 t_{r}\right)=-\ln \left(1-\tau_{r}^{2}\right) / 2$ (see (51) and (7)), then (64) takes the form

$$
\begin{equation*}
\frac{1}{n} \ln \frac{1}{P_{\mathrm{e}}} \leq \min _{r \leq R} f\left(r, \tau_{r}\right)=\min _{r \leq R} C(v(r))-R, \quad R \leq \min \left\{\bar{R}_{1}(A), \bar{R}_{2}\right\} \tag{68}
\end{equation*}
$$

where

$$
\begin{equation*}
C(v)=\frac{A v}{4}-\frac{1}{2} \ln [v(2-v)], \quad v(r)=\left(1-\tau_{r}\right) e^{2(r-R)} \tag{69}
\end{equation*}
$$

Note that for $r=R$ the inequality (68) reduces to the previous bound (10). We show that such $r$ is optimal in (68). We have

$$
4 v(2-v) C_{v}^{\prime}=-A v^{2}+2(A+2) v-4, \quad C_{v^{2}}^{\prime \prime}>0
$$

Since $0 \leq v \leq 1$, the equation $C_{v}^{\prime}=0$ has the unique root $v_{1}$, where

$$
\begin{equation*}
v_{1}=\frac{4}{A+2+\sqrt{A^{2}+4}}=e^{-2 R_{\text {crit }}(A)} . \tag{70}
\end{equation*}
$$

The function $C(v), 0 \leq v \leq 1$, monotone decreases on $0 \leq v<v_{1}$ and monotone increases on $v>v_{1}$. Note that since $v(r)=e^{f_{2}(r)-2 R}$, then (see lemma 3) the function $v(r)$ monotone decreases on $0 \leq r<\bar{R}_{2}$ and monotone increases on $r>\bar{R}_{2}$.

If now $R \leq \min \left\{\bar{R}_{1}(A), \bar{R}_{2}\right\}$, then $v(r) \geq v_{1}$ for $r \leq R$. Therefore $r=R$ is optimal in (68), and then (68) reduces to the previous bound (10).

C as e $\bar{R}_{2}<R<\bar{R}_{1}(A)$ (i.e. $A>A_{0}$ ). Then $\bar{R}_{2}<\bar{R}_{3}(A)<\bar{R}_{1}(A)$, where $\bar{R}_{3}(A)$ is defined in (14). Consider first the case $\bar{R}_{2} \leq R \leq \bar{R}_{3}(A)$. It is simple to check that then the inequality (66) is again satisfied (see (14)). Therefore $\rho=\tau_{r}$ is optimal in the right-hand side of (64), and (64) takes the form (68). Since $R \leq \bar{R}_{3}(A)$, then $v(r) \geq v_{1}$ for $r \leq R$. Since $R \geq \bar{R}_{2}$ then $r=\bar{R}_{2}$ is optimal in (68), and then from (68) the second of bounds (16) follows.

It remains to consider the case $\bar{R}_{2} \leq \bar{R}_{3}(A) \leq R \leq \bar{R}_{1}(A)$. Since minimum of $C(v)$ over $0 \leq v \leq 1$ is attained for $v=v_{1}($ see (70)), then

$$
\begin{equation*}
\min _{0 \leq v \leq 1} C(v)=C\left(v_{1}\right)=E_{\text {sp }}\left(R_{\text {crit }}, A\right)+R_{\text {crit }} \tag{71}
\end{equation*}
$$

where the formula was used

$$
E_{\text {sp }}\left(R_{\text {crit }}, A\right)+R_{\text {crit }}=\frac{A v_{1}}{4}-\frac{1}{2} \ln v_{1}-\frac{1}{2} \ln \left(2-v_{1}\right) .
$$

Now in the right-hand side of (64) we set $r$ such that $v(r)=v_{1}$ (it is possible when $R \geq \bar{R}_{3}$ ). Then again the inequality (66) is fulfilled and $\rho=\tau_{r}$ is optimal in the right-hand side of (64). From (68) and (71) the first of upper bounds (15) follows. The upper bound (15) can also be proved applying the "straight-line bound" to the sphere-packing bound and the second of upper bounds (16) at $R=\bar{R}_{3}$, and the formula

$$
E_{\mathrm{sp}}\left(R_{\text {crit }}, A\right)+R_{\text {crit }}-\bar{R}_{3}=\frac{A a e^{-2 \bar{R}_{3}}}{4}-\frac{1}{2} \ln \left(2-a e^{-2 \bar{R}_{3}}\right)-\frac{1}{2} \ln a,
$$

which is simple to check using the relations (12). It completes the proof of theorem 2.

## APPENDIX

Proofof formula (32). Without loss of generality we may assume that $\boldsymbol{x}_{i}, \boldsymbol{x}_{j}, \boldsymbol{y}$ have the form

$$
\boldsymbol{x}_{i}=\left(x_{1}, x_{2}, 0, \ldots, 0\right), \quad \boldsymbol{x}_{j}=\left(-x_{1}, x_{2}, 0, \ldots, 0\right), \quad \boldsymbol{y}=\left(0, y_{2}, y_{3}, \ldots, y_{n}\right)
$$

from which we have

$$
\begin{gathered}
d\left(\boldsymbol{x}_{i}, \boldsymbol{x}_{j}\right)=4 x_{1}^{2}=2 A n\left(1-\rho_{i j}\right) \\
d\left(\boldsymbol{x}_{i}, \boldsymbol{y}\right)=x_{1}^{2}+\left(y_{2}-x_{2}\right)^{2}+\sum_{k=3}^{n} y_{k}^{2}=s n, \\
x_{1}^{2}+x_{2}^{2}=A n, \quad \sum_{k=2}^{n} y_{k}^{2}=r n .
\end{gathered}
$$

Solving those equations we get

$$
\begin{equation*}
x_{1}=\sqrt{\frac{A n\left(1-\rho_{i j}\right)}{2}}, \quad x_{2}=\sqrt{\frac{A n\left(1+\rho_{i j}\right)}{2}}, \quad y_{2}=\frac{(A+r-s) n}{\sqrt{2 A n\left(1+\rho_{i j}\right)}}, \tag{72}
\end{equation*}
$$

and therefore

$$
\sum_{k=3}^{n} y_{k}^{2}=r n-y_{2}^{2}=r n-\frac{(A+r-s)^{2} n}{2 A\left(1+\rho_{i j}\right)}=r_{1} n
$$

from which the formula (32) follows.
Optimality of $s(\rho), r(\rho)$ from the formulas (37) also follows from (72).
Proof of formula (53). For the function $f(\rho)=J\left(t_{R}, \rho\right)-A \rho / 4$ from (51) we have

$$
f^{\prime}=\frac{4 t_{R}\left(1+t_{R}\right)}{\rho+\sqrt{\left(1+2 t_{R}\right)^{2} \rho^{2}-4 t_{R}\left(1+t_{R}\right)}}-\frac{A}{4}, \quad f^{\prime \prime}(t, \rho)<0 .
$$

Then for $\rho \geq \tau_{R}$ we have

$$
f^{\prime} \leq\left. f^{\prime}\right|_{\rho=\tau_{R}}=\frac{4 t_{R}\left(1+t_{R}\right)}{\tau_{R}}-\frac{A}{4}=\frac{\tau_{R}}{1-\tau_{R}^{2}}-\frac{A}{4} \leq 0
$$

if $\tau_{R} \leq \bar{\tau}_{1}(A)$, which proves the formula (53).
Proof of lemma 1 . Let $\left\{\boldsymbol{x}_{1}, \ldots, \boldsymbol{x}_{M}\right\} \subset S^{n-1}(\sqrt{A n})$ be a code such that $\max _{i \neq j}\left(\boldsymbol{x}_{i}, \boldsymbol{x}_{j}\right) \leq 0$, i.e. $\min _{i \neq j}\left\|\boldsymbol{x}_{i}-\boldsymbol{x}_{j}\right\|^{2} \geq 2 A$. Then, clearly, $M \leq 2 n$.

In lemma 1 for all $i$ we have $\left(\boldsymbol{x}_{i}, \boldsymbol{y}\right)=(A+r-s) n / 2$. Consider $M$ vectors $\left\{\boldsymbol{x}_{i}^{\prime}=\boldsymbol{x}_{i}-a \boldsymbol{y}\right\}$, where $a=(A+r-s) /(2 r)$. Then due to the condition (48) we have

$$
\max _{i \neq j}\left(\boldsymbol{x}_{i}^{\prime}, \boldsymbol{x}_{j}^{\prime}\right) \leq\left[4 \operatorname{Ar} \rho-(A+r-s)^{2}\right] n /(4 r) \leq 0
$$

and therefore $M \leq 2 n$.
Proofoflemma 2. To prove lemma we reduce it to the case $\rho \approx 0$, and then use lemma 4 (see below). We set some integer $m$ such that $1<m<M$, and introduce the vector

$$
\boldsymbol{z}=a \sum_{k=1}^{m} \boldsymbol{x}_{k}, \quad a=\frac{\rho}{1+(m-1) \rho} .
$$

After simple calculations we get

$$
\begin{gather*}
\rho-\delta-\frac{1}{m} \leq\|\boldsymbol{z}\|^{2} \leq \rho+\delta \\
\frac{\rho-\delta}{1+(1-\rho) /(m \rho)} \leq\left(\boldsymbol{x}_{i}, \boldsymbol{z}\right) \leq \frac{\rho+\delta}{1+(1-\rho) /(m \rho)}, \quad i=m+1, \ldots, M \tag{73}
\end{gather*}
$$

Consider the normalized vectors

$$
\boldsymbol{u}_{i}=\frac{\boldsymbol{x}_{i}-\boldsymbol{z}}{\left\|\boldsymbol{x}_{i}-\boldsymbol{z}\right\|}, \quad i=m+1, \ldots, M
$$

Using the formulas (73), for any $i, j \geq m+1, i \neq j$, we get

$$
\begin{equation*}
\left(\boldsymbol{u}_{i}, \boldsymbol{u}_{j}\right) \leq \frac{2}{(1-\rho)}\left(\delta+\frac{1}{m}\right)=o(1), \quad n \rightarrow \infty \tag{74}
\end{equation*}
$$

if we set $m \rightarrow \infty$ as $n \rightarrow \infty$. To upperbound the maximal possible number $M-m$ of vectors $\left\{\boldsymbol{u}_{i}\right\}$ satisfying the condition (74), we use a modification of [16, Theorem 2].

L em m a 4 . Let $\mathcal{C}=\left\{\boldsymbol{x}_{1}, \ldots, \boldsymbol{x}_{M}\right\} \subset S^{n-1}(1)$ be a code with $\left(\boldsymbol{x}_{i}, \boldsymbol{x}_{j}\right) \leq \mu, i \neq j$. Then for $n \geq 1$ the upper bound holds

$$
\begin{equation*}
M \leq 2 n^{3 / 2}(1-\mu)^{-n / 2}, \quad 0 \leq \mu<1 \tag{75}
\end{equation*}
$$

Proof. Denote $\mu=\cos (2 \varphi)$, and let $M(\varphi)$ be the maximal cardinality of such a code. For $M(\varphi)$ the upper bound holds [16, Theorem 2]

$$
\begin{equation*}
M(\varphi) \leq \frac{(n-1) \sqrt{\pi} \Gamma\left(\frac{n-1}{2}\right) \sin \beta \tan \beta}{2 \Gamma\left(\frac{n}{2}\right)\left[\sin ^{n-1} \beta-f(\beta, n-2) \cos \beta\right]}, \quad 0<\varphi<\frac{\pi}{4} \tag{76}
\end{equation*}
$$

where $\beta=\arcsin (\sqrt{2} \sin \varphi)$ and

$$
f(\beta, n-2)=(n-1) \int_{0}^{\beta} \sin ^{n-2} z d z
$$

Integrating by parts, for the function $f(\beta, n-2)$ we have

$$
\begin{gathered}
f(\beta, n-2)=\frac{\sin ^{n-1} \beta}{\cos \beta}-\frac{\sin ^{n+1} \beta}{(n+1) \cos ^{3} \beta}-\frac{3}{(n+1)} \int_{0}^{\beta} \frac{\sin ^{n+2} z}{\cos ^{4} z} d z \geq \\
\geq \frac{\sin ^{n-1} \beta}{\cos \beta}-\frac{\sin ^{n+1} \beta}{(n+1) \cos ^{3} \beta}-\frac{3 \tan ^{4} \beta}{(n+1)} f(\beta, n-2)
\end{gathered}
$$

and therefore

$$
\begin{equation*}
1 /\left[1+\frac{3 \tan ^{4} \beta}{n^{2}-1}\right] \leq f(\beta, n-2) /\left\{\frac{\sin ^{n-1} \beta}{\cos \beta}\left[1-\frac{\tan ^{2} \beta}{n+1}\right]\right\} \leq 1 \tag{77}
\end{equation*}
$$

if $\tan ^{2} \beta<n+1$, i.e. if $2 \sin ^{2} \varphi<(n+1) /(n+2)$. From (76) and (77) we get

$$
\begin{equation*}
M(\varphi) \leq \frac{\sqrt{\pi} \Gamma\left(\frac{n-1}{2}\right)\left(n^{2}-1\right) \cos \beta}{2 \Gamma\left(\frac{n}{2}\right) \sin ^{n-1} \beta}<\frac{n \sqrt{\pi n\left(1-2 \sin ^{2} \varphi\right)}}{\sqrt{2}(\sqrt{2} \sin \varphi)^{n-1}} \tag{78}
\end{equation*}
$$

since

$$
\Gamma\left(\frac{z-1}{2}\right)\left(z^{2}-1\right) / \Gamma\left(\frac{z}{2}\right)<\sqrt{2} z^{3 / 2} e^{1 / z}, \quad z \geq 0 .
$$

From (78) the inequality (75) follows provided $2 \sin ^{2} \varphi<(n+1) /(n+2)$, i.e. if $\mu>1 /(n+2)$. Since the function $M(\varphi)$ is continuous on the left for $\varphi \in(0, \pi]$, the upper bound (78) remains valid for $\mu=1 /(n+2)$ as well. For $\mu=1 /(n+2), n \geq 1$, the right-hand side of (78) does not exceed $n \sqrt{\pi e / 2}$, which in turn does not exceed the right-hand side of (75) for any $\mu \geq 0, n \geq 2$. Since $M(\varphi)$ is a decreasing function, it proves the inequality (75) for any $\mu \geq 0, n \geq 2$. Clearly, (75) remains valid for $n=1$ as well.

Now from (74) and (75) we get lemma 2.
The author thanks L.A.Bassalygo, G.A.Kabatyansky and V.V.Prelov for useful discussions and constructive critical remarks.

## REFERENCES

1. Shannon C. E. Probability of Error for Optimal Codes in Gaussian Channel // Bell System Techn. J. 1959. V. 38. № 3. P. 611-656.
2. Shannon C. E., Gallager R. G.. Berlekamp E. R. Lower Bounds to Error Probability for Codes on Discrete Memoryless Channels. I, II // Inform. and Control. 1967. V. 10. № 1. P. 65-103; № 5. P. 522-552.
3. Gallager R. G. Information theory and reliable communication. Wiley, NY, 1968.
4. Kabatyansky G. A., Levenshtein V. I. Bounds for packings on the sphere and in space // Probl. Inform. Transm. 1978. V. 14. № 1. P. 3-25.
5. Ashikhmin A., Barg A., Litsyn S. A New Upper Bound on the Reliability Function of the Gaussian Channel // IEEE Trans. Inform. Theory. 2000. V. 46. № 6. P. 19451961.
6. Burnashev M. V. On the Relation Between the Code Spectrum and the Decoding Error Probability // Probl. Inform. Transm. 2000. V. 36. № 4. P. 3-24.
7. Burnashev M. V. On Relation Between Code Geometry and Decoding Error Probability // Proc. IEEE Int. Sympos. on Information Theory (ISIT). Washington, DC, USA. June 24-29, 2001. P. 133.
8. Barg A., McGregor A. Distance Distribution of Binary Codes and the Error Probability of Decoding // IEEE Trans. Inform. Theory. 2005. V. 51. № 12. P. 42374246.
9. Ben-Haim Y., Litsyn S. Improved Upper Bounds on the Reliability Function of the Gaussian Channel // IEEE Trans. Inform. Theory (submitted).
10. Burnashev M. V. Bounds for Achievable Accuracy in Parameter Transmission over the White Gaussian Channel // Probl. Inform. Transm. 1977. V. 13. № 4. P. 9-24.
11. Burnashev M. V. A New Lower Bound for the $\alpha$-Mean Error of Parameter Transmission over the White Gaussian Channel // IEEE Trans. Inform. Theory. 1984. V. 30. № 1. P. 23-34.
12. Burnashev M. V. On a Minimum Attainable Mean-Square Error for Parameter Transmission over the White Gaussian Channel // Probl. Inform. Transm. 1985. V. 21. № 4. P. 3-16.
13. Burnashev M. V. Upper Bound Sharpening on Reliability Function of Binary Symmetric Channel // Probl. Inform. Transm. 2005. V. 41. № 4. P. 3-22.
14. Burnashev M. V. Code Spectrum and Reliability Function: Binary Symmetric Channel // Probl. Inform. Transm. 2006. V. 42. № 4. P. 3-22.
15. Burnashev M. V. Supplement to the Paper: Code Spectrum and Reliability Function: Binary Symmetric Channel // Probl. Inform. Transm. 2006. V. 43. № 1. P. 28-31.
16. Rankin R. A. The Closest Packing of Spherical Caps in $n$ Dimensions // Proc. Glasgow Math. Assoc. 1955. V. 2. P. 139-144.

Burnashev Marat Valievich
Institute for Information Transmission Problems RAS
burn@iitp.ru


Figure. Upper (15),(16) and lower (17) bounds for $E(R, A)$ and $A=4$


[^0]:    ${ }^{1}$ The research described in this publication was made possible in part by the Russian Fund for Fundamental Research (project number 06-01-00226).

