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## ON ZERO-RATE ERROR EXPONENT FOR BSC WITH NOISY FEEDBACK <sup>1</sup>

For the information transmission a binary symmetric channel is used. There is also another noisy binary symmetric channel (feedback channel), and the transmitter observes without delay all the outputs of the forward channel via that feedback channel. The transmission of a nonexponential number of messages (i.e. the transmission rate equals zero) is considered. The achievable decoding error exponent for such a combination of channels is investigated. It is shown that if the crossover probability of the feedback channel is less than a certain positive value, then the achievable error exponent is better than the similar error exponent of the no-feedback channel.

The transmission method described and the corresponding lower bound for the error exponent can be strengthened, and also extended to the positive transmission rates.

### § 1. Introduction and main results

The binary symmetric channel  $BSC(p)$  with crossover probability  $0 < p < 1/2$  (and  $q = 1 - p$ ) is considered. It is assumed that there is the feedback  $BSC(p_1)$  channel, and the transmitter observes (without delay) all outputs of the forward  $BSC(p)$  channel via that noisy feedback channel. No coding is used in the feedback channel (i.e. the receiver simply re-transmits all received outputs to the transmitter). In words, the feedback channel is “passive”.

Since the Shannon’s paper [1] it has been known that even the noiseless feedback does not increase the capacity of the BSC (or any other memoryless channel). However, the feedback can improve the decoding error probability (or simplify the effective transmission method). In the case of BSC with noiseless feedback investigations of the decoding error probability (or its best error exponent - *channel reliability function*) have been actively studied since Dobrushin [2], Horstein [3] and Berlekamp [4]. Some characteristics of a number of efficient transmission methods have been investigated (see, for example, [1–10]). Generally, the case of BSC with noiseless feedback is reasonably well investigated (although there are still some important open problems).

The case of noisy feedback was not investigated. It was not even known whether such feedback can improve the error exponent of the no-feedback case. In this respect, only two recent papers [11, 12] can probably be mentioned, but both of them consider different

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problems. In the paper [11] the *variable-length* coding (i.e. non-block codes) is used under a different error criterion. Moreover, it is assumed that at certain moments an error-free mechanism in the feedback is available. In the paper [12] Gaussian channel with only the *average power* constraint is considered. Such constraint allows using some methods which are unavailable in the case of discrete channels.

We try to explain the reason why the noisy feedback case is so badly investigated, and what creates the main difficulty (how we see it). In the noiseless feedback case the transmitter *at any moment* may change its coding function (transmission method), and the receiver will know exactly about this change. Such an ideal mutual understanding (mutual coordination) between the transmitter and the receiver was very important for all results on the noiseless feedback case [1–10]. If we try to apply any of the transmission methods from [1–10] to a noisy feedback case, we find that the transmitter and the receiver rather quickly lose their mutual coordination. Due to noise in the feedback link they can achieve mutual coordination only in some probabilistic sense. In particular, if the transmitter wants to change its coding function at some moment  $t$ , it should know with high reliability the current output values of some functions (e.g. posterior message probabilities) at the receiver. Of course, it takes a certain time to achieve high reliability of such knowledge. For that reason, the transmitter should probably change the coding function not very often (i.e. only after accumulating some very reliable information on the receiver uncertainty).

The following geometrical picture explains that description. Let  $\mathcal{D}_1, \dots, \mathcal{D}_M$  be the optimal decoding regions of messages  $\theta_1, \dots, \theta_M$ , respectively. The boundary part of each region  $\mathcal{D}_i$  gives the main contribution to the decoding error. The transmitter aim is to “push” the output into the corresponding region  $\mathcal{D}_i$ . The best transmitter strategy is to “push” the current output in the direction “orthogonal” to the closest boundary of the true region  $\mathcal{D}_i$ . Then, essentially, two cases are possible.

1) If all  $\mathcal{D}_i$  are “round-shaped” (i.e. similar to “balls”), then they have the centers, and therefore the best transmitting strategy is to send the center of the corresponding “ball” (and that strategy does not depend on the output signals). It automatically pushes the output in the direction “orthogonal” to the closest boundary. This situation takes place for sufficiently high transmission rates  $R$ . Then, even noiseless feedback cannot improve the error exponent.

2) The situation becomes quite different if the optimal decoding regions  $\{\mathcal{D}_i\}$  are not “round-shaped” (and so, they do not have the natural centers). Now the best transmitter strategy depends on the current output location. For the case of three messages, it is depicted in Fig. 1. Let the message  $\theta_1$  be transmitted, and then the transmitter pushes the output into the region  $\mathcal{D}_1$ . If the current output is close to the point  $A$  (i.e. to two other possible regions), then best is to push the output simultaneously away from both competitive regions. On the contrary, if the current output is close to the point  $B$  (i.e. it is much closer to the competitive region  $\mathcal{D}_2$  than to  $\mathcal{D}_3$ ), then best is to push the output mainly away from the region  $\mathcal{D}_2$ , paying less attention to the other region  $\mathcal{D}_3$ .

That best strategy is possible only if the transmitter knows exactly the current output location (i.e. if there is noiseless feedback). If there is no any feedback then the transmitter

knows nothing on the current output location, and there is no sense to change the push direction. The situation becomes “fuzzy”, if the transmitter knows only approximately the current output location (i.e. if there is noisy feedback).

In this paper we realize those arguments, allowing only one fixed time moment when the transmitter may change the coding function. At that moment the transmitter, using observations over the feedback channel, finds two messages which are the most probable for the receiver. After that the transmitter only helps the receiver to decide between those two messages. Of course, an error is possible when choosing those two most probable messages. However, we show that if the crossover probability of the feedback channel is less than the certain positive value, then the probability of making an error in that choice is sufficiently small. Such simple transmission method (together with the properly chosen decoding) allows already to improve the decoding error probability in comparison with the no-feedback case.

Of course, if the feedback channel noise is rather small then it is possible to use a larger number of such “switching” moments, and to improve further the error probability exponent. In the limit (if the feedback channel noise is very small), using a growing number of switching moments, we can achieve the noiseless feedback case performance.

We consider the case when the overall transmission time  $n$  and  $M = M_n$  equiprobable messages  $\{\theta_1, \dots, \theta_M\}$  are given. It is assumed that  $M_n \rightarrow \infty$ , but  $\ln M_n = o(n)$  as  $n \rightarrow \infty$ , i.e. the transmission rate  $R = 0$ . After the moment  $n$ , the receiver makes a decision  $\hat{\theta}$  on the message transmitted. We limit ourselves here only to the case  $R = 0$ , since in that case the difficulties of using noisy feedback are seen most clearly. In the case of a positive transmission rate  $R$  (it will be considered in another publication) some additional technical difficulties appear, which we want to avoid for a while. It should also be mentioned that the investigation of the best error exponent for  $R = 0$  even for the noiseless feedback case is not a simple task [4].

As a result, we show that if the crossover probability  $p_1$  of the feedback channel  $\text{BSC}(p_1)$  is less than the certain positive value  $p_0(p)$ , then it is possible to improve the best error exponent  $E(p)$  of  $\text{BSC}(p)$  without feedback. The transmission method with one “switching” moment, giving such an improvement, is described in §3.

Denote by  $E(p)$  the best error exponent for  $M_n$  codewords over  $\text{BSC}(p)$  without feedback, i.e.

$$E(p) = \limsup_{n \rightarrow \infty} \frac{1}{n} \ln \frac{1}{P_e(M_n, n, p)}, \quad \ln M_n = o(n), \quad (1)$$

where  $P_e(M_n, n, p)$  is the minimal possible decoding error probability  $P_e$  for all codes of length  $n$ . Clearly, we have

$$E(p) = \frac{1}{4} \ln \frac{1}{4pq}. \quad (2)$$

Indeed, the minimal Hamming distance of any such code does not exceed  $n/2$  (Plotkin bound). On the other hand, due to the Varshamov-Gilbert bound there exist codes with approximately such minimal distance. If  $E(R, p)$  – the reliability function of the  $\text{BSC}(p)$  without feedback, then  $E(p) = E(0, p)$ .

Denote by  $E_2(p)$  the best error exponent for two codewords over BSC( $p$ ) (it remains the same for the channel with noiseless feedback, as well). Clearly, we have

$$E_2(p) = \frac{1}{2} \ln \frac{1}{4pq}.$$

Denote by  $F(p)$  the best error exponent for  $M_n$  messages over BSC( $p$ ) with noiseless feedback. It is defined similarly to (1), where  $P_e(M_n, n, p)$  is the minimal possible decoding error probability for all transmission methods. Denote also by  $F_3(p)$  the best error exponent for three messages over BSC( $p$ ) with noiseless feedback. Then [4]

$$F(p) = F_3(p) = -\ln(p^{1/3}q^{2/3} + q^{1/3}p^{2/3}). \quad (3)$$

If  $F(R, p)$  – the reliability function of such channel, then  $F(p) = F(0, p)$ .

Denote by  $F(p, p_1)$  the best error exponent for  $M_n$  messages transmitted over the BSC( $p$ ) with the noisy BSC( $p_1$ ) feedback channel. Clearly,  $E(p) \leq F(p, p_1) \leq F(p)$  for all  $p, p_1$ . In particular,  $F(p, 0) = F(p)$ ,  $F(p, 1/2) = E(p)$ . Moreover,  $E(p) < F(p) < E_2(p)$ ,  $0 < p < 1/2$ . Let  $r(p) = F(p)/E(p)$ . The function  $r(p)$  monotonically increases on  $p$ , and, in particular,

$$r(0) = \lim_{p \rightarrow 0} r(p) = 4/3, \quad r(0.01) \approx 1.67, \quad r(1/2) = \lim_{p \uparrow 1/2} = 16/9 \approx 1.78.$$

More exactly, if  $p = (1 - \varepsilon)/2$  then ( $\varepsilon \rightarrow 0$ )

$$E(p) = \varepsilon^2/4 + O(\varepsilon^4), \quad F(p) = 4\varepsilon^2/9 + O(\varepsilon^4).$$

Below in the paper  $f \sim g$  means  $n^{-1} \ln f = n^{-1} \ln g + o(1)$ ,  $n \rightarrow \infty$ , and  $f \lesssim g$  means  $n^{-1} \ln f \leq n^{-1} \ln g + o(1)$ ,  $n \rightarrow \infty$ .

To formulate the paper main result, introduce the functions:

$$\begin{aligned} h(x) &= -x \ln x - (1-x) \ln(1-x), \\ z &= q/p, \quad z_1 = q_1/p_1, \\ 3G_1(t, p) &= \ln \frac{1}{qp^2} - \max_a \{2h(a) + h(a+t) + (a+t) \ln z\}, \\ 3G_2(t, p, p_1) &= (2c_0 + t) \ln z - h(c_0 + t) - h(c_0) + [2 + t - 2(1+t)b_1] \ln z_1 - \\ &\quad - (1+t)h(b_1) - (1-t)h\left[\frac{(1+t)b_1 - t}{1-t}\right] - 2 \ln(qq_1), \\ c_0(t, p) &= \frac{2(1-t)}{2 + t(z^2 - 1) + \sqrt{4z^2 + t^2(z^2 - 1)^2}}, \\ b_1(t, p_1) &= \frac{2z_1^2}{(2+t)z_1^2 - t + \sqrt{4z_1^2 + (z_1^2 - 1)^2 t^2}}. \end{aligned} \quad (4)$$

The optimal  $a_0 = a_0(p, t)$  in (4) is defined as the unique root of the equation

$$q(1-a)^2(1-a-t) = pa^2(a+t). \quad (5)$$

We have  $a_0(p, 0) = a_0(p)$ , where  $a_0(p)$  is the same as defined below in (15).

Introduce also the function  $p_0(p)$  as the unique root of the equation

$$3G_2(1/2 - p, p, p_0) = \ln \frac{1}{4pq}, \quad 0 < p < 1/2. \quad (6)$$

Denote by  $F_1(p, p_1)$  the error exponent for the transmission method with one switching moment, described in §3. Clearly,  $F_1(p, p_1) \leq F(p, p_1)$  for all  $p, p_1$ . The paper main result is

**T h e o r e m.** *If  $p_1 < p_0(p)$ , then*

$$F(p, p_1) \geq F_1(p, p_1) = \max_t \frac{6 \min \{G_1(t, p), G_2(t, p, p_1)\} E(p)}{3 \min \{G_1(t, p), G_2(t, p, p_1)\} + 4E(p)} > E(p). \quad (7)$$

The function  $G_1(t, p)$  monotonically decreases on  $t$ , and  $G_1(0, p) = F(p)$ . On the other hand, the function  $G_2(t, p, p_1)$  monotonically increases on  $t$ . Moreover,  $G_2(0, p, p_1) = 0$ , and  $G_2(t, p, 0) = \infty$ ,  $t > 0$ .

The function  $p_0(p)$ ,  $0 < p < 1/2$ , is positive and monotonically increases on  $p$ . Its plot is shown in Fig. 2.

**E x a m p l e 1.** Consider the case  $p \rightarrow 0$ . Then

$$p_0(p) = \frac{16p}{27} (1 + o(1)).$$

The approximation  $p_0(p) \approx p/2$  is quite accurate for  $p \leq 0.01$ .

**E x a m p l e 2.** Consider the opposite asymptotic case  $p = (1 - \varepsilon)/2$ ,  $\varepsilon \rightarrow 0$ , and  $t \leq 1/2 - p = \varepsilon/2$ . Then  $a = a_0(p, t) = 1/2 - \rho$ ,  $\rho \rightarrow 0$ , and after standard algebra we get

$$\rho = \frac{2t - \varepsilon}{6} + O(\varepsilon^2),$$

which gives

$$G_1(t, p) = \frac{4(\varepsilon^2 - \varepsilon t + t^2)}{9} + O(\varepsilon^4),$$

$$G_2(t, p, p_1) = \frac{t^2}{12p_1q_1} + O(\varepsilon^3), \quad t \leq \frac{\varepsilon}{2}.$$

If  $G_1(t, p) = G_2(t, p, p_1)$ , then

$$t = \frac{4\varepsilon\sqrt{p_1q_1}}{\sqrt{3 - 12p_1q_1} + 2\sqrt{p_1q_1}} + O(\varepsilon^2),$$

which gives

$$\min \{G_1(t, p), G_2(t, p, p_1)\} = \frac{4\varepsilon^2}{3 [\sqrt{3 - 12p_1q_1} + 2\sqrt{p_1q_1}]^2} + O(\varepsilon^3).$$

The condition  $t \leq \varepsilon/2$  is equivalent to the inequality  $16p_1q_1 \leq 1$ , which means that

$$\lim_{p \rightarrow 1/2} p_0(p) = \frac{1}{4(2 + \sqrt{3})} \approx \frac{1}{14.93} \approx 0.067.$$

For  $p_1 \rightarrow 0$  we get

$$F(p, p_1) \geq \frac{8E(p)}{7} \left[ 1 - \frac{4\sqrt{3p_1}}{7} + \frac{104p_1}{49} + O\left(p_1^{3/2}\right) \right]. \quad (8)$$

In words, for small  $p_1$  the strategy described in §3 gives 14% gain over the no-feedback channel.

*C o r o l l a r y.* If  $p_1 = 0$ , then

$$F_1(p, 0) = \frac{6E(p)F(p)}{4E(p) + 3F(p)} > E(p), \quad 0 < p < 1/2. \quad (9)$$

*E x a m p l e 3.* We have  $F_1(p, p_1) \rightarrow F_1(p, 0)$  as  $p_1 \rightarrow 0$ . We investigate the rate of that convergence since it gives some idea on when the noisy feedback behaves like the noiseless feedback. If  $p_1 \rightarrow 0$ , then the optimal  $t \rightarrow 0$ . For a fixed  $0 < p < 1/2$  and  $t \rightarrow 0$  for the root  $a(t, p)$  of the equation (5) we have

$$a(t, p) = a_0(p) - \frac{t}{3} + O(t^2),$$

which gives

$$G_1(t, p) = F(p) - \frac{2t}{9} \ln z + O(t^2).$$

We also can get as  $p_1, t \rightarrow 0$

$$c_0(t, p) = p - \frac{t}{2} + \frac{(q-p)t^2}{8qp} + O(t^3),$$

$$b_1(t, p_1) = 1 - t + O\left(\frac{p_1^2}{p_1 + t}\right) + O(t^2),$$

which gives

$$3G_2(t, p, p_1) = -t \ln p_1 + O(p_1 \ln p_1) + O(t \ln t) + O(t^2 \ln p_1).$$

If  $G_1(t, p) = G_2(t, p, p_1)$ , then

$$t = \frac{3F(p)}{\ln(1/p_1)} [1 + o(1)],$$

and

$$\min \{G_1(t, p), G_2(t, p, p_1)\} = F(p) \left[ 1 - \frac{2 \ln z}{3 \ln(1/p_1)} + o\left(\frac{1}{\ln(1/p_1)}\right) \right].$$

As a result, we get as  $p_1 \rightarrow 0$

$$F_1(p, p_1) = F_1(p, 0) \left[ 1 - \frac{8E \ln z}{3(4E + 3F) \ln(1/p_1)} + o\left(\frac{1}{\ln(1/p_1)}\right) \right].$$

*Remark 1.* The transmission method described in §3, reduces the problem to testing of two most probable messages (at the fixed moment). Such strategy is not optimal even for one switching moment. But it is relatively simple for investigation, and it gives already a reasonable improvement over the no-feedback case.

In §2 the transmission method with one switching moment for the channel with noiseless feedback is described and investigated. In particular, the formula (9) is proved. In §3 that transmission method (slightly modified) is investigated for the channel with noisy feedback, and the theorem is proved. In §4 the simple transmission method with active feedback is considered.

The preliminary (and simplified) paper version (without detailed proofs) for  $M = 3$  messages was published as [13].

## §2. Channel with noiseless feedback. Proof of the formula (9).

We start with the noiseless feedback case and describe the transmission method which will be used for noisy feedback as well. Moreover, in the noisy feedback case we will need some formulas from that case.

Consider the BSC( $p$ ) with noiseless feedback and  $M$  messages  $\theta_1, \dots, \theta_M$ . We assume that  $M_n \rightarrow \infty$ , but  $\ln M_n = o(n)$  as  $n \rightarrow \infty$ . We set some  $\gamma \in [0, 1]$  (it will be chosen later) and divide the total transmission period  $[0, n]$  on two phases:  $[0, \gamma n]$  (phase I) and  $(\gamma n, n]$  (phase II). We perform as follows:

1) On phase I (i.e. on  $[0, \gamma n]$ ) we use a code of  $M$  codewords  $\{\mathbf{x}_i\}$  such that  $d(\mathbf{x}_i, \mathbf{x}_j) = \gamma n/2 + o(n)$ ,  $i \neq j$  (existence of such “almost” a simplex code can be shown using random choice of codewords). On that phase the transmitter only observes via the feedback channel outputs of the forward channel, but does not change the transmission method.

2) Let  $\mathbf{x}$  be the transmitted codeword (of length  $\gamma n$ ) and  $\mathbf{y}$  be the received (by the receiver) block. After phase I, based on the block  $\mathbf{y}$ , the transmitter selects two messages  $\theta_i, \theta_j$  (codewords  $\mathbf{x}_i, \mathbf{x}_j$ ) which are the most probable for the receiver, and ignore all the remaining messages  $\{\theta_k\}$ . Then, on phase II (i.e. on  $(\gamma n, n]$ ) the transmitter helps the receiver only to decide between those two most probable messages  $\theta_i, \theta_j$ , using two opposite codewords of length  $(1 - \gamma)n$ . After moment  $n$  the receiver makes a decision between those two remaining messages  $\theta_i, \theta_j$  (based on all received on  $[0, n]$  signals).

Clearly, a decoding error occurs in the following two cases.

1) After phase I the true message is not among two most probable messages. We denote that probability  $P_1$ .

2) After phase I the true message is among two most probable, but after phase II the true message is not the most probable. We denote that probability  $P_2$ .

Then for the total decoding error probability  $P_e$  we have

$$P_e \leq P_1 + P_2. \quad (10)$$

To evaluate the probabilities  $P_1$  and  $P_2$ , without loss of generality, we assume that the message  $\theta_1$  is transmitted. We start with the probability  $P_1$ . Denote  $d(\mathbf{x}, \mathbf{y})$  the Hamming distance between  $\mathbf{x}$  and  $\mathbf{y}$ , and  $d_i = d(\mathbf{x}_i, \mathbf{y})$ . Then

$$P_1 \leq \sum_{i>j>1} \mathbf{P}\{d_1 \geq \max\{d_i, d_j\} | \mathbf{x}_1\}. \quad (11)$$

We use the following auxiliary result (see proof in Appendix).

*L e m m a. 1) Let  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$  be the codewords of length  $m$ . Denote  $d_{ij} = d(\mathbf{x}_i, \mathbf{x}_j)$ ,  $d_i = d(\mathbf{x}_i, \mathbf{y})$ . Assuming that  $d_{12} = d_{13} = d_{23} = 2m/3 + o(m)$ ,  $m \rightarrow \infty$ , consider the probability*

$$P_1(t, t_1) = \mathbf{P}\left(d_2 = d_1 + \frac{2tm}{3} + o(m); d_3 = d_1 + \frac{2t_1m}{3} + o(m) \middle| \mathbf{x}_1\right).$$

Then

$$\frac{3}{m} \ln P_1(t, t_1) = \ln(p^2q) + f(t, t_1) + o(1), \quad |t| \leq 1, \quad |t_1| \leq 1, \quad (12)$$

where

$$\begin{aligned} f(t, t_1) &= \max_a f(a, t, t_1) = f(a_0, t, t_1), \\ f(a, t, t_1) &= h(a) + h(a+t) + h(a+t_1) + (a+t_1+t) \ln z, \end{aligned} \quad (13)$$

and  $a_0 = a_0(t, t_1)$  is the unique root of the equation

$$f'_a = \ln \frac{1-a}{a} + \ln \frac{1-a+t}{a-t} + \ln \frac{1-a-t_1}{a+t_1} + \ln z = 0.$$

The function  $f(t, t_1)$  monotone increases on  $t_1 \leq (1-2p+t)/2$ , and monotone decreases on  $t_1 \geq (1-2p+t)/2$ .

2) For any  $|t| \leq 1$  and  $t_1 \leq (1-2p+t)/2$ , we have

$$\mathbf{P}\left(d_2 \leq d_1 + \frac{2tm}{3}; d_3 \leq d_1 + \frac{2t_1m}{3} \middle| \mathbf{x}_1\right) = P_1(t, t_1)e^{o(m)}, \quad m \rightarrow \infty. \quad (14)$$

Note that the number of summation terms in the right-hand side of (11) does not exceed  $M^2 = e^{o(n)}$ . Any three codewords  $\mathbf{x}_1, \mathbf{x}_i, \mathbf{x}_j$  have the effective length  $m = 3\gamma n/4 + o(n)$  (on the remaining  $\gamma n/4 + o(n)$  positions they have equal coordinates) and mutual distances  $d(\mathbf{x}_k, \mathbf{x}_l) = 2m/3 + o(m)$ ,  $k \neq l$ . Then using the formulas (13) and (14) with  $t = t_1 = 0$ , we have

$$\begin{aligned} m^{-1} \ln P_1 &= \frac{1}{3} \ln(p^2q) + \frac{1}{3} \max_a \{3h(a) + a \ln z\} + o(1) = \\ &= \ln(p^{1/3}q^{2/3} + p^{2/3}q^{1/3}) = -F(p) + o(1), \end{aligned}$$

where  $F(p)$  is defined in (3), and the optimal  $a = a_0$  is given by

$$a_0 = a_0(p) = \frac{q^{1/3}}{p^{1/3} + q^{1/3}}. \quad (15)$$

As a result, from (11) we get

$$\ln \frac{1}{P_1} = \frac{3}{4}\gamma F(p)n + o(n), \quad 0 \leq p \leq 1/2. \quad (16)$$

*Remark 2.* Let  $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$  be a simplex code of length  $n$ . Then

$$-\frac{1}{n} \ln \mathbf{P}\{d(\mathbf{x}_1, \mathbf{y}) \geq \max\{d(\mathbf{x}_2, \mathbf{y}), d(\mathbf{x}_3, \mathbf{y})\} | \mathbf{x}_1\} = F_3(p) + o(1), \quad n \rightarrow \infty. \quad (17)$$

It explains the meaning of the value  $F(p) = F_3(p)$ .

Now we evaluate the probability  $P_2$ . On phase I (of length  $\gamma n$ ) all the distances among codewords are equal to  $\gamma n/2 + o(n)$ . On phase II (of length  $(1 - \gamma)n$ ) the distance between two remaining codewords equals  $(1 - \gamma)n$ . Therefore the total distance between the true and any concurrent codeword equals  $(1 - \gamma/2)n$ . Therefore

$$P_2 \leq M\mathbf{P}\{\text{error when testing two codewords on distance } (1 - \gamma/2)n\},$$

and then

$$\frac{1}{n} \ln P_2 = \frac{(1 - \gamma/2)}{2} \ln(4pq) + o(1) = -(2 - \gamma)E(p) + o(1). \quad (18)$$

As a result, from (10), (16) and (18) for the decoding error probability  $P_e$  we have

$$\frac{1}{n} \ln P_e \leq \frac{1}{n} \max\{\ln P_1, \ln P_2\} \leq -\min\left\{\frac{3}{4}\gamma F(p), (2 - \gamma)E(p)\right\} + o(1).$$

We choose  $\gamma = \gamma_0$  such that  $P_1 = P_2$ , i.e. set

$$\gamma_0 = \frac{8E(p)}{4E(p) + 3F(p)},$$

and then for  $0 < p < 1/2$  get the formula (9).

If  $p = (1 - \varepsilon)/2$ ,  $\varepsilon \rightarrow 0$ , then

$$F(p, 0) \rightarrow \frac{8}{7}E(p), \quad p \rightarrow 1/2,$$

i.e. such strategy with one switching moment gives 14% gain over the no-feedback case (the best strategy without limit on the number of switching moments gives 78% gain).

### § 3. Channel with noisy feedback. Proof of theorem

In the noisy feedback case, still using one switching moment, we will slightly modify the transmission method from § 2 (especially, its decoding method).

**Transmission.** Again we set a number  $0 < \gamma < 1$ . On phase I, of length  $\gamma n$ , we use an “almost” simplex code. Let  $\mathbf{x}$  be the transmitted codeword (of length  $\gamma n$ ),  $\mathbf{y}$  be the received (by the receiver) block, and  $\mathbf{x}'$  be the received (by the transmitter) block. Based on the transmitted codeword  $\mathbf{x}$  and the received block  $\mathbf{x}'$ , the transmitter selects two messages  $\theta_i, \theta_j$  which look most probable for the receiver.

If the true message is among those two selected messages  $\theta_i, \theta_j$ , then, on phase II (i.e. on  $(\gamma n, n]$ ) the transmitter uses the two opposite codewords of length  $(1 - \gamma)n$  to help the receiver to decide between those two most probable messages. For example, the transmitter uses all-zeros and all-ones codewords.

If the true message is not among two selected messages  $\theta_i, \theta_j$ , then, on phase II the transmitter sends an intermediate block (say, half-zeros and half-ones). In any case, such event will be treated as an error.

**Decoding.** We set an additional number  $t > 0$ . Arrange the distances  $\{d(\mathbf{x}_i, \mathbf{y}), i = 1, \dots, M\}$  in the increasing order, denoting

$$d^{(1)} = \min_i d(\mathbf{x}_i, \mathbf{y}) \leq d^{(2)} \leq \dots \leq d^{(M)} = \max_i d(\mathbf{x}_i, \mathbf{y}),$$

(in case of tie we use any order). Let also  $\mathbf{x}^1, \dots, \mathbf{x}^M$  be the ranking of codewords after phase I, i.e.  $\mathbf{x}^1$  is the most probable codeword, etc. There are possible two cases.

**C a s e 1.** If  $d^{(3)} \leq d^{(2)} + t\gamma n/2$ , then the receiver makes the decoding immediately after phase I (in favor of the closest to  $\mathbf{y}$  codeword). Although the transmitter still continues transmission, the receiver has already made its decision.

**C a s e 2.** If  $d^{(3)} > d^{(2)} + t\gamma n/2$ , then after phase I the receiver selects two most probable messages  $\theta_i, \theta_j$ , and after transmission on phase II (i.e. after moment  $n$ ) makes a decision between those two remaining messages  $\theta_i, \theta_j$  in favor of more probable of them.

In order to perform in agreement with the receiver, in the case 2 it is important that the transmitter can correctly identify two messages  $\theta_i, \theta_j$  which are most probable for the receiver. Of course, an error in such selection is possible, but its probability should be sufficiently small (which will be secured below).

*Remark 3.* We separate the case 1 since after phase I, with relatively high probability the second  $\mathbf{x}^2$  and the third  $\mathbf{x}^3$  ranked codewords will be approximately equiprobable, and then it will be difficult to the transmitter to rank them correctly. But in that case (with high probability) the first message  $\mathbf{x}^1$  will be much more probable than  $\mathbf{x}^2$  and  $\mathbf{x}^3$ .

To evaluate the decoding error probability  $P_e$ , denote  $P_1$  and  $P_2$  the decoding error probability in the case 1 (i.e. after phase I), and in the case 2 (i.e. after the moment  $n$ ) for the noiseless feedback channel, respectively. Similarly, denote  $P_{2n}$  the decoding error probability in the case 2 for the noisy feedback case. Then for the decoding error probability  $P_e$  we have

$$P_e \leq P_1 + P_2 + P_{2n}. \quad (19)$$

We evaluate the probabilities  $P_1, P_2, P_{2n}$  in the right-hand side of (19). For  $P_1$  we have

$$P_1 \leq M^2 (P_{11} + P_{12}), \quad (20)$$

where

$$\begin{aligned} P_{11} &= \mathbf{P}(d_2 \leq d_1 \leq d_3 \leq d_1 + t\gamma n/2 | \mathbf{x}_1), \\ P_{12} &= \mathbf{P}(d_1 \geq \max\{d_2, d_3\} | \mathbf{x}_1) \end{aligned} \quad (21)$$

and  $d_i = d(\mathbf{x}_i, \mathbf{y})$ ,  $i = 1, \dots, M$ .

The value  $P_{12}$  was already estimated in (16) (denoted there  $P_1$ ). The main contribution to  $P_1$  is given by the value  $P_{11}$ . To evaluate  $P_{11}$  it is sufficient to consider the case when the codewords  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$  have length  $m = 3\gamma n/4$  (on the remaining  $\gamma n/4$  positions they have equal coordinates) and mutual distances  $d(\mathbf{x}_i, \mathbf{x}_j) = 2m/3$ ,  $i \neq j$ . Then from (14) we have

$$P_{11} \leq \mathbf{P}\left(d_2 \leq d_1; d_3 \leq d_1 + \frac{2tm}{3} \middle| \mathbf{x}_1\right) e^{o(n)} = P_1(0, t) e^{o(n)}. \quad (22)$$

For the value  $P_1(0, t)$  we get from (14) and (13)

$$\frac{1}{m} \ln \frac{1}{P_1(0, t)} = G_1(t, p) + o(1), \quad t \leq \frac{1}{2} - p, \quad (23)$$

where  $G_1(t, p)$  is defined in (4). Moreover,

$$\frac{1}{m} \ln \frac{1}{P_1(0, t)} = \frac{1}{3} \ln \frac{1}{4pq} + o(1) = \frac{4}{3} E(p) + o(1), \quad t \geq \frac{1}{2} - p. \quad (24)$$

The function  $G_1(t, p)$  monotonically decreases on  $t \leq 1/2 - p$ . Moreover,  $G_1(0, p) = F(p)$ . For  $t \geq 1/2 - p$  the value  $P_1(0, t)$  is essentially defined only by the event  $\{d_1 \geq d_2\}$ .

Since  $P_{12} \lesssim P_{11}$ , we get from (20), (16) and (23)

$$\frac{4}{3\gamma n} \ln \frac{1}{P_1} = G_1(t, p) + o(1), \quad t \leq \frac{1}{2} - p. \quad (25)$$

For the value  $P_2$  the formula (18) remains valid.

It remains us to evaluate  $P_{2n}$ , which is the probability that the true codeword  $\mathbf{x}_1$  is among two most probable codewords for the receiver, but it is not such one for the transmitter. Introduce the random event

$$\mathcal{A} = \left\{ \begin{array}{l} d(\mathbf{x}_3, \mathbf{y}) > \max\{d(\mathbf{x}_1, \mathbf{y}), d(\mathbf{x}_2, \mathbf{y})\} + t\gamma n/2; \\ d(\mathbf{x}_3, \mathbf{x}') \leq \max\{d(\mathbf{x}_1, \mathbf{x}'), d(\mathbf{x}_2, \mathbf{x}')\} \end{array} \right\}. \quad (26)$$

Then

$$P_{2n} \leq M^2 \mathbf{P}(\mathcal{A} | \mathbf{x}_1) e^{o(n)}.$$

To evaluate  $\mathbf{P}(\mathcal{A}|\mathbf{x}_1)$  it is convenient to use two related random events

$$\begin{aligned}\mathcal{A}_1 &= \left\{ \begin{array}{l} d(\mathbf{x}_3, \mathbf{y}) \geq d(\mathbf{x}_2, \mathbf{y}) + t\gamma n/2; \\ d(\mathbf{x}_3, \mathbf{x}') \leq d(\mathbf{x}_2, \mathbf{x}') \end{array} \right\}, \\ \mathcal{A}_2 &= \left\{ \begin{array}{l} d(\mathbf{x}_3, \mathbf{y}) \geq d(\mathbf{x}_2, \mathbf{y}) + t\gamma n/2; \\ d(\mathbf{x}_2, \mathbf{x}') \leq d(\mathbf{x}_3, \mathbf{x}') \leq d(\mathbf{x}_1, \mathbf{x}') \end{array} \right\}.\end{aligned}\quad (27)$$

Since  $\mathcal{A} \subseteq \mathcal{A}_1 \cup \mathcal{A}_2$ , we have

$$\mathbf{P}(\mathcal{A}|\mathbf{x}_1) \leq \mathbf{P}(\mathcal{A}_1|\mathbf{x}_1) + \mathbf{P}(\mathcal{A}_2|\mathbf{x}_1). \quad (28)$$

We may assume that the codewords  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$  have length  $m = 3\gamma n/4$  (on the remaining  $\gamma n/4$  positions they have equal coordinates) and mutual distances  $d(\mathbf{x}_i, \mathbf{x}_j) = 2m/3$ ,  $i \neq j$ . All blocks  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{y}, \mathbf{x}'$  are shown in Fig. 3, where  $a, b, c, a_1, a_2, b_1, b_2, c_1, c_2$  denote the fractions of 1's in the corresponding parts of the received blocks  $\mathbf{y}$  and  $\mathbf{x}'$ . Then in addition to the formulas (46) (see Appendix) we have

$$\begin{aligned}d(\mathbf{x}_1, \mathbf{x}') &= [aa_1 + (1-a)a_2 + bb_1 + (1-b)b_2 + cc_1 + (1-c)c_2]m/3, \\ d(\mathbf{x}_2, \mathbf{x}') &= [a(1-a_1) + (1-a)(1-a_2) + b(1-b_1) + (1-b)(1-b_2) + cc_1 + (1-c)c_2]m/3, \\ d(\mathbf{x}_3, \mathbf{x}') &= [a(1-a_1) + (1-a)(1-a_2) + bb_1 + (1-b)b_2 + c(1-c_1) + (1-c)(1-c_2)]m/3, \\ d(\mathbf{y}, \mathbf{x}') &= [a(1-a_1) + (1-a)a_2 + b(1-b_1) + (1-b)b_2 + c(1-c_1) + (1-c)c_2]m/3.\end{aligned}$$

We start with the probability  $\mathbf{P}(\mathcal{A}_1|\mathbf{x}_1)$ . Since

$$\begin{aligned}d(\mathbf{x}_3, \mathbf{y}) \geq d(\mathbf{x}_2, \mathbf{y}) + t\gamma n/2 &\Leftrightarrow b \geq c + t, \\ d(\mathbf{x}_3, \mathbf{x}') \leq d(\mathbf{x}_2, \mathbf{x}') &\Leftrightarrow cc_1 + (1-c)c_2 \geq bb_1 + (1-b)b_2,\end{aligned}$$

for  $\mathbf{P}(\mathcal{A}_1|\mathbf{x}_1)$  we have with  $z = q/p$ ,  $z_1 = q_1/p_1$  (omitting the parts, where  $\mathbf{x}_2, \mathbf{x}_3$  coincide on all positions)

$$\mathbf{P}(\mathcal{A}_1|\mathbf{x}_1) = (qq_1)^{2m/3} \max_{b, \dots, c_2} \{AB\} [1 + o(1)] \leq (qq_1)^{2m/3} \max_{b, \dots, c_2} A \cdot \max_{b, \dots, c_2} B [1 + o(1)], \quad (29)$$

where

$$\begin{aligned}A &= \binom{m/3}{bm/3} \binom{m/3}{cm/3} z^{-m(b+c)/3}, \\ B &= \binom{bm/3}{b_1bm/3} \binom{(1-b)m/3}{b_2(1-b)m/3} \binom{cm/3}{c_1cm/3} \binom{(1-c)m/3}{c_2(1-c)m/3} z_1^{-\delta(\mathbf{y}, \mathbf{x}')m/3}, \\ \delta(\mathbf{y}, \mathbf{x}') &= b(1-b_1) + (1-b)b_2 + c(1-c_1) + (1-c)c_2,\end{aligned}\quad (30)$$

and where maximum is taken provided

$$\begin{aligned}b &\geq c + t, \\ cc_1 + (1-c)c_2 &\geq bb_1 + (1-b)b_2.\end{aligned}\quad (31)$$

From the definition (27) of the set  $\mathcal{A}_1$  it is clear that maximum of  $\{AB\}$  in (29) is attained when there are equalities in both relations (31). Moreover, there is no loss when we maximize the values  $A, B$  separately. Then we have

$$3m^{-1} \ln \mathbf{P}(\mathcal{A}_1 | \mathbf{x}_1) \leq 2 \ln(qq_1) + \max f + \max g + o(1), \quad (32)$$

where

$$\begin{aligned} f &= 3m^{-1} \ln A = h(b) + h(c) - (b+c) \ln z, \\ g &= 3m^{-1} \ln B = bh(b_1) + (1-b)h(b_2) + ch(c_1) + (1-c)h(c_2) - \delta(\mathbf{y}, \mathbf{x}') \ln z_1, \end{aligned} \quad (33)$$

and where maximum is taken provided

$$\begin{aligned} b &= c + t, \\ cc_1 + (1-c)c_2 &= bb_1 + (1-b)b_2. \end{aligned} \quad (34)$$

Note that both functions  $f, g$  are  $\cap$ -concave on all variables.

For the maximum of  $f$  we have

$$\begin{aligned} \max_{(34)} f &\leq \max_{b=c+t} f = \max_c \{h(c) + h(c+t) - (2c+t) \ln z\} = \\ &= h(c_0 + t) + h(c_0) - (2c_0 + t) \ln z, \end{aligned} \quad (35)$$

where  $c_0(t, p)$  is defined in (4). In fact, there is equality in (35).

To maximize the function  $g$  we use the standard Lagrange multipliers. Then for the optimal parameter values we get

$$c_1 = 1 - b_2, \quad c_2 = 1 - b_1, \quad b_2 = \frac{1 - (1+t)b_1}{1-t},$$

where  $b_1 = b_1(t, p_1)$  is defined in (4). It gives

$$\max_{(34)} g = (1+t)h(b_1) + (1-t)h\left[\frac{(1+t)b_1 - t}{1-t}\right] - [2+t - 2(1+t)b_1] \ln z_1. \quad (36)$$

Note that (since  $z_1 > 1$ )

$$2+t - 2(1+t)b_1 = \frac{4 + (z_1^2 - 1)t^2}{2 + \sqrt{4z_1^2 + (z_1^2 - 1)^2 t^2}} > 0.$$

Therefore from (32), (35) and (36) we get

$$\ln \mathbf{P}(\mathcal{A}_1 | \mathbf{x}_1) = -G_2(t, p, p_1)m + o(n), \quad (37)$$

where  $G_2(t, p, p_1)$  is defined in (4).

Finally consider the probability  $\mathbf{P}(\mathcal{A}_2|\mathbf{x}_1)$  from (27), (28). We show that

$$\ln \mathbf{P}(\mathcal{A}_2|\mathbf{x}_1) \leq \ln \mathbf{P}(\mathcal{A}_1|\mathbf{x}_1) + o(n). \quad (38)$$

For that purpose introduce the random events

$$\begin{aligned} \mathcal{C} &= \{d(\mathbf{x}_3, \mathbf{y}) \geq d(\mathbf{x}_2, \mathbf{y}) + t\gamma n/2\}, \\ \mathcal{D} &= \{d(\mathbf{x}_2, \mathbf{x}') \leq d(\mathbf{x}_3, \mathbf{x}') \leq d(\mathbf{x}_1, \mathbf{x}')\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{C}_1 &= \{d(\mathbf{x}_3, \mathbf{y}) = d(\mathbf{x}_2, \mathbf{y}) + t\gamma n/2 + o(n)\}, \\ \mathcal{D}_1 &= \{d(\mathbf{x}_2, \mathbf{x}') = d(\mathbf{x}_3, \mathbf{x}') + o(n) = d(\mathbf{x}_1, \mathbf{x}') + o(n)\}. \end{aligned}$$

Then  $\mathcal{A}_2 = \mathcal{C} \cap \mathcal{D}$ , and we have for any  $t \geq 0$

$$\begin{aligned} \mathbf{P}(\mathcal{A}_2|\mathbf{x}_1) &= \mathbf{P}(\mathcal{C} \cap \mathcal{D}|\mathbf{x}_1) \sim \mathbf{P}(\mathcal{C}_1 \cap \mathcal{D}_1|\mathbf{x}_1) \leq \\ &\leq \mathbf{P}(\{d(\mathbf{x}_3, \mathbf{x}') \leq d(\mathbf{x}_2, \mathbf{x}')\} \cap \mathcal{C}_1|\mathbf{x}_1) \sim \mathbf{P}(\mathcal{A}_1|\mathbf{x}_1), \end{aligned}$$

which proves the inequality (38).

As a result, from (28), (37) and (38) we have

$$\frac{1}{n} \ln P_{2n} = -\frac{3\gamma}{4} G_2(t, p, p_1) + o(1). \quad (39)$$

For the decoding error probability  $P_e$  from (19), (25), (18) and (39) we get

$$\begin{aligned} \frac{1}{n} \ln \frac{1}{P_e} &= \max_{\gamma, t} \min \left\{ \frac{3\gamma}{4} \min \{G_1(t, p), G_2(t, p, p_1)\}, (2 - \gamma) E(p) \right\} = \\ &= \max_t \frac{6 \min \{G_1(t, p), G_2(t, p, p_1)\} E(p)}{3 \min \{G_1(t, p), G_2(t, p, p_1)\} + 4E(p)}, \end{aligned} \quad (40)$$

where we set

$$\gamma = \frac{8E(p)}{3 \min \{G_1(t, p), G_2(t, p, p_1)\} + 4E(p)}.$$

The right-hand side of (40) exceeds  $E(p)$ , if for some  $t$  the inequality holds

$$3 \min \{G_1(t, p), G_2(t, p, p_1)\} > 4E(p). \quad (41)$$

Moreover,  $t \leq 1/2 - p$  (otherwise,  $3G_1(t, p) = 4E(p)$ ). Since  $G_2(t, p, p_1)$  monotonically increases in  $t$ , in order to have the inequality (41) fulfilled, we need to have  $3G_2(1/2 - p, p, p_1) > 4E(p)$ . Therefore introduce the function  $p_0(p)$  as the unique root of the equation (6). Then for any  $p_1 < p_0(p)$  and some  $t < 1/2 - p$  the inequality (41) is fulfilled, and therefore the right-hand side of (40) exceeds  $E(p)$ . As a result, from (40) we get the formula (7), which proves the theorem.  $\square$

## § 4. Channel with active feedback. Example

Using of coding in the feedback channel enlarges transmission possibilities. As an example, we consider the simplest of such transmission methods, proposed by G.A. Kabatyansky. The transmitter and the receiver will send information by turns.

We set some numbers  $\gamma, \gamma_1 > 0$ , such that  $\gamma + \gamma_1 < 1$ , and divide the total transmission period  $[0, n]$  on intervals  $[0, \gamma n]$ ,  $(\gamma n, (\gamma + \gamma_1)n]$  and  $((\gamma + \gamma_1)n, n]$ . We call those intervals phases I, II and III, respectively.

The transmitter will send information on phases I and III, while the receiver will send information only on phase II. On phase I of length  $\gamma n$  we use “almost” a simplex code. After phase I, based on the received block  $\mathbf{y}$ , the receiver selects two most probable messages. Then, during the phase II of length  $\gamma_1 n$ , it informs the transmitter on those two messages. On phase III, the transmitter uses two opposite codewords of length  $(1 - \gamma - \gamma_1)n$  to help the receiver to decide between those two most probable messages.

A decoding error occurs in the following three cases:

- 1) After phase I the true message is not among two most probable messages. We denote that probability  $P_1$ .
- 2) After phase I the true message is among two most probable, but on phase II the decoding error occurs on the transmitter. We denote that probability  $P_2$ .
- 3) After phase II the transmitter identified correctly two most probable messages (and the true message is among them), but after phase III the true message is not the most probable one among two possible messages. We denote that probability  $P_3$ .

Then for the decoding error probability  $P_e$  we have

$$P_e \leq P_1 + P_2 + P_3. \quad (42)$$

Similarly to § 3, for the values  $P_1, P_2, P_3$  in the right-hand side of (42) we have (as  $n \rightarrow \infty$ )

$$\begin{aligned} \frac{1}{n} \ln \frac{1}{P_1} &= \frac{3}{4} \gamma F(p) + o(1), \\ \frac{1}{n} \ln \frac{1}{P_2} &= \gamma_1 E(p_1) + o(1), \\ \frac{1}{n} \ln P_3 &= (2 - \gamma - 2\gamma_1) E(p) + o(1). \end{aligned} \quad (43)$$

We choose parameters  $\gamma, \gamma_1$  such that the values  $P_1, P_2, P_3$  become equal, i.e. we set

$$\gamma_1 = \frac{3\gamma F(p)}{4E(p_1)}, \quad \gamma = \frac{8E(p)}{3F(p) + 4E(p) + 6F(p)E(p)/E(p_1)}.$$

Then we get

**P r o p o s i t i o n.** *For the decoding error probability  $P_e$  of the transmission method described the relation holds*

$$\frac{1}{n} \ln \frac{1}{P_e} \geq \frac{E(p)}{1/2 + 2E(p)/(3F(p)) + E(p)/E(p_1)} + o(1). \quad (44)$$

Since  $E(p)/F(p) \rightarrow 3/4$ ,  $p \rightarrow 0$ , such transmission method, essentially, does not improve  $E(p)$  for small  $p$  (and any  $p_1$ ).

But if  $p = (1 - \varepsilon)/2$ ,  $\varepsilon \rightarrow 0$ , then  $E(p)/F(p) \rightarrow 9/16$ ,  $p \rightarrow 1/2$ , and (44) takes the form

$$\frac{1}{n} \ln \frac{1}{P_e} \geq \frac{E(p)}{7/8 + E(p)/E(p_1)} + o(1). \quad (45)$$

In that case, such transmission method improves  $E(p)$ , if  $E(p_1) > 8E(p)$ . In particular, if  $p_1 = (1 - \varepsilon_1)/2$ ,  $\varepsilon_1 \rightarrow 0$ , then  $E(p)/E(p_1) \approx \varepsilon^2/\varepsilon_1^2$ . Therefore the right-hand side of (45) is better than  $E(p)$ , if  $\varepsilon_1 > \varepsilon\sqrt{8}$ . It is better than the relation (7) (where it is demanded  $p_1 < 0.067$ ).

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## APPENDIX

**P r o o f o f l e m m a.** Since the part 2) follows from the part 1), it is sufficient to prove the part 1). To simplify formulas we assume that  $d_{12} = d_{13} = d_{23} = 2m/3$  (i.e. that  $\{\mathbf{x}_i\}$  is a simplex code). Such codewords  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$  are shown in Fig. 4, where  $a, b, c$  denote the fractions of 1's in the corresponding parts of the received block  $\mathbf{y}$ . Since

$$\begin{aligned} d_1 &= d(\mathbf{x}_1, \mathbf{y}) = (a + b + c)m/3, \\ d_2 &= d(\mathbf{x}_2, \mathbf{y}) = (2 + c - a - b)m/3, \\ d_3 &= d(\mathbf{x}_3, \mathbf{y}) = (2 + b - a - c)m/3, \end{aligned} \tag{46}$$

for the corresponding random events we have

$$\begin{aligned} \{d_2 = d_1 + 2tm/3\} &\Leftrightarrow \{a + b = 1 - t\}, \\ \{d_3 = d_1 + 2t_1m/3\} &\Leftrightarrow \{a + c = 1 - t_1\}. \end{aligned}$$

Therefore

$$P_1(t, t_1) \sim q^m \max_{\substack{a+b=1-t \\ a+c=1-t_1}} \left\{ \binom{m/3}{am/3} \binom{m/3}{bm/3} \binom{m/3}{cm/3} z^{-(a+b+c)m/3} \right\},$$

and then

$$\frac{3}{m} \ln P_1(t, t_1) = \ln(p^2q) + \max_a f(a, t, t_1) + o(1),$$

where

$$\begin{aligned} f(a, t, t_1) &= h(a) + h(a + t) + h(a + t_1) + (a + t_1 + t) \ln z, \\ f'_a &= \ln \frac{1-a}{a} + \ln \frac{1-a-t}{a+t} + \ln \frac{1-a-t_1}{a+t_1} + \ln z, \\ f'_t &= \ln \frac{1-a-t}{a+t} + \ln z, \quad f'_{t_1} = \ln \frac{1-a-t_1}{a+t_1} + \ln z. \end{aligned}$$

The function  $f(a, t, t_1)$  is  $\cap$ -concave on all arguments. Therefore, the function  $\max_a f(a, t, t_1)$  (and similar ones) is also  $\cap$ -concave on all arguments. In particular,  $\max_{a, t, t_1} f(a, t, t_1)$  is attained for  $a = p, t = t_1 = 1 - 2p$ . Similarly,  $\max_{a, t_1} f(a, t, t_1)$  is attained for  $a = (1 - t)/2, t_1 = (1 - 2p + t)/2$ . Then we get the part 1) of the lemma.  $\square$

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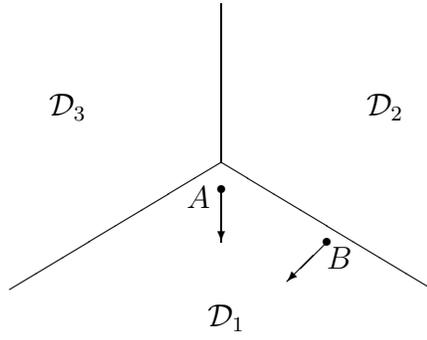


Fig 1. Decoding regions  $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$  and directions of output drives

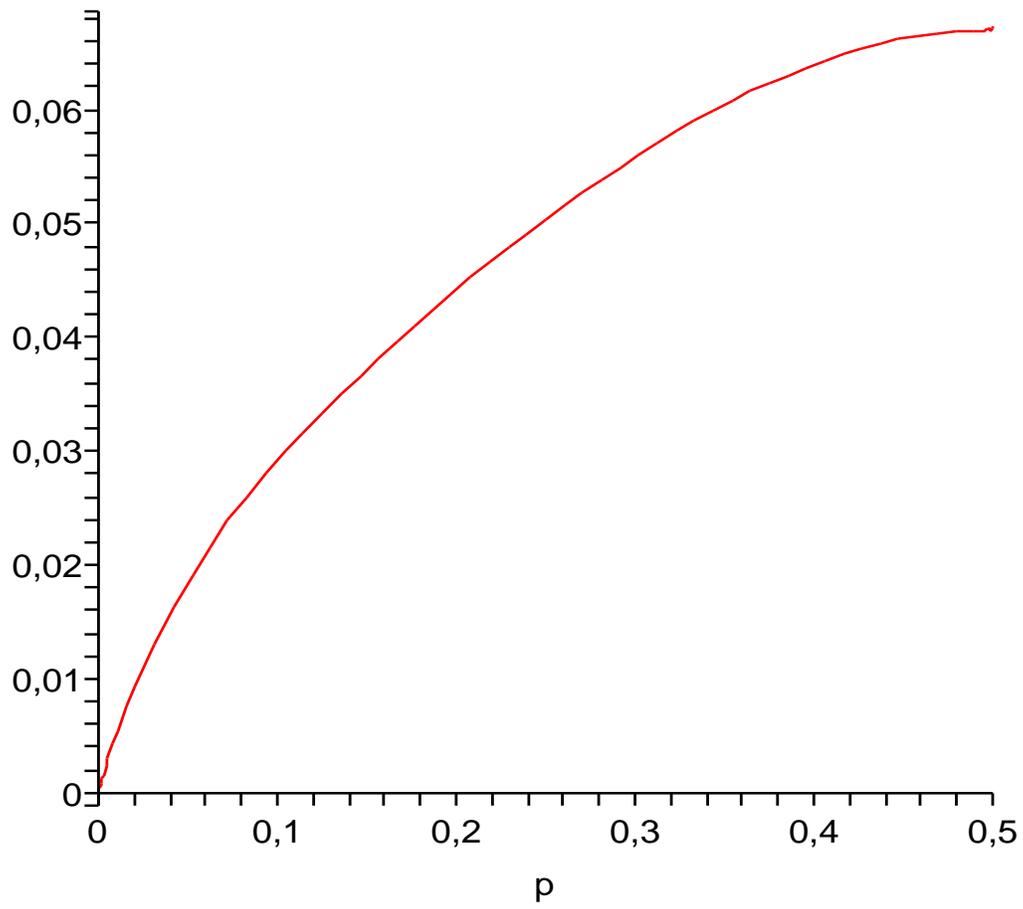


Fig. 2. Plot of the function  $p_0(p)$

	$a$		$b$		$c$	
$\mathbf{x}_1$	00	00	00	00	00	00
$\mathbf{x}_2$	11	11	11	11	00	00
$\mathbf{x}_3$	11	11	00	00	11	11
$\mathbf{y}$	11	00	11	00	11	00
$\mathbf{x}'$	$a_1$	$a_2$	$b_1$	$b_2$	$c_1$	$c_2$
	0		$m/3$		$2m/3$	$m$

Fig 3. Blocks  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{y}, \mathbf{x}'$

$\mathbf{x}_1$	00	00	00
$\mathbf{x}_2$	11	11	00
$\mathbf{x}_3$	11	00	11
$\mathbf{y}$	$a$	$b$	$c$
	0	$m/3$	$2m/3$
		$m$	

Fig 4. Blocks  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{y}$