# On weak isometries of Preparata codes 

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#### Abstract

Let $C_{1}$ and $C_{2}$ be codes with code distance $d$. Codes $C_{1}$ and $C_{2}$ are called weakly isometric, if there exists a mapping $J: C_{1} \rightarrow C_{2}$, such that for any $x, y$ from $C_{1}$ the equality $d(x, y)=d$ holds if and only if $d(J(x), J(y))=d$. Obviously two codes are weakly isometric if and only if the minimal distance graphs of these codes are isomorphic. In this paper we prove that Preparata codes of length $n \geq 2^{12}$ are weakly isometric if and only if these codes are equivalent. The analogous result is obtained for punctured Preparata codes of length not less than $2^{10}-1$.


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## 1 Introduction

Let $E^{n}$ denote all binary vectors of length n . The Hamming distance between two vectors from $E^{n}$ is the number of places where they differ. The weight of vector $x \in E^{n}$ is the distance between this vector and the all-zero vector $0^{n}$, and the support of $x$ is the set $\operatorname{supp}(x)=\{i \in$ $\left.\{1, \ldots, n\}: x_{i}=1\right\}$.

A set $C, C \subset E^{n}$, is called a code with parameters $(n, M, d)$, if $|C|=M$ and the minimal distance between two codewords from $C$ equals $d$. We say that a code $C$ is reduced if it contains all-zero vector.

A collection of $k$-subsets (referred to as blocks) of a $n$-set such that any $t$-subset occurs in $\lambda$ blocks precisely is called a $(\lambda, n, k, t)$-design.

The minimal distance graph of a code $C$ is defined as the graph with all codewords of $C$ as vertices, with two vertices being connected if and only if the Hamming distance between corresponding codewords equals to the code distance of the code $C$. Two codeword of $C$ are called $d$-adjacent if the Hamming distance equals code distance $d$ of the code $C$.

Two codes $C_{1}$ and $C_{2}$ of length $n$ are called equivalent, if an automorphism $F$ of $E^{n}$ exists such that $F\left(C_{1}\right)=C_{2}$. A mapping $I: C_{1} \rightarrow C_{2}$ of two codes $C_{1}$ and $C_{2}$ is called an isometry between codes $C_{1}$ and $C_{2}$, if the equality $d(x, y)=d(I(x), I(y))$ holds for all $x$ and $y$ from $C_{1}$. Then codes $C_{1}$ and $C_{2}$ are called isometric. A mapping $J: C_{1} \rightarrow C_{2}$ is called a weak isometry of codes $C_{1}$ and $C_{2}$ (and codes $C_{1}$ and $C_{2}$ weakly isometrical), if for any $x, y$ from $C_{1}$ the equality $d(x, y)=d$ holds if and only if $d(J(x), J(y))=d$ where $d$ is the code distance of code $C_{1}$. Obviously two codes are weakly isometric if and only if the minimal distance graphs of these codes are isomorphic. In [2] Avgustinovich established that any two weakly isometric 1-perfect codes are equivalent. In [5] it was proved that this result also holds for extended 1-perfect codes.

In this paper any weak isometry of two Preparata codes (punctured Preparata codes) is proved to be an isometry of these codes. Moreover, weakly isometric Preparata codes (punctured Preparata codes) of length $n \geq 2^{12}$ (of length $n \geq 2^{10}-1$ respectively) are proved to be equivalent.

This topic is closely related with problem of metrical rigidity of codes. A code $C$ is called metrically rigid if any isometry $I: C \rightarrow E^{n}$ can be extended to an isometry (automorphism) of the whole space $E^{n}$. Obviously any two metrically rigid isometric codes are equivalent. In [4] it was established that any reduced binary code of length $n$ containing 2-( $n, k, \lambda)$-design is metrically rigid for any $n \geq k^{4}$.

A maximal binary code of length $n=2^{m}$ for even $m, m \geq 4$ with code distance 6 is called a Preparata code $\overline{P^{n}}$. Punctured Preparata code is a code obtained from Preparata code by deleting one coordinate. By $P^{n}$ we denote a punctured Preparata code of length $n$. Preparata codes and punctured Preparata codes have some useful properties. All of them are distance invariant [1], strongly distance invariant [3]. Also a punctured Preparata code is contained in the unique 1-perfect code [6]. An arbitrary punctured Preparata code is uniformly packed [1]. As a consequence of this property, codewords of minimal weight of a Preparata code (punctured Preparata code) form a design. The last property is crucial in proving the main result of this paper.

## 2 Weak isomery of punctured Preparata codes

In this section we prove that any two punctured Preparata codes of length $n$ with isomorphic minimum distance graphs are isometric. Moreover, these codes are equivalent for $n \geq 2^{10}-1$. First we give some preliminary statements.

Lemma 1. [1]. Let $P^{n}$ be an arbitrary reduced punctured Preparata code. Then codewords of weight 5 of the code $P^{n}$ form 2-(n,5,(n-3)/3) design.

Taking into account a structure of the design from this lemma we obtain
Corollary 1. Let $P^{n}$ be an arbitrary reduced punctured Preparata code and $r, s$ be arbitrary elements of the set $\{1, \ldots, n\}$. Then there exists exactly one coordinate $t$ such that all codewords of minimal weight of the code $P^{n}$ with ones in coordinates $r$ and $s$ has zero in the coordinate $t$.

Let $C$ be a code with code distance $d$ and $x$ be an arbitrary codeword of $C$ of weight $i$. Denote by $D_{i, j}(x)$ the set of all codewords of $C$ of weight $j$ which are $d$-adjacent with vector $x$. In case when $C$ is a punctured Preparata code we give some properties of the set $D_{i, j}(x)$ that make the structure of minimal distance graph of this code more clear.

Lemma 2. Let $x$ be an arbitrary codeword of a punctured Preparata code $P^{n}$. Then any vector from $D_{i, i-1}(x)\left(D_{i, i-3}(x) \quad D_{i, i-5}(x)\right.$ respectively) has exactly 3 (4 and 5 resp.) zero coordinates from $\operatorname{supp}(x)$ and exactly $2(1$ and 0 resp.) nonzero coordinates from $\{1, \ldots, n\} \backslash \operatorname{supp}(x)$.

Proof. Suppose a vector $y \in D_{i, i-k}(x)$ has $m_{k}$ zero coordinates from $\operatorname{supp}(x)$. Then it has exactly $m_{k}-k$ nonzero coordinates from the set $\{1, \ldots, n\} \backslash \operatorname{supp}(x)$. Since $d(x, y)=5$ we have $m_{k}=(5+k) / 2$, which implies the required property for $k=1,3,5$.

Let $x$ be a codeword of weight $i$ from a $P^{n} ; m, l$ be arbitrary coordinates from $\operatorname{supp}(x)$. We denote by $A_{m, l}(x)\left(B_{m, l}(x)\right.$ and $\left.C_{m, l}(x)\right)$ the sets $D_{i, i-1}(x)\left(D_{i, i-3}(x)\right.$ and $D_{i, i-5}(x)$ respectively $)$ with coordinates $m$ and $l$ equal to zero.

Lemma 3. Let $x \in P^{n}, m, l \in \operatorname{supp}(x)$ and $u, v$ be arbitrary codewords of $P^{n}$ with zeros in coordinates $m$ and $l$ that are at distance 5 from $x$. Then $u, v$ do not share zero coordinates in $\operatorname{supp}(x) \backslash\{m, l\}$ and do not share coordinates equal to one in the set $\{1, \ldots, n\} \backslash \operatorname{supp}(x)$.

Proof. Let us suppose the opposite. Then the vectors $x+u$ and $x+v$ of weight five share at least three coordinates with ones in them and therefore

$$
d(u, v)=d(x+u, x+v) \leq 4
$$

holds. Since code distance of the code $P^{n}$ equals 5 we get a contradiction.

Lemma 4. Let $x$ be an arbitrary codeword of weight $i$ from a punctured Preparata code. Then the following inequalities hold:

$$
\begin{equation*}
(i-3) C_{i}^{2} \leq 3\left|D_{i, i-1}(x)\right|+12\left|D_{i, i-3}(x)\right|+30\left|D_{i, i-5}(x)\right| \leq(i-2) C_{i}^{2} \tag{1}
\end{equation*}
$$

Proof. Fix two coordinates $m$ and $l$ from $\operatorname{supp}(x)$. By Lemma 2 an arbitrary vector from $A_{m, l}(x)\left(B_{m, l}\right.$ and $\left.C_{m, l}\right)$ has exactly one zero coordinate (two and three respectively) from $\operatorname{supp}(x) \backslash\{m, l\}$. Then taking into account Lemma 3 the number of coordinates from $\operatorname{supp}(x) \backslash\{m, l\}$ which are zero for vectors from $A_{m, l}, B_{m, l}$ and $C_{m, l}$ equals $\left|A_{m, l}(x)\right|, 2\left|B_{m, l}\right|$ and $3\left|C_{m, l}\right|$ respectively. Therefore the number of coordinates from the $\operatorname{supp}(x) \backslash\{m, l\}$ which are zero for vectors from $A_{m, l} \cup B_{m, l} \cup C_{m, l}$ equals

$$
\left|A_{m, l}(x)\right|+2\left|B_{m, l}(x)\right|+3\left|C_{m, l}(x)\right|
$$

Since $x$ is a vector of weight $i$ and $m, l \in \operatorname{supp}(x)$, this number does not exceed $i-2$. From the other hand by Corollary 1 there exists at most one coordinate from $\operatorname{supp}(x) \backslash\{m, l\}$ such that all vectors from $A_{m, l} \cup B_{m, l} \cup C_{m, l}$ have one in it. Thus we have:

$$
i-3 \leq\left|A_{m, l}(x)\right|+2\left|B_{m, l}(x)\right|+3\left|C_{m, l}(x)\right| \leq i-2 .
$$

Summing these inequalities for all $m, l \in \operatorname{supp}(x)$ we obtain

$$
\begin{equation*}
(i-3) C_{i}^{2} \leq \sum_{m, l \in \operatorname{supp}(x)}\left|A_{m, l}(x)\right|+2 \sum_{m, l \in \operatorname{supp}(x)}\left|B_{m, l}(x)\right|+3 \sum_{m, l \in \operatorname{supp}(x)}\left|C_{m, l}(x)\right| \leq(i-2) C_{i}^{2} \tag{2}
\end{equation*}
$$

As an arbitrary vector from $D_{i, i-1}(x)$ has exactly 3 zero coordinates from $\operatorname{supp}(x)$, any such vector is counted $C_{3}^{2}$ times in the sum $\sum_{m, l \in \operatorname{supp}(x)}\left|A_{m, l}(x)\right|$. Then

$$
\sum_{m, l \in \operatorname{supp}(x)}\left|A_{m, l}(x)\right|=C_{3}^{2}\left|D_{i, i-1}(x)\right|
$$

Analogously we get:

$$
\begin{aligned}
\sum_{m, l \in \operatorname{supp}(x)}\left|B_{m, l}(x)\right| & =C_{4}^{2}\left|D_{i, i-3}(x)\right| \\
\sum_{m, l \in \operatorname{supp}(x)}\left|C_{m, l}(x)\right| & =C_{5}^{2}\left|D_{i, i-5}(x)\right|
\end{aligned}
$$

So from (2) we get (11).

Now we prove the main result using Lemmas 2 and 4.
Theorem 1. The minimal distance graphs of two punctured Preparata codes are isomorphic if and only if these codes are isometric.

Proof. It is obvious that if two punctured Preparata codes are isometric then they are weakly isometric.

Let $J: P_{1}^{n} \rightarrow P_{2}^{n}$ be a weak isometry of two punctured Preparata codes $P_{1}^{n}$ and $P_{2}^{n}$ of length $n$. Without loss of generality suppose that $0^{n} \in P_{1}^{n}, J\left(0^{n}\right)=0^{n}$. We now show that mapping $J$ is an isometry. For proving this it is sufficient to show that $w t(J(x))=w t(x)$ for all $x \in P_{1}^{n}$.

Suppose $z$ is a codeword of the code $P_{1}^{n}$, such that $w t(J(z)) \neq w t(z)=i$ holds and the mapping $J$ preserves weight of all codewords of weight smaller that $i$. The vector $z$ satisfying these conditions we call critical Since $J\left(0^{n}\right)=0^{n}$ and the mapping $J$ preserves the distance
between all codewords at distance 5 , we have $i \geq 6$. We prove that there is no critical codewords in $P_{1}^{n}$. From $0^{n} \in P_{1}^{n}$ holds that the weak isometry $J$ preserves a parity of weight of a vector and therefore $w t(J(z))$ equals either $i+2$ or $i+4$.

Suppose $w t(J(z))=i+2$. Since $J$ is a weak isometry and $z$ is a critical vector we have the following: $\left|D_{i+2, i-1}(J(z))\right|=\left|D_{i, i-1}(z)\right|,\left|D_{i+2, i-3}(J(z))\right|=\left|D_{i, i-3}(z)\right|,\left|D_{i, i-5}(z)\right|=0$. Taking into account these equalities, from the inequalities of Lemma 4 for vectors $z$ and $J(z)$ we get

$$
\begin{gather*}
(i-3) C_{i}^{2} \leq 3\left|D_{i, i-1}(z)\right|+12\left|D_{i, i-3}(z)\right|,  \tag{3}\\
3\left|D_{i+2, i+1}(J(z))\right|+12\left|D_{i, i-1}(z)\right|+30\left|D_{i, i-3}(z)\right| \leq i C_{i+2}^{2} . \tag{4}
\end{gather*}
$$

Multiplying both sides of inequality (3) by -4 we get

$$
-12\left|D_{i, i-1}(z)\right|-48\left|D_{i, i-3}(z)\right| \leq-4(i-3) C_{i}^{2} .
$$

Summing this inequality with (4) we get

$$
3\left|D_{i+2, i+1(J(z))}\right|-18\left|D_{i, i-3}(z)\right| \leq i C_{i+2}^{2}-4(i-3) C_{i}^{2}
$$

and therefore

$$
\begin{equation*}
\left|D_{i, i-3}(z)\right| \geq \frac{4(i-3) C_{i}^{2}-i C_{i+2}^{2}}{18} \tag{5}
\end{equation*}
$$

In particular, from the inequality (5) we have $\left|D_{i, i-3}(z)\right| \geq 1$ for $i=6$ and $i=7$. But there is no codewords of weight 3 and 4 in the $P_{1}$ since $P_{1}$ is reduced code with code distance 5. Therefore $i \geq 8$. From Lemma 4 we have the following

$$
\begin{equation*}
\left|D_{i, i-3}(z)\right| \leq \frac{(i-2) C_{i}^{2}}{12} \tag{6}
\end{equation*}
$$

But for $i \geq 10$ the inequality $3(i-2) C_{i}^{2}<2\left(4(i-3) C_{i}^{2}-i C_{i+2}^{2}\right)$ holds. This contradicts with (5) and (6).

So it is only remains to prove that there are no codewords of weight 8 and 9 , such that their images under the mapping $J$ have weights 10 and 11 respectively. Obviously the Hamming distance between any two vectors from $D_{i+2, i-3}(J(z))$ is not less than 6. By Lemma 2 all ones coordinates of each vector from $D_{i+2, i-3}(J(z))$ are in set $\operatorname{supp}(J(z))$. So $\left|D_{i+2, i-3}(J(z))\right|$ does not exceed the cardinality of maximal constant weight code of length $i+2$, with all code words of weight being equal $i-3$ and being at distance not less than 6 pairwise. For $i=8$ and $i=9$ the cardinalities of such codes equal to 6 and 11 respectively, but from (5) we have

$$
\left|D_{10,5}(J(z))\right|=\left|D_{8,5}(z)\right| \geq 12,\left|D_{11,6}(J(z))\right|=\left|D_{9,6}(z)\right| \geq 21,
$$

a contradiction. Therefore there is no critical vectors $z$ in $P_{1}^{n}, w t(z)=i$, such that $w t(J(z))=$ $i+2$.

Suppose $w t(J(z))=i+4$. In this case we have $\left|D_{i, i-3}(z)\right|=\left|D_{i, i-5}(z)\right|=0,\left|D_{i+4, i-1}(J(z))\right|=$ $\left|D_{i, i-1}(z)\right|$. Using these equalities we have from the inequalities of Lemma 4 for the vectors $z$ and $J(z)$ the following:

$$
\begin{gathered}
(i-3) C_{i}^{2} \leq 3\left|D_{i, i-1}(z)\right|, \\
30\left|D_{i, i-1}(z)\right| \leq(i+2) C_{i+4}^{2} .
\end{gathered}
$$

From these last two inequalities we obtain

$$
\frac{(i-3) C_{i}^{2}}{3} \leq \frac{(i+2) C_{i+4}^{2}}{30}
$$

and therefore

$$
10 i(i-1)(i-3) \leq(i+4)(i+3)(i+2)
$$

that implies

$$
10 i(i-1)(i-3) \leq 2 i(i+3)(i+2)
$$

Since last inequality does not hold for $i \geq 6$ there is no critical vectors in $P_{1}^{n}$ and therefore the mapping $J$ is an isometry.

In [4] the following theorem was proved
Theorem 2. Any reduced code of length $n$, that contains a $2-(n, k, \lambda)$-design is metrically rigid for $n \geq k^{4}$.

Taking into account that by Lemma 1 any punctured reduced Preparata code contains 2$(n, 5,(n-3) / 4)$-design applying Theorems 1 and 2 we get
Corollary 2. Let $n \geq 2^{10}-1$. Two punctured Preparata codes of length $n$ are equivalent if and only if the minimal distance graphs of these codes are isomorphic.

## 3 Weak isometry of Preparata codes

Using the analogous considerations, Theorems 1,2 and Corollary 2 can easily be extended for extended Preparata codes. We now give the analogues of Lemmas 1-4 omitting their proofs.
Lemma 5. ([1]) Let $\overline{P^{n}}$ be an arbitrary reduced Preparata code. Then codewords of weight 6 of code $\overline{P^{n}}$ form 3-(n,6,(n-4)/3)-design.
Lemma 6. Let $x$ be an arbitrary codeword of a Preparata code $\overline{P^{n}}, w t(x)=i$. Then any vector from $D_{i, i-2}(x)\left(D_{i, i-4}(x) \quad D_{i, i-6}(x)\right.$ respectively) has exactly 4 ( 5 and 6 respectively) zero coordinates from $\operatorname{supp}(x)$ and exactly 2 (1 and 0 respectively) nonzero coordinates from $\{1, \ldots, n\} \backslash \operatorname{supp}(x)$.
Lemma 7. Let $x \in \overline{P^{n}}, m, l, k \in \operatorname{supp}(x)$, and $u, v$ be arbitrary codewords of $\overline{P^{n}}$ at distance 6 from the vector $x$ with zero coordinates in positions $m, l, k$. Then there is no coordinate from $\operatorname{supp}(x) \backslash\{m, l, k\}$ such that $u, v$ have zeros in it and there is no coordinate from $\{1, \ldots, n\} \backslash \operatorname{supp}(x)$ such that $u, v$ have ones in it.
Lemma 8. Let $x$ be an arbitrary codeword of weight $i$ from a Preparata code. Then the following inequalities hold:

$$
\begin{equation*}
C_{i}^{3}(i-4) \leq 4\left|D_{i, i-2}(x)\right|+20\left|D_{i, i-4}(x)\right|+60\left|D_{i, i-6}(x)\right| \leq C_{i}^{3}(i-3) \tag{7}
\end{equation*}
$$

Using Lemmas 5-8 and the same arguments as in the proof of Theorem 1 the following theorem it is not difficult to prove

Theorem 3. The minimal distance graphs of two Preparata codes are isomorphic if and only if the codes are isometric.

From this theorem, Lemma 5 and Theorem 2 we get
Corollary 3. Let $n \geq 2^{12}$. Two Preparata codes of length $n$ are equivalent if and only if the minimal distance graphs of these codes are isomorphic.

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