

On frequency estimation of periodic ergodic diffusion process

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Abstract

We consider the problem of frequency estimation by observations of the periodic diffusion process possessing ergodic properties in two different situations. The first one corresponds to continuously differentiable with respect to parameter trend coefficient and the second - to discontinuous trend coefficient. It is shown that in the first case the maximum likelihood and bayesian estimators are asymptotically normal with rate $T^{3/2}$ and in the second case these estimators have different limit distributions with the rate T^2 .

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1 Introduction

Let us consider the model “signal in noise” of the following type

$$x(t) = S(\vartheta, t) + n(t), \quad 0 \leq t \leq T, \quad (1)$$

where $S(\vartheta, \cdot)$ is the signal transmitting the “information” ϑ and observed in the presence of additive “noise” $n(\cdot)$. This is a typical model for the theory of telecommunications. There is a large diversity of statistical problems related to this model. One way is to study the different noises (white Gaussian, “colored” Gaussian, stationary *etc.*) and another way is to study the different types of “modulations”, i.e.; to choose the signals $S(\cdot)$, like

amplitude modulation $S(\vartheta, t) = \vartheta h(t)$, *phase modulation* $S(\vartheta, t) = h(t - \vartheta)$ or *frequency modulation* $S(\vartheta, t) = h(\vartheta t)$. Here the function $h(\cdot)$ is usually supposed to be periodic. The most developed is the theory of estimation of periodic signals observed in white Gaussian noise.

The problem of period (or frequency) estimation has particularities which put it in some sense out of traditional $(\sqrt{n})$ statistical framework. Let us remind some known properties of the maximum likelihood estimator (MLE) of the frequency $\vartheta \in (\alpha, \beta)$, $0 < \alpha < \beta < \infty$ obtained by Ibragimov and Khasminskii [9] for the model *signal in white Gaussian noise* (SWN) and some related problems for inhomogeneous Poisson processes.

Suppose that the observed process is

$$dX_t = A \sin(\vartheta t) dt + \sigma dW_t, \quad X_0 = 0, \quad 0 \leq t \leq T$$

(SWN) and we have to estimate the frequency ϑ by the observations $X^T = (X_t, 0 \leq t \leq T)$. We are interested by the properties of estimators in asymptotics of “large samples”: $T \rightarrow \infty$. The Fisher information is

$$I_T(\vartheta) = \frac{A^2}{\sigma^2} \int_0^T t^2 \cos^2(\vartheta t) dt = \frac{A^2 T^3}{3\sigma^2} (1 + o(1))$$

and the MLE $\hat{\vartheta}_T$ is asymptotically normal with the rate $T^{3/2}$, i.e.;

$$T^{3/2} (\hat{\vartheta}_T - \vartheta) \Rightarrow \mathcal{N}\left(0, \frac{3\sigma^2}{A^2}\right), \quad \mathbf{E}_\vartheta (\hat{\vartheta}_T - \vartheta)^2 = \frac{3\sigma^2}{A^2 T^3} (1 + o(1)). \quad (2)$$

Note that if $\beta = \infty$, then the (uniformly in ϑ) consistent estimation is impossible. Even if we allow $\beta_T \rightarrow \infty$, then for $\beta_T < \exp\left\{\left(\frac{A^2}{4\sigma^2} - \varepsilon\right) T\right\}$ (any $\varepsilon > 0$) the MLE is consistent and for $\beta_T > \exp\left\{\left(\frac{A^2}{4\sigma^2} + \varepsilon\right) T\right\}$ the uniformly consistent estimation of ϑ is impossible (see [9], Section 7.1 for exact statements and proofs).

If we consider the problem of parameter estimation by observations

$$dX_t = S(t - \vartheta) dt + \sigma dW_t, \quad X_0 = 0, \quad 0 \leq t \leq T,$$

where $S(\cdot)$ is periodic function of period 1 having a discontinuity at points $\tau_* + k$, $k = 0, 1, 2, \dots$, then the rate of convergence of the MLE $\hat{\vartheta}_T$ is different. Let us denote $S(\tau_* -)$ and $S(\tau_* +)$ the left and right limits, $S(\tau_* +) - S(\tau_* -) = r \neq 0$. Then

$$T (\hat{\vartheta}_T - \vartheta) \Rightarrow \eta, \quad \mathbf{E}_\vartheta (\hat{\vartheta}_T - \vartheta)^2 = \frac{26\sigma^2}{r^2 T^2} (1 + o(1)), \quad (3)$$

where η is a random variable (see [9], Section 7.2).

The similar problems of parameter estimation were considered in [10] for the model of periodic Poisson process. It was supposed that the observed inhomogeneous Poisson process $X^T = (X_t, 0 \leq t \leq T)$ has intensity function

$$S(\vartheta, t) = S(\vartheta t)$$

where $S(t)$ is τ -periodic smooth function. It was shown that the MLE $\hat{\vartheta}_T$ is asymptotically normal with the rate $T^{3/2}$:

$$T^{3/2}(\hat{\vartheta}_T - \vartheta) \implies \mathcal{N}(0, a^2), \quad \mathbf{E}_\vartheta(\hat{\vartheta}_T - \vartheta)^2 = \frac{a^2}{T^3}(1 + o(1)), \quad (4)$$

see [10], Section 2.3 for details.

In the case of discontinuous periodic intensity $S(t - \vartheta)$ (shift parameter) the rate is (like (3)) T , i.e.;

$$T(\hat{\vartheta}_T - \vartheta) \implies \xi, \quad \mathbf{E}_\vartheta(\hat{\vartheta}_T - \vartheta)^2 = \frac{c^2}{T^2}(1 + o(1)). \quad (5)$$

See [10], Section 5.1 for details. Moreover, the problem of frequency estimation of periodic discontinuous intensity function $S(\vartheta t)$ was also considered and it was shown that the rate of convergence of the MLE is T^2 , i.e.; we have the limits

$$T^2(\hat{\vartheta}_T - \vartheta) \implies \zeta, \quad \mathbf{E}_\vartheta(\hat{\vartheta}_T - \vartheta)^2 = \frac{b^2}{T^4}(1 + o(1)). \quad (6)$$

In the present work we consider the problem of frequency estimation in the case of periodic discontinuous trend coefficient of ergodic diffusion process. This work is a continuation of our study of parameter estimation problems for periodic diffusion processes started in [6]-[7]. In all these works we suppose that the observed diffusion process is given by the equation

$$dX_t = [S(\vartheta, t) + b(X_t)]dt + \sigma(X_t) dW_t, \quad X_0, \quad 0 \leq t \leq T, \quad (7)$$

where the function $S(\vartheta, t), t \geq 0$ is (known) periodic of period τ , i.e., $S(\vartheta, t + k\tau) = S(\vartheta, t)$ the functions $b(\cdot)$ and $\sigma(\cdot)$ are known and smooth.

This problem of parameter estimation can be considered as particular case of (1) with “diffusion noise” $n(t) = b(X_t) + \sigma(X_t) \dot{W}_t$. Therefore once more we have a problem of the theory of telecommunication (transmission of signals) but this model of observations is as well interesting in some biological experiments related to membrane potential data sets (see Höpfner [3]). We suppose that the diffusion process has ergodic properties and describe

the asymptotics of the MLE and BE in regular and singular (discontinuous) situations. The existence of periodic solution for Markov processes with periodic coefficients were studied by Khasminskii [3] and the ergodic properties (law of large numbers, periodic invariant density...) used in the present work were obtained by Höpfner and Löcherbach [8].

We have to note that if the diffusion coefficient is a deterministic function, say, $\sigma(x) \equiv \sigma > 0$, then the simple transformation (we suppose always that $b(x)$ is known)

$$Y_t = X_0 - \int_0^t b(X_s) ds$$

reduces the equation (7) to the well known signal in WGN model

$$dY_t = S(\vartheta, t) dt + \sigma dW_t, \quad Y_0 = 0, \quad 0 \leq t \leq T,$$

and for this model all mentioned above problems are already well studied.

Let us denote by $\{\mathbf{P}_\vartheta^{(T)}, \vartheta \in \Theta\}$ the family of measures induced by the solutions of (7) in the measurable space $(\mathcal{C}[0, T], \mathcal{B}[0, T])$ and put

$$L(\vartheta, X^T) = \frac{d\mathbf{P}_\vartheta^{(T)}}{d\mathbf{P}^{(T)}}(X^T),$$

where $\mathbf{P}^{(T)}$ is the measure corresponding to the process (7) with $S(\vartheta, t) \equiv 0$. The likelihood ratio is

$$L(\vartheta, X^T) = \exp \left\{ \int_0^T \frac{S(\vartheta, t)}{\sigma(X_t)^2} dX_t - \int_0^T \frac{S(\vartheta, t)^2 + 2S(\vartheta, t)b(X_t)}{2\sigma(X_t)^2} dt \right\} \quad (8)$$

and the estimators are defined by the usual formulas: for the MLE $\hat{\vartheta}_T$ we have

$$L(\hat{\vartheta}_T, X^T) = \sup_{\theta \in \Theta} L(\theta, X^T), \quad (9)$$

and for Bayesian estimator $\tilde{\vartheta}_T$ we write

$$\tilde{\vartheta}_T = \frac{\int_{\Theta} \theta p(\theta) L(\theta, X^T) d\theta}{\int_{\Theta} p(\theta) L(\theta, X^T) d\theta}, \quad (10)$$

i.e., we suppose that the loss function is quadratic and the density a priory $p(\cdot)$ is given. We study the asymptotic properties of these estimators with the help of the methode by Ibragimov and Khasminskii [9] which consists in the establishing some properties of the normalized likelihood ratio process

$$Z_T(u) = \frac{L(\vartheta + \varphi_T u, X^T)}{L(\vartheta, X^T)}, \quad u \in \mathbb{U}_T = \left[\frac{\alpha - \vartheta}{\varphi_T}, \frac{\beta - \vartheta}{\varphi_T} \right]$$

where ϑ is the true value and the choice of the normalizing function $\varphi_T \rightarrow 0$ depends on the “smoothness” of the problem.

In the first work [6] we supposed that the function $S(\vartheta, t) = S(t - \vartheta)$ and $S(t), t \geq 0$ is periodic function having a jump $r = S(\tau_*+) - S(\tau_*-) \neq 0$ at the point $\tau_* \in (0, \tau)$. It is shown that the choice of the function $\varphi_T = T^{-1}$ provides the limit

$$Z_T(u) \Rightarrow Z(u) = \exp \left\{ \gamma W(u) - \frac{|u|}{2} \gamma^2 \right\}, \quad u \in \mathcal{R} \quad (11)$$

with some constant γ . Here $W(u)$ is double sided Wiener process. The estimators have the following properties. Let us put

$$Z(\hat{u}) = \sup_u Z(u), \quad \tilde{u} = \frac{\int_{-\infty}^{\infty} u Z(u) du}{\int_{-\infty}^{\infty} Z(u) du}, \quad (12)$$

then we can write

$$T(\hat{\vartheta}_T - \vartheta) \Rightarrow \hat{u}, \quad T(\tilde{\vartheta}_T - \vartheta) \Rightarrow \tilde{u}, \quad (13)$$

and convergence of all polynomial moments of these estimators (like (3)) take place (see [6]).

In the second work [5] we considered the usual (regular) estimation problem with smooth periodic function $S(\vartheta, t)$ such that its derivative is periodic too. It is shown that with classical normalization $\varphi_T = T^{-1/2}$ the corresponding family of measures is LAN :

$$Z_T(u) \Rightarrow Z(u) = \exp \left\{ u\Delta - \frac{u^2}{2} I(\vartheta) \right\}, \quad (14)$$

where $I(\vartheta)$ is the Fisher information (on one period) and $\Delta \sim \mathcal{N}(0, I(\vartheta))$. For the estimators we obtain, as usual, asymptotic normality

$$\sqrt{T}(\hat{\vartheta}_T - \vartheta) \Rightarrow \frac{\Delta}{I(\vartheta)}, \quad \sqrt{T}(\tilde{\vartheta}_T - \vartheta) \Rightarrow \frac{\Delta}{I(\vartheta)} \quad (15)$$

and convergence of all polynomial moments

$$\mathbf{E}_{\vartheta} \left(\hat{\vartheta}_T - \vartheta \right)^2 = \frac{1}{T I(\vartheta)} (1 + o(1)).$$

The last work is devoted to the study of the local structure of the family of measures corresponding to the observations (7) with $S(\vartheta, t) = S(\vartheta t)$ where $S(t)$ is a periodic function. We describe the asymptotic behavior of

the normalized likelihood ratio $Z_T(u)$ in two situations: when the periodic function $S(t)$ is smooth and when it is discontinuous. It is shown that in the first case the normalizing function $\varphi_T = T^{-3/2}$ and the limit is like (14) and in the second case $\varphi_T = T^{-2}$ and the limit is like (11). In the present work we describe the properties of the MLE and Bayesian estimators of the frequency in the same two situations. We show that in the smooth case the MLE and BE are asymptotically normal similar to (4) and in the discontinuous case we have convergence like (6) but with the different limit distributions.

2 Main results

The observed diffusion process $X^T = (X_t, 0 \leq t \leq T)$ satisfies the stochastic differential equation

$$dX_t = [S(\vartheta t) + b(X_t)] dt + \sigma(X_t) dW_t, \quad X_0, \quad 0 \leq t \leq T, \quad (16)$$

and we study the properties of estimators of the parameter ϑ in two situations : when the function $S(t)$ is smooth (regular estimation problem) and when the function $S(t)$ has a discontinuity (singular estimation problem).

In both cases we suppose that the following conditions are fulfilled.

A1. *The function $S(t)$ is bounded and periodic of period $\tau = 1$. The functions $b(\cdot), \sigma(\cdot) \in \mathcal{C}_b^3$, i.e.; have three continuous bounded derivatives. The parameter $\vartheta \in (\alpha, \beta) = \Theta$.*

Let us introduce the constants m, M by the relation

$$m \leq S(t) \leq M, \quad t \in [0, 1],$$

and put $S_- = \min(m, 0)$ and $S_+ = \max(M, 0)$.

A2. *There exist constants $A > 0$ and $\varepsilon > 0$ such that for $|x| > A$ we have*

$$2x\mathbb{I}_{\{x < -A\}}S_- + 2x\mathbb{I}_{\{x > A\}}S_+ + 2xb(x) + \sigma(x)^2 < -\varepsilon.$$

Note that by condition **A1** these functions satisfy the global Lipschitz condition

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \leq L |x - y|$$

and the linear growth condition

$$|b(x)| + |\sigma(x)| \leq L_1 (1 + |x|).$$

Therefore the equation (16) has a unique strong solution [12].

A3. *The diffusion coefficient is a bounded function separated from zero : there exist two constants k, K such that*

$$0 < \kappa \leq \sigma(x)^2 \leq K \quad (17)$$

Under conditions **A1,A2,A3** the diffusion process has ergodic properties (Höpfner and Löcherbach, [8]), i.e., there exists an invariant (periodic) density function $f_\vartheta(t, x)$ such that for any absolutely integrable $\tau = 1/\vartheta$ -periodic in time function $h(\vartheta, t, x)$ we have (with probability 1) the following limits

$$\frac{1}{T} \int_0^T h(\vartheta, t, X_t) dt \longrightarrow \frac{1}{\tau} \int_{-\infty}^{\infty} \int_0^{\tau} h(\vartheta, t, x) f_\vartheta(t, x) dt dx, \quad (18)$$

$$\frac{1}{n} \sum_{k=1}^n h(\vartheta, t_* + k\tau, X_{t_*+k\tau}) \longrightarrow \int_{-\infty}^{\infty} h(\vartheta, t_*, x) f_\vartheta(t_*, x) dx. \quad (19)$$

To prove the asymptotic efficiency we need the following uniform law of large numbers.

A4. *The convergence in (18), (8) is uniform on compacts $\mathbb{K} \in \Theta$.*

A sufficient for **A4** condition is given in the Section 3. Below the condition **A= (A1,A2,A3,A4)**.

2.1 Smooth trend

We consider the problem of frequency ϑ estimation by observations X^T of the periodic diffusion process (16).

B. *The periodic function $S(t), t \geq 0$ is nonconstant and continuously differentiable.*

The role of Fisher information in our problem plays the quantity

$$I(\vartheta) = \frac{1}{3\tau} \int_0^{\tau} \dot{S}(\vartheta t)^2 \int_{-\infty}^{\infty} \frac{f_\vartheta(t, x)}{\sigma(x)^2} dx dt,$$

where dot means derivation : $\dot{S}(t) = dS(t)/dt$.

Introduce the lower bound on the meansquare risk of all estimators $\bar{\vartheta}_T$:

$$\lim_{\delta \rightarrow 0} \lim_{T \rightarrow \infty} \sup_{|\vartheta - \vartheta_0| \leq \delta} T^3 \mathbf{E}_\vartheta |\bar{\vartheta}_T - \vartheta|^2 \geq I(\vartheta_0)^{-1}. \quad (20)$$

This is a version of the well-known Hajek-Le Cam lower bound (see for example [9]). We call an estimator ϑ_T^* asymptotically efficient if for all $\vartheta_0 \in \Theta$ we have the equality

$$\lim_{\delta \rightarrow 0} \lim_{T \rightarrow \infty} \sup_{|\vartheta - \vartheta_0| \leq \delta} T^3 \mathbf{E}_\vartheta |\vartheta_T^* - \vartheta|^2 = I(\vartheta_0)^{-1}. \quad (21)$$

Theorem 1 *Let the conditions **A** and **B** be fulfilled. Then the MLE $\hat{\vartheta}_T$ and BE $\tilde{\vartheta}_T$ have the following properties uniformly on compacts $\mathbb{K} \subset \Theta$.*

- *These estimators are consistent: for any $\delta > 0$*

$$\sup_{\vartheta \in \mathbb{K}} \mathbf{P}_{\vartheta} \left\{ \left| \hat{\vartheta}_T - \vartheta \right| > \delta \right\} \rightarrow 0, \quad \sup_{\vartheta \in \mathbb{K}} \mathbf{P}_{\vartheta} \left\{ \left| \tilde{\vartheta}_T - \vartheta \right| > \delta \right\} \rightarrow 0.$$

- *These estimators are asymptotically normal*

$$T^{3/2} \left(\hat{\vartheta}_T - \vartheta \right) \Longrightarrow \zeta, \quad T^{3/2} \left(\tilde{\vartheta}_T - \vartheta \right) \Longrightarrow \zeta, \quad \zeta \sim \mathcal{N} \left(0, \mathbf{I}(\vartheta)^{-1} \right).$$

- *We have the convergence of moments : for any $p > 0$*

$$\lim_{T \rightarrow \infty} T^{\frac{3p}{2}} \mathbf{E}_{\vartheta} \left| \hat{\vartheta}_T - \vartheta \right|^p = \mathbf{E}_{\vartheta} |\zeta|^p, \quad \lim_{T \rightarrow \infty} T^{\frac{3p}{2}} \mathbf{E}_{\vartheta} \left| \tilde{\vartheta}_T - \vartheta \right|^p = \mathbf{E}_{\vartheta} |\zeta|^p,$$

- *The both estimators are asymptotically efficient in the sense (21) .*

Proof. Let us introduce the normalized likelihood ratio

$$Z_T(u) = \frac{L(\vartheta + T^{-3/2}u, X^T)}{L(\vartheta, X^T)}, \quad u \in \mathbb{U}_T = (T^{3/2}(\alpha - \vartheta), T^{3/2}(\beta - \vartheta)).$$

According to (8) it has the form (below $\vartheta_u = \vartheta + T^{-3/2}u$)

$$\ln Z_T(u) = \int_0^T \frac{S(\vartheta_u t) - S(\vartheta t)}{\sigma(X_t)} dW_t - \frac{1}{2} \int_0^T \left(\frac{S(\vartheta_u t) - S(\vartheta t)}{\sigma(X_t)} \right)^2 dt.$$

This proces can be written as

$$\ln Z_T(u) = \frac{u}{T^{3/2}} \int_0^T \frac{t \dot{S}(\vartheta t)}{\sigma(X_t)} dW_t - \frac{u^2}{2T^3} \int_0^T \left(\frac{t \dot{S}(\vartheta t)}{\sigma(X_t)} \right)^2 dt + r_T.$$

It was shown (see [7]) that

$$\frac{1}{T^3} \int_0^T \left(\frac{t \dot{S}(\vartheta t)}{\sigma(X_t)} \right)^2 dt \longrightarrow \mathbf{I}(\vartheta), \quad r_T \rightarrow 0$$

and

$$\Delta_T(\vartheta) = \frac{1}{T^{3/2}} \int_0^T \frac{t \dot{S}(\vartheta t)}{\sigma(X_t)} dW_t \Longrightarrow \Delta(\vartheta) \sim \mathcal{N}(0, \mathbf{I}(\vartheta)).$$

Moreover, as we suppose uniform law of large numbers (8), this convergence is uniform on compacts $\mathbb{K} \subset \Theta$. Therefore, if we introduce the random process (see (14))

$$Z(u) = \exp \left\{ u \Delta(\vartheta) - \frac{u^2}{2} \mathbf{I}(\vartheta) \right\}, \quad u \in \mathcal{R},$$

then the following result take place.

Lemma 1 *The finite dimensional distributions of the random process $Z_T(\cdot)$ converge to the finite dimensional distributions of the process $Z(\cdot)$ uniformly in $\vartheta \in \mathbb{K}$.*

For the proof see [7], Theorem 1.1. Just note that in [7] we do not supposed the uniformity of this convergence and at present it follows from the uniform law of large numbers.

Lemma 2 *For any $R > 0$ the following inequaulity holds*

$$\sup_{\vartheta \in \mathbb{K}} \sup_{|u_1|+|u_2| \leq R} |u_2 - u_1|^{-2} \mathbf{E}_{\vartheta} \left| Z_T^{1/2}(u_2) - Z_T^{1/2}(u_1) \right|^2 \leq C (1 + R^2). \quad (22)$$

Proof. Let us put $\vartheta_1 = \vartheta + T^{-3/2}u_1$, $\vartheta_2 = \vartheta + T^{-3/2}u_2$ and $\delta(t, x)$ is defined below in (27). Then the estimate (28) with $m = 1$ allows us to write

$$\begin{aligned} & \mathbf{E}_{\vartheta} \left| Z_T^{1/2}(u_2) - Z_T^{1/2}(u_1) \right|^2 \\ & \leq C_1 \mathbf{E}_{\vartheta_1} \left(\int_0^T V_t \delta(t, X_t)^2 dt \right)^2 + C_2 \mathbf{E}_{\vartheta_1} \int_0^T V_t^2 \delta(t, X_t)^2 dt \\ & \leq C_1 T \int_0^T \mathbf{E}_{\vartheta_1} V_t^2 \delta(t, X_t)^4 dt + C_2 \int_0^T \mathbf{E}_{\vartheta_1} V_t^2 \delta(t, X_t)^2 dt \\ & = C_1 T \int_0^T \mathbf{E}_{\vartheta_2} \delta(t, X_t)^4 dt + C_2 \int_0^T \mathbf{E}_{\vartheta_2} \delta(t, X_t)^2 dt. \end{aligned}$$

As the derivative of the function $S(t)$ is bounded and we have (17) we can write

$$T \int_0^T \mathbf{E}_{\vartheta_2} \delta(t, X_t)^4 dt \leq CT (u_2 - u_1)^4 T^{-6} \int_0^T t^4 dt \leq C |u_2 - u_1|^4$$

and similary

$$\int_0^T \mathbf{E}_{\vartheta_2} \delta(t, X_t)^2 dt \leq C (u_2 - u_1)^2 T^{-3} \int_0^T t^2 dt \leq C |u_2 - u_1|^2.$$

Therefore this lemma is proved.

Lemma 3 For sufficiently large T we have

$$\sup_{\vartheta \in \mathbb{K}} \mathbf{E}_{\vartheta} Z_T^{1/2}(u) \leq e^{-\kappa|u|^{2/3}} \quad (23)$$

Proof. Let us put $\vartheta_2 = \vartheta + T^{-3/2}u$, $\vartheta_1 = \vartheta$. We can write

$$\begin{aligned} & \mathbf{E}_{\vartheta} Z_T^{1/2}(u) \\ &= \mathbf{E}_{\vartheta} \exp \left\{ \int_0^T \frac{\delta(t, X_t)}{2} dW_t - \int_0^T \frac{\delta(t, X_t)^2}{8} dt - \int_0^T \frac{\delta(t, X_t)^2}{8} dt \right\} \\ &\leq \exp \left\{ -\frac{1}{8K} \int_0^T [S(\vartheta t + T^{-3/2}ut) - S(\vartheta t)]^2 dt \right\} \end{aligned}$$

because

$$\mathbf{E}_{\vartheta} \exp \left\{ \int_0^T \frac{\delta(t, X_t)}{2} dW_t - \frac{1}{2} \int_0^T \frac{\delta(t, X_t)^2}{4} dt \right\} = 1$$

and by condition (17) we have as well

$$\int_0^T \delta(t, X_t)^2 dt \geq \frac{1}{K} \int_0^T [S(\vartheta t + T^{-3/2}ut) - S(\vartheta t)]^2 dt.$$

For the last integral according to (29) we have ($z = \vartheta^{-1}T^{-1/2}u$)

$$\int_0^T [S(\vartheta t + T^{-3/2}ut) - S(\vartheta t)]^2 dt \geq cT \frac{\frac{u^2}{\vartheta^2 T}}{1 + \frac{u^2}{\vartheta^2 T}}.$$

Further, if $u^2 \leq \vartheta^2 T$, then

$$T \frac{\frac{u^2}{\vartheta^2 T}}{1 + \frac{u^2}{\vartheta^2 T}} \geq \frac{u^2}{2\vartheta^2},$$

and if $u^2 > \vartheta^2 T$, then

$$T \frac{\frac{u^2}{\vartheta^2 T}}{1 + \frac{u^2}{\vartheta^2 T}} \geq \frac{T}{2} \geq \frac{|u|^{2/3}}{2(\beta - \alpha)^{2/3}}$$

because $|u| \leq T^{3/2}(\beta - \alpha)$. Therefore

$$\frac{1}{8K} \int_0^T \delta(t, X_t)^2 dt \geq \kappa |u|^{2/3}$$

with some positive κ .

The properties of the likelihood ratio process established in Lemmas 1-3 allow us to apply the Theorems 3.1.1, 3.1.3 and 3.2.1 in [9] and to obtain all properties of the MLE and BE announced in the Theorem 1.

2.2 Discontinuous trend

We have the same model of observed periodic diffusion process (16) but the function $S(t), t \geq 0$ is now discontinuous. More precisely, the following condition holds.

C. *The function $S(\cdot)$ is periodic with period 1, is continuously differentiable everywhere except the points $\tau_* + k$ ($\tau_* \in (0, 1), k = 0, 1, 2, \dots$) and at the points $\tau_* + k$ it has the left and right limits $S(\tau_* -)$ and $S(\tau_* +)$ respectively, $S(\tau_* +) - S(\tau_* -) = r \neq 0$.*

The likelihood ratio random function $L(\vartheta, X^T), \vartheta \in \Theta$ is continuous with probability 1 (see Lemma 5 below), hence the solution of equation (9) exists and the both estimators $\hat{\vartheta}_T$ and $\tilde{\vartheta}_T$ are well defined.

The limit distributions of these estimators are described with the help of the random variables $\hat{u}$ and $\tilde{u}$ defined in (12) where $Z(u)$ is given by (11) with

$$\gamma^2 = [S(\tau_* +) - S(\tau_* -)]^2 \int_{-\infty}^{\infty} \frac{f_{\vartheta}(\tau_*, x)}{2\sigma(x)^2} dx.$$

The lower bound on the meansquare risk of all estimators is similar to (20) :

$$\lim_{\delta \rightarrow 0} \lim_{T \rightarrow \infty} \sup_{|\vartheta - \vartheta_0| \leq \delta} T^4 \mathbf{E}_{\vartheta} |\bar{\vartheta}_T - \vartheta|^2 \geq \mathbf{E}_{\vartheta_0} \tilde{u}^2.$$

For the proof of more general result see [9], Section 1.9. In our case this inequality can be proved in three lines if we suppose that we have already proved the uniform convergence of moments of the BE

$$T^4 \mathbf{E}_{\vartheta} |\tilde{\vartheta}_T - \vartheta|^2 \longrightarrow \mathbf{E}_{\vartheta} |\tilde{u}|^2$$

(Theorem 2 below) as follows. Let us denote $p_{\delta}(\theta), \theta \in [\vartheta_0 - \delta, \vartheta_0 + \delta]$ a density function and $\tilde{\theta}_T$ the corresponding bayesian estimator. Then we can write

$$\begin{aligned} \sup_{|\vartheta - \vartheta_0| \leq \delta} T^4 \mathbf{E}_{\vartheta} |\bar{\vartheta}_T - \vartheta|^2 &\geq T^4 \int_{\vartheta_0 - \delta}^{\vartheta_0 + \delta} \mathbf{E}_{\vartheta} |\bar{\vartheta}_T - \theta|^2 p_{\delta}(\theta) d\theta \\ &\geq T^4 \int_{\vartheta_0 - \delta}^{\vartheta_0 + \delta} \mathbf{E}_{\vartheta} |\tilde{\theta}_T - \theta|^2 p_{\delta}(\theta) d\theta \longrightarrow \int_{\vartheta_0 - \delta}^{\vartheta_0 + \delta} \mathbf{E}_{\vartheta} |\tilde{u}|^2 p_{\delta}(\theta) d\theta = \mathbf{E}_{\vartheta_0} |\tilde{u}|^2. \end{aligned}$$

This bound allows us to call an estimator ϑ_T^* asymptotically efficient if for all $\vartheta_0 \in \Theta$ we have the equality

$$\lim_{\delta \rightarrow 0} \lim_{T \rightarrow \infty} \sup_{|\vartheta - \vartheta_0| \leq \delta} T^4 \mathbf{E}_{\vartheta} |\vartheta_T^* - \vartheta|^2 = \mathbf{E}_{\vartheta_0} |\tilde{u}|^2. \quad (24)$$

Remind that the last value is known : $\mathbf{E}_\vartheta |\tilde{u}|^2 \approx 19,3 \gamma^{-4}$ [13] and is less than the similar quantity for the MLE $\mathbf{E}_\vartheta |\hat{u}|^2 = 26 \gamma^{-4}$ [14].

Theorem 2 *Let the conditions **A** and **C** be fulfilled. Then the MLE $\hat{\vartheta}_T$ and BE $\tilde{\vartheta}_T$ have the following properties uniformly on compacts $\mathbb{K} \subset \Theta$:*

- *The both estimators are consistent.*
- *They have different limit distributions*

$$T^2 \left(\hat{\vartheta}_T - \vartheta \right) \Longrightarrow \hat{u}, \quad T^2 \left(\tilde{\vartheta}_T - \vartheta \right) \Longrightarrow \tilde{u}.$$

- *The convergence of moments take place : for any $p > 0$*

$$\lim_{T \rightarrow \infty} T^{2p} \mathbf{E}_\vartheta \left| \hat{\vartheta}_T - \vartheta \right|^p = \mathbf{E}_\vartheta |\hat{u}|^p, \quad \lim_{T \rightarrow \infty} T^{2p} \mathbf{E}_\vartheta \left| \tilde{\vartheta}_T - \vartheta \right|^p = \mathbf{E}_\vartheta |\tilde{u}|^p,$$

- *BE are asymptotically efficient in the sense (24).*

Proof. The normalized likelihood ratio is now

$$Z_T(u) = \frac{L(\vartheta + T^{-2}u, X^T)}{L(\vartheta, X^T)}, \quad u \in \mathbb{U}_T = (T^2(\alpha - \vartheta), T^2(\beta - \vartheta)).$$

We have the following result (see [7], Theorem 1.2)

Lemma 4 *The finite dimensional distributions of the random process $Z_T(u)$ converge to the finite dimensional distributions of the process $Z(u)$ uniformly on compacts $\mathbb{K} \in \Theta$.*

To explain this convergence we can write the log-likelihood ratio as follows:
(below $\vartheta_u = \vartheta + T^{-2}u, u > 0$)

$$\begin{aligned} \ln Z_T(u) &= \int_0^T \frac{S(\vartheta_u t) - S(\vartheta t)}{\sigma(X_t)} dW_t - \frac{1}{2} \int_0^T \left(\frac{S(\vartheta_u t) - S(\vartheta t)}{\sigma(X_t)} \right)^2 dt \\ &= \sum_{k=0}^{[T\vartheta]} \int_{\frac{\tau_*+k}{\vartheta_u}}^{\frac{\tau_*+k}{\vartheta}} \frac{S(\vartheta_u t) - S(\vartheta t)}{\sigma(X_t)} dW_t - \sum_{k=0}^{[T\vartheta]} \int_{\frac{\tau_*+k}{\vartheta_u}}^{\frac{\tau_*+k}{\vartheta}} \frac{(S(\vartheta_u t) - S(\vartheta t))^2}{2\sigma(X_t)^2} dt + o(1) \\ &= \sum_{k=0}^{[T\vartheta]} \left[r \int_{\frac{\tau_*+k}{\vartheta_u}}^{\frac{\tau_*+k}{\vartheta}} \frac{dW_t}{\sigma(X_t)} - \frac{r^2}{2} \int_{\frac{\tau_*+k}{\vartheta_u}}^{\frac{\tau_*+k}{\vartheta}} \frac{dt}{\sigma(X_t)^2} \right] + o(1) \Longrightarrow \gamma W(u) - \frac{\gamma^2}{2} |u| \end{aligned}$$

because

$$\begin{aligned} \sum_{k=0}^{[T\vartheta]} \int_{\frac{\tau_*+k}{\vartheta}}^{\frac{\tau_*+k+1}{\vartheta}} \frac{dt}{\sigma(X_t)^2} &= \frac{u}{\vartheta^2 T^2} \sum_{k=0}^{[T\vartheta]} \frac{k}{\sigma\left(X_{\frac{\tau_*+k}{\vartheta}}\right)^2} + o(1) \\ &\rightarrow \frac{u}{2} \int_{-\infty}^{\infty} \frac{f_{\vartheta}\left(\frac{\tau_*}{\vartheta}, x\right)}{\sigma(x)^2} dx = \frac{u}{2} \int_{-\infty}^{\infty} \frac{f_1(\tau_*, x)}{\sigma(x)^2} dx \end{aligned}$$

Here $[T\vartheta]$ is the integer part of $T\vartheta$. The asymptotic normality of the stochastic integral follows from this convergence (central limit theorem). For the details see [7].

Lemma 5 *The following inequality holds*

$$\sup_{\vartheta \in \mathbb{K}} \mathbf{E}_{\vartheta} \left| Z_T^{1/4}(u_2) - Z_T^{1/4}(u_1) \right|^4 \leq C |u_2 - u_1|^2 \quad (25)$$

Proof. According to (28) with $m = 2$ we have

$$\begin{aligned} \mathbf{E}_{\vartheta} \left| Z_T^{1/4}(u_2) - Z_T^{1/4}(u_1) \right|^4 \\ \leq C_1 \mathbf{E}_{\vartheta_1} \left(\int_0^T V_t \delta(t, X_t)^2 dt \right)^4 + C_2 \mathbf{E}_{\vartheta_1} \left(\int_0^T V_t^2 \delta(t, X_t)^2 dt \right)^2. \end{aligned}$$

Let us put $\vartheta_{u_1} = \vartheta + T^{-2}u_1$, $\vartheta_{u_2} = \vartheta + T^{-2}u_2$, $N = [T\vartheta_{u_1}]$ and consider the case $0 < u_1 < u_2$. Then we can write

$$\int_0^T V_t \delta(t, X_t)^2 dt = \sum_{k=0}^{N-1} \left[\int_{\frac{k}{\vartheta_{u_1}}}^{\frac{\tau_*+k}{\vartheta_{u_2}}} + \int_{\frac{\tau_*+k}{\vartheta_{u_2}}}^{\frac{\tau_*+k}{\vartheta_{u_1}}} + \int_{\frac{\tau_*+k}{\vartheta_{u_1}}}^{\frac{k+1}{\vartheta_{u_1}}} \right] V_t \delta(t, X_t)^2 dt.$$

The function $\delta(t, X_t)^2$ on the intervals

$$\left[\frac{k}{\vartheta_{u_1}}, \frac{\tau_*+k}{\vartheta_{u_2}} \right] \quad \text{and} \quad \left[\frac{\tau_*+k}{\vartheta_{u_1}}, \frac{k+1}{\vartheta_{u_1}} \right]$$

is continuously differentiable on ϑ and therefore is majorated as follows

$$\delta(t, X_t)^2 \leq C t^2 \frac{(u_2 - u_1)^2}{T^4}.$$

Further, we have

$$\int_{\frac{\tau_*+k}{\vartheta_{u_2}}}^{\frac{\tau_*+k}{\vartheta_{u_1}}} V_t \delta(t, X_t)^2 dt \leq C [S(\tau_*) - S(\tau_*)]^2 \int_{\frac{\tau_*+k}{\vartheta_{u_2}}}^{\frac{\tau_*+k}{\vartheta_{u_1}}} V_t dt.$$

Hence

$$\begin{aligned}
\mathbf{E}_{\vartheta_1} \left(\sum_{k=0}^{N-1} \int_{\frac{\tau_*+k}{\vartheta u_2}}^{\frac{\tau_*+k}{\vartheta u_1}} V_t dt \right)^4 &\leq C N^3 \sum_{k=0}^{N-1} \mathbf{E}_{\vartheta_1} \left(\int_{\frac{\tau_*+k}{\vartheta u_2}}^{\frac{\tau_*+k}{\vartheta u_1}} V_t dt \right)^4 \\
&\leq C N^3 \sum_{k=0}^{N-1} \frac{(\tau_* + k)^3 (u_2 - u_1)^3}{T^6} \int_{\frac{\tau_*+k}{\vartheta u_2}}^{\frac{\tau_*+k}{\vartheta u_1}} \mathbf{E}_{\vartheta_1} V_t^4 dt \\
&\leq C N^3 \sum_{k=0}^{N-1} \frac{(\tau_* + k)^4 (u_2 - u_1)^4}{T^8} \leq C (u_2 - u_1)^4.
\end{aligned}$$

Remind that $\mathbf{E}_{\vartheta_1} V_t^4 = 1$. For the second integral the similar arguments provide

$$\mathbf{E}_{\vartheta_1} \left(\int_0^T V_t^2 \delta(t, X_t)^2 dt \right)^2 \leq C (u_2 - u_1)^2.$$

Now (25) follows from the last two estimates.

Lemma 6 *For sufficiently large T we have*

$$\sup_{\vartheta \in \mathbb{K}} \mathbf{E}_{\vartheta} Z_T^{1/2}(u) \leq e^{-\kappa|u|^{1/2}} \quad (26)$$

Proof. Following the proof of Lemma 3 (with $\vartheta_u = \vartheta + T^{-2}u$) we obtain the estimates

$$\mathbf{E}_{\vartheta} Z_T^{1/2}(u) \leq \exp \left\{ -\frac{1}{8K} \int_0^T [S(\vartheta t + T^{-2}ut) - S(\vartheta t)]^2 dt \right\}$$

and

$$\int_0^T \delta(t, X_t)^2 dt \geq \frac{1}{K} \int_0^T [S(\vartheta t + T^{-2}ut) - S(\vartheta t)]^2 dt.$$

Further, the estimate (30) allows us to write ($z = \vartheta^{-1}T^{-1}u$)

$$\int_0^T [S(\vartheta t + T^{-2}ut) - S(\vartheta t)]^2 dt \geq cT \frac{\frac{|u|}{\vartheta T}}{1 + \frac{|u|}{\vartheta T}}.$$

If $|u| \leq \vartheta T$, then

$$T \frac{\frac{|u|}{\vartheta T}}{1 + \frac{|u|}{\vartheta T}} \geq \frac{|u|}{2\vartheta},$$

and if $|u| > \vartheta T$, then

$$T \frac{\frac{|u|}{\vartheta T}}{1 + \frac{|u|}{\vartheta T}} \geq \frac{T}{2} \geq \frac{|u|^{1/2}}{2(\beta - \alpha)^{1/2}}$$

because $|u| \leq T^2(\beta - \alpha)$. Therefore

$$\frac{1}{8K} \int_0^T \delta(t, X_t)^2 dt \geq \kappa |u|^{1/2}$$

with some positive κ .

The convergence of the finite-dimensional distributions of the random function $Z_T(\cdot)$ together with (25) and (26) allow us to cite the Theorems 1.10.1, 1.10.2, where the mentioned in the Theorem 2 properties of estimators are proved.

3 Auxiliary results

Two lemmæ. We remind here one estimate for the increaments of the likelihood ratio and two lemmæ which allowed us to prove the Lemma 2,3,5,6.

Let us introduce three diffusion processes

$$dX_t = [S(\vartheta_i t) + b(X_t)] dt + \sigma(X_t) dW_t, \quad X_0, 0 \leq t \leq T, \quad i = 0, 1, 2,$$

and denote by $\mathbf{P}_{\vartheta_i}^{(T)}, i = 0, 1, 2$, the corresponding measures induced by these processes in $(\mathcal{C}[0, T], \mathcal{B}[0, T])$. The Radon-Nikodym derivatives are denoted as

$$Z_i = \frac{d\mathbf{P}_{\vartheta_i}^{(T)}}{d\mathbf{P}_{\vartheta_0}^{(T)}}(X^T), \quad i = 1, 2, \quad V_t = \left(\frac{d\mathbf{P}_{\vartheta_2}^{(t)}}{d\mathbf{P}_{\vartheta_1}^{(t)}}(X^t) \right)^{1/2m}$$

where $m \geq 1$ is some integer. Below we put

$$\delta(t, x) = \frac{S(\vartheta_2 t) - S(\vartheta_1 t)}{\sigma(x)}. \quad (27)$$

Remind that we suppose (17) and that the function $S(t), t \geq 0$ is bounded, hence the function $\delta(t, x)$ is bounded too and by Lemma 1.13 in [11] we have the following result.

There exist constants $C_1(m)$ and $C_2(m)$ such that

$$\begin{aligned} \mathbf{E}_{\vartheta_0} \left| Z_2^{1/2m} - Z_1^{1/2m} \right|^{2m} &\leq C_1(m) \mathbf{E}_{\vartheta_1} \left(\int_0^T V_t \delta(t, X_t)^2 dt \right)^{2m} \\ &\quad + C_2(m) \mathbf{E}_{\vartheta_1} \left(\int_0^T V_t^2 \delta(t, X_t)^2 dt \right)^m \end{aligned} \quad (28)$$

The exponential decreasing of the tails of $Z_T(u)$ are verified with the help of the following two lemmas.

Lemma 7 (Ibragimov and Khasminskii) *Let $h(t)$ be a nonconstant continuously differentiable periodic function. Then for all T sufficiently large and for some constant $c > 0$, the inequality*

$$\frac{1}{T} \int_0^T \left[h\left(t + \frac{z}{T}t\right) - h(t) \right]^2 dt \geq c \frac{z^2}{1 + z^2} \quad (29)$$

is valid.

For the proof see [9], Lemma 3.5.3.

The similar result for discontinuous function is given in the following Lemma.

Lemma 8 *Let $S(t)$ satisfies the condition **B**. Then for all T sufficiently large and for some constant $c > 0$, the inequality*

$$\frac{1}{T} \int_0^T \left[S\left(t + \frac{z}{T}t\right) - S(t) \right]^2 dt \geq c \frac{|z|}{1 + |z|} \quad (30)$$

is valid.

The proof of this lemma is a modification of the proof of lemma 7, which can be found in [10], Lemma 5.7.

On uniform convergence. The uniform in $\vartheta \in \Theta$ convergence (condition **A4**)

$$\frac{1}{T} \int_0^T h(\vartheta, t, X_t) dt \longrightarrow \frac{1}{\tau} \int_{-\infty}^{\infty} \int_0^{\tau} h(\vartheta, t, x) f_{\vartheta}(t, x) dt dx \equiv A(\vartheta).$$

means, that for any $\varepsilon > 0$ we have

$$\sup_{\vartheta \in \Theta} \mathbf{P}_{\vartheta} \left\{ \left| \frac{1}{T} \int_0^T [h(\vartheta, t, X_t) - A(\vartheta)] dt \right| > \varepsilon \right\} \longrightarrow 0. \quad (31)$$

Let us denote by $H(\vartheta, t, x)$ the solution of the following equation

$$\frac{\partial H}{\partial t} + [S(\vartheta t) + b(x)] \frac{\partial H}{\partial x} + \frac{\sigma(x)^2}{2} \frac{\partial^2 H}{\partial x^2} = h(\vartheta, t, x) - A(\vartheta)$$

Then we can write

$$\begin{aligned} \frac{1}{T} \int_0^T [h(\vartheta, t, X_t) - A(\vartheta)] dt &= \frac{H(\vartheta, T, X_T) - H(\vartheta, 0, X_0)}{T} \\ &\quad - \frac{1}{T} \int_0^T H'_x(\vartheta, t, X_t) \sigma(X_t) dW_t \end{aligned}$$

Hence

$$\begin{aligned} \mathbf{E}_\vartheta \left(\frac{1}{T} \int_0^T [h(\vartheta, t, X_t) - A(\vartheta)] dt \right)^2 &\leq 2 \frac{\mathbf{E}_\vartheta (H(\vartheta, T, X_T) - H(\vartheta, 0, X_0))^2}{T^2} \\ &+ \frac{2}{T^2} \int_0^T \mathbf{E}_\vartheta (H'_x(\vartheta, t, X_t) \sigma(X_t))^2 dt \end{aligned}$$

It is sufficient to suppose that the last expectations are bounded uniformly in $\vartheta \in \Theta$ and apply in (31) the Tchebyshev inequality.

4 Discussion

Choice of the signal. Let us consider the equation (7) with two types of modulation : p) phase $S(\vartheta, t) = S(t - \vartheta)$ and f) frequency $S(\vartheta, t) = S(\vartheta t)$ in two situations : smooth and discontinuous. Then from the results obtained in [5]-[7] and presented in this work it follows that we have four problems with four different rates

$$\begin{aligned} \text{smooth} \quad (p) \quad \mathbf{E}_\vartheta \left(\hat{\vartheta}_T - \vartheta \right)^2 &\sim \frac{c}{T}, & (f) \quad \mathbf{E}_\vartheta \left(\hat{\vartheta}_T - \vartheta \right)^2 &\sim \frac{c}{T^3} \\ \text{discontinuous} \quad (p) \quad \mathbf{E}_\vartheta \left(\hat{\vartheta}_T - \vartheta \right)^2 &\sim \frac{c}{T^2}, & (f) \quad \mathbf{E}_\vartheta \left(\hat{\vartheta}_T - \vartheta \right)^2 &\sim \frac{c}{T^4}. \end{aligned}$$

It is natural to ask: *how far can we go in the rate of convergence? What is the best choice of the signal and what is the best rate?*

The similar statement for the *signal in white Gaussian noise* problem

$$dX_t = S(\vartheta, t) dt + dW_t, \quad X_0 = 0, \quad 0 \leq t \leq T, \quad \vartheta \in [0, 1]$$

was considered by M. Burnashev [1]. It was shown that for signals satisfying

$$\frac{1}{T} \int_0^T S(\vartheta, t)^2 dt \leq L \tag{32}$$

the best choice yields ($T \rightarrow \infty$)

$$\inf_{S, \hat{\vartheta}_T} \sup_{\vartheta \in [0, 1]} \mathbf{E}_\vartheta \left(\hat{\vartheta}_T - \vartheta \right)^2 = \exp \left\{ -\frac{L}{6} T (1 + o(1)) \right\}.$$

Therefore the rate can be even exponential. The similar result was obtained for inhomogeneous Poisson processes too [2]. It follows that if the diffusion coefficient $\sigma(x)^2 \equiv 1$ and the signal $S(\vartheta, t)$ in the equation (7) satisfies the condition (32), then we have the same result with exponential rate.

Generalisations. There are several generalisations which can be done by the direct calculations similar to one given above.

If the function $S(t)$ has jumps in k points $0 < \tau_1, \dots, \tau_k < \tau$ and is continuously differentiable between these points, then the estimators have the same asymptotic properties as described in the Theorem 2 but the constant is

$$\gamma^2 = \sum_{l=1}^k \int \frac{[S(\tau_l+) - S(\tau_l-)]^2}{\sigma(x)^2} f_{\vartheta}(\tau_l + \vartheta, x) dx.$$

The problem becomes a bit more complicate if

$$dX_t = \sum_{l=1}^k S_l(t - \vartheta_l) dt + b(X_t) dt + \sigma(X_t) dW_t$$

and $\vartheta = (\vartheta_1, \dots, \vartheta_k)$ but can be done too. The limit likelihood ratio is a product of k one-dimensional likelihood ratios. See the details in [10], where the similar problems were considered for periodic Poisson processes.

Three-dimensional parameter. In both problems studied above we supposed that the unknown parameter is one-dimensional. It is interesting to see the properties of estimators, say, in smooth case when the we have three dimensional parameter $\vartheta = (\rho, \omega, \varphi)$ in the model of observations

$$dX_t = \rho \sin(2\pi\omega t + \varphi) dt + b(X_t)dt + \sigma(X_t) dW_t, \quad X_0 = 0, 0 \leq t \leq T,$$

i.e.: we have to estimate the amplitude ρ , frequency ω and phase φ of the signal $S(\vartheta, t) = \rho \sin(\omega t + \varphi)$.

The functions $b(\cdot)$ and $\sigma(\cdot)$ satisfy the condition **A**. This problem of parameter estimation is regular and the technique developped in this work together with calculus presented in Example 4 Section 3.5 in [9] allows us to show that the MLE $\hat{\vartheta}_T$ is consistent and asymptotically normal :

$$\sqrt{T}(\hat{\rho}_T - \rho) \Longrightarrow \eta, \quad T^{3/2}(\hat{\omega}_T - \omega) \Longrightarrow \xi, \quad \sqrt{T}(\hat{\varphi}_T - \varphi) \Longrightarrow \zeta$$

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