On n-partite superactivation of quantum channel capacities

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Abstract

A generalization of the superactivation of quantum channel capacities to the case of n > 2 channels is considered. An explicit example of such superactivation for the 1-shot quantum zero-error capacity is constructed for n = 3.

Some implications of this example and its reformulation on terms of quantum measurements are described.

1 General observations

The superactivation of quantum channel capacities is one of the most impressive quantum effects having no classical counterpart. It means that the particular capacity C of the tensor product of two quantum channels Φ_1 and Φ_2 may be positive despite the same capacity of each of these channels is zero, i.e.

$$C(\Phi_1 \otimes \Phi_2) > 0 \quad \text{while} \quad C(\Phi_1) = C(\Phi_2) = 0. \tag{1}$$

This effect was originally observed by G.Smith and J.Yard for the case of quantum ε -error capacity [12]. Then the possibility of superactivation of other capacities, in particular, zero-error capacities was shown [2, 3, 4, 11].

It seems reasonable to consider the generalization of the above effect to the case of n > 2 channels $\Phi_1, ..., \Phi_n$ consisting in the following property

$$C(\Phi_1 \otimes \ldots \otimes \Phi_n) > 0$$
 while $C(\Phi_{i_1} \otimes \ldots \otimes \Phi_{i_k}) = 0$ (2)

for any proper subset $\Phi_{i_1}, ..., \Phi_{i_k}$ (k < n) of the set $\Phi_1, ..., \Phi_n$. This property can be called *n*-partite superactivation of the capacity C.

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Property (2) means that all the channels $\Phi_1, ..., \Phi_n$ are required to transmit (classical or quantum) information by using the protocol corresponding to the capacity C, i.e. excluding any channel from the set $\Phi_1, ..., \Phi_n$ makes other channels useless for information transmission.

The obvious difficulty in finding channels $\Phi_1, ..., \Phi_n$ demonstrating property (2) for given capacity C consists in necessity to prove the vanishing of $C(\Phi_{i_1} \otimes ... \otimes \Phi_{i_k})$ for any subset $\Phi_{i_1}, ..., \Phi_{i_k}$.

If C is the 1-shot capacity of some protocol of information transmission and $\Phi_i = \Phi$ for all $i = \overline{1, n}$ then (2) means that the n-shot capacity of this protocol is positive while the corresponding (n - 1)-shot capacity is zero.

In [10] it is shown how to construct for any n a channel Ψ_n such that

$$\bar{Q}_0(\Psi_n^{\otimes n}) = 0 \quad \text{and} \quad \bar{Q}_0(\Psi_n^{\otimes m}) > 0, \tag{3}$$

where Q_0 is the 1-shot quantum zero-error capacity and m is a natural number satisfying the inequality $n/m \leq 2\ln(3/2)/\pi$ (implying m > n). It follows that there is $\tilde{n} > n$ not exceeding m such that (2) holds for $n = \tilde{n}$, $C = \bar{Q}_0$ and $\Phi_1 = \ldots = \Phi_{\tilde{n}} = \Psi_n$. Unfortunately, we can not specify the number \tilde{n} in that construction.

In this paper we modify the example in [10] (by extending its noncommutative graph) to construct a family of channels $\{\Phi_{\theta}\}$ with $d_A = 4$ and $d_E = 3$ having the following property

$$\bar{Q}_0(\Phi_{\theta_1} \otimes \Phi_{\theta_2} \otimes \Phi_{\theta_3}) > 0 \quad \text{while} \quad \bar{Q}_0(\Phi_{\theta_i} \otimes \Phi_{\theta_j}) = 0 \quad \forall i \neq j, \qquad (4)$$

where $\theta_1, \theta_2, \theta_3$ are positive numbers such that $\theta_1 + \theta_2 + \theta_3 = \pi$. Thus, the channels $\Phi_{\theta_1}, \Phi_{\theta_2}, \Phi_{\theta_3}$ demonstrate the 3-partite superactivation of the 1-shot quantum zero-error capacity.

Property (4) means that all the channels Φ_{θ_i} and all the bipartite channels $\Phi_{\theta_i} \otimes \Phi_{\theta_j}$ have no ideal (noiseless or reversible) subchannels, but the tripartite channel $\Phi_{\theta_1} \otimes \Phi_{\theta_2} \otimes \Phi_{\theta_3}$ has.

By using the observation in [9, Section 4] superactivation property (4) can be reformulated in terms of quantum measurements theory as the existence of quantum observables $\mathcal{M}_{\theta_1}, \mathcal{M}_{\theta_2}, \mathcal{M}_{\theta_3}$ such that all the observables \mathcal{M}_{θ_i} and all the bipartite observables $\mathcal{M}_{\theta_i} \otimes \mathcal{M}_{\theta_j}$ have no indistinguishable subspaces but the tripartite observable $\mathcal{M}_{\theta_1} \otimes \mathcal{M}_{\theta_2} \otimes \mathcal{M}_{\theta_3}$ has.

2 Preliminaries

Let $\Phi : \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ be a quantum channel, i.e. a completely positive trace-preserving linear map [6, 8]. It has the Kraus representation

$$\Phi(\rho) = \sum_{k} V_k \rho V_k^*,\tag{5}$$

where V_k are linear operators from \mathcal{H}_A into \mathcal{H}_B such that $\sum_k V_k^* V_k = I_{\mathcal{H}_A}$. The minimal number of summands in such representation is called *Choi rank* of Φ and is denoted d_E (while $d_A \doteq \dim \mathcal{H}_A$ and $d_B \doteq \dim \mathcal{H}_B$).

The 1-shot quantum zero-error capacity $\bar{Q}_0(\Phi)$ of a channel Φ is defined as $\sup_{\mathcal{H}\in q_0(\Phi)} \log_2 \dim \mathcal{H}$, where $q_0(\Phi)$ is the set of all subspaces \mathcal{H}_0 of \mathcal{H}_A on which the channel Φ is perfectly reversible (in the sense that there is a channel Θ such that $\Theta(\Phi(\rho)) = \rho$ for all states ρ supported by \mathcal{H}_0). Any subspace $\mathcal{H}_0 \in q_0(\Phi)$ is called *error correcting code* for the channel Φ [5, 6].

The (asymptotic) quantum zero-error capacity is defined by regularization: $Q_0(\Phi) = \sup_n n^{-1} \bar{Q}_0(\Phi^{\otimes n})$ [1, 3, 5].

The quantum zero-error capacity of a channel Φ is determined by its *non-commutative graph* $\mathcal{G}(\Phi)$, which can be defined as the subspace of $\mathfrak{B}(\mathcal{H}_A)$ spanned by the operators $V_k^* V_l$, where V_k are operators from any Kraus representation (5) of Φ [5]. In particular, the Knill-Laflamme error-correcting condition [7] implies the following lemma.

Lemma 1. A set $\{\varphi_k\}_{k=1}^d$ of unit orthogonal vectors in \mathcal{H}_A is a basis of error-correcting code for a channel $\Phi : \mathfrak{S}(\mathcal{H}_A) \to \mathfrak{S}(\mathcal{H}_B)$ if and only if

 $\langle \varphi_l | A | \varphi_k \rangle = 0 \quad and \quad \langle \varphi_l | A | \varphi_l \rangle = \langle \varphi_k | A | \varphi_k \rangle \quad \forall A \in \mathfrak{L}, \ \forall k \neq l,$ (6)

where \mathfrak{L} is any subset of $\mathfrak{B}(\mathcal{H}_A)$ such that $\lim \mathfrak{L} = \mathcal{G}(\Phi)$.

Since a subspace \mathfrak{L} of the algebra \mathfrak{M}_n of $n \times n$ matrices is a noncommutative graph of a particular channel if and only if

$$\mathfrak{L}$$
 is symmetric ($\mathfrak{L} = \mathfrak{L}^*$) and contains the unit matrix (7)

(see Lemma 2 in [4] or the Appendix in [9]), Lemma 1 shows that one can "construct" a channel Φ with dim $\mathcal{H}_A = n$ having positive (correspondingly, zero) 1-shot quantum zero-error capacity by taking a subspace $\mathfrak{L} \subset \mathfrak{M}_n$ satisfying (7) for which the following condition is valid (correspondingly, not valid)

 $\exists \varphi, \psi \in [\mathbb{C}^n]_1 \text{ s.t. } \langle \psi | A | \varphi \rangle = 0 \text{ and } \langle \varphi | A | \varphi \rangle = \langle \psi | A | \psi \rangle \quad \forall A \in \mathfrak{L}, \quad (8)$ where $[\mathbb{C}^n]_1$ is the unit sphere of \mathbb{C}^n .

3 Example of 3-partite superactivation

For given $\theta \in (-\pi, \pi]$ consider the 8-D subspace

$$\mathfrak{N}_{\theta} = \left\{ M = \begin{bmatrix} a & b & e & f \\ c & d & f & \bar{\gamma}e \\ g & h & a & b \\ h & \gamma g & c & d \end{bmatrix}, \quad a, b, c, d, e, f, g, h \in \mathbb{C} \right\}$$
(9)

of \mathfrak{M}_4 satisfying condition (7), where $\gamma = \exp[i\theta]$. This subspace is an extension of the 4-D subspace \mathfrak{L}_{θ} used in [10], i.e. $\mathfrak{L}_{\theta} \subset \mathfrak{N}_{\theta}$ for each θ .¹

Denote by $\widehat{\mathfrak{N}}_{\theta}$ the set of all channels whose noncommutative graph coincides with \mathfrak{N}_{θ} . In [9, the Appendix] it is shown how to explicitly construct pseudo-diagonal channels in $\widehat{\mathfrak{N}}_{\theta}$ with $d_A = 4$ and $d_E \geq 3$ (since $\dim \mathfrak{N}_{\theta} = 8 \leq 3^2$).

Theorem 1. Let Φ_{θ} be a channel in $\widehat{\mathfrak{N}}_{\theta}$ and $n \in \mathbb{N}$ be arbitrary.

A) $\bar{Q}_0(\Phi_{\theta}) > 0$ if and only if $\theta = \pi$ and $\bar{Q}_0(\Phi_{\pi}) = 1$.

B) If $\theta_1 + \ldots + \theta_n = \pi \pmod{2\pi}$ then $\bar{Q}_0(\Phi_{\theta_1} \otimes \ldots \otimes \Phi_{\theta_n}) > 0$ and 2-D error-correcting code for the channel $\Phi_{\theta_1} \otimes \ldots \otimes \Phi_{\theta_n}$ is spanned by the vectors

$$|\varphi\rangle = \frac{1}{\sqrt{2}} \left[|1\dots1\rangle + i|2\dots2\rangle \right], \ |\psi\rangle = \frac{1}{\sqrt{2}} \left[|3\dots3\rangle + i|4\dots4\rangle \right], \tag{10}$$

where $\{|1\rangle, \ldots, |4\rangle\}$ is the canonical basis in \mathbb{C}^4 .

C₂) If $|\theta_1| + |\theta_2| < \pi$ then $\bar{Q}_0(\Phi_{\theta_1} \otimes \Phi_{\theta_2}) = 0$.

 C_n) If $|\theta_1| + \ldots + |\theta_n| \le 2\ln(3/2)$ then $\overline{Q}_0(\Phi_{\theta_1} \otimes \ldots \otimes \Phi_{\theta_n}) = 0$.

Assertion C₂ is the main progress of this theorem in comparison with Theorem 1 in [10]. It complements assertion B with n = 2. It is the proof of assertion C₂ that motivates the extension $\mathfrak{L}_{\theta} \to \mathfrak{N}_{\theta}$.

Remark 1. Since assertion C_n is proved by using quite coarse estimates, the other assertions of Theorem 1 make it reasonable to conjecture validity of the following strengthened version:

 C_n^*) If $|\theta_1| + \ldots + |\theta_n| < \pi$ then $\bar{Q}_0(\Phi_{\theta_1} \otimes \ldots \otimes \Phi_{\theta_n}) = 0$. The below proof of C_2 shows difficulty of the direct proof of this conjecture.

¹In contrast to this paper $\gamma = \exp[i\theta/2]$ is used in [10].

Theorem 1 implies the following example of 3-partite superactivation of 1-shot quantum zero-error capacity.

Corollary 1. Let $\theta_1, \theta_2, \theta_3$ be positive numbers such that $\theta_1 + \theta_2 + \theta_3 = \pi$. Then

$$\bar{Q}_0(\Phi_{\theta_1} \otimes \Phi_{\theta_2} \otimes \Phi_{\theta_3}) > 0 \quad while \quad \bar{Q}_0(\Phi_{\theta_i} \otimes \Phi_{\theta_j}) = 0 \quad \forall i \neq j.$$

2-D error-correcting code for the channel $\Phi_{\theta_1} \otimes \Phi_{\theta_2} \otimes \Phi_{\theta_3}$ is spanned by the vectors

$$|\varphi\rangle = \frac{1}{\sqrt{2}} \left[|111\rangle + i |222\rangle \right], \ |\psi\rangle = \frac{1}{\sqrt{2}} \left[|333\rangle + i |444\rangle \right].$$
(11)

If conjecture C_n^* in Remark 1 is valid for some n > 2 then the similar assertion holds for n+1 channels $\Phi_{\theta_1}, \ldots, \Phi_{\theta_{n+1}}$. This would give an example of (n+1)-partite superactivation of 1-shot quantum zero-error capacity.

For each θ one can choose (non-uniquely) a basis $\{M_k^{\theta}\}_{k=1}^8$ of the subspace \mathfrak{N}_{θ} consisting of positive operators such that $\sum_{k=1}^8 M_k^{\theta} = I_{\mathcal{H}_A}$ (since the subspace \mathfrak{N}_{θ} satisfies condition (7), see [9]). This basis can be considered as a quantum observable \mathcal{M}_{θ} . By using Proposition 1 in [9] and Lemma 1 Corollary 1 can be reformulated in terms of theory of quantum measurements.

Corollary 2. Let $\theta_1, \theta_2, \theta_3$ be positive numbers such that $\theta_1 + \theta_2 + \theta_3 = \pi$. Then all the observables \mathcal{M}_{θ_i} and all the bipartite observables $\mathcal{M}_{\theta_i} \otimes \mathcal{M}_{\theta_j}$ have no indistinguishable subspaces but the tripartite observable $\mathcal{M}_{\theta_1} \otimes \mathcal{M}_{\theta_2} \otimes \mathcal{M}_{\theta_3}$ has indistinguishable subspace spanned by the vectors (11).²

Note also that Theorem 1 implies the following example of superactivation of 2-shot quantum zero-error capacity.

Corollary 3. Let θ_1, θ_2 be positive numbers such that $\theta_1 + \theta_2 = \pi/2$. Then

$$\bar{Q}_0\left(\left[\Phi_{\theta_1}\otimes\Phi_{\theta_2}\right]^{\otimes 2}\right) > 0 \quad while \quad \bar{Q}_0\left(\Phi_{\theta_1}^{\otimes 2}\right) = \bar{Q}_0\left(\Phi_{\theta_2}^{\otimes 2}\right) = \bar{Q}_0\left(\Phi_{\theta_1}\otimes\Phi_{\theta_2}\right) = 0.$$

The proof of Theorem 1. The equality $\bar{Q}_0(\Phi_\theta) = 0$ for $\theta \neq \pi$, the inequality $\bar{Q}_0(\Phi_\pi) \leq 1$ and assertion C_n follows from Theorem 1 in [10], since the inclusion $\mathfrak{L}_{\theta} \subset \mathfrak{N}_{\theta}$ implies $\bar{Q}_0(\Phi_{\theta_1} \otimes \ldots \otimes \Phi_{\theta_n}) \leq \bar{Q}_0(\Psi_{\theta_1} \otimes \ldots \otimes \Psi_{\theta_n})$ for any channels $\Psi_{\theta_1} \in \widehat{\mathfrak{L}}_{\theta_1}, \ldots, \Psi_{\theta_n} \in \widehat{\mathfrak{L}}_{\theta_n}$.

²We call a subspace \mathcal{H}_0 indistinguishable for an observable \mathcal{M} if application of \mathcal{M} to all states supported by \mathcal{H}_0 leads to the same outcomes probability distribution.

To prove that $\bar{Q}_0(\Phi_{\pi}) \geq 1$ it suffices to show, by using Lemma 1, that the vectors $|\varphi\rangle = [1, i, 0, 0]^{\top}$ and $|\psi\rangle = [0, 0, 1, i]^{\top}$ generate error-correcting code for the channel Φ_{π} .

B) Let $M_1 \in \mathfrak{N}_{\theta_1}, \ldots, M_n \in \mathfrak{N}_{\theta_n}$ be arbitrary, $X = M_1 \otimes \ldots \otimes M_n$ and φ, ψ be the vectors defined in (10). By Lemma 1 it suffices to show that

 $\langle \psi | X | \varphi \rangle = 0$ and $\langle \psi | X | \psi \rangle = \langle \varphi | X | \varphi \rangle.$ (12)

Let $a_k, b_k, ..., h_k$ be elements of the matrix M_k . We have

$$2\langle \psi | X | \varphi \rangle = \langle 3 \dots 3 | X | 1 \dots 1 \rangle + i \langle 3 \dots 3 | X | 2 \dots 2 \rangle - i \langle 4 \dots 4 | X | 1 \dots 1 \rangle$$
$$+ \langle 4 \dots 4 | X | 2 \dots 2 \rangle = g_1 \dots g_n (1 + \gamma_1 \dots \gamma_n) + h_1 \dots h_n (i - i) = 0,$$

since $\gamma_1 \dots \gamma_n = -1$ by the condition $\theta_1 + \dots + \theta_n = \pi \pmod{2\pi}$,

$$2\langle \varphi | X | \varphi \rangle = \langle 1 \dots 1 | X | 1 \dots 1 \rangle + i \langle 1 \dots 1 | X | 2 \dots 2 \rangle - i \langle 2 \dots 2 | X | 1 \dots 1 \rangle$$
$$+ \langle 2 \dots 2 | X | 2 \dots 2 \rangle = a_1 \dots a_n + i (b_1 \dots b_n - c_1 \dots c_n) + d_1 \dots d_n$$

and

$$2\langle \psi | X | \psi \rangle = \langle 3 \dots 3 | X | 3 \dots 3 \rangle + i \langle 3 \dots 3 | X | 4 \dots 4 \rangle - i \langle 4 \dots 4 | X | 3 \dots 3 \rangle$$
$$+ \langle 4 \dots 4 | X | 4 \dots 4 \rangle = a_1 \dots a_n + i (b_1 \dots b_n - c_1 \dots c_n) + d_1 \dots d_n.$$

Thus the both equalities in (12) are valid.

C₂) To prove this assertion we have to show that the subspace $\mathfrak{N}_{\theta_1} \otimes \mathfrak{N}_{\theta_2}$ does not satisfy condition (8) if $|\theta_1| + |\theta_2| < \pi$. In the case $\theta_1 = \theta_2 = 0$ this follows from assertion C_n. So, we may assume, by symmetry, that $\theta_2 \neq 0$.

Throughout the proof we will use the isomorphism

$$\mathbb{C}^n \otimes \mathbb{C}^m \ni x \otimes y \iff [x_1y, \dots, x_ny]^\top \in \underbrace{\mathbb{C}^m \oplus \dots \oplus \mathbb{C}^m}_n$$

and the corresponding isomorphism

$$\mathfrak{M}_n \otimes \mathfrak{M}_m \ni A \otimes B \iff [a_{ij}B] \in \mathfrak{M}_{nm}.$$
(13)

Let U_1, U_2, V_1, V_2 be the unitary operators in \mathbb{C}^2 determined (in the canonical basis) by the matrices

$$U_1 = \begin{bmatrix} 1 & 0 \\ 0 & \gamma_1 \end{bmatrix}, \quad V_1 = \begin{bmatrix} 1 & 0 \\ 0 & \gamma_2 \end{bmatrix}, \quad U_2 = V_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

We will identify \mathbb{C}^4 with $\mathbb{C}^2 \oplus \mathbb{C}^2$. So, arbitrary matrices $M_1 \in \mathfrak{N}_{\theta_1}$ and $M_2 \in \mathfrak{N}_{\theta_2}$ can be represented as follows

$$M_1 = \begin{bmatrix} A_1 & e_1 U_1^* + f_1 U_2^* \\ g_1 U_1 + h_1 U_2 & A_1 \end{bmatrix}, M_2 = \begin{bmatrix} A_2 & e_2 V_1^* + f_2 V_2^* \\ g_2 V_1 + h_2 V_2 & A_2 \end{bmatrix}$$

or, according to (13), as

$$M_1 = I_2 \otimes A_1 + |2\rangle \langle 1| \otimes [g_1 U_1 + h_1 U_2] + |1\rangle \langle 2| \otimes [e_1 U_1^* + f_1 U_2^*]$$

and

$$M_2 = I_2 \otimes A_2 + |2\rangle \langle 1| \otimes [g_2 V_1 + h_2 V_2] + |1\rangle \langle 2| \otimes [e_2 V_1^* + f_2 V_2^*],$$

where A_1 and A_2 are arbitrary matrices in \mathfrak{M}_2 .

Assume the existence of orthogonal unit vectors φ and ψ in $\mathbb{C}^4\otimes\mathbb{C}^4$ such that

$$\langle \psi | M_1 \otimes M_2 | \varphi \rangle = 0$$
 and $\langle \psi | M_1 \otimes M_2 | \psi \rangle = \langle \varphi | M_1 \otimes M_2 | \varphi \rangle$ (14)

for all $M_1 \in \mathfrak{N}_{\theta_1}$ and $M_2 \in \mathfrak{N}_{\theta_2}$.

By using the above representations of M_1 and M_2 we have

$$M_1 \otimes M_2 = [I_2 \otimes I_2] \otimes [A_1 \otimes A_2] + [I_2 \otimes |2\rangle \langle 1|] \otimes [A_1 \otimes [g_2 V_1 + h_2 V_2]] +$$

$$[I_2 \otimes |1\rangle \langle 2|] \otimes [A_1 \otimes [e_2 V_1^* + f_2 V_2^*]] + [|2\rangle \langle 1| \otimes I_2] \otimes [[g_1 U_1 + h_1 U_2] \otimes A_2] + \dots$$

Since $\mathfrak{M}_2 \otimes \mathfrak{M}_2 = \mathfrak{M}_4$, by choosing $e_i = f_i = g_i = h_i = 0, i = 1, 2$, we obtain from (14) that

$$\langle \psi | I_4 \otimes A | \varphi \rangle = 0$$
 and $\langle \psi | I_4 \otimes A | \psi \rangle = \langle \varphi | I_4 \otimes A | \varphi \rangle \quad \forall A \in \mathfrak{M}_4.$

According to (13) we have

$$I_4 \otimes A = \begin{bmatrix} A & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & 0 & A \end{bmatrix}, \quad |\varphi\rangle = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad |\psi\rangle = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix},$$

where x_i, y_i are vectors in \mathbb{C}^4 . So, the above relations can be written as the following ones

$$\sum_{i=1}^{4} \langle y_i | A | x_i \rangle = 0 \quad \text{and} \quad \sum_{i=1}^{4} \langle y_i | A | y_i \rangle = \sum_{i=1}^{4} \langle x_i | A | x_i \rangle \quad \forall A \in \mathfrak{M}_4$$

which are equivalent to the operator equalities

$$\sum_{i=1}^{4} |y_i\rangle \langle x_i| = 0.$$
(15)

and

$$\sum_{i=1}^{4} |y_i\rangle\langle y_i| = \sum_{i=1}^{4} |x_i\rangle\langle x_i|$$
(16)

By choosing $e_i = f_i = g_1 = h_1 = 0$, $i = 1, 2, A_2 = 0$, $(g_2, h_2) = (1, 0)$ and $(g_2, h_2) = (0, 1)$ we obtain from (14) that

$$\langle \psi | [I_2 \otimes | 2 \rangle \langle 1 |] \otimes [A_1 \otimes V_k] | \varphi \rangle = 0$$

and

$$\langle \psi | [I_2 \otimes | 2 \rangle \langle 1 |] \otimes [A_1 \otimes V_k] | \psi \rangle = \langle \varphi | [I_2 \otimes | 2 \rangle \langle 1 |] \otimes [A_1 \otimes V_k] | \varphi \rangle$$

for all A_1 in \mathfrak{M}_2 and k = 1, 2. According to (13) we have

$$[I_2 \otimes |2\rangle \langle 1|] \otimes [A_1 \otimes V_k] = \begin{bmatrix} 0 & 0 & 0 & 0\\ A_1 \otimes V_k & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & A_1 \otimes V_k & 0 \end{bmatrix}$$

and hence the above equalities imply

$$\langle y_2 | A \otimes V_k | x_1 \rangle + \langle y_4 | A \otimes V_k | x_3 \rangle = 0 \quad \forall A \in \mathfrak{M}_2, \ k = 1, 2,$$
(17)

and

$$\langle y_2 | A \otimes V_k | y_1 \rangle + \langle y_4 | A \otimes V_k | y_3 \rangle =$$

$$\langle x_2 | A \otimes V_k | x_1 \rangle + \langle x_4 | A \otimes V_k | x_3 \rangle \qquad \forall A \in \mathfrak{M}_2, \ k = 1, 2.$$

$$(18)$$

Similarly, by choosing $e_i = f_i = g_2 = h_2 = 0$, $i = 1, 2, A_1 = 0$, $(g_1, h_1) = (1, 0)$ and $(g_1, h_1) = (0, 1)$ we obtain from (14) the equalities

$$\langle y_3 | U_k \otimes A | x_1 \rangle + \langle y_4 | U_k \otimes A | x_2 \rangle = 0, \quad \forall A \in \mathfrak{M}_2, \ k = 1, 2,$$
(19)

and

$$\langle y_3 | U_k \otimes A | y_1 \rangle + \langle y_4 | U_k \otimes A | y_2 \rangle =$$

$$\langle x_3 | U_k \otimes A | x_1 \rangle + \langle x_4 | U_k \otimes A | x_2 \rangle, \qquad \forall A \in \mathfrak{M}_2, \ k = 1, 2.$$

$$(20)$$

By the symmetry of condition (14) with respect to φ and ψ relations (17) and (19) imply respectively

$$\langle x_2 | A \otimes V_k | y_1 \rangle + \langle x_4 | A \otimes V_k | y_3 \rangle = 0 \quad \forall A \in \mathfrak{M}_2, \ k = 1, 2,$$
(21)

and

$$\langle x_3 | U_k \otimes A | y_1 \rangle + \langle x_4 | U_k \otimes A | y_2 \rangle = 0, \quad \forall A \in \mathfrak{M}_2, \ k = 1, 2.$$
(22)

Finally, by choosing $A_1 = A_2 = 0$ and appropriate values of e_i, f_i, g_i, h_i , i = 1, 2, one can obtain from (14) the following equalities

$$\langle y_4 | U_k \otimes V_l | x_1 \rangle = \langle x_4 | U_k \otimes V_l | y_1 \rangle = 0 \qquad k, l = 1, 2, \tag{23}$$

$$\langle y_4 | U_k \otimes V_l | y_1 \rangle = \langle x_4 | U_k \otimes V_l | x_1 \rangle, \qquad k, l = 1, 2, \qquad (24)$$

$$\langle y_3 | U_k \otimes V_l^* | x_2 \rangle = \langle x_3 | U_k \otimes V_l^* | y_2 \rangle = 0 \qquad k, l = 1, 2, \tag{25}$$

$$\langle y_3 | U_k \otimes V_l^* | y_2 \rangle = \langle x_3 | U_k \otimes V_l^* | x_2 \rangle, \qquad k, l = 1, 2.$$
(26)

We will prove below that the system (15)-(26) has no nontrivial solutions.

We will use the following lemmas.

Lemma 2. A) Equations (15) and (16) imply that all the vectors x_i, y_i , $i = \overline{1, 4}$, lie in some 2-D subspace of \mathbb{C}^4 .

B) If $x_{i_0} = y_{i_0} = 0$ for some i_0 then equations (15) and (16) imply that all the vectors $x_i, y_i, i = \overline{1, 4}$, are collinear.

Proof. A) Consider the 4×4 - matrices

$$X = [\langle x_i | x_j \rangle], \quad Y = [\langle y_i | y_j \rangle], \quad Z = [\langle x_i | y_j \rangle].$$

It is easy to see that (15) implies XY = 0 while (16) shows that $X^2 = ZZ^*$ and $Y^2 = Z^*Z$. Hence rank $X = \operatorname{rank} Y \leq 2$. Since (16) implies that the sets $\{x_i\}_{i=1}^4$ and $\{y_i\}_{i=1}^4$ have the same linear hull, the above inequality shows that this linear hull has dimension ≤ 2 .

B) This assertion is proved similarly, since the same argumentation with 3×3 - matrices X, Y, Z implies rank $X = \operatorname{rank} Y \leq 1$.

Lemma 3. A) The condition

$$\langle z_4 | U_k \otimes V_l | z_1 \rangle = 0 \qquad k, l = 1, 2, \tag{27}$$

holds if and only if the pair (z_1, z_4) has one of the following forms:

1)
$$z_1 = \begin{bmatrix} \mu_1 \\ s \end{bmatrix} \otimes \begin{bmatrix} a \\ b \end{bmatrix}$$
, $z_4 = \begin{bmatrix} \bar{\mu}_1 \\ -s \end{bmatrix} \otimes \begin{bmatrix} c \\ d \end{bmatrix}$;
2) $z_1 = \begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} \mu_2 \\ s \end{bmatrix}$, $z_4 = \begin{bmatrix} c \\ d \end{bmatrix} \otimes \begin{bmatrix} \bar{\mu}_2 \\ -s \end{bmatrix}$;
3) $z_1 = a \begin{bmatrix} \mu_1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} \mu_2 \\ s \end{bmatrix} + b \begin{bmatrix} \mu_1 \\ -1 \end{bmatrix} \otimes \begin{bmatrix} \mu_2 \\ -s \end{bmatrix}$, $z_4 = c \begin{bmatrix} \bar{\mu}_1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} \bar{\mu}_2 \\ -s \end{bmatrix} + d \begin{bmatrix} \bar{\mu}_1 \\ -1 \end{bmatrix} \otimes \begin{bmatrix} \bar{\mu}_2 \\ s \end{bmatrix}$;
4) $z_1 = h \begin{bmatrix} \mu_1 \\ s \end{bmatrix} \otimes \begin{bmatrix} \mu_2 \\ t \end{bmatrix}$, $z_4 = \begin{bmatrix} \bar{\mu}_1 \\ -s \end{bmatrix} \otimes \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} \otimes \begin{bmatrix} \bar{\mu}_2 \\ -t \end{bmatrix}$;
5) $z_1 = \begin{bmatrix} \mu_1 \\ -s \end{bmatrix} \otimes \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} \otimes \begin{bmatrix} \mu_2 \\ -t \end{bmatrix}$, $z_4 = h \begin{bmatrix} \bar{\mu}_1 \\ s \end{bmatrix} \otimes \begin{bmatrix} \bar{\mu}_2 \\ t \end{bmatrix}$;
where $\mu_1 = \sqrt{\mu_1} = \sqrt{\mu_1} = 1$, and $a, b, c, d, b, c \in C$, $c = \pm 1$, $t = \pm 1$.

where $\mu_k = \sqrt{\gamma_k}, \ k = 1, 2, \ and \ a, b, c, d, h \in \mathbb{C}, \ s = \pm 1, \ t = \pm 1.$

B) Validity of (23) and (24) for vectors $x_i, y_i, i = 1, 4$, implies

$$\langle y_4|U_k\otimes V_l|y_1\rangle = \langle x_4|U_k\otimes V_l|x_1\rangle = 0.$$

Lemma 4. A) The condition

$$\langle z_3 | U_k \otimes V_l^* | z_2 \rangle = 0 \qquad k, l = 1, 2,$$
 (28)

holds if and only if the pair (z_2, z_3) has one of the following forms:

1)
$$z_2 = \begin{bmatrix} \mu_1 \\ s \end{bmatrix} \otimes \begin{bmatrix} a \\ b \end{bmatrix}$$
, $z_3 = \begin{bmatrix} \bar{\mu}_1 \\ -s \end{bmatrix} \otimes \begin{bmatrix} c \\ d \end{bmatrix}$;
2) $z_2 = \begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} \bar{\mu}_2 \\ s \end{bmatrix}$, $z_3 = \begin{bmatrix} c \\ d \end{bmatrix} \otimes \begin{bmatrix} \mu_2 \\ -s \end{bmatrix}$;
3) $z_2 = a \begin{bmatrix} \mu_1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} \bar{\mu}_2 \\ s \end{bmatrix} + b \begin{bmatrix} \mu_1 \\ -1 \end{bmatrix} \otimes \begin{bmatrix} \bar{\mu}_2 \\ -s \end{bmatrix}$, $z_3 = c \begin{bmatrix} \bar{\mu}_1 \\ -1 \end{bmatrix} \otimes \begin{bmatrix} \mu_2 \\ s \end{bmatrix} + d \begin{bmatrix} \bar{\mu}_1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} \mu_2 \\ -s \end{bmatrix}$;
4) $z_2 = h \begin{bmatrix} \mu_1 \\ s \end{bmatrix} \otimes \begin{bmatrix} \bar{\mu}_2 \\ t \end{bmatrix}$, $z_3 = \begin{bmatrix} \bar{\mu}_1 \\ -s \end{bmatrix} \otimes \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} \otimes \begin{bmatrix} \mu_2 \\ -t \end{bmatrix}$;
5) $z_2 = \begin{bmatrix} \mu_1 \\ -s \end{bmatrix} \otimes \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} \otimes \begin{bmatrix} \bar{\mu}_2 \\ -t \end{bmatrix}$, $z_3 = h \begin{bmatrix} \bar{\mu}_1 \\ s \end{bmatrix} \otimes \begin{bmatrix} \mu_2 \\ t \end{bmatrix}$;

where $\mu_k = \sqrt{\gamma_k}, \ k = 1, 2, \ and \ a, b, c, d, h \in \mathbb{C}, \ s = \pm 1, \ t = \pm 1.$

B) Validity of (25) and (26) for vectors $x_i, y_i, i = 2, 3$, implies

$$\langle y_3|U_k\otimes V_l^*|y_2\rangle = \langle x_3|U_k\otimes V_l^*|x_2\rangle = 0.$$

Lemmas 3 and 4 are proved in the Appendix.

Lemma 5. Let $|\theta_1| + |\theta_2| < \pi$ then

A) $\langle x|U_1|x\rangle \neq 0$ and $\langle x|V_1|x\rangle \neq 0$ for any nonzero $x \in \mathbb{C}^2$;

B) $\langle y|U_1 \otimes V_1|y \rangle \neq 0$ and $\langle y|U_1 \otimes V_1^*|y \rangle \neq 0$ for any nonzero $y \in \mathbb{C}^2 \otimes \mathbb{C}^2$.

Proof. A) Let $|x\rangle = [x_1, x_2]^\top \neq 0$ then $\langle x|U_1|x\rangle = |x_1|^2 + \gamma_1|x_2|^2 \neq 0$ and $\langle x|V_1|x\rangle = |x_1|^2 + \gamma_2|x_2|^2 \neq 0$, since $\theta_1, \theta_2 \neq \pi$.

B) Since $U_1 \otimes V_1 = \text{diag}\{1, \gamma_2, \gamma_1, \gamma_1\gamma_2\}$, the equality $\langle y|U_1 \otimes V_1|y \rangle = 0$ for vector $|y\rangle = [y_1, y_2, y_3, y_4]^{\top}$ means that $|y_1|^2 + |y_2|^2\gamma_2 + |y_3|^2\gamma_1 + |y_4|^2\gamma_1\gamma_2 = 0$. By the condition $|\theta_1| + |\theta_2| < \pi$ the numbers $0, 1, \gamma_2, \gamma_1, \gamma_1\gamma_2$ are extreme points of a convex polygon in complex plane, so the last equality can be valid only if $y_i = 0$ for all i.

Similarly one can show that $\langle y|U_1 \otimes V_1^*|y\rangle = 0$ implies y = 0.

Lemma 6. Let p and q be complex numbers such that $|p|^2 + |q|^2 = 1$. If $\{|x_i\rangle\}_{i=1}^4$ and $\{|y_i\rangle\}_{i=1}^4$ satisfy the system (15)-(26) then $\{|px_i - qy_i\rangle\}_{i=1}^4$ and $\{|\bar{q}x_i + \bar{p}y_i\rangle\}_{i=1}^4$ also satisfy the system (15)-(26).

Proof. It suffices to note that the condition

$$\langle \varphi | A | \psi \rangle = \langle \psi | A | \varphi \rangle = \langle \psi | A | \psi \rangle - \langle \varphi | A | \varphi \rangle = 0$$

is invariant under the rotation $|\varphi\rangle \mapsto p|\varphi\rangle - q|\psi\rangle, \ |\psi\rangle \mapsto \bar{q}|\varphi\rangle + \bar{p}|\psi\rangle.$

Lemma 7. If $|\theta_1| + |\theta_2| < \pi$ then the system (15)-(26) has no nontrivial solution of the form $|x_i\rangle = \alpha_i |z\rangle$ and $|y_i\rangle = \beta_i |z\rangle$, $i = \overline{1, 4}$.

Proof. Assume that $|x_i\rangle = \alpha_i |z\rangle$ and $|y_i\rangle = \beta_i |z\rangle$, $i = \overline{1, 4}$, form a nontrivial solution of the system (15)-(26). Then (15) implies that $|\alpha\rangle = [\alpha_1, \ldots, \alpha_4]^{\top}$ and $|\beta\rangle = [\beta_1, \ldots, \beta_4]^{\top}$ are orthogonal nonzero vectors of the same norm. By Lemma 5B it follows from (23) and (25) that

$$\alpha_1\beta_4 = \alpha_4\beta_1 = \alpha_2\beta_3 = \alpha_3\beta_2 = 0. \tag{29}$$

If $|\alpha_i| + |\beta_i| > 0$ for all *i* then we may consider by Lemma 6 that $\alpha_i \neq 0$ for all *i* and hence (29) implies $\beta_i = 0$ for all *i*, i.e. $|\beta\rangle = 0$.

Assume that $\alpha_1 = \beta_1 = 0$ and consider the following two cases.

1) If $\alpha_4 = \beta_4 = 0$ then the condition $\langle \beta | \alpha \rangle = 0$ implies $|\alpha_i| + |\beta_i| > 0$ for i = 2, 3 and we may consider by Lemma 6 that $\alpha_i \neq 0$ for i = 2, 3. Hence (29) implies $\beta_i = 0$ for i = 2, 3 and hence $|\beta\rangle = 0$.

2) If $|\alpha_4| + |\beta_4| > 0$ then we may consider by Lemma 6 that $\alpha_4 \neq 0$. By Lemma 5B it follows from (21) with $A = U_1$ and (22) with $A = V_1$ that $\alpha_4\beta_2 = \alpha_4\beta_3 = 0$, which implies $\beta_2 = \beta_3 = 0$. Hence the condition $\langle \beta | \alpha \rangle = 0$ can be valid only if $|\beta\rangle = 0$.

By the similar way one can show that the assumption $\alpha_i = \beta_i = 0$ for i = 2, 3, 4 leads to a contradiction.

Assume now that the collections $\{x_i\}_1^4$ and $\{y_i\}_1^4$ form a nontrivial solution of the system (15)-(26).

If $x_i \not\models y_i$ for some *i* then (23)-(26) and Lemmas 3B, 4B imply

$$\langle y_{5-i}|W_i|x_i\rangle = \langle x_{5-i}|W_i|y_i\rangle = \langle x_{5-i}|W_i|x_i\rangle = \langle y_{5-i}|W_i|y_i\rangle = 0,$$

where $W_1 = U_1 \otimes V_1$, $W_2 = U_1 \otimes V_1^*$, $W_3 = U_1^* \otimes V_1$, $W_4 = U_1^* \otimes V_1^*$. By Lemma 2A we have $x_{5-i}, y_{5-i} \in \lim\{x_i, y_i\}$, so the above equalities show that $\langle x_{5-i}|W_i|x_{5-i}\rangle = \langle y_{5-i}|W_i|y_{5-i}\rangle = 0$. Lemma 5B implies $x_{5-i} = y_{5-i} = 0$, which contradicts to the assumption $x_i \not| y_i$ by Lemma 2B.

Thus, $x_i \parallel y_i$ for all $i = \overline{1, 4}$. By Lemma 7 we may assume in what follows that

 $|x_i\rangle = \alpha_i |z_i\rangle$ and $|y_i\rangle = \beta_i |z_i\rangle$, where $|z_i\rangle$ are non-collinear vectors³. (30)

Then Lemma 2B implies $|\alpha_i| + |\beta_i| > 0$, $i = \overline{1, 4}$, and equations (15), (16) can be rewritten as follows

$$\sum_{i=1}^{4} \bar{\beta}_i \alpha_i |z_i\rangle \langle z_i| = 0, \qquad (31)$$

$$\sum_{i=1}^{4} \left[|\beta_i|^2 - |\alpha_i|^2 \right] |z_i\rangle \langle z_i| = 0.$$
(32)

By Lemma 6 we may assume that $\beta_1 = 0$ and hence $\alpha_1 \neq 0$. There are two cases:

1) If $\beta_i \alpha_i \neq 0$ for all i > 1 then (31) and Lemma 8 in the Appendix imply $z_2 \parallel z_3 \parallel z_4$. Then it follows from (32) that

$$|\alpha_1|^2 |z_1\rangle \langle z_1| + [\ldots] |z_2\rangle \langle z_2| = 0$$

and hence $z_1 \parallel z_2 \parallel z_3 \parallel z_4$ contradicting to the assumption (30).

2) If there is k > 1 such that $\beta_k \alpha_k = 0$ then (31) implies that either $\beta_i \alpha_i \neq 0$ and $\beta_j \alpha_j \neq 0$ or $\beta_i \alpha_i = \beta_j \alpha_j = 0$, where *i* and j > i are complementary indexes to 1 and *k*.

If $\beta_i \alpha_i \neq 0$ and $\beta_j \alpha_j \neq 0$ then it follows from (31) that $z_i \parallel z_j$ and (32) implies

$$|\alpha_1|^2 |z_1\rangle \langle z_1| + p |z_k\rangle \langle z_k| + [\ldots] |z_i\rangle \langle z_i| = 0,$$

where p is a nonzero number (equal either to $|\alpha_k|^2$ or to $-|\beta_k|^2$). Hence $z_1 \parallel z_k$ by Lemma 8 in the Appendix.

Thus $z_1 \parallel z_k$ and $z_i \parallel z_j$. By Lemma 5B it follows from (23) and (25) that $k \neq 4$ and $(i, j) \neq (2, 3)$. So, we have only two possibilities:

a) k = 2, i = 3, j = 4. In this case $z_3 \parallel z_4$ and (21) with $A = U_1$ implies

$$\bar{\alpha}_4\beta_3\langle z_4|U_1\otimes V_1|z_3\rangle = -\bar{\alpha}_2\beta_1\langle z_2|U_1\otimes V_1|z_1\rangle = 0 \quad (\text{since } \beta_1 = 0).$$

³In the sense that all the vectors $|z_i\rangle$, $i = \overline{1, 4}$, are not collinear to each other, but some two of them may be collinear.

Hence Lemma 5 shows that $\alpha_4\beta_3 = 0$ contradicting to the assumption.

b) k = 3, i = 2, j = 4. In this case $z_2 \parallel z_4$ and (22) with $A = V_1$ implies

$$\bar{\alpha}_4\beta_2\langle z_4|U_1\otimes V_1|z_2\rangle = -\bar{\alpha}_3\beta_1\langle z_3|U_1\otimes V_1|z_1\rangle = 0 \quad (\text{since } \beta_1 = 0).$$

Hence Lemma 5B shows that $\alpha_4\beta_2 = 0$ contradicting to the assumption.

So, we necessarily have $\beta_i \alpha_i = 0$ for all $i = \overline{1, 4}$. Since the vectors z_1, \ldots, z_4 are not collinear by assumption (30), equality (32) and Lemma 2B imply that there are two nonzero α_i and two nonzero β_i . Thus, we have (up to permutation) the following cases

$$\mathbf{a}) |\varphi\rangle, |\psi\rangle = \begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ y_3 \\ y_4 \end{bmatrix}; \mathbf{b}) |\varphi\rangle, |\psi\rangle = \begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ y_2 \\ 0 \\ y_4 \end{bmatrix}; \mathbf{c}) |\varphi\rangle, |\psi\rangle = \begin{bmatrix} x_1 \\ 0 \\ 0 \\ x_4 \end{bmatrix}, \begin{bmatrix} 0 \\ y_2 \\ y_3 \\ 0 \end{bmatrix},$$

where $x_1 \not\parallel x_k$ and $y_i \not\parallel y_j$ (since otherwise (32) implies $x_1 \parallel x_k \parallel y_i \parallel y_j$).

Show first that case c) is not possible. It follows from (17) with $A = U_1$ and (19) with $A = V_1$ that

$$\langle y_2|U_1\otimes V_1|x_1\rangle = \langle y_3|U_1\otimes V_1|x_1\rangle = 0.$$

Since $y_2 \not\parallel y_3$, Lemma 2A shows that $x_1 \in \lim\{y_2, y_3\}$ and the above equalities imply $\langle x_1 | U_1 \otimes V_1 | x_1 \rangle = 0$. By Lemma 5B $x_1 = 0$.

It is more difficult to show impossibility of cases a) and b). We will consider these cases simultaneously by denoting $z_2 = x_2, z_3 = y_3$ in case a), $z_2 = y_2, z_3 = x_3$ in case b) and $z_1 = x_1, z_4 = y_4$ in the both cases. The system (15)-(26) implies the following equations:

$$|x_1\rangle\langle x_1| + |x_i\rangle\langle x_i| = |y_j\rangle\langle y_j| + |y_4\rangle\langle y_4|$$
(33)

where (i, j) = (2, 3) in case a) and (i, j) = (3, 2) in case b),

$$\langle z_3 | U_k \otimes A | x_1 \rangle = -\sigma_* \langle y_4 | U_k \otimes A | z_2 \rangle \quad \forall A \in \mathfrak{M}_2, \ k = 1, 2,$$
(34)

$$\langle z_2 | A \otimes V_k | x_1 \rangle = +\sigma_* \langle y_4 | A \otimes V_k | z_3 \rangle \quad \forall A \in \mathfrak{M}_2, \ k = 1, 2,$$
(35)

where $\sigma_* = 1$ in case a) and $\sigma_* = -1$ in case b),

$$\langle y_4 | U_k \otimes V_l | x_1 \rangle = 0 \quad k, l = 1, 2, \tag{36}$$

$$\langle z_3 | U_k \otimes V_l^* | z_2 \rangle = 0 \quad k, l = 1, 2.$$
 (37)

It follows from (36) and (37) that the pairs (x_1, y_4) and (z_2, z_3) must have one of the forms 1-5 presented in part A of Lemmas 3 and 4 correspondingly.

Assume first that the both pairs (x_1, y_4) and (z_2, z_3) have forms 1-2. In this case x_1, z_2, z_3, y_4 are tensor product vectors. By Lemma 9 in the Appendix (33) can be valid only in the following cases (1-4):

1) $|z_i\rangle = |p\rangle \otimes |a_i\rangle$, $i = \overline{1, 4}$. It follows from (34) that

$$\langle p|U_1|p\rangle\langle a_3|A|a_1\rangle = -\sigma_*\langle p|U_1|p\rangle\langle a_4|A|a_2\rangle \quad \forall A \in \mathfrak{M}_2.$$

Since $\langle p|U_1|p \rangle \neq 0$ by Lemma 5A, we have $a_1 \parallel a_2$ and $a_3 \parallel a_4$. In case a) this and (33) implies $x_1 \parallel x_2 \parallel y_3 \parallel y_4$ contradicting to assumption (30). In case b) it means $x_1 \parallel y_2$ and $x_3 \parallel y_4$. The assumption $x_1 \not\models x_3$ and (33) show that this can be valid only if $|x_1\rangle\langle x_1| = |y_2\rangle\langle y_2|$ and $|x_3\rangle\langle x_3| = |y_4\rangle\langle y_4|$. So, this case is reduced to case 4) considered below.

2) $|z_i\rangle = |a_i\rangle \otimes |p\rangle$, $i = \overline{1, 4}$. Similarly to case 1) this case is reduced to case 4) by using (35) instead of (34).

3) $|x_1\rangle\langle x_1| = |y_4\rangle\langle y_4|$ and $|z_2\rangle\langle z_2| = |z_3\rangle\langle z_3|$. This is not possible due to (36)-(37) and Lemma 5B.

4) $|x_1\rangle\langle x_1| = |y_i\rangle\langle y_i|$ and $|x_{5-i}\rangle\langle x_{5-i}| = |y_4\rangle\langle y_4|$, where i = 3 in case a) and i = 2 in case b).

If i = 3 then $y_3 = \alpha x_1$, $y_4 = \beta x_2$, $|\alpha| = |\beta| = 1$, and (34) with $\sigma_* = 1$ implies

$$\bar{\alpha}\langle x_1|U_1\otimes A|x_1\rangle = -\bar{\beta}\langle x_2|U_1\otimes A|x_2\rangle \quad \forall A\in\mathfrak{M}_2.$$
(38)

Since x_1 and x_2 are product vectors, it follows from (38) and Lemma 5A that

$$x_1 = a \otimes p$$
 and $x_2 = b \otimes p$

for some nonzero vectors a, b, p. Hence (36), (37) and Lemma 5A imply

$$\langle b|U_k|a\rangle = \langle b|U_k^*|a\rangle = 0, \quad k = 1, 2.$$

If $\gamma_1 \neq 1$ (i.e. $\theta_1 \neq 0$) then this can not be valid for nonzero vectors a and b. If $\gamma_1 = 1$ then (38) shows that $\bar{\alpha} ||a||^2 = -\bar{\beta} ||b||^2$ while (35) with $\sigma_* = 1$ and Lemma 5A imply $\bar{\beta}\alpha = 1$, i.e. $\alpha = \beta$.

Similarly, if i = 2 then by using Lemma 5A one can obtain from (35) that

 $x_1 \parallel y_2 \parallel p \otimes a \quad \text{and} \quad x_3 \parallel y_4 \parallel p \otimes b$

for some nonzero vectors a, b, p. Hence (36), (37) and Lemma 5A imply

$$\langle b|V_k|a\rangle = \langle b|V_k^*|a\rangle = 0, \quad k = 1, 2,$$

which can not be valid for nonzero vectors a and b (since $\theta_2 \neq 0 \Rightarrow \gamma_2 \neq \overline{\gamma}_2$).

Assume now that the pair (x_1, y_4) have form 3 in Lemma 3A, i.e.

$$x_1 = a \begin{bmatrix} \mu_1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} \mu_2 \\ s \end{bmatrix} + b \begin{bmatrix} \mu_1 \\ -1 \end{bmatrix} \otimes \begin{bmatrix} \mu_2 \\ -s \end{bmatrix}, \ y_4 = c \begin{bmatrix} \bar{\mu}_1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} \bar{\mu}_2 \\ -s \end{bmatrix} + d \begin{bmatrix} \bar{\mu}_1 \\ -1 \end{bmatrix} \otimes \begin{bmatrix} \bar{\mu}_2 \\ s \end{bmatrix},$$

where $s = \pm 1$, and show incompatibility of the system (33)-(37) if the pair (z_2, z_3) has forms 1,2,3 in Lemma 4A. We will do this by reducing to the case of tensor product vectors x_1, z_2, z_3, y_4 considered before.

1) The pair (z_2, z_3) has form 1, i.e.

$$z_2 = \begin{bmatrix} \mu_1 \\ t \end{bmatrix} \otimes \begin{bmatrix} p \\ q \end{bmatrix}, \ z_3 = \begin{bmatrix} \bar{\mu}_1 \\ -t \end{bmatrix} \otimes \begin{bmatrix} x \\ y \end{bmatrix}, \ t = \pm 1, \ |p| + |q| \neq 0, \ |x| + |y| \neq 0.$$

By substituting the expressions for x_1, z_2, z_3, y_4 into (34) and by noting that

$$\left\langle \begin{array}{c} \bar{\mu}_1\\ s \end{array} \middle| U_k \left| \begin{array}{c} \mu_1\\ -s \end{array} \right\rangle = 0, \quad s = \pm 1, \quad k = 1, 2 \tag{39}$$

we obtain

$$b\left\langle \begin{array}{c} \bar{\mu}_{1} \\ -1 \end{array} \middle| U_{k} \middle| \begin{array}{c} \mu_{1} \\ -1 \end{array} \right\rangle \left\langle \begin{array}{c} x \\ y \end{array} \middle| A \middle| \begin{array}{c} \mu_{2} \\ -s \end{array} \right\rangle = -\sigma_{*}\bar{c}\left\langle \begin{array}{c} \bar{\mu}_{1} \\ 1 \end{array} \middle| U_{k} \middle| \begin{array}{c} \mu_{1} \\ 1 \end{array} \right\rangle \left\langle \begin{array}{c} \bar{\mu}_{2} \\ -s \end{array} \middle| A \middle| \begin{array}{c} p \\ q \end{array} \right\rangle \quad \text{if } t = 1$$

and

$$a\left\langle \begin{array}{c} \bar{\mu}_{1} \\ 1 \end{array} \middle| U_{k} \middle| \begin{array}{c} \mu_{1} \\ 1 \end{array} \right\rangle \left\langle \begin{array}{c} x \\ y \end{array} \middle| A \middle| \begin{array}{c} \mu_{2} \\ s \end{array} \right\rangle = -\sigma_{*}\bar{d}\left\langle \begin{array}{c} \bar{\mu}_{1} \\ -1 \end{array} \middle| U_{k} \middle| \begin{array}{c} \mu_{1} \\ -1 \end{array} \right\rangle \left\langle \begin{array}{c} \bar{\mu}_{2} \\ s \end{array} \middle| A \middle| \begin{array}{c} p \\ q \end{array} \right\rangle \quad \text{if } t = -1.$$

Validity of this equality for all $A \in \mathfrak{M}_2$ implies

$$b \lambda_k^- \left| \begin{array}{c} \mu_2 \\ -s \end{array} \right\rangle \left\langle \begin{array}{c} x \\ y \end{array} \right| = -\sigma_* \bar{c} \lambda_k^+ \left| \begin{array}{c} p \\ q \end{array} \right\rangle \left\langle \begin{array}{c} \bar{\mu}_2 \\ -s \end{array} \right| \quad \text{if } t = 1$$

and

$$a \lambda_k^+ \begin{vmatrix} \mu_2 \\ s \end{vmatrix} \begin{pmatrix} x \\ y \end{vmatrix} = -\sigma_* \bar{d} \lambda_k^- \begin{vmatrix} p \\ q \end{pmatrix} \begin{pmatrix} \bar{\mu}_2 \\ s \end{vmatrix} \quad \text{if } t = -1,$$

where $\lambda_1^{\pm} = \left\langle \begin{array}{c} \bar{\mu}_1 \\ \pm 1 \end{array} \middle| U_1 \middle| \begin{array}{c} \mu_1 \\ \pm 1 \end{array} \right\rangle = 2\mu_1^2$ and $\lambda_2^{\pm} = \left\langle \begin{array}{c} \bar{\mu}_1 \\ \pm 1 \end{array} \middle| U_2 \middle| \begin{array}{c} \mu_1 \\ \pm 1 \end{array} \right\rangle = \pm 2\mu_1$. Since $\lambda_1^+ = \lambda_1^- \neq 0$ and $\lambda_2^+ = -\lambda_2^- \neq 0$, validity of the above equalities for k = 1, 2 implies b = c = 0 if t = 1 and a = d = 0 if t = -1. So, x_1, z_2, z_3, y_4 are product vectors.

2) The pair (z_2, z_3) has form 2, i.e.

$$z_2 = \begin{bmatrix} p \\ q \end{bmatrix} \otimes \begin{bmatrix} \bar{\mu}_2 \\ t \end{bmatrix}, \ z_3 = \begin{bmatrix} x \\ y \end{bmatrix} \otimes \begin{bmatrix} \mu_2 \\ -t \end{bmatrix}, \ t = \pm 1, \ |p| + |q| \neq 0, \ |x| + |y| \neq 0.$$

By substituting the expressions for x_1, z_2, z_3, y_4 into (35) and by noting that

$$\left\langle \begin{array}{c} \bar{\mu}_2 \\ t \end{array} \middle| V_k \left| \begin{array}{c} \mu_2 \\ -t \end{array} \right\rangle = 0, \ t = \pm 1, \ k = 1, 2$$

we obtain

$$a \left\langle \begin{array}{c} p \\ q \end{array} \middle| A \left| \begin{array}{c} \mu_1 \\ 1 \end{array} \right\rangle \left\langle \begin{array}{c} \bar{\mu}_2 \\ t \end{array} \right| V_k \left| \begin{array}{c} \mu_2 \\ t \end{array} \right\rangle = \sigma_* \bar{c} \left\langle \begin{array}{c} \bar{\mu}_1 \\ 1 \end{array} \middle| A \left| \begin{array}{c} x \\ y \end{array} \right\rangle \left\langle \begin{array}{c} \bar{\mu}_2 \\ -t \end{array} \middle| V_k \left| \begin{array}{c} \mu_2 \\ -t \end{array} \right\rangle \quad \text{if } t = s$$

and

$$b\left\langle \begin{array}{c} p\\ q \end{array} \middle| A \left| \begin{array}{c} \mu_1\\ -1 \end{array} \right\rangle \left\langle \begin{array}{c} \bar{\mu}_2\\ t \end{array} \right| V_k \left| \begin{array}{c} \mu_2\\ t \end{array} \right\rangle = \sigma_* \bar{d} \left\langle \begin{array}{c} \bar{\mu}_1\\ -1 \end{array} \middle| A \left| \begin{array}{c} x\\ y \end{array} \right\rangle \left\langle \begin{array}{c} \bar{\mu}_2\\ -t \end{array} \middle| V_k \left| \begin{array}{c} \mu_2\\ -t \end{array} \right\rangle \quad \text{if } t = -s.$$

Validity of this equality for all $A \in \mathfrak{M}_2$ implies

$$a \nu_k^t \begin{vmatrix} \mu_1 \\ 1 \end{vmatrix} \left\langle \begin{array}{c} p \\ q \end{vmatrix} = \sigma_* \bar{c} \nu_k^{-t} \begin{vmatrix} x \\ y \end{vmatrix} \left\langle \begin{array}{c} \bar{\mu}_1 \\ 1 \end{vmatrix} \quad \text{if } t = s$$

and

$$b\nu_k^t \begin{vmatrix} \mu_1 \\ -1 \end{vmatrix} \begin{pmatrix} p \\ q \end{vmatrix} = \sigma_* \bar{d}\nu_k^{-t} \begin{vmatrix} x \\ y \end{pmatrix} \begin{pmatrix} \bar{\mu}_1 \\ -1 \end{vmatrix} \quad \text{if } t = -s,$$

where $\nu_1^t = \left\langle \begin{array}{c} \bar{\mu}_2 \\ t \end{array} \middle| V_1 \middle| \begin{array}{c} \mu_2 \\ t \end{array} \right\rangle = 2\mu_2^2$ and $\nu_2^t = \left\langle \begin{array}{c} \bar{\mu}_2 \\ t \end{array} \middle| V_2 \middle| \begin{array}{c} \mu_2 \\ t \end{array} \right\rangle = 2t\mu_2$. Since $\nu_1^t = \nu_1^{-t} \neq 0$ and $\nu_2^t = -\nu_2^{-t} \neq 0$, validity of the above equalities for k = 1, 2 implies a = c = 0 if t = s and b = d = 0 if t = -s. So, x_1, z_2, z_3, y_4 are product vectors.

3) The pair (z_2, z_3) has form 3, i.e.

$$z_2 = p \begin{bmatrix} \mu_1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} \bar{\mu}_2 \\ t \end{bmatrix} + q \begin{bmatrix} \mu_1 \\ -1 \end{bmatrix} \otimes \begin{bmatrix} \bar{\mu}_2 \\ -t \end{bmatrix}, \ z_3 = x \begin{bmatrix} \bar{\mu}_1 \\ -1 \end{bmatrix} \otimes \begin{bmatrix} \mu_2 \\ t \end{bmatrix} + y \begin{bmatrix} \bar{\mu}_1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} \mu_2 \\ -t \end{bmatrix},$$

where $t = \pm 1$. If we substitute the expressions for x_1, z_2, z_3, y_4 into (34) (by using (39)) then the left and the right hand sides of this equality will be equal respectively to

$$\bar{x}b\left\langle \begin{array}{c} \bar{\mu}_{1} \\ -1 \end{array} \middle| U_{k} \middle| \begin{array}{c} \mu_{1} \\ -1 \end{array} \right\rangle \left\langle \begin{array}{c} \mu_{2} \\ t \end{array} \middle| A \middle| \begin{array}{c} \mu_{2} \\ -s \end{array} \right\rangle + \bar{y}a\left\langle \begin{array}{c} \bar{\mu}_{1} \\ 1 \end{array} \middle| U_{k} \middle| \begin{array}{c} \mu_{1} \\ 1 \end{array} \right\rangle \left\langle \begin{array}{c} \mu_{2} \\ -t \end{array} \middle| A \middle| \begin{array}{c} \mu_{2} \\ s \end{array} \right\rangle$$

and to

$$-\sigma_{*}\bar{c}p\left\langle \begin{array}{c} \bar{\mu}_{1} \\ 1 \end{array} \middle| U_{k} \middle| \begin{array}{c} \mu_{1} \\ 1 \end{array} \right\rangle \left\langle \begin{array}{c} \bar{\mu}_{2} \\ -s \end{array} \middle| A \middle| \begin{array}{c} \bar{\mu}_{2} \\ t \end{array} \right\rangle - \sigma_{*}\bar{d}q\left\langle \begin{array}{c} \bar{\mu}_{1} \\ -1 \end{array} \middle| U_{k} \middle| \begin{array}{c} \mu_{1} \\ -1 \end{array} \right\rangle \left\langle \begin{array}{c} \bar{\mu}_{2} \\ s \end{array} \middle| A \middle| \begin{array}{c} \bar{\mu}_{2} \\ -t \end{array} \right\rangle$$

So, validity of the equality for all $A \in \mathfrak{M}_2$ implies

$$\begin{bmatrix} \bar{y}a & \mu_2 \\ s & -t \end{bmatrix} + \sigma_* \bar{c}p & \bar{\mu}_2 \\ t & -s \end{bmatrix} = \varsigma_k \begin{bmatrix} \sigma_* \bar{d}q & \bar{\mu}_2 \\ -t & s \end{bmatrix} + \bar{x}b & \mu_2 \\ -s & t \end{bmatrix}$$

where $\varsigma_k \doteq -\lambda_k^-/\lambda_k^+ = (-1)^k$. This equality can be valid for k = 1, 2 only if the operators in the squared brackets are equal to zero. Since $\mu_2 \neq \pm \bar{\mu}_2$ by the assumption $\theta_2 \neq 0, \pi$, it follows that ya = cp = dq = xb = 0. It is easy to see that this implies that x_1, z_2, z_3, y_4 are product vectors.

The similar argumentation shows incompatibility of the system (33)-(37) (by reducing to the case of tensor product vectors) if the pair (z_2, z_3) has form 3 and the pair (x_1, y_4) has form 1 or 2.

Assume finally that the pair (x_1, y_4) has form 4, i.e.

$$x_1 = h \begin{bmatrix} \mu_1 \\ s \end{bmatrix} \otimes \begin{bmatrix} \mu_2 \\ t \end{bmatrix}, \ y_4 = \begin{bmatrix} \bar{\mu}_1 \\ -s \end{bmatrix} \otimes \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} \otimes \begin{bmatrix} \bar{\mu}_2 \\ -t \end{bmatrix}, \quad s, t = \pm 1,$$

and the pair (z_2, z_3) is arbitrary. We will show that (33)-(35) imply that y_4 is a product vector. So, in fact the pair (x_1, y_4) has form 1 or 2.

Assume that y_4 is not a product vector and denote the vectors $[\mu_1, s]^{\top}$ and $[\mu_2, t]^{\top}$ by $|s\rangle$ and $|t\rangle$. In this notations $|x_1\rangle = h|s \otimes t\rangle$.

In case a) it follows from (34) and Lemma 10 in the Appendix that $|x_2\rangle = |p \otimes t\rangle$ for some vector $|p\rangle$. Hence the left hand side of (33) has the form

$$|h|^{2}|s\rangle\langle s|\otimes|t\rangle\langle t|+|p\rangle\langle p|\otimes|t\rangle\langle t|=\left[|h|^{2}|s\rangle\langle s|+|p\rangle\langle p|\right]\otimes|t\rangle\langle t|$$

and (33) implies $|y_4\rangle\langle y_4| \leq [|h|^2|s\rangle\langle s| + |p\rangle\langle p|] \otimes |t\rangle\langle t|$. This operator inequality can be valid only if y_4 is a product vector. In case b) it follows from (35) and Lemma 10 in the Appendix that $|x_3\rangle = |s \otimes q\rangle$ for some vector $|q\rangle$. Hence the left hand side of (33) has the form

$$|h|^{2}|s\rangle\langle s|\otimes|t\rangle\langle t|+|s\rangle\langle s|\otimes|q\rangle\langle q|=|s\rangle\langle s|\otimes\left[|h|^{2}|t\rangle\langle t|+|q\rangle\langle q|\right]$$

and similarly to the case a) we conclude that y_4 is a product vector.

By using the same argumentation exploiting (33)-(35) and Lemma 10 one can show that neither (x_1, y_4) nor (z_2, z_3) can be a pair of form 4 or 5 (not coinciding with form 1 or 2).

Thus, we have shown that the system (15)-(26) has no nontrivial solutions. This completes the proof of assertion C_2 . \Box

Appendix

3.1 Proofs of Lemmas 3 and 4

Proof of Lemma 3. A) Let $\langle z_4 | = [a, b, c, d]$ and

$$W = \begin{bmatrix} a & \gamma_2 b & \gamma_1 c & \gamma_1 \gamma_2 d \\ b & a & \gamma_1 d & \gamma_1 c \\ c & \gamma_2 d & a & \gamma_2 b \\ d & c & b & a \end{bmatrix}, \quad S = \begin{bmatrix} \mu_1 \mu_2 & \mu_1 \mu_2 & \mu_1 \mu_2 & \mu_1 \mu_2 \\ \mu_1 & -\mu_1 & \mu_1 & -\mu_1 \\ \mu_2 & \mu_2 & -\mu_2 & -\mu_2 \\ +1 & -1 & -1 & +1 \end{bmatrix},$$

where $\mu_k = \sqrt{\gamma_k}$, k = 1, 2. By identifying $A \otimes B$ with the matrix $||a_{ij}B||$ the equalities $\langle z_4 | U_k \otimes V_l | z_1 \rangle = 0$, k, l = 1, 2 can be rewritten as the system of linear equations

$$W|z_1\rangle = 0 \tag{40}$$

and it is easy to see that $S^{-1}WS = \text{diag}\{p_1, p_2, p_3, p_4\}$, where

$$p_1 = a + \mu_2 b + \mu_1 c + \mu_1 \mu_2 d, \qquad p_2 = a - \mu_2 b + \mu_1 c - \mu_1 \mu_2 d, p_3 = a + \mu_2 b - \mu_1 c - \mu_1 \mu_2 d, \qquad p_4 = a - \mu_2 b - \mu_1 c + \mu_1 \mu_2 d.$$
(41)

So, system (40) can be rewritten as the system $p_k u_k = 0$, $k = \overline{1, 4}$, where $[u_1, u_2, u_3, u_4]^{\top} = S^{-1} |z_1\rangle$. Hence this system has nontrivial solutions if and only if $p_1 p_2 p_3 p_4 = 0$ and

$$\{p_k = 0\} \Leftrightarrow \{W|q_k\rangle = 0\},\$$

where $|q_k\rangle$ is the k-th column of the matrix S.

Thus, by choosing some of p_1, \ldots, p_4 equal to zero we obtain all pairs (z_1, z_4) such that $\langle z_4 | U_k \otimes V_l | z_1 \rangle = 0$, k, l = 1, 2. We have

a) $C_4^2 = 6$ possibilities to take $p_k = p_l = 0$ and $p_i \neq 0, i \neq k, l;$

b) $C_4^1 = 4$ possibilities to take $p_k = 0$ and $p_i \neq 0, i \neq k$;

c)
$$C_4^3 = 4$$
 possibilities to take $p_k = p_l = p_j = 0$ and $p_i \neq 0, i \neq k, l, j$.

(the case $p_1 = p_2 = p_3 = p_4 = 0$ means that a = b = c = d = 0, so it gives only trivial solution).

By identifying the vectors $x \otimes y$ and $[x_1y, x_2y]^{\top}$ it is easy to see that

$$|q_1\rangle = \begin{bmatrix} \mu_1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} \mu_2 \\ 1 \end{bmatrix}, |q_2\rangle = \begin{bmatrix} \mu_1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} \mu_2 \\ -1 \end{bmatrix}, |q_3\rangle = \begin{bmatrix} \mu_1 \\ -1 \end{bmatrix} \otimes \begin{bmatrix} \mu_2 \\ 1 \end{bmatrix}, |q_4\rangle = \begin{bmatrix} \mu_1 \\ -1 \end{bmatrix} \otimes \begin{bmatrix} \mu_2 \\ -1 \end{bmatrix}$$

and that

$$p_{1} = 0 \quad \Leftrightarrow \quad |z_{4}\rangle = \begin{bmatrix} c_{1} \\ c_{2} \end{bmatrix} \otimes \begin{bmatrix} \bar{\mu}_{2} \\ -1 \end{bmatrix} + \begin{bmatrix} \bar{\mu}_{1} \\ -1 \end{bmatrix} \otimes \begin{bmatrix} c_{3} \\ c_{4} \end{bmatrix}, \quad c_{1}, \dots c_{4} \in \mathbb{C},$$

$$p_{2} = 0 \quad \Leftrightarrow \quad |z_{4}\rangle = \begin{bmatrix} c_{1} \\ c_{2} \end{bmatrix} \otimes \begin{bmatrix} \bar{\mu}_{2} \\ 1 \end{bmatrix} + \begin{bmatrix} \bar{\mu}_{1} \\ -1 \end{bmatrix} \otimes \begin{bmatrix} c_{3} \\ c_{4} \end{bmatrix}, \quad c_{1}, \dots c_{4} \in \mathbb{C},$$

$$p_{3} = 0 \quad \Leftrightarrow \quad |z_{4}\rangle = \begin{bmatrix} c_{1} \\ c_{2} \end{bmatrix} \otimes \begin{bmatrix} \bar{\mu}_{2} \\ -1 \end{bmatrix} + \begin{bmatrix} \bar{\mu}_{1} \\ 1 \end{bmatrix} \otimes \begin{bmatrix} c_{3} \\ c_{4} \end{bmatrix}, \quad c_{1}, \dots c_{4} \in \mathbb{C},$$

$$p_{4} = 0 \quad \Leftrightarrow \quad |z_{4}\rangle = \begin{bmatrix} c_{1} \\ c_{2} \end{bmatrix} \otimes \begin{bmatrix} \bar{\mu}_{2} \\ 1 \end{bmatrix} + \begin{bmatrix} \bar{\mu}_{1} \\ 1 \end{bmatrix} \otimes \begin{bmatrix} c_{3} \\ c_{4} \end{bmatrix}, \quad c_{1}, \dots c_{4} \in \mathbb{C},$$

Hence the above six possibilities in a) correspond to cases 1)-3) in Lemma 3A (for example, the choice $p_1 = p_2 = 0$, $p_3, p_4 \neq 0$ corresponds to case 1) with s = 1), while the four possibilities in b) and in c) correspond respectively to cases 4) and 5).

B) Denote by W_x and W_y the above matrix W determined respectively via $z_4 = x_4$ and $z_4 = y_4$. Then the equalities in (23) and (24) can be rewritten as the system

$$W_x|y_1\rangle = W_y|x_1\rangle = 0, \qquad W_x|x_1\rangle = W_y|y_1\rangle = |c\rangle, \qquad |c\rangle \in \mathbb{C}^4.$$
 (42)

Since $S^{-1}W_xS = \text{diag}\{p_1^x, p_2^x, p_3^x, p_4^x\}$ and $S^{-1}W_yS = \text{diag}\{p_1^y, p_2^y, p_3^y, p_4^y\}$, where $p_1^x, p_2^x, p_3^x, p_4^x$ and $p_1^y, p_2^y, p_3^y, p_4^y$ are defined in (41) with $z_4 = x_4$ and $z_4 = y_4$ correspondingly, system (42) is equivalent to the following one

$$p_k^x v_k = p_k^y u_k = 0, \qquad p_k^x u_k = p_k^y v_k = \tilde{c}_k, \qquad k = \overline{1, 4},$$
 (43)

where $[u_1, u_2, u_3, u_4]^{\top} = S^{-1} |x_1\rangle$, $[v_1, v_2, v_3, v_4]^{\top} = S^{-1} |y_1\rangle$ and $[\tilde{c}_1, \tilde{c}_2, \tilde{c}_3, \tilde{c}_4]^{\top} = S^{-1} |c\rangle$. It follows that $\tilde{c}_k = 0$ for all k. Indeed, if $p_k^y \neq 0$ for some k then the first equality in (43) implies $u_k = 0$ and the second equality in (43) shows that $\tilde{c}_k = 0$. Hence $|c\rangle = S |\tilde{c}\rangle = 0$. \Box

Lemma 4 follows from Lemma 3 with γ_2 replaced by $\bar{\gamma}_2$.

3.2 Auxiliary lemmas

Lemma 8. If $|a\rangle\langle x| + |b\rangle\langle y| + |c\rangle\langle z| = 0$ then either a ||b||c or x ||y||z.

Proof. We may assume that all the vectors are nonzero (since otherwise the assertion is trivial).

Let $p \perp x$. Then $\langle y|p\rangle |b\rangle + \langle z|p\rangle |c\rangle = 0$ and hence either $b \parallel c$ or $\langle y|p\rangle = \langle z|p\rangle = 0$.

If $b \parallel c$ then we have $|a\rangle\langle x| = -|b\rangle\langle y + \lambda z|, \lambda \in \mathbb{C}$, and hence $a \parallel b \parallel c$. If $\langle y|p\rangle = \langle z|p\rangle = 0$ then $x \parallel y \parallel z$, since the vector p is arbitrary. \Box

Lemma 9. The equality

$$X_1 \otimes Y_1 + X_2 \otimes Y_2 = X_3 \otimes Y_3 + X_4 \otimes Y_4, \tag{44}$$

where $X_i = |x_i\rangle\langle x_i|$, $Y_i = |y_i\rangle\langle y_i|$, $i = \overline{1, 4}$, can be valid only in the following cases:

- 1) $x_i \parallel x_j$ for all i, j and $Y_1 \parallel x_1 \parallel^2 + Y_2 \parallel x_2 \parallel^2 = Y_3 \parallel x_3 \parallel^2 + Y_4 \parallel x_4 \parallel^2;$
- 2) $y_i \parallel y_j$ for all i, j and $X_1 \parallel y_1 \parallel^2 + X_2 \parallel y_2 \parallel^2 = X_3 \parallel y_3 \parallel^2 + X_4 \parallel y_4 \parallel^2$;
- 3) $X_1 \otimes Y_1 = X_4 \otimes Y_4$ and $X_2 \otimes Y_2 = X_3 \otimes Y_3$;
- 4) $X_1 \otimes Y_1 = X_3 \otimes Y_3$ and $X_2 \otimes Y_2 = X_4 \otimes Y_4$.

Proof. We may assume that all the vectors x_i, y_i are nonzero (since otherwise the assertion is trivial).

Let $p \perp x_1$. By multiplying the both sides of (44) by $|p\rangle\langle p|\otimes I$ we obtain

$$|\langle x_2|p\rangle|^2 Y_2 = |\langle x_3|p\rangle|^2 Y_3 + |\langle x_4|p\rangle|^2 Y_4.$$
(45)

If $x_2 \parallel x_1$ then $\langle x_3 | p \rangle = \langle x_4 | p \rangle = 0$ and hence $x_1 \parallel x_2 \parallel x_3 \parallel x_4$, since the vector p is arbitrary. So, case 1) holds.

If $x_2 \not\parallel x_1$ then one can choose p such that $\langle x_2 | p \rangle \neq 0$. So, (45) implies that either $x_3 \not\parallel x_1$ or $x_4 \not\parallel x_1$. Thus, we have the following possibilities:

a) If $x_i \not\parallel x_1$ for i = 2, 3, 4 then one can choose p such that $\langle x_i | p \rangle \neq 0$, i = 2, 3, 4, and (45) implies $y_2 \parallel y_3 \parallel y_4$. Hence (44) leads to $X_1 \otimes Y_1 = [\ldots] \otimes Y_2$, which gives $y_1 \parallel y_2$. So, we have $y_1 \parallel y_2 \parallel y_3 \parallel y_4$, i.e. case 2).

b) If $x_i \not\parallel x_1$ for i = 2, 3 but $x_4 \parallel x_1$ then one can choose p such that $\langle x_i | p \rangle \neq 0$, i = 2, 3 and (45) implies $y_2 \parallel y_3$. So, we have $x_4 = \alpha x_1$ and $y_3 = \beta y_2, \alpha, \beta \in \mathbb{C}$. It follows from (44) that

$$X_1 \otimes [Y_1 - |\alpha|^2 Y_4] = [X_3|\beta|^2 - X_2] \otimes Y_2$$

and hence $Y_1 - |\alpha|^2 Y_4 = \lambda Y_2$, $\lambda \in \mathbb{C}$. If $\lambda \neq 0$ then Lemma 8 implies $y_1 \parallel y_2 \parallel y_3 \parallel y_4$, i.e. case 2) holds. If $\lambda = 0$ then $y_1 \parallel y_4$ and $x_2 \parallel x_3$. Thus we have

$$X_4 \otimes Y_4 = \gamma X_1 \otimes Y_1, \quad X_3 \otimes Y_3 = \delta X_2 \otimes Y_2, \quad \gamma, \delta \in \mathbb{C}$$

and (44) implies $(1 - \gamma)X_1 \otimes Y_1 = (\delta - 1)X_2 \otimes Y_2$. Since $x_1 \not\parallel x_2$, we have $\gamma = \delta = 1$, i.e. case 3) holds.

c) If $x_i \not\parallel x_1$ for i = 2, 4 but $x_3 \parallel x_1$ then the similar arguments (with the permutation $3 \leftrightarrow 4$) shows that case 4) holds.

Lemma 10. Let $U_1 = \text{diag}\{1, \gamma\}$ and x, y be nonzero vectors in \mathbb{C}^2 . If $\langle a | U_1 \otimes A | x \otimes y \rangle = \langle c | U_1 \otimes A | d \rangle$ for all $A \in \mathfrak{M}_2$ then either $|d\rangle = |z\rangle \otimes |y\rangle$ or $|c\rangle = |p\rangle \otimes |q\rangle$ for some vectors p, q, z in \mathbb{C}^2 .

Proof. By using the isomorphism $\mathbb{C}^2 \otimes \mathbb{C}^2 \ni u \otimes v \leftrightarrow [u_1 v, u_2 v]^\top \in \mathbb{C}^2 \oplus \mathbb{C}^2$ the condition of the lemma can be rewritten as follows

$$\left\langle \begin{array}{c|c} a_1 & A & 0 \\ a_2 & 0 & \gamma A \end{array} \middle| \begin{array}{c} x_1 y \\ x_2 y \end{array} \right\rangle = \left\langle \begin{array}{c|c} c_1 & A & 0 \\ c_2 & 0 & \gamma A \end{array} \middle| \begin{array}{c} d_1 \\ d_2 \end{array} \right\rangle, \quad \forall A \in \mathfrak{M}_2,$$

where a_1, a_2 are components of the vector a, etc. So, we have

$$x_1 \langle a_1 | A | y \rangle + x_2 \gamma \langle a_2 | A | y \rangle = \langle c_1 | A | d_1 \rangle + \gamma \langle c_2 | A | d_2 \rangle \quad \forall A \in \mathfrak{M}_2,$$

which is equivalent to the equality $|y\rangle\langle \bar{x}_1a_1+\bar{x}_2\bar{\gamma}a_2| = |d_1\rangle\langle c_1|+\gamma|d_2\rangle\langle c_2|$. By Lemma 8 it follows that either $d_1 \parallel d_2 \parallel y$, which means that $|d\rangle = |z\rangle \otimes |y\rangle$, or $c_1 \parallel c_2$, which means that $|c\rangle = |p\rangle \otimes |q\rangle$.

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