# On n-partite superactivation of quantum channel capacities 

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#### Abstract

A generalization of the superactivation of quantum channel capacities to the case of $n>2$ channels is considered. An explicit example of such superactivation for the 1 -shot quantum zero-error capacity is constructed for $n=3$.

Some implications of this example and its reformulation on terms of quantum measurements are described.


## 1 General observations

The superactivation of quantum channel capacities is one of the most impressive quantum effects having no classical counterpart. It means that the particular capacity $C$ of the tensor product of two quantum channels $\Phi_{1}$ and $\Phi_{2}$ may be positive despite the same capacity of each of these channels is zero, i.e.

$$
\begin{equation*}
C\left(\Phi_{1} \otimes \Phi_{2}\right)>0 \quad \text { while } \quad C\left(\Phi_{1}\right)=C\left(\Phi_{2}\right)=0 \tag{1}
\end{equation*}
$$

This effect was originally observed by G.Smith and J.Yard for the case of quantum $\varepsilon$-error capacity [12]. Then the possibility of superactivation of other capacities, in particular, zero-error capacities was shown [2, 3, 4, 11].

It seems reasonable to consider the generalization of the above effect to the case of $n>2$ channels $\Phi_{1}, \ldots, \Phi_{n}$ consisting in the following property

$$
\begin{equation*}
C\left(\Phi_{1} \otimes \ldots \otimes \Phi_{n}\right)>0 \quad \text { while } \quad C\left(\Phi_{i_{1}} \otimes \ldots \otimes \Phi_{i_{k}}\right)=0 \tag{2}
\end{equation*}
$$

for any proper subset $\Phi_{i_{1}}, \ldots, \Phi_{i_{k}}(k<n)$ of the set $\Phi_{1}, \ldots, \Phi_{n}$. This property can be called $n$-partite superactivation of the capacity $C$.

[^0]Property (2) means that all the channels $\Phi_{1}, \ldots, \Phi_{n}$ are required to transmit (classical or quantum) information by using the protocol corresponding to the capacity $C$, i.e. excluding any channel from the set $\Phi_{1}, \ldots, \Phi_{n}$ makes other channels useless for information transmission.

The obvious difficulty in finding channels $\Phi_{1}, \ldots, \Phi_{n}$ demonstrating property (2) for given capacity $C$ consists in necessity to prove the vanishing of $C\left(\Phi_{i_{1}} \otimes \ldots \otimes \Phi_{i_{k}}\right)$ for any subset $\Phi_{i_{1}}, \ldots, \Phi_{i_{k}}$.

If $C$ is the 1 -shot capacity of some protocol of information transmission and $\Phi_{i}=\Phi$ for all $i=\overline{1, n}$ then (22) means that the $n$-shot capacity of this protocol is positive while the corresponding $(n-1)$-shot capacity is zero.

In [10] it is shown how to construct for any $n$ a channel $\Psi_{n}$ such that

$$
\begin{equation*}
\bar{Q}_{0}\left(\Psi_{n}^{\otimes n}\right)=0 \quad \text { and } \quad \bar{Q}_{0}\left(\Psi_{n}^{\otimes m}\right)>0 \tag{3}
\end{equation*}
$$

where $\bar{Q}_{0}$ is the 1 -shot quantum zero-error capacity and $m$ is a natural number satisfying the inequality $n / m \leq 2 \ln (3 / 2) / \pi$ (implying $m>n$ ). It follows that there is $\tilde{n}>n$ not exceeding $m$ such that (2) holds for $n=\tilde{n}$, $C=\bar{Q}_{0}$ and $\Phi_{1}=\ldots=\Phi_{\tilde{n}}=\Psi_{n}$. Unfortunately, we can not specify the number $\tilde{n}$ in that construction.

In this paper we modify the example in [10] (by extending its noncommutative graph) to construct a family of channels $\left\{\Phi_{\theta}\right\}$ with $d_{A}=4$ and $d_{E}=3$ having the following property

$$
\begin{equation*}
\bar{Q}_{0}\left(\Phi_{\theta_{1}} \otimes \Phi_{\theta_{2}} \otimes \Phi_{\theta_{3}}\right)>0 \quad \text { while } \quad \bar{Q}_{0}\left(\Phi_{\theta_{i}} \otimes \Phi_{\theta_{j}}\right)=0 \quad \forall i \neq j \tag{4}
\end{equation*}
$$

where $\theta_{1}, \theta_{2}, \theta_{3}$ are positive numbers such that $\theta_{1}+\theta_{2}+\theta_{3}=\pi$. Thus, the channels $\Phi_{\theta_{1}}, \Phi_{\theta_{2}}, \Phi_{\theta_{3}}$ demonstrate the 3-partite superactivation of the 1-shot quantum zero-error capacity.

Property (4) means that all the channels $\Phi_{\theta_{i}}$ and all the bipartite channels $\Phi_{\theta_{i}} \otimes \Phi_{\theta_{j}}$ have no ideal (noiseless or reversible) subchannels, but the tripartite channel $\Phi_{\theta_{1}} \otimes \Phi_{\theta_{2}} \otimes \Phi_{\theta_{3}}$ has.

By using the observation in [9, Section 4] superactivation property (4) can be reformulated in terms of quantum measurements theory as the existence of quantum observables $\mathcal{M}_{\theta_{1}}, \mathcal{M}_{\theta_{2}}, \mathcal{M}_{\theta_{3}}$ such that all the observables $\mathcal{M}_{\theta_{i}}$ and all the bipartite observables $\mathcal{M}_{\theta_{i}} \otimes \mathcal{M}_{\theta_{j}}$ have no indistinguishable subspaces but the tripartite observable $\mathcal{M}_{\theta_{1}} \otimes \mathcal{M}_{\theta_{2}} \otimes \mathcal{M}_{\theta_{3}}$ has.

## 2 Preliminaries

Let $\Phi: \mathfrak{S}\left(\mathcal{H}_{A}\right) \rightarrow \mathfrak{S}\left(\mathcal{H}_{B}\right)$ be a quantum channel, i.e. a completely positive trace-preserving linear map [6, 8]. It has the Kraus representation

$$
\begin{equation*}
\Phi(\rho)=\sum_{k} V_{k} \rho V_{k}^{*} \tag{5}
\end{equation*}
$$

where $V_{k}$ are linear operators from $\mathcal{H}_{A}$ into $\mathcal{H}_{B}$ such that $\sum_{k} V_{k}^{*} V_{k}=I_{\mathcal{H}_{A}}$. The minimal number of summands in such representation is called Choi rank of $\Phi$ and is denoted $d_{E}$ (while $d_{A} \doteq \operatorname{dim} \mathcal{H}_{A}$ and $d_{B} \doteq \operatorname{dim} \mathcal{H}_{B}$ ).

The 1 -shot quantum zero-error capacity $\bar{Q}_{0}(\Phi)$ of a channel $\Phi$ is defined as $\sup _{\mathcal{H} \in q_{0}(\Phi)} \log _{2} \operatorname{dim} \mathcal{H}$, where $q_{0}(\Phi)$ is the set of all subspaces $\mathcal{H}_{0}$ of $\mathcal{H}_{A}$ on which the channel $\Phi$ is perfectly reversible (in the sense that there is a channel $\Theta$ such that $\Theta(\Phi(\rho))=\rho$ for all states $\rho$ supported by $\mathcal{H}_{0}$ ). Any subspace $\mathcal{H}_{0} \in q_{0}(\Phi)$ is called error correcting code for the channel $\Phi$ [5, 6].

The (asymptotic) quantum zero-error capacity is defined by regularization: $Q_{0}(\Phi)=\sup _{n} n^{-1} \bar{Q}_{0}\left(\Phi^{\otimes n}\right)$ [1, 3, 5].

The quantum zero-error capacity of a channel $\Phi$ is determined by its noncommutative graph $\mathcal{G}(\Phi)$, which can be defined as the subspace of $\mathfrak{B}\left(\mathcal{H}_{A}\right)$ spanned by the operators $V_{k}^{*} V_{l}$, where $V_{k}$ are operators from any Kraus representation (5) of $\Phi$ (5]. In particular, the Knill-Laflamme error-correcting condition [7] implies the following lemma.

Lemma 1. A set $\left\{\varphi_{k}\right\}_{k=1}^{d}$ of unit orthogonal vectors in $\mathcal{H}_{A}$ is a basis of error-correcting code for a channel $\Phi: \mathfrak{S}\left(\mathcal{H}_{A}\right) \rightarrow \mathfrak{S}\left(\mathcal{H}_{B}\right)$ if and only if

$$
\begin{equation*}
\left\langle\varphi_{l}\right| A\left|\varphi_{k}\right\rangle=0 \quad \text { and } \quad\left\langle\varphi_{l}\right| A\left|\varphi_{l}\right\rangle=\left\langle\varphi_{k}\right| A\left|\varphi_{k}\right\rangle \quad \forall A \in \mathfrak{L}, \quad \forall k \neq l, \tag{6}
\end{equation*}
$$

where $\mathfrak{L}$ is any subset of $\mathfrak{B}\left(\mathcal{H}_{A}\right)$ such that $\operatorname{lin} \mathfrak{L}=\mathcal{G}(\Phi)$.
Since a subspace $\mathfrak{L}$ of the algebra $\mathfrak{M}_{n}$ of $n \times n$ matrices is a noncommutative graph of a particular channel if and only if

$$
\begin{equation*}
\mathfrak{L} \text { is symmetric }\left(\mathfrak{L}=\mathfrak{L}^{*}\right) \text { and contains the unit matrix } \tag{7}
\end{equation*}
$$

(see Lemma 2 in [4] or the Appendix in [9]), Lemma [1 shows that one can "construct" a channel $\Phi$ with $\operatorname{dim} \mathcal{H}_{A}=n$ having positive (correspondingly, zero) 1-shot quantum zero-error capacity by taking a subspace $\mathfrak{L} \subset \mathfrak{M}_{n}$ satisfying (7) for which the following condition is valid (correspondingly, not valid)

$$
\begin{equation*}
\exists \varphi, \psi \in\left[\mathbb{C}^{n}\right]_{1} \text { s.t. }\langle\psi| A|\varphi\rangle=0 \text { and }\langle\varphi| A|\varphi\rangle=\langle\psi| A|\psi\rangle \quad \forall A \in \mathfrak{L}, \tag{8}
\end{equation*}
$$

where $\left[\mathbb{C}^{n}\right]_{1}$ is the unit sphere of $\mathbb{C}^{n}$.

## 3 Example of 3-partite superactivation

For given $\theta \in(-\pi, \pi]$ consider the 8 -D subspace

$$
\mathfrak{N}_{\theta}=\left\{M=\left[\begin{array}{cccc}
a & b & e & f  \tag{9}\\
c & d & f & \bar{\gamma} e \\
g & h & a & b \\
h & \gamma g & c & d
\end{array}\right], \quad a, b, c, d, e, f, g, h \in \mathbb{C}\right\}
$$

of $\mathfrak{M}_{4}$ satisfying condition (7), where $\gamma=\exp [i \theta]$. This subspace is an extension of the 4-D subspace $\mathfrak{L}_{\theta}$ used in [10], i.e. $\mathfrak{L}_{\theta} \subset \mathfrak{N}_{\theta}$ for each $\left.\theta 1\right]$

Denote by $\widehat{\mathfrak{N}}_{\theta}$ the set of all channels whose noncommutative graph coincides with $\mathfrak{N}_{\theta}$. In [9, the Appendix] it is shown how to explicitly construct pseudo-diagonal channels in $\widehat{\mathfrak{N}}_{\theta}$ with $d_{A}=4$ and $d_{E} \geq 3$ (since $\operatorname{dim} \mathfrak{N}_{\theta}=8 \leq 3^{2}$ )

Theorem 1. Let $\Phi_{\theta}$ be a channel in $\widehat{\mathfrak{N}}_{\theta}$ and $n \in \mathbb{N}$ be arbitrary.
A) $\bar{Q}_{0}\left(\Phi_{\theta}\right)>0$ if and only if $\theta=\pi$ and $\bar{Q}_{0}\left(\Phi_{\pi}\right)=1$.
B) If $\theta_{1}+\ldots+\theta_{n}=\pi(\bmod 2 \pi)$ then $\bar{Q}_{0}\left(\Phi_{\theta_{1}} \otimes \ldots \otimes \Phi_{\theta_{n}}\right)>0$ and $2-D$ error-correcting code for the channel $\Phi_{\theta_{1}} \otimes \ldots \otimes \Phi_{\theta_{n}}$ is spanned by the vectors

$$
\begin{equation*}
|\varphi\rangle=\frac{1}{\sqrt{2}}[|1 \ldots 1\rangle+\mathrm{i}|2 \ldots 2\rangle],|\psi\rangle=\frac{1}{\sqrt{2}}[|3 \ldots 3\rangle+\mathrm{i}|4 \ldots 4\rangle], \tag{10}
\end{equation*}
$$

where $\{|1\rangle, \ldots,|4\rangle\}$ is the canonical basis in $\mathbb{C}^{4}$.
$\mathbf{C}_{2}$ ) If $\left|\theta_{1}\right|+\left|\theta_{2}\right|<\pi$ then $\bar{Q}_{0}\left(\Phi_{\theta_{1}} \otimes \Phi_{\theta_{2}}\right)=0$.
$\mathrm{C}_{\mathrm{n}}$ ) If $\left|\theta_{1}\right|+\ldots+\left|\theta_{n}\right| \leq 2 \ln (3 / 2)$ then $\bar{Q}_{0}\left(\Phi_{\theta_{1}} \otimes \ldots \otimes \Phi_{\theta_{n}}\right)=0$.
Assertion $\mathrm{C}_{2}$ is the main progress of this theorem in comparison with Theorem 1 in [10]. It complements assertion B with $n=2$. It is the proof of assertion $\mathrm{C}_{2}$ that motivates the extension $\mathfrak{L}_{\theta} \rightarrow \mathfrak{N}_{\theta}$.

Remark 1. Since assertion $\mathrm{C}_{\mathrm{n}}$ is proved by using quite coarse estimates, the other assertions of Theorem 1 make it reasonable to conjecture validity of the following strengthened version:

$$
\left.\mathrm{C}_{\mathrm{n}}^{*}\right) \text { If }\left|\theta_{1}\right|+\ldots+\left|\theta_{n}\right|<\pi \text { then } \bar{Q}_{0}\left(\Phi_{\theta_{1}} \otimes \ldots \otimes \Phi_{\theta_{n}}\right)=0 .
$$

The below proof of $\mathrm{C}_{2}$ shows difficulty of the direct proof of this conjecture.

[^1]Theorem 1 implies the following example of 3-partite superactivation of 1 -shot quantum zero-error capacity.

Corollary 1. Let $\theta_{1}, \theta_{2}, \theta_{3}$ be positive numbers such that $\theta_{1}+\theta_{2}+\theta_{3}=\pi$. Then

$$
\bar{Q}_{0}\left(\Phi_{\theta_{1}} \otimes \Phi_{\theta_{2}} \otimes \Phi_{\theta_{3}}\right)>0 \quad \text { while } \quad \bar{Q}_{0}\left(\Phi_{\theta_{i}} \otimes \Phi_{\theta_{j}}\right)=0 \quad \forall i \neq j .
$$

2-D error-correcting code for the channel $\Phi_{\theta_{1}} \otimes \Phi_{\theta_{2}} \otimes \Phi_{\theta_{3}}$ is spanned by the vectors

$$
\begin{equation*}
|\varphi\rangle=\frac{1}{\sqrt{2}}[|111\rangle+\mathrm{i}|222\rangle],|\psi\rangle=\frac{1}{\sqrt{2}}[|333\rangle+\mathrm{i}|444\rangle] . \tag{11}
\end{equation*}
$$

If conjecture $\mathrm{C}_{\mathrm{n}}^{*}$ in Remark 1 is valid for some $n>2$ then the similar assertion holds for $n+1$ channels $\Phi_{\theta_{1}}, \ldots, \Phi_{\theta_{n+1}}$. This would give an example of $(n+1)$-partite superactivation of 1 -shot quantum zero-error capacity.

For each $\theta$ one can choose (non-uniquely) a basis $\left\{M_{k}^{\theta}\right\}_{k=1}^{8}$ of the subspace $\mathfrak{N}_{\theta}$ consisting of positive operators such that $\sum_{k=1}^{8} M_{k}^{\theta}=I_{\mathcal{H}_{A}}$ (since the subspace $\mathfrak{N}_{\theta}$ satisfies condition (7), see [9]). This basis can be considered as a quantum observable $\mathcal{M}_{\theta}$. By using Proposition 1 in [9] and Lemma 1 Corollary 1 can be reformulated in terms of theory of quantum measurements.

Corollary 2. Let $\theta_{1}, \theta_{2}, \theta_{3}$ be positive numbers such that $\theta_{1}+\theta_{2}+\theta_{3}=\pi$. Then all the observables $\mathcal{M}_{\theta_{i}}$ and all the bipartite observables $\mathcal{M}_{\theta_{i}} \otimes \mathcal{M}_{\theta_{j}}$ have no indistinguishable subspaces but the tripartite observable $\mathcal{M}_{\theta_{1}} \otimes \mathcal{M}_{\theta_{2}} \otimes \mathcal{M}_{\theta_{3}}$ has indistinguishable subspace spanned by the vectors (11). $2^{2}$

Note also that Theorem 1 implies the following example of superactivation of 2 -shot quantum zero-error capacity.

Corollary 3. Let $\theta_{1}, \theta_{2}$ be positive numbers such that $\theta_{1}+\theta_{2}=\pi / 2$. Then
$\bar{Q}_{0}\left(\left[\Phi_{\theta_{1}} \otimes \Phi_{\theta_{2}}\right]^{\otimes 2}\right)>0 \quad$ while $\quad \bar{Q}_{0}\left(\Phi_{\theta_{1}}^{\otimes 2}\right)=\bar{Q}_{0}\left(\Phi_{\theta_{2}}^{\otimes 2}\right)=\bar{Q}_{0}\left(\Phi_{\theta_{1}} \otimes \Phi_{\theta_{2}}\right)=0$.
The proof of Theorem 1. The equality $\bar{Q}_{0}\left(\Phi_{\theta}\right)=0$ for $\theta \neq \pi$, the inequality $\bar{Q}_{0}\left(\Phi_{\pi}\right) \leq 1$ and assertion $\mathrm{C}_{\mathrm{n}}$ follows from Theorem 1 in [10], since the inclusion $\mathfrak{L}_{\theta} \subset \mathfrak{N}_{\theta}$ implies $\bar{Q}_{0}\left(\Phi_{\theta_{1}} \otimes \ldots \otimes \Phi_{\theta_{n}}\right) \leq \bar{Q}_{0}\left(\Psi_{\theta_{1}} \otimes \ldots \otimes \Psi_{\theta_{n}}\right)$ for any channels $\Psi_{\theta_{1}} \in \widehat{\mathfrak{L}}_{\theta_{1}}, \ldots, \Psi_{\theta_{n}} \in \widehat{\mathfrak{L}}_{\theta_{n}}$.

[^2]To prove that $\bar{Q}_{0}\left(\Phi_{\pi}\right) \geqq 1$ it suffices to show, by using Lemma 1, that the vectors $|\varphi\rangle=[1, \mathrm{i}, 0,0]^{\top}$ and $|\psi\rangle=[0,0,1, \mathrm{i}]^{\top}$ generate error-correcting code for the channel $\Phi_{\pi}$.
B) Let $M_{1} \in \mathfrak{N}_{\theta_{1}}, \ldots, M_{n} \in \mathfrak{N}_{\theta_{n}}$ be arbitrary, $X=M_{1} \otimes \ldots \otimes M_{n}$ and $\varphi, \psi$ be the vectors defined in (10). By Lemma 1 it suffices to show that

$$
\begin{equation*}
\langle\psi| X|\varphi\rangle=0 \quad \text { and } \quad\langle\psi| X|\psi\rangle=\langle\varphi| X|\varphi\rangle . \tag{12}
\end{equation*}
$$

Let $a_{k}, b_{k}, \ldots, h_{k}$ be elements of the matrix $M_{k}$. We have

$$
\begin{aligned}
& 2\langle\psi| X|\varphi\rangle=\langle 3 \ldots 3| X|1 \ldots 1\rangle+\mathrm{i}\langle 3 \ldots 3| X|2 \ldots 2\rangle-\mathrm{i}\langle 4 \ldots 4| X|1 \ldots 1\rangle \\
& \quad+\langle 4 \ldots 4| X|2 \ldots 2\rangle=g_{1} \ldots g_{n}\left(1+\gamma_{1} \ldots \gamma_{n}\right)+h_{1} \ldots h_{n}(\mathrm{i}-\mathrm{i})=0,
\end{aligned}
$$

since $\gamma_{1} \ldots \gamma_{n}=-1$ by the condition $\theta_{1}+\ldots+\theta_{n}=\pi(\bmod 2 \pi)$,

$$
\begin{aligned}
& 2\langle\varphi| X|\varphi\rangle=\langle 1 \ldots 1| X|1 \ldots 1\rangle+\mathrm{i}\langle 1 \ldots 1| X|2 \ldots 2\rangle-\mathrm{i}\langle 2 \ldots 2| X|1 \ldots 1\rangle \\
& \quad+\langle 2 \ldots 2| X|2 \ldots 2\rangle=a_{1} \ldots a_{n}+\mathrm{i}\left(b_{1} \ldots b_{n}-c_{1} \ldots c_{n}\right)+d_{1} \ldots d_{n}
\end{aligned}
$$

and

$$
\begin{aligned}
& 2\langle\psi| X|\psi\rangle=\langle 3 \ldots 3| X|3 \ldots 3\rangle+\mathrm{i}\langle 3 \ldots 3| X|4 \ldots 4\rangle-\mathrm{i}\langle 4 \ldots 4| X|3 \ldots 3\rangle \\
& \quad+\langle 4 \ldots 4| X|4 \ldots 4\rangle=a_{1} \ldots a_{n}+\mathrm{i}\left(b_{1} \ldots b_{n}-c_{1} \ldots c_{n}\right)+d_{1} \ldots d_{n} .
\end{aligned}
$$

Thus the both equalities in (12) are valid.
$\mathrm{C}_{2}$ ) To prove this assertion we have to show that the subspace $\mathfrak{N}_{\theta_{1}} \otimes \mathfrak{N}_{\theta_{2}}$ does not satisfy condition (8) if $\left|\theta_{1}\right|+\left|\theta_{2}\right|<\pi$. In the case $\theta_{1}=\theta_{2}=0$ this follows from assertion $\mathrm{C}_{\mathrm{n}}$. So, we may assume, by symmetry, that $\theta_{2} \neq 0$.

Throughout the proof we will use the isomorphism

$$
\mathbb{C}^{n} \otimes \mathbb{C}^{m} \ni x \otimes y \leftrightarrow\left[x_{1} y, \ldots, x_{n} y\right]^{\top} \in \underbrace{\mathbb{C}^{m} \oplus \ldots \oplus \mathbb{C}^{m}}_{n}
$$

and the corresponding isomorphism

$$
\begin{equation*}
\mathfrak{M}_{n} \otimes \mathfrak{M}_{m} \ni A \otimes B \leftrightarrow\left[a_{i j} B\right] \in \mathfrak{M}_{n m} \tag{13}
\end{equation*}
$$

Let $U_{1}, U_{2}, V_{1}, V_{2}$ be the unitary operators in $\mathbb{C}^{2}$ determined (in the canonical basis) by the matrices

$$
U_{1}=\left[\begin{array}{cc}
1 & 0 \\
0 & \gamma_{1}
\end{array}\right], \quad V_{1}=\left[\begin{array}{rr}
1 & 0 \\
0 & \gamma_{2}
\end{array}\right], \quad U_{2}=V_{2}=\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right] .
$$

We will identify $\mathbb{C}^{4}$ with $\mathbb{C}^{2} \oplus \mathbb{C}^{2}$. So, arbitrary matrices $M_{1} \in \mathfrak{N}_{\theta_{1}}$ and $M_{2} \in \mathfrak{N}_{\theta_{2}}$ can be represented as follows

$$
M_{1}=\left[\begin{array}{cc}
A_{1} & e_{1} U_{1}^{*}+f_{1} U_{2}^{*} \\
g_{1} U_{1}+h_{1} U_{2} & A_{1}
\end{array}\right], M_{2}=\left[\begin{array}{cc}
A_{2} & e_{2} V_{1}^{*}+f_{2} V_{2}^{*} \\
g_{2} V_{1}+h_{2} V_{2} & A_{2}
\end{array}\right]
$$

or, according to (13), as

$$
M_{1}=I_{2} \otimes A_{1}+|2\rangle\langle 1| \otimes\left[g_{1} U_{1}+h_{1} U_{2}\right]+|1\rangle\langle 2| \otimes\left[e_{1} U_{1}^{*}+f_{1} U_{2}^{*}\right]
$$

and

$$
M_{2}=I_{2} \otimes A_{2}+|2\rangle\langle 1| \otimes\left[g_{2} V_{1}+h_{2} V_{2}\right]+|1\rangle\langle 2| \otimes\left[e_{2} V_{1}^{*}+f_{2} V_{2}^{*}\right]
$$

where $A_{1}$ and $A_{2}$ are arbitrary matrices in $\mathfrak{M}_{2}$.
Assume the existence of orthogonal unit vectors $\varphi$ and $\psi$ in $\mathbb{C}^{4} \otimes \mathbb{C}^{4}$ such that

$$
\begin{equation*}
\langle\psi| M_{1} \otimes M_{2}|\varphi\rangle=0 \quad \text { and } \quad\langle\psi| M_{1} \otimes M_{2}|\psi\rangle=\langle\varphi| M_{1} \otimes M_{2}|\varphi\rangle \tag{14}
\end{equation*}
$$

for all $M_{1} \in \mathfrak{N}_{\theta_{1}}$ and $M_{2} \in \mathfrak{N}_{\theta_{2}}$.
By using the above representations of $M_{1}$ and $M_{2}$ we have

$$
\begin{gathered}
M_{1} \otimes M_{2}=\left[I_{2} \otimes I_{2}\right] \otimes\left[A_{1} \otimes A_{2}\right]+\left[I_{2} \otimes|2\rangle\langle 1|\right] \otimes\left[A_{1} \otimes\left[g_{2} V_{1}+h_{2} V_{2}\right]\right]+ \\
{\left[I_{2} \otimes|1\rangle\langle 2|\right] \otimes\left[A_{1} \otimes\left[e_{2} V_{1}^{*}+f_{2} V_{2}^{*}\right]\right]+\left[|2\rangle\langle 1| \otimes I_{2}\right] \otimes\left[\left[g_{1} U_{1}+h_{1} U_{2}\right] \otimes A_{2}\right]+\ldots}
\end{gathered}
$$

Since $\mathfrak{M}_{2} \otimes \mathfrak{M}_{2}=\mathfrak{M}_{4}$, by choosing $e_{i}=f_{i}=g_{i}=h_{i}=0, i=1,2$, we obtain from (14) that

$$
\langle\psi| I_{4} \otimes A|\varphi\rangle=0 \quad \text { and } \quad\langle\psi| I_{4} \otimes A|\psi\rangle=\langle\varphi| I_{4} \otimes A|\varphi\rangle \quad \forall A \in \mathfrak{M}_{4} .
$$

According to (13) we have

$$
I_{4} \otimes A=\left[\begin{array}{cccc}
A & 0 & 0 & 0 \\
0 & A & 0 & 0 \\
0 & 0 & A & 0 \\
0 & 0 & 0 & A
\end{array}\right], \quad|\varphi\rangle=\left[\begin{array}{c}
x_{1} \\
x_{2} \\
x_{3} \\
x_{4}
\end{array}\right], \quad|\psi\rangle=\left[\begin{array}{l}
y_{1} \\
y_{2} \\
y_{3} \\
y_{4}
\end{array}\right],
$$

where $x_{i}, y_{i}$ are vectors in $\mathbb{C}^{4}$. So, the above relations can be written as the following ones

$$
\sum_{i=1}^{4}\left\langle y_{i}\right| A\left|x_{i}\right\rangle=0 \quad \text { and } \quad \sum_{i=1}^{4}\left\langle y_{i}\right| A\left|y_{i}\right\rangle=\sum_{i=1}^{4}\left\langle x_{i}\right| A\left|x_{i}\right\rangle \quad \forall A \in \mathfrak{M}_{4}
$$

which are equivalent to the operator equalities

$$
\begin{equation*}
\sum_{i=1}^{4}\left|y_{i}\right\rangle\left\langle x_{i}\right|=0 \tag{15}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{i=1}^{4}\left|y_{i}\right\rangle\left\langle y_{i}\right|=\sum_{i=1}^{4}\left|x_{i}\right\rangle\left\langle x_{i}\right| \tag{16}
\end{equation*}
$$

By choosing $e_{i}=f_{i}=g_{1}=h_{1}=0, i=1,2, A_{2}=0,\left(g_{2}, h_{2}\right)=(1,0)$ and $\left(g_{2}, h_{2}\right)=(0,1)$ we obtain from (14) that

$$
\langle\psi|\left[I_{2} \otimes|2\rangle\langle 1|\right] \otimes\left[A_{1} \otimes V_{k}\right]|\varphi\rangle=0
$$

and

$$
\langle\psi|\left[I_{2} \otimes|2\rangle\langle 1|\right] \otimes\left[A_{1} \otimes V_{k}\right]|\psi\rangle=\langle\varphi|\left[I_{2} \otimes|2\rangle\langle 1|\right] \otimes\left[A_{1} \otimes V_{k}\right]|\varphi\rangle
$$

for all $A_{1}$ in $\mathfrak{M}_{2}$ and $k=1,2$. According to (13) we have

$$
\left[I_{2} \otimes|2\rangle\langle 1|\right] \otimes\left[A_{1} \otimes V_{k}\right]=\left[\begin{array}{cccc}
0 & 0 & 0 & 0 \\
A_{1} \otimes V_{k} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & A_{1} \otimes V_{k} & 0
\end{array}\right]
$$

and hence the above equalities imply

$$
\begin{equation*}
\left\langle y_{2}\right| A \otimes V_{k}\left|x_{1}\right\rangle+\left\langle y_{4}\right| A \otimes V_{k}\left|x_{3}\right\rangle=0 \quad \forall A \in \mathfrak{M}_{2}, k=1,2, \tag{17}
\end{equation*}
$$

and

$$
\begin{align*}
& \left\langle y_{2}\right| A \otimes V_{k}\left|y_{1}\right\rangle+\left\langle y_{4}\right| A \otimes V_{k}\left|y_{3}\right\rangle=  \tag{18}\\
& \left\langle x_{2}\right| A \otimes V_{k}\left|x_{1}\right\rangle+\left\langle x_{4}\right| A \otimes V_{k}\left|x_{3}\right\rangle \quad \forall A \in \mathfrak{M}_{2}, k=1,2 .
\end{align*}
$$

Similarly, by choosing $e_{i}=f_{i}=g_{2}=h_{2}=0, i=1,2, A_{1}=0,\left(g_{1}, h_{1}\right)=$ $(1,0)$ and $\left(g_{1}, h_{1}\right)=(0,1)$ we obtain from (14) the equalities

$$
\begin{equation*}
\left\langle y_{3}\right| U_{k} \otimes A\left|x_{1}\right\rangle+\left\langle y_{4}\right| U_{k} \otimes A\left|x_{2}\right\rangle=0, \quad \forall A \in \mathfrak{M}_{2}, k=1,2, \tag{19}
\end{equation*}
$$

and

$$
\begin{align*}
& \left\langle y_{3}\right| U_{k} \otimes A\left|y_{1}\right\rangle+\left\langle y_{4}\right| U_{k} \otimes A\left|y_{2}\right\rangle=  \tag{20}\\
& \left\langle x_{3}\right| U_{k} \otimes A\left|x_{1}\right\rangle+\left\langle x_{4}\right| U_{k} \otimes A\left|x_{2}\right\rangle, \quad \forall A \in \mathfrak{M}_{2}, k=1,2 .
\end{align*}
$$

By the symmetry of condition (14) with respect to $\varphi$ and $\psi$ relations (17) and (19) imply respectively

$$
\begin{equation*}
\left\langle x_{2}\right| A \otimes V_{k}\left|y_{1}\right\rangle+\left\langle x_{4}\right| A \otimes V_{k}\left|y_{3}\right\rangle=0 \quad \forall A \in \mathfrak{M}_{2}, k=1,2, \tag{21}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\langle x_{3}\right| U_{k} \otimes A\left|y_{1}\right\rangle+\left\langle x_{4}\right| U_{k} \otimes A\left|y_{2}\right\rangle=0, \quad \forall A \in \mathfrak{M}_{2}, k=1,2 . \tag{22}
\end{equation*}
$$

Finally, by choosing $A_{1}=A_{2}=0$ and appropriate values of $e_{i}, f_{i}, g_{i}, h_{i}$, $i=1,2$, one can obtain from (14) the following equalities

$$
\begin{align*}
\left\langle y_{4}\right| U_{k} \otimes V_{l}\left|x_{1}\right\rangle & =\left\langle x_{4}\right| U_{k} \otimes V_{l}\left|y_{1}\right\rangle=0 & & k, l=1,2,  \tag{23}\\
\left\langle y_{4}\right| U_{k} \otimes V_{l}\left|y_{1}\right\rangle & =\left\langle x_{4}\right| U_{k} \otimes V_{l}\left|x_{1}\right\rangle, & & k, l=1,2,  \tag{24}\\
\left\langle y_{3}\right| U_{k} \otimes V_{l}^{*}\left|x_{2}\right\rangle & =\left\langle x_{3}\right| U_{k} \otimes V_{l}^{*}\left|y_{2}\right\rangle=0 & & k, l=1,2,  \tag{25}\\
\left\langle y_{3}\right| U_{k} \otimes V_{l}^{*}\left|y_{2}\right\rangle & =\left\langle x_{3}\right| U_{k} \otimes V_{l}^{*}\left|x_{2}\right\rangle, & & k, l=1,2 . \tag{26}
\end{align*}
$$

We will prove below that the system (15)-(26) has no nontrivial solutions.
We will use the following lemmas.
Lemma 2. A) Equations (15) and (16) imply that all the vectors $x_{i}, y_{i}$, $i=\overline{1,4}$, lie in some 2-D subspace of $\mathbb{C}^{4}$.
B) If $x_{i_{0}}=y_{i_{0}}=0$ for some $i_{0}$ then equations (15) and (16) imply that all the vectors $x_{i}, y_{i}, i=\overline{1,4}$, are collinear.

Proof. A) Consider the $4 \times 4$ - matrices

$$
X=\left[\left\langle x_{i} \mid x_{j}\right\rangle\right], \quad Y=\left[\left\langle y_{i} \mid y_{j}\right\rangle\right], \quad Z=\left[\left\langle x_{i} \mid y_{j}\right\rangle\right]
$$

It is easy to see that (15) implies $X Y=0$ while (16) shows that $X^{2}=Z Z^{*}$ and $Y^{2}=Z^{*} Z$. Hence $\operatorname{rank} X=\operatorname{rank} Y \leq 2$.

Since (16) implies that the sets $\left\{x_{i}\right\}_{i=1}^{4}$ and $\left\{y_{i}\right\}_{i=1}^{4}$ have the same linear hull, the above inequality shows that this linear hull has dimension $\leq 2$.
B) This assertion is proved similarly, since the same argumentation with $3 \times 3$ - matrices $X, Y, Z$ implies $\operatorname{rank} X=\operatorname{rank} Y \leq 1$.

Lemma 3. A) The condition

$$
\begin{equation*}
\left\langle z_{4}\right| U_{k} \otimes V_{l}\left|z_{1}\right\rangle=0 \quad k, l=1,2, \tag{27}
\end{equation*}
$$

holds if and only if the pair $\left(z_{1}, z_{4}\right)$ has one of the following forms:

1) $z_{1}=\left[\begin{array}{c}\mu_{1} \\ s\end{array}\right] \otimes\left[\begin{array}{l}a \\ b\end{array}\right], z_{4}=\left[\begin{array}{c}\bar{\mu}_{1} \\ -s\end{array}\right] \otimes\left[\begin{array}{l}c \\ d\end{array}\right]$;
2) $z_{1}=\left[\begin{array}{l}a \\ b\end{array}\right] \otimes\left[\begin{array}{c}\mu_{2} \\ s\end{array}\right], z_{4}=\left[\begin{array}{l}c \\ d\end{array}\right] \otimes\left[\begin{array}{c}\bar{\mu}_{2} \\ -s\end{array}\right]$;
3) $z_{1}=a\left[\begin{array}{c}\mu_{1} \\ 1\end{array}\right] \otimes\left[\begin{array}{c}\mu_{2} \\ s\end{array}\right]+b\left[\begin{array}{c}\mu_{1} \\ -1\end{array}\right] \otimes\left[\begin{array}{c}\mu_{2} \\ -s\end{array}\right], z_{4}=c\left[\begin{array}{c}\bar{\mu}_{1} \\ 1\end{array}\right] \otimes\left[\begin{array}{c}\bar{\mu}_{2} \\ -s\end{array}\right]+d\left[\begin{array}{c}\bar{\mu}_{1} \\ -1\end{array}\right] \otimes\left[\begin{array}{c}\bar{\mu}_{2} \\ s\end{array}\right]$;
4) $z_{1}=h\left[\begin{array}{c}\mu_{1} \\ s\end{array}\right] \otimes\left[\begin{array}{c}\mu_{2} \\ t\end{array}\right], z_{4}=\left[\begin{array}{c}\bar{\mu}_{1} \\ -s\end{array}\right] \otimes\left[\begin{array}{l}a \\ b\end{array}\right]+\left[\begin{array}{c}c \\ d\end{array}\right] \otimes\left[\begin{array}{c}\bar{\mu}_{2} \\ -t\end{array}\right]$;
5) $z_{1}=\left[\begin{array}{c}\mu_{1} \\ -s\end{array}\right] \otimes\left[\begin{array}{l}a \\ b\end{array}\right]+\left[\begin{array}{l}c \\ d\end{array}\right] \otimes\left[\begin{array}{c}\mu_{2} \\ -t\end{array}\right], z_{4}=h\left[\begin{array}{c}\bar{\mu}_{1} \\ s\end{array}\right] \otimes\left[\begin{array}{c}\bar{\mu}_{2} \\ t\end{array}\right]$;
where $\mu_{k}=\sqrt{\gamma_{k}}, k=1,2$, and $a, b, c, d, h \in \mathbb{C}, s= \pm 1, t= \pm 1$.
B) Validity of (23) and (24) for vectors $x_{i}, y_{i}, i=1,4$, implies

$$
\left\langle y_{4}\right| U_{k} \otimes V_{l}\left|y_{1}\right\rangle=\left\langle x_{4}\right| U_{k} \otimes V_{l}\left|x_{1}\right\rangle=0 .
$$

Lemma 4. A) The condition

$$
\begin{equation*}
\left\langle z_{3}\right| U_{k} \otimes V_{l}^{*}\left|z_{2}\right\rangle=0 \quad k, l=1,2, \tag{28}
\end{equation*}
$$

holds if and only if the pair $\left(z_{2}, z_{3}\right)$ has one of the following forms:

1) $z_{2}=\left[\begin{array}{c}\mu_{1} \\ s\end{array}\right] \otimes\left[\begin{array}{l}a \\ b\end{array}\right], z_{3}=\left[\begin{array}{c}\bar{\mu}_{1} \\ -s\end{array}\right] \otimes\left[\begin{array}{l}c \\ d\end{array}\right]$;
2) $z_{2}=\left[\begin{array}{l}a \\ b\end{array}\right] \otimes\left[\begin{array}{c}\bar{\mu}_{2} \\ s\end{array}\right], z_{3}=\left[\begin{array}{l}c \\ d\end{array}\right] \otimes\left[\begin{array}{c}\mu_{2} \\ -s\end{array}\right]$;
3) $z_{2}=a\left[\begin{array}{c}\mu_{1} \\ 1\end{array}\right] \otimes\left[\begin{array}{c}\bar{\mu}_{2} \\ s\end{array}\right]+b\left[\begin{array}{c}\mu_{1} \\ -1\end{array}\right] \otimes\left[\begin{array}{c}\bar{\mu}_{2} \\ -s\end{array}\right], z_{3}=c\left[\begin{array}{c}\bar{\mu}_{1} \\ -1\end{array}\right] \otimes\left[\begin{array}{c}\mu_{2} \\ s\end{array}\right]+d\left[\begin{array}{c}\bar{\mu}_{1} \\ 1\end{array}\right] \otimes\left[\begin{array}{c}\mu_{2} \\ -s\end{array}\right]$;
4) $z_{2}=h\left[\begin{array}{c}\mu_{1} \\ s\end{array}\right] \otimes\left[\begin{array}{c}\bar{\mu}_{2} \\ t\end{array}\right], z_{3}=\left[\begin{array}{c}\bar{\mu}_{1} \\ -s\end{array}\right] \otimes\left[\begin{array}{l}a \\ b\end{array}\right]+\left[\begin{array}{l}c \\ d\end{array}\right] \otimes\left[\begin{array}{c}\mu_{2} \\ -t\end{array}\right]$;
5) $z_{2}=\left[\begin{array}{c}\mu_{1} \\ -s\end{array}\right] \otimes\left[\begin{array}{l}a \\ b\end{array}\right]+\left[\begin{array}{l}c \\ d\end{array}\right] \otimes\left[\begin{array}{c}\bar{\mu}_{2} \\ -t\end{array}\right], z_{3}=h\left[\begin{array}{c}\bar{\mu}_{1} \\ s\end{array}\right] \otimes\left[\begin{array}{c}\mu_{2} \\ t\end{array}\right]$;
where $\mu_{k}=\sqrt{\gamma_{k}}, k=1,2$, and $a, b, c, d, h \in \mathbb{C}, s= \pm 1, t= \pm 1$.
B) Validity of (25) and (26) for vectors $x_{i}, y_{i}, i=2,3$, implies

$$
\left\langle y_{3}\right| U_{k} \otimes V_{l}^{*}\left|y_{2}\right\rangle=\left\langle x_{3}\right| U_{k} \otimes V_{l}^{*}\left|x_{2}\right\rangle=0 .
$$

Lemmas 3 and 4 are proved in the Appendix.
Lemma 5. Let $\left|\theta_{1}\right|+\left|\theta_{2}\right|<\pi$ then
A) $\langle x| U_{1}|x\rangle \neq 0$ and $\langle x| V_{1}|x\rangle \neq 0$ for any nonzero $x \in \mathbb{C}^{2}$;
B) $\langle y| U_{1} \otimes V_{1}|y\rangle \neq 0$ and $\langle y| U_{1} \otimes V_{1}^{*}|y\rangle \neq 0$ for any nonzero $y \in \mathbb{C}^{2} \otimes \mathbb{C}^{2}$.

Proof. A) Let $|x\rangle=\left[x_{1}, x_{2}\right]^{\top} \neq 0$ then $\langle x| U_{1}|x\rangle=\left|x_{1}\right|^{2}+\gamma_{1}\left|x_{2}\right|^{2} \neq 0$ and $\langle x| V_{1}|x\rangle=\left|x_{1}\right|^{2}+\gamma_{2}\left|x_{2}\right|^{2} \neq 0$, since $\theta_{1}, \theta_{2} \neq \pi$.
B) Since $U_{1} \otimes V_{1}=\operatorname{diag}\left\{1, \gamma_{2}, \gamma_{1}, \gamma_{1} \gamma_{2}\right\}$, the equality $\langle y| U_{1} \otimes V_{1}|y\rangle=0$ for vector $|y\rangle=\left[y_{1}, y_{2}, y_{3}, y_{4}\right]^{\top}$ means that $\left|y_{1}\right|^{2}+\left|y_{2}\right|^{2} \gamma_{2}+\left|y_{3}\right|^{2} \gamma_{1}+\left|y_{4}\right|^{2} \gamma_{1} \gamma_{2}=0$. By the condition $\left|\theta_{1}\right|+\left|\theta_{2}\right|<\pi$ the numbers $0,1, \gamma_{2}, \gamma_{1}, \gamma_{1} \gamma_{2}$ are extreme points of a convex polygon in complex plane, so the last equality can be valid only if $y_{i}=0$ for all $i$.

Similarly one can show that $\langle y| U_{1} \otimes V_{1}^{*}|y\rangle=0$ implies $y=0$.

Lemma 6. Let $p$ and $q$ be complex numbers such that $|p|^{2}+|q|^{2}=1$. If $\left\{\left|x_{i}\right\rangle\right\}_{i=1}^{4}$ and $\left\{\left|y_{i}\right\rangle\right\}_{i=1}^{4}$ satisfy the system (15)- (26) then $\left\{\left|p x_{i}-q y_{i}\right\rangle\right\}_{i=1}^{4}$ and $\left\{\left|\bar{q} x_{i}+\bar{p} y_{i}\right\rangle\right\}_{i=1}^{4}$ also satisfy the system (15)-(26).

Proof. It suffices to note that the condition

$$
\langle\varphi| A|\psi\rangle=\langle\psi| A|\varphi\rangle=\langle\psi| A|\psi\rangle-\langle\varphi| A|\varphi\rangle=0
$$

is invariant under the rotation $|\varphi\rangle \mapsto p|\varphi\rangle-q|\psi\rangle,|\psi\rangle \mapsto \bar{q}|\varphi\rangle+\bar{p}|\psi\rangle$.
Lemma 7. If $\left|\theta_{1}\right|+\left|\theta_{2}\right|<\pi$ then the system (15)-(26) has no nontrivial solution of the form $\left|x_{i}\right\rangle=\alpha_{i}|z\rangle$ and $\left|y_{i}\right\rangle=\beta_{i}|z\rangle, i=\overline{1,4}$.

Proof. Assume that $\left|x_{i}\right\rangle=\alpha_{i}|z\rangle$ and $\left|y_{i}\right\rangle=\beta_{i}|z\rangle, i=\overline{1,4}$, form a nontrivial solution of the system (15)-(26). Then (15) implies that $|\alpha\rangle=\left[\alpha_{1}, \ldots, \alpha_{4}\right]^{\top}$ and $|\beta\rangle=\left[\beta_{1}, \ldots, \beta_{4}\right]^{\top}$ are orthogonal nonzero vectors of the same norm. By Lemma 5B it follows from (23) and (25) that

$$
\begin{equation*}
\alpha_{1} \beta_{4}=\alpha_{4} \beta_{1}=\alpha_{2} \beta_{3}=\alpha_{3} \beta_{2}=0 \tag{29}
\end{equation*}
$$

If $\left|\alpha_{i}\right|+\left|\beta_{i}\right|>0$ for all $i$ then we may consider by Lemma that $\alpha_{i} \neq 0$ for all $i$ and hence (29) implies $\beta_{i}=0$ for all $i$, i.e. $|\beta\rangle=0$.

Assume that $\alpha_{1}=\beta_{1}=0$ and consider the following two cases.

1) If $\alpha_{4}=\beta_{4}=0$ then the condition $\langle\beta \mid \alpha\rangle=0$ implies $\left|\alpha_{i}\right|+\left|\beta_{i}\right|>0$ for $i=2,3$ and we may consider by Lemma 6 that $\alpha_{i} \neq 0$ for $i=2,3$. Hence (29) implies $\beta_{i}=0$ for $i=2,3$ and hence $|\beta\rangle=0$.
2) If $\left|\alpha_{4}\right|+\left|\beta_{4}\right|>0$ then we may consider by Lemma 6 that $\alpha_{4} \neq 0$. By Lemma 5B it follows from (21) with $A=U_{1}$ and (22) with $A=V_{1}$ that $\alpha_{4} \beta_{2}=\alpha_{4} \beta_{3}=0$, which implies $\beta_{2}=\beta_{3}=0$. Hence the condition $\langle\beta \mid \alpha\rangle=0$ can be valid only if $|\beta\rangle=0$.

By the similar way one can show that the assumption $\alpha_{i}=\beta_{i}=0$ for $i=2,3,4$ leads to a contradiction.

Assume now that the collections $\left\{x_{i}\right\}_{1}^{4}$ and $\left\{y_{i}\right\}_{1}^{4}$ form a nontrivial solution of the system (15)-(26).

If $x_{i} \nVdash y_{i}$ for some $i$ then (23)-(26) and Lemmas 3B, 4 B imply

$$
\left\langle y_{5-i}\right| W_{i}\left|x_{i}\right\rangle=\left\langle x_{5-i}\right| W_{i}\left|y_{i}\right\rangle=\left\langle x_{5-i}\right| W_{i}\left|x_{i}\right\rangle=\left\langle y_{5-i}\right| W_{i}\left|y_{i}\right\rangle=0,
$$

where $W_{1}=U_{1} \otimes V_{1}, W_{2}=U_{1} \otimes V_{1}^{*}, W_{3}=U_{1}^{*} \otimes V_{1}, W_{4}=U_{1}^{*} \otimes V_{1}^{*}$. By Lemma 2A we have $x_{5-i}, y_{5-i} \in \operatorname{lin}\left\{x_{i}, y_{i}\right\}$, so the above equalities show that
$\left\langle x_{5-i}\right| W_{i}\left|x_{5-i}\right\rangle=\left\langle y_{5-i}\right| W_{i}\left|y_{5-i}\right\rangle=0$. Lemma 5B implies $x_{5-i}=y_{5-i}=0$, which contradicts to the assumption $x_{i} \nVdash y_{i}$ by Lemma 2B.

Thus, $x_{i} \| y_{i}$ for all $i=\overline{1,4}$. By Lemma 7 we may assume in what follows that

$$
\begin{equation*}
\left.\left|x_{i}\right\rangle=\alpha_{i}\left|z_{i}\right\rangle \text { and }\left|y_{i}\right\rangle=\beta_{i}\left|z_{i}\right\rangle \text {, where }\left|z_{i}\right\rangle \text { are non-collinear vectors }\right]^{3} \text {. } \tag{30}
\end{equation*}
$$

Then Lemma 2 B implies $\left|\alpha_{i}\right|+\left|\beta_{i}\right|>0, i=\overline{1,4}$, and equations (15), (16) can be rewritten as follows

$$
\begin{gather*}
\sum_{i=1}^{4} \bar{\beta}_{i} \alpha_{i}\left|z_{i}\right\rangle\left\langle z_{i}\right|=0,  \tag{31}\\
\sum_{i=1}^{4}\left[\left|\beta_{i}\right|^{2}-\left|\alpha_{i}\right|^{2}\right]\left|z_{i}\right\rangle\left\langle z_{i}\right|=0 . \tag{32}
\end{gather*}
$$

By Lemma 6 we may assume that $\beta_{1}=0$ and hence $\alpha_{1} \neq 0$. There are two cases:

1) If $\beta_{i} \alpha_{i} \neq 0$ for all $i>1$ then (31) and Lemma 8 in the Appendix imply $z_{2}\left\|z_{3}\right\| z_{4}$. Then it follows from (32) that

$$
\left|\alpha_{1}\right|^{2}\left|z_{1}\right\rangle\left\langle z_{1}\right|+[\ldots]\left|z_{2}\right\rangle\left\langle z_{2}\right|=0
$$

and hence $z_{1}\left\|z_{2}\right\| z_{3} \| z_{4}$ contradicting to the assumption (30).
2) If there is $k>1$ such that $\beta_{k} \alpha_{k}=0$ then (31) implies that either $\beta_{i} \alpha_{i} \neq 0$ and $\beta_{j} \alpha_{j} \neq 0$ or $\beta_{i} \alpha_{i}=\beta_{j} \alpha_{j}=0$, where $i$ and $j>i$ are complementary indexes to 1 and $k$.

If $\beta_{i} \alpha_{i} \neq 0$ and $\beta_{j} \alpha_{j} \neq 0$ then it follows from (31) that $z_{i} \| z_{j}$ and (32) implies

$$
\left|\alpha_{1}\right|^{2}\left|z_{1}\right\rangle\left\langle z_{1}\right|+p\left|z_{k}\right\rangle\left\langle z_{k}\right|+[\ldots]\left|z_{i}\right\rangle\left\langle z_{i}\right|=0
$$

where $p$ is a nonzero number (equal either to $\left|\alpha_{k}\right|^{2}$ or to $-\left|\beta_{k}\right|^{2}$ ). Hence $z_{1} \| z_{k}$ by Lemma 8 in the Appendix.

Thus $z_{1} \| z_{k}$ and $z_{i} \| z_{j}$. By Lemma 5B it follows from (23) and (25) that $k \neq 4$ and $(i, j) \neq(2,3)$. So, we have only two possibilities:
a) $k=2, i=3, j=4$. In this case $z_{3} \| z_{4}$ and (21) with $A=U_{1}$ implies

$$
\bar{\alpha}_{4} \beta_{3}\left\langle z_{4}\right| U_{1} \otimes V_{1}\left|z_{3}\right\rangle=-\bar{\alpha}_{2} \beta_{1}\left\langle z_{2}\right| U_{1} \otimes V_{1}\left|z_{1}\right\rangle=0 \quad\left(\text { since } \beta_{1}=0\right) .
$$

[^3]Hence Lemma 5 shows that $\alpha_{4} \beta_{3}=0$ contradicting to the assumption.
b) $k=3, i=2, j=4$. In this case $z_{2} \| z_{4}$ and (22) with $A=V_{1}$ implies

$$
\bar{\alpha}_{4} \beta_{2}\left\langle z_{4}\right| U_{1} \otimes V_{1}\left|z_{2}\right\rangle=-\bar{\alpha}_{3} \beta_{1}\left\langle z_{3}\right| U_{1} \otimes V_{1}\left|z_{1}\right\rangle=0 \quad\left(\text { since } \beta_{1}=0\right) .
$$

Hence Lemma 5 B shows that $\alpha_{4} \beta_{2}=0$ contradicting to the assumption.
So, we necessarily have $\beta_{i} \alpha_{i}=0$ for all $i=\overline{1,4}$. Since the vectors $z_{1}, \ldots, z_{4}$ are not collinear by assumption (30), equality (32) and Lemma 2B imply that there are two nonzero $\alpha_{i}$ and two nonzero $\beta_{i}$. Thus, we have (up to permutation) the following cases
a) $|\varphi\rangle,|\psi\rangle=\left[\begin{array}{c}x_{1} \\ x_{2} \\ 0 \\ 0\end{array}\right],\left[\begin{array}{c}0 \\ 0 \\ y_{3} \\ y_{4}\end{array}\right] ;$ b) $|\varphi\rangle,|\psi\rangle=\left[\begin{array}{c}x_{1} \\ 0 \\ x_{3} \\ 0\end{array}\right],\left[\begin{array}{c}0 \\ y_{2} \\ 0 \\ y_{4}\end{array}\right] ; \quad$ c) $|\varphi\rangle,|\psi\rangle=\left[\begin{array}{c}x_{1} \\ 0 \\ 0 \\ x_{4}\end{array}\right],\left[\begin{array}{c}0 \\ y_{2} \\ y_{3} \\ 0\end{array}\right]$,
where $x_{1} \nVdash x_{k}$ and $y_{i} \nVdash y_{j}$ (since otherwise (32) implies $x_{1}\left\|x_{k}\right\| y_{i} \| y_{j}$ ).
Show first that case c) is not possible. It follows from (17) with $A=U_{1}$ and (19) with $A=V_{1}$ that

$$
\left\langle y_{2}\right| U_{1} \otimes V_{1}\left|x_{1}\right\rangle=\left\langle y_{3}\right| U_{1} \otimes V_{1}\left|x_{1}\right\rangle=0 .
$$

Since $y_{2} \nVdash y_{3}$, Lemma 2 A shows that $x_{1} \in \operatorname{lin}\left\{y_{2}, y_{3}\right\}$ and the above equalities imply $\left\langle x_{1}\right| U_{1} \otimes V_{1}\left|x_{1}\right\rangle=0$. By Lemma 5B $x_{1}=0$.

It is more difficult to show impossibility of cases a) and b). We will consider these cases simultaneously by denoting $z_{2}=x_{2}, z_{3}=y_{3}$ in case a), $z_{2}=y_{2}, z_{3}=x_{3}$ in case b) and $z_{1}=x_{1}, z_{4}=y_{4}$ in the both cases. The system (15)-(26) implies the following equations:

$$
\begin{equation*}
\left|x_{1}\right\rangle\left\langle x_{1}\right|+\left|x_{i}\right\rangle\left\langle x_{i}\right|=\left|y_{j}\right\rangle\left\langle y_{j}\right|+\left|y_{4}\right\rangle\left\langle y_{4}\right| \tag{33}
\end{equation*}
$$

where $(i, j)=(2,3)$ in case a) and $(i, j)=(3,2)$ in case b),

$$
\begin{align*}
\left\langle z_{3}\right| U_{k} \otimes A\left|x_{1}\right\rangle=-\sigma_{*}\left\langle y_{4}\right| U_{k} \otimes A\left|z_{2}\right\rangle & \forall A \in \mathfrak{M}_{2}, k=1,2,  \tag{34}\\
\left\langle z_{2}\right| A \otimes V_{k}\left|x_{1}\right\rangle=+\sigma_{*}\left\langle y_{4}\right| A \otimes V_{k}\left|z_{3}\right\rangle & \forall A \in \mathfrak{M}_{2}, k=1,2, \tag{35}
\end{align*}
$$

where $\sigma_{*}=1$ in case a) and $\sigma_{*}=-1$ in case b),

$$
\begin{equation*}
\left\langle y_{4}\right| U_{k} \otimes V_{l}\left|x_{1}\right\rangle=0 \quad k, l=1,2, \tag{36}
\end{equation*}
$$

$$
\begin{equation*}
\left\langle z_{3}\right| U_{k} \otimes V_{l}^{*}\left|z_{2}\right\rangle=0 \quad k, l=1,2 . \tag{37}
\end{equation*}
$$

It follows from (36) and (37) that the pairs $\left(x_{1}, y_{4}\right)$ and $\left(z_{2}, z_{3}\right)$ must have one of the forms 1-5 presented in part A of Lemmas 3 and 4 correspondingly.

Assume first that the both pairs $\left(x_{1}, y_{4}\right)$ and $\left(z_{2}, z_{3}\right)$ have forms 1-2. In this case $x_{1}, z_{2}, z_{3}, y_{4}$ are tensor product vectors. By Lemma 9 in the Appendix (33) can be valid only in the following cases (1-4):

1) $\left|z_{i}\right\rangle=|p\rangle \otimes\left|a_{i}\right\rangle, i=\overline{1,4}$. It follows from (34) that

$$
\langle p| U_{1}|p\rangle\left\langle a_{3}\right| A\left|a_{1}\right\rangle=-\sigma_{*}\langle p| U_{1}|p\rangle\left\langle a_{4}\right| A\left|a_{2}\right\rangle \quad \forall A \in \mathfrak{M}_{2}
$$

Since $\langle p| U_{1}|p\rangle \neq 0$ by Lemma 5A, we have $a_{1} \| a_{2}$ and $a_{3} \| a_{4}$. In case a) this and (33) implies $x_{1}\left\|x_{2}\right\| y_{3} \| y_{4}$ contradicting to assumption (30). In case b) it means $x_{1} \| y_{2}$ and $x_{3} \| y_{4}$. The assumption $x_{1} \nVdash x_{3}$ and (33) show that this can be valid only if $\left|x_{1}\right\rangle\left\langle x_{1}\right|=\left|y_{2}\right\rangle\left\langle y_{2}\right|$ and $\left|x_{3}\right\rangle\left\langle x_{3}\right|=\left|y_{4}\right\rangle\left\langle y_{4}\right|$. So, this case is reduced to case 4) considered below.
2) $\left|z_{i}\right\rangle=\left|a_{i}\right\rangle \otimes|p\rangle, i=\overline{1,4}$. Similarly to case 1) this case is reduced to case 4) by using (35) instead of (34).
3) $\left|x_{1}\right\rangle\left\langle x_{1}\right|=\left|y_{4}\right\rangle\left\langle y_{4}\right|$ and $\left|z_{2}\right\rangle\left\langle z_{2}\right|=\left|z_{3}\right\rangle\left\langle z_{3}\right|$. This is not possible due to (36)-(37) and Lemma 5B.
4) $\left|x_{1}\right\rangle\left\langle x_{1}\right|=\left|y_{i}\right\rangle\left\langle y_{i}\right|$ and $\left|x_{5-i}\right\rangle\left\langle x_{5-i}\right|=\left|y_{4}\right\rangle\left\langle y_{4}\right|$, where $i=3$ in case a) and $i=2$ in case b).

If $i=3$ then $y_{3}=\alpha x_{1}, y_{4}=\beta x_{2},|\alpha|=|\beta|=1$, and (34) with $\sigma_{*}=1$ implies

$$
\begin{equation*}
\bar{\alpha}\left\langle x_{1}\right| U_{1} \otimes A\left|x_{1}\right\rangle=-\bar{\beta}\left\langle x_{2}\right| U_{1} \otimes A\left|x_{2}\right\rangle \quad \forall A \in \mathfrak{M}_{2} . \tag{38}
\end{equation*}
$$

Since $x_{1}$ and $x_{2}$ are product vectors, it follows from (38) and Lemma 5 A that

$$
x_{1}=a \otimes p \quad \text { and } \quad x_{2}=b \otimes p
$$

for some nonzero vectors $a, b, p$. Hence (36), (37) and Lemma (5A imply

$$
\langle b| U_{k}|a\rangle=\langle b| U_{k}^{*}|a\rangle=0, \quad k=1,2 .
$$

If $\gamma_{1} \neq 1$ (i.e. $\theta_{1} \neq 0$ ) then this can not be valid for nonzero vectors $a$ and $b$. If $\gamma_{1}=1$ then (38) shows that $\bar{\alpha}\|a\|^{2}=-\bar{\beta}\|b\|^{2}$ while (35) with $\sigma_{*}=1$ and Lemma 5A imply $\bar{\beta} \alpha=1$, i.e. $\alpha=\beta$.

Similarly, if $i=2$ then by using Lemma 5A one can obtain from (35) that

$$
x_{1}\left\|y_{2}\right\| p \otimes a \quad \text { and } \quad x_{3}\left\|y_{4}\right\| p \otimes b
$$

for some nonzero vectors $a, b, p$. Hence (36), (37) and Lemma 5A imply

$$
\langle b| V_{k}|a\rangle=\langle b| V_{k}^{*}|a\rangle=0, \quad k=1,2,
$$

which can not be valid for nonzero vectors $a$ and $b$ (since $\theta_{2} \neq 0 \Rightarrow \gamma_{2} \neq \bar{\gamma}_{2}$ ).
Assume now that the pair $\left(x_{1}, y_{4}\right)$ have form 3 in Lemma 3A, i.e.
$x_{1}=a\left[\begin{array}{c}\mu_{1} \\ 1\end{array}\right] \otimes\left[\begin{array}{c}\mu_{2} \\ s\end{array}\right]+b\left[\begin{array}{c}\mu_{1} \\ -1\end{array}\right] \otimes\left[\begin{array}{c}\mu_{2} \\ -s\end{array}\right], y_{4}=c\left[\begin{array}{c}\bar{\mu}_{1} \\ 1\end{array}\right] \otimes\left[\begin{array}{c}\bar{\mu}_{2} \\ -s\end{array}\right]+d\left[\begin{array}{c}\bar{\mu}_{1} \\ -1\end{array}\right] \otimes\left[\begin{array}{c}\bar{\mu}_{2} \\ s\end{array}\right]$,
where $s= \pm 1$, and show incompatibility of the system (33)-(37) if the pair $\left(z_{2}, z_{3}\right)$ has forms $1,2,3$ in Lemma 4A. We will do this by reducing to the case of tensor product vectors $x_{1}, z_{2}, z_{3}, y_{4}$ considered before.

1) The pair $\left(z_{2}, z_{3}\right)$ has form 1, i.e.
$z_{2}=\left[\begin{array}{c}\mu_{1} \\ t\end{array}\right] \otimes\left[\begin{array}{l}p \\ q\end{array}\right], z_{3}=\left[\begin{array}{c}\bar{\mu}_{1} \\ -t\end{array}\right] \otimes\left[\begin{array}{l}x \\ y\end{array}\right], \quad t= \pm 1,|p|+|q| \neq 0,|x|+|y| \neq 0$.
By substituting the expressions for $x_{1}, z_{2}, z_{3}, y_{4}$ into (34) and by noting that

$$
\left\langle\begin{array}{c|c|c}
\bar{\mu}_{1}  \tag{39}\\
s
\end{array}\right| U_{k}\left|\begin{array}{c}
\mu_{1} \\
-s
\end{array}\right\rangle=0, \quad s= \pm 1, \quad k=1,2
$$

we obtain
$b\left\langle\begin{array}{c}\bar{\mu}_{1} \\ -1\end{array}\right| U_{k}\left|\begin{array}{c}\mu_{1} \\ -1\end{array}\right\rangle\left\langle\begin{array}{c}x \\ y\end{array}\right| A\left|\begin{array}{c}\mu_{2} \\ -s\end{array}\right\rangle=-\sigma_{*} \bar{c}\left\langle\begin{array}{c}\bar{\mu}_{1} \\ 1\end{array}\right| U_{k}\left|\begin{array}{c}\mu_{1} \\ 1\end{array}\right\rangle\left\langle\begin{array}{c}\bar{\mu}_{2} \\ -s\end{array}\right| A\left|\begin{array}{c}p \\ q\end{array}\right\rangle \quad$ if $t=1$
and
$a\left\langle\begin{array}{c}\bar{\mu}_{1} \\ 1\end{array}\right| U_{k}\left|\begin{array}{c}\mu_{1} \\ 1\end{array}\right\rangle\left\langle\begin{array}{l}x \\ y\end{array}\right| A\left|\begin{array}{c}\mu_{2} \\ s\end{array}\right\rangle=-\sigma_{*} \bar{d}\left\langle\begin{array}{c}\bar{\mu}_{1} \\ -1\end{array}\right| U_{k}\left|\begin{array}{c}\mu_{1} \\ -1\end{array}\right\rangle\left\langle\begin{array}{c}\bar{\mu}_{2} \\ s\end{array}\right| A\left|\begin{array}{c}p \\ q\end{array}\right\rangle \quad$ if $t=-1$.
Validity of this equality for all $A \in \mathfrak{M}_{2}$ implies

$$
b \lambda_{k}^{-}\left|\begin{array}{c}
\mu_{2} \\
-s
\end{array}\right\rangle\left\langle\begin{array}{l}
x \\
y
\end{array}\right|=-\sigma_{*} \bar{c} \lambda_{k}^{+}\left|\begin{array}{c}
p \\
q
\end{array}\right\rangle\left\langle\begin{array}{c}
\bar{\mu}_{2} \\
-s
\end{array}\right| \quad \text { if } t=1
$$

and

$$
a \lambda_{k}^{+}\left|\begin{array}{c}
\mu_{2} \\
s
\end{array}\right\rangle\left\langle\begin{array}{l}
x \\
y
\end{array}\right|=-\sigma_{*} \bar{d} \lambda_{k}^{-}\left|\begin{array}{c}
p \\
q
\end{array}\right\rangle\left\langle\begin{array}{c}
\bar{\mu}_{2} \\
s
\end{array}\right| \quad \text { if } t=-1
$$

where $\lambda_{1}^{ \pm}=\left\langle\left.\begin{array}{c|c|c}\bar{\mu}_{1} \\ \pm 1\end{array} \right\rvert\, \begin{array}{c}U_{1} \\ \mu_{1} \\ \pm 1\end{array}\right\rangle=2 \mu_{1}^{2}$ and $\lambda_{2}^{ \pm}=\left\langle\begin{array}{c}\bar{\mu}_{1} \\ \pm 1\end{array}\right| \begin{gathered}U_{2}\end{gathered}\left|\begin{array}{c}\mu_{1} \\ \pm 1\end{array}\right\rangle= \pm 2 \mu_{1}$.
Since $\lambda_{1}^{+}=\lambda_{1}^{-} \neq 0$ and $\lambda_{2}^{+}=-\lambda_{2}^{-} \neq 0$, validity of the above equalities for $k=1,2$ implies $b=c=0$ if $t=1$ and $a=d=0$ if $t=-1$. So, $x_{1}, z_{2}, z_{3}, y_{4}$ are product vectors.
2) The pair $\left(z_{2}, z_{3}\right)$ has form 2, i.e.
$z_{2}=\left[\begin{array}{c}p \\ q\end{array}\right] \otimes\left[\begin{array}{c}\bar{\mu}_{2} \\ t\end{array}\right], z_{3}=\left[\begin{array}{l}x \\ y\end{array}\right] \otimes\left[\begin{array}{c}\mu_{2} \\ -t\end{array}\right], \quad t= \pm 1,|p|+|q| \neq 0,|x|+|y| \neq 0$.
By substituting the expressions for $x_{1}, z_{2}, z_{3}, y_{4}$ into (35) and by noting that

$$
\left\langle\begin{array}{c|c|c}
\bar{\mu}_{2} & V_{k} & \mu_{2} \\
t & & -t
\end{array}\right\rangle=0, \quad t= \pm 1, \quad k=1,2
$$

we obtain

$$
a\left\langle\begin{array}{c}
p \\
q
\end{array}\right| A\left|\begin{array}{c}
\mu_{1} \\
1
\end{array}\right\rangle\left\langle\begin{array}{c}
\bar{\mu}_{2} \\
t
\end{array}\right| V_{k}\left|\begin{array}{c}
\mu_{2} \\
t
\end{array}\right\rangle=\sigma_{*} \bar{c}\left\langle\begin{array}{c}
\bar{\mu}_{1} \\
1
\end{array}\right| A\left|\begin{array}{l}
x \\
y
\end{array}\right\rangle\left\langle\begin{array}{c|c|c}
\bar{\mu}_{2} \\
-t & \mid & V_{k}\left|\begin{array}{c}
\mu_{2} \\
-t
\end{array}\right\rangle \quad \text { if } t=s, ~
\end{array}\right.
$$

and
$b\left\langle\begin{array}{c}p \\ q\end{array}\right| A\left|\begin{array}{c}\mu_{1} \\ -1\end{array}\right\rangle\left\langle\begin{array}{c}\bar{\mu}_{2} \\ t\end{array}\right| V_{k}\left|\begin{array}{c}\mu_{2} \\ t\end{array}\right\rangle=\sigma_{*} \bar{d}\left\langle\begin{array}{c}\bar{\mu}_{1} \\ -1\end{array}\right| A\left|\begin{array}{c}x \\ y\end{array}\right\rangle\left\langle\left.\begin{array}{c|c|c}\bar{\mu}_{2} \\ -t\end{array} \right\rvert\, \begin{array}{c}V_{k} \\ \mu_{2} \\ -t\end{array}\right\rangle \quad$ if $t=-s$.
Validity of this equality for all $A \in \mathfrak{M}_{2}$ implies

$$
a \nu_{k}^{t}\left|\begin{array}{c}
\mu_{1} \\
1
\end{array}\right\rangle\left\langle\begin{array}{c}
p \\
q
\end{array}\right|=\sigma_{*} \bar{c} \nu_{k}^{-t}\left|\begin{array}{l}
x \\
y
\end{array}\right\rangle\left\langle\begin{array}{c}
\bar{\mu}_{1} \\
1
\end{array}\right| \quad \text { if } t=s
$$

and

$$
b \nu_{k}^{t}\left|\begin{array}{c}
\mu_{1} \\
-1
\end{array}\right\rangle\left\langle\begin{array}{c}
p \\
q
\end{array}\right|=\sigma_{*} \bar{d} \nu_{k}^{-t}\left|\begin{array}{c}
x \\
y
\end{array}\right\rangle\left\langle\begin{array}{c}
\bar{\mu}_{1} \\
-1
\end{array}\right| \quad \text { if } t=-s,
$$

where $\nu_{1}^{t}=\left\langle\begin{array}{c|c|c}\bar{\mu}_{2} & V_{1} & \mu_{2} \\ t & t\end{array}\right\rangle=2 \mu_{2}^{2}$ and $\nu_{2}^{t}=\left\langle\begin{array}{c|c|c}\bar{\mu}_{2} & V_{2} & \mu_{2} \\ t & t\end{array}\right\rangle=2 t \mu_{2}$. Since $\nu_{1}^{t}=\nu_{1}^{-t} \neq 0$ and $\nu_{2}^{t}=-\nu_{2}^{-t} \neq 0$, validity of the above equalities for $k=1,2$ implies $a=c=0$ if $t=s$ and $b=d=0$ if $t=-s$. So, $x_{1}, z_{2}, z_{3}, y_{4}$ are product vectors.
3) The pair $\left(z_{2}, z_{3}\right)$ has form 3, i.e.
$z_{2}=p\left[\begin{array}{c}\mu_{1} \\ 1\end{array}\right] \otimes\left[\begin{array}{c}\bar{\mu}_{2} \\ t\end{array}\right]+q\left[\begin{array}{c}\mu_{1} \\ -1\end{array}\right] \otimes\left[\begin{array}{c}\bar{\mu}_{2} \\ -t\end{array}\right], z_{3}=x\left[\begin{array}{c}\bar{\mu}_{1} \\ -1\end{array}\right] \otimes\left[\begin{array}{c}\mu_{2} \\ t\end{array}\right]+y\left[\begin{array}{c}\bar{\mu}_{1} \\ 1\end{array}\right] \otimes\left[\begin{array}{c}\mu_{2} \\ -t\end{array}\right]$,
where $t= \pm 1$. If we substitute the expressions for $x_{1}, z_{2}, z_{3}, y_{4}$ into (34) (by using (39)) then the left and the right hand sides of this equality will be equal respectively to

$$
\bar{x} b\left\langle\begin{array}{c|c}
\bar{\mu}_{1} \\
-1
\end{array}\right| U_{k}\left|\begin{array}{c}
\mu_{1} \\
-1
\end{array}\right\rangle\left\langle\begin{array}{c}
\mu_{2} \\
t
\end{array}\right| A\left|\begin{array}{c}
\mu_{2} \\
-s
\end{array}\right\rangle+\bar{y} a\left\langle\begin{array}{c|c|c}
\bar{\mu}_{1} & U_{k} & \mu_{1} \\
1
\end{array}\right\rangle\left\langle\begin{array}{c|c|c}
\mu_{2} & A \left\lvert\, \begin{array}{c}
\mu_{2} \\
-t
\end{array}\right. & 1
\end{array}\right\rangle
$$

and to

$$
-\sigma_{*} \bar{c} p\left\langle\begin{array}{c|c|c}
\bar{\mu}_{1} & U_{k} & \mu_{1} \\
1
\end{array}\right\rangle\left\langle\begin{array}{c}
\bar{\mu}_{2} \\
-s
\end{array}\right| A\left|\begin{array}{c}
\bar{\mu}_{2} \\
t
\end{array}\right\rangle-\sigma_{*} \bar{d} q\left\langle\begin{array}{c|c|c}
\bar{\mu}_{1} \\
-1 & U_{k} & \mu_{1} \\
-1
\end{array}\right\rangle\left\langle\begin{array}{c|c|c}
\bar{\mu}_{2} & \bar{\mu}_{2} \\
s & A & -t
\end{array}\right\rangle .
$$

So, validity of the equality for all $A \in \mathfrak{M}_{2}$ implies
where $\varsigma_{k} \doteq-\lambda_{k}^{-} / \lambda_{k}^{+}=(-1)^{k}$. This equality can be valid for $k=1,2$ only if the operators in the squared brackets are equal to zero. Since $\mu_{2} \neq \pm \bar{\mu}_{2}$ by the assumption $\theta_{2} \neq 0, \pi$, it follows that $y a=c p=d q=x b=0$. It is easy to see that this implies that $x_{1}, z_{2}, z_{3}, y_{4}$ are product vectors.

The similar argumentation shows incompatibility of the system (33)-(37) (by reducing to the case of tensor product vectors) if the pair $\left(z_{2}, z_{3}\right)$ has form 3 and the pair $\left(x_{1}, y_{4}\right)$ has form 1 or 2 .

Assume finally that the pair $\left(x_{1}, y_{4}\right)$ has form 4 , i.e.

$$
x_{1}=h\left[\begin{array}{c}
\mu_{1} \\
s
\end{array}\right] \otimes\left[\begin{array}{c}
\mu_{2} \\
t
\end{array}\right], y_{4}=\left[\begin{array}{c}
\bar{\mu}_{1} \\
-s
\end{array}\right] \otimes\left[\begin{array}{l}
a \\
b
\end{array}\right]+\left[\begin{array}{l}
c \\
d
\end{array}\right] \otimes\left[\begin{array}{c}
\bar{\mu}_{2} \\
-t
\end{array}\right], \quad s, t= \pm 1
$$

and the pair $\left(z_{2}, z_{3}\right)$ is arbitrary. We will show that (33))-(35) imply that $y_{4}$ is a product vector. So, in fact the pair $\left(x_{1}, y_{4}\right)$ has form 1 or 2 .

Assume that $y_{4}$ is not a product vector and denote the vectors $\left[\mu_{1}, s\right]^{\top}$ and $\left[\mu_{2}, t\right]^{\top}$ by $|s\rangle$ and $|t\rangle$. In this notations $\left|x_{1}\right\rangle=h|s \otimes t\rangle$.

In case a) it follows from (34) and Lemma 10 in the Appendix that $\left|x_{2}\right\rangle=|p \otimes t\rangle$ for some vector $|p\rangle$. Hence the left hand side of (33) has the form

$$
|h|^{2}|s\rangle\langle s| \otimes|t\rangle\langle t|+|p\rangle\langle p| \otimes|t\rangle\langle t|=\left[|h|^{2}|s\rangle\langle s|+|p\rangle\langle p|\right] \otimes|t\rangle\langle t|
$$

and (33) implies $\left|y_{4}\right\rangle\left\langle y_{4}\right| \leq\left[|h|^{2}|s\rangle\langle s|+|p\rangle\langle p|\right] \otimes|t\rangle\langle t|$. This operator inequality can be valid only if $y_{4}$ is a product vector.

In case b) it follows from (35) and Lemma 10 in the Appendix that $\left|x_{3}\right\rangle=|s \otimes q\rangle$ for some vector $|q\rangle$. Hence the left hand side of (33) has the form

$$
|h|^{2}|s\rangle\langle s| \otimes|t\rangle\langle t|+|s\rangle\langle s| \otimes|q\rangle\langle q|=|s\rangle\langle s| \otimes\left[|h|^{2}|t\rangle\langle t|+|q\rangle\langle q|\right]
$$

and similarly to the case a) we conclude that $y_{4}$ is a product vector.
By using the same argumentation exploiting (33)-(35) and Lemma 10 one can show that neither $\left(x_{1}, y_{4}\right)$ nor $\left(z_{2}, z_{3}\right)$ can be a pair of form 4 or 5 (not coinciding with form 1 or 2 ).

Thus, we have shown that the system (15)-(26) has no nontrivial solutions. This completes the proof of assertion $\mathrm{C}_{2}$.

## Appendix

### 3.1 Proofs of Lemmas 3 and 4

Proof of Lemma [3. A) Let $\left\langle z_{4}\right|=[a, b, c, d]$ and

$$
W=\left[\begin{array}{cccc}
a & \gamma_{2} b & \gamma_{1} c & \gamma_{1} \gamma_{2} d \\
b & a & \gamma_{1} d & \gamma_{1} c \\
c & \gamma_{2} d & a & \gamma_{2} b \\
d & c & b & a
\end{array}\right], \quad S=\left[\begin{array}{rrrr}
\mu_{1} \mu_{2} & \mu_{1} \mu_{2} & \mu_{1} \mu_{2} & \mu_{1} \mu_{2} \\
\mu_{1} & -\mu_{1} & \mu_{1} & -\mu_{1} \\
\mu_{2} & \mu_{2} & -\mu_{2} & -\mu_{2} \\
+1 & -1 & -1 & +1
\end{array}\right]
$$

where $\mu_{k}=\sqrt{\gamma_{k}}, k=1,2$. By identifying $A \otimes B$ with the matrix $\left\|a_{i j} B\right\|$ the equalities $\left\langle z_{4}\right| U_{k} \otimes V_{l}\left|z_{1}\right\rangle=0, k, l=1,2$ can be rewritten as the system of linear equations

$$
\begin{equation*}
W\left|z_{1}\right\rangle=0 \tag{40}
\end{equation*}
$$

and it is easy to see that $S^{-1} W S=\operatorname{diag}\left\{p_{1}, p_{2}, p_{3}, p_{4}\right\}$, where

$$
\begin{array}{ll}
p_{1}=a+\mu_{2} b+\mu_{1} c+\mu_{1} \mu_{2} d, & p_{2}=a-\mu_{2} b+\mu_{1} c-\mu_{1} \mu_{2} d \\
p_{3}=a+\mu_{2} b-\mu_{1} c-\mu_{1} \mu_{2} d, & p_{4}=a-\mu_{2} b-\mu_{1} c+\mu_{1} \mu_{2} d . \tag{41}
\end{array}
$$

So, system (40) can be rewritten as the system $p_{k} u_{k}=0, k=\overline{1,4}$, where $\left[u_{1}, u_{2}, u_{3}, u_{4}\right]^{\top}=S^{-1}\left|z_{1}\right\rangle$. Hence this system has nontrivial solutions if and only if $p_{1} p_{2} p_{3} p_{4}=0$ and

$$
\left\{p_{k}=0\right\} \Leftrightarrow\left\{W\left|q_{k}\right\rangle=0\right\}
$$

where $\left|q_{k}\right\rangle$ is the $k$-th column of the matrix $S$.
Thus, by choosing some of $p_{1}, \ldots, p_{4}$ equal to zero we obtain all pairs $\left(z_{1}, z_{4}\right)$ such that $\left\langle z_{4}\right| U_{k} \otimes V_{l}\left|z_{1}\right\rangle=0, k, l=1,2$. We have
a) $C_{4}^{2}=6$ possibilities to take $p_{k}=p_{l}=0$ and $p_{i} \neq 0, i \neq k, l$;
b) $C_{4}^{1}=4$ possibilities to take $p_{k}=0$ and $p_{i} \neq 0, i \neq k$;
c) $C_{4}^{3}=4$ possibilities to take $p_{k}=p_{l}=p_{j}=0$ and $p_{i} \neq 0, i \neq k, l, j$.
(the case $p_{1}=p_{2}=p_{3}=p_{4}=0$ means that $a=b=c=d=0$, so it gives only trivial solution).

By identifying the vectors $x \otimes y$ and $\left[x_{1} y, x_{2} y\right]^{\top}$ it is easy to see that

$$
\left|q_{1}\right\rangle=\left[\begin{array}{c}
\mu_{1} \\
1
\end{array}\right] \otimes\left[\begin{array}{c}
\mu_{2} \\
1
\end{array}\right],\left|q_{2}\right\rangle=\left[\begin{array}{c}
\mu_{1} \\
1
\end{array}\right] \otimes\left[\begin{array}{c}
\mu_{2} \\
-1
\end{array}\right],\left|q_{3}\right\rangle=\left[\begin{array}{c}
\mu_{1} \\
-1
\end{array}\right] \otimes\left[\begin{array}{c}
\mu_{2} \\
1
\end{array}\right],\left|q_{4}\right\rangle=\left[\begin{array}{c}
\mu_{1} \\
-1
\end{array}\right] \otimes\left[\begin{array}{c}
\mu_{2} \\
-1
\end{array}\right]
$$

and that

$$
\begin{array}{ll}
p_{1}=0 \quad \Leftrightarrow \quad\left|z_{4}\right\rangle=\left[\begin{array}{l}
c_{1} \\
c_{2}
\end{array}\right] \otimes\left[\begin{array}{c}
\bar{\mu}_{2} \\
-1
\end{array}\right]+\left[\begin{array}{c}
\bar{\mu}_{1} \\
-1
\end{array}\right] \otimes\left[\begin{array}{l}
c_{3} \\
c_{4}
\end{array}\right], \quad c_{1}, \ldots c_{4} \in \mathbb{C}, \\
p_{2}=0 \quad \Leftrightarrow \quad\left|z_{4}\right\rangle=\left[\begin{array}{l}
c_{1} \\
c_{2}
\end{array}\right] \otimes\left[\begin{array}{c}
\bar{\mu}_{2} \\
1
\end{array}\right]+\left[\begin{array}{c}
\bar{\mu}_{1} \\
-1
\end{array}\right] \otimes\left[\begin{array}{l}
c_{3} \\
c_{4}
\end{array}\right], \quad c_{1}, \ldots c_{4} \in \mathbb{C}, \\
p_{3}=0 \quad \Leftrightarrow \quad\left|z_{4}\right\rangle=\left[\begin{array}{l}
c_{1} \\
c_{2}
\end{array}\right] \otimes\left[\begin{array}{c}
\bar{\mu}_{2} \\
-1
\end{array}\right]+\left[\begin{array}{c}
\bar{\mu}_{1} \\
1
\end{array}\right] \otimes\left[\begin{array}{l}
c_{3} \\
c_{4}
\end{array}\right], \quad c_{1}, \ldots c_{4} \in \mathbb{C}, \\
p_{4}=0 \quad \Leftrightarrow \quad\left|z_{4}\right\rangle=\left[\begin{array}{c}
c_{1} \\
c_{2}
\end{array}\right] \otimes\left[\begin{array}{c}
\bar{\mu}_{2} \\
1
\end{array}\right]+\left[\begin{array}{c}
\bar{\mu}_{1} \\
1
\end{array}\right] \otimes\left[\begin{array}{l}
c_{3} \\
c_{4}
\end{array}\right], \quad c_{1}, \ldots c_{4} \in \mathbb{C},
\end{array}
$$

Hence the above six possibilities in a) correspond to cases 1)-3) in Lemma 3A (for example, the choice $p_{1}=p_{2}=0, p_{3}, p_{4} \neq 0$ corresponds to case 1 ) with $s=1$ ), while the four possibilities in b) and in c) correspond respectively to cases 4) and 5).
B) Denote by $W_{x}$ and $W_{y}$ the above matrix $W$ determined respectively via $z_{4}=x_{4}$ and $z_{4}=y_{4}$. Then the equalities in (23) and (24) can be rewritten as the system

$$
\begin{equation*}
W_{x}\left|y_{1}\right\rangle=W_{y}\left|x_{1}\right\rangle=0, \quad W_{x}\left|x_{1}\right\rangle=W_{y}\left|y_{1}\right\rangle=|c\rangle, \quad|c\rangle \in \mathbb{C}^{4} \tag{42}
\end{equation*}
$$

Since $S^{-1} W_{x} S=\operatorname{diag}\left\{p_{1}^{x}, p_{2}^{x}, p_{3}^{x}, p_{4}^{x}\right\}$ and $S^{-1} W_{y} S=\operatorname{diag}\left\{p_{1}^{y}, p_{2}^{y}, p_{3}^{y}, p_{4}^{y}\right\}$, where $p_{1}^{x}, p_{2}^{x}, p_{3}^{x}, p_{4}^{x}$ and $p_{1}^{y}, p_{2}^{y}, p_{3}^{y}, p_{4}^{y}$ are defined in (41) with $z_{4}=x_{4}$ and $z_{4}=y_{4}$ correspondingly, system (42) is equivalent to the following one

$$
\begin{equation*}
p_{k}^{x} v_{k}=p_{k}^{y} u_{k}=0, \quad p_{k}^{x} u_{k}=p_{k}^{y} v_{k}=\tilde{c}_{k}, \quad k=\overline{1,4} \tag{43}
\end{equation*}
$$

where $\left[u_{1}, u_{2}, u_{3}, u_{4}\right]^{\top}=S^{-1}\left|x_{1}\right\rangle,\left[v_{1}, v_{2}, v_{3}, v_{4}\right]^{\top}=S^{-1}\left|y_{1}\right\rangle$ and $\left[\tilde{c}_{1}, \tilde{c}_{2}, \tilde{c}_{3}, \tilde{c}_{4}\right]^{\top}=$ $S^{-1}|c\rangle$. It follows that $\tilde{c}_{k}=0$ for all $k$. Indeed, if $p_{k}^{y} \neq 0$ for some $k$ then the first equality in (43) implies $u_{k}=0$ and the second equality in (43) shows that $\tilde{c}_{k}=0$. Hence $|c\rangle=S|\tilde{c}\rangle=0$.

Lemma 4 follows from Lemma 3 with $\gamma_{2}$ replaced by $\bar{\gamma}_{2}$.

### 3.2 Auxiliary lemmas

Lemma 8. If $|a\rangle\langle x|+|b\rangle\langle y|+|c\rangle\langle z|=0$ then either $a\|b\| c$ or $x\|y\| z$.
Proof. We may assume that all the vectors are nonzero (since otherwise the assertion is trivial).

Let $p \perp x$. Then $\langle y \mid p\rangle|b\rangle+\langle z \mid p\rangle|c\rangle=0$ and hence either $b \| c$ or $\langle y \mid p\rangle=\langle z \mid p\rangle=0$.

If $b \| c$ then we have $|a\rangle\langle x|=-|b\rangle\langle y+\lambda z|, \lambda \in \mathbb{C}$, and hence $a\|b\| c$. If $\langle y \mid p\rangle=\langle z \mid p\rangle=0$ then $x\|y\| z$, since the vector $p$ is arbitrary.

Lemma 9. The equality

$$
\begin{equation*}
X_{1} \otimes Y_{1}+X_{2} \otimes Y_{2}=X_{3} \otimes Y_{3}+X_{4} \otimes Y_{4} \tag{44}
\end{equation*}
$$

where $X_{i}=\left|x_{i}\right\rangle\left\langle x_{i}\right|, Y_{i}=\left|y_{i}\right\rangle\left\langle y_{i}\right|, i=\overline{1,4}$, can be valid only in the following cases:

1) $x_{i} \| x_{j}$ for all $i, j$ and $Y_{1}\left\|x_{1}\right\|^{2}+Y_{2}\left\|x_{2}\right\|^{2}=Y_{3}\left\|x_{3}\right\|^{2}+Y_{4}\left\|x_{4}\right\|^{2}$;
2) $y_{i} \| y_{j}$ for all $i, j$ and $X_{1}\left\|y_{1}\right\|^{2}+X_{2}\left\|y_{2}\right\|^{2}=X_{3}\left\|y_{3}\right\|^{2}+X_{4}\left\|y_{4}\right\|^{2}$;
3) $X_{1} \otimes Y_{1}=X_{4} \otimes Y_{4}$ and $X_{2} \otimes Y_{2}=X_{3} \otimes Y_{3}$;
4) $X_{1} \otimes Y_{1}=X_{3} \otimes Y_{3}$ and $X_{2} \otimes Y_{2}=X_{4} \otimes Y_{4}$.

Proof. We may assume that all the vectors $x_{i}, y_{i}$ are nonzero (since otherwise the assertion is trivial).

Let $p \perp x_{1}$. By multiplying the both sides of (44) by $|p\rangle\langle p| \otimes I$ we obtain

$$
\begin{equation*}
\left|\left\langle x_{2} \mid p\right\rangle\right|^{2} Y_{2}=\left|\left\langle x_{3} \mid p\right\rangle\right|^{2} Y_{3}+\left|\left\langle x_{4} \mid p\right\rangle\right|^{2} Y_{4} . \tag{45}
\end{equation*}
$$

If $x_{2} \| x_{1}$ then $\left\langle x_{3} \mid p\right\rangle=\left\langle x_{4} \mid p\right\rangle=0$ and hence $x_{1}\left\|x_{2}\right\| x_{3} \| x_{4}$, since the vector $p$ is arbitrary. So, case 1) holds.

If $x_{2} \nVdash x_{1}$ then one can choose $p$ such that $\left\langle x_{2} \mid p\right\rangle \neq 0$. So, (45) implies that either $x_{3} \nVdash x_{1}$ or $x_{4} \nVdash x_{1}$. Thus, we have the following possibilities:
a) If $x_{i} \nVdash x_{1}$ for $i=2,3,4$ then one can choose $p$ such that $\left\langle x_{i} \mid p\right\rangle \neq 0$, $i=2,3,4$, and (45) implies $y_{2}\left\|y_{3}\right\| y_{4}$. Hence (44) leads to $X_{1} \otimes Y_{1}=$ $[\ldots] \otimes Y_{2}$, which gives $y_{1} \| y_{2}$. So, we have $y_{1}\left\|y_{2}\right\| y_{3} \| y_{4}$, i.e. case 2).
b) If $x_{i} \nVdash x_{1}$ for $i=2,3$ but $x_{4} \| x_{1}$ then one can choose $p$ such that $\left\langle x_{i} \mid p\right\rangle \neq 0, i=2,3$ and (45) implies $y_{2} \| y_{3}$. So, we have $x_{4}=\alpha x_{1}$ and $y_{3}=\beta y_{2}, \alpha, \beta \in \mathbb{C}$. It follows from (44) that

$$
X_{1} \otimes\left[Y_{1}-|\alpha|^{2} Y_{4}\right]=\left[X_{3}|\beta|^{2}-X_{2}\right] \otimes Y_{2}
$$

and hence $Y_{1}-|\alpha|{ }^{2} Y_{4}=\lambda Y_{2}, \lambda \in \mathbb{C}$. If $\lambda \neq 0$ then Lemma 8 implies $y_{1}\left\|y_{2}\right\| y_{3} \| y_{4}$, i.e. case 2) holds. If $\lambda=0$ then $y_{1} \| y_{4}$ and $x_{2} \| x_{3}$. Thus we have

$$
X_{4} \otimes Y_{4}=\gamma X_{1} \otimes Y_{1}, \quad X_{3} \otimes Y_{3}=\delta X_{2} \otimes Y_{2}, \quad \gamma, \delta \in \mathbb{C}
$$

and (44) implies $(1-\gamma) X_{1} \otimes Y_{1}=(\delta-1) X_{2} \otimes Y_{2}$. Since $x_{1} \nVdash x_{2}$, we have $\gamma=\delta=1$, i.e. case 3) holds.
c) If $x_{i} \nVdash x_{1}$ for $i=2,4$ but $x_{3} \| x_{1}$ then the similar arguments (with the permutation $3 \leftrightarrow 4$ ) shows that case 4 ) holds.

Lemma 10. Let $U_{1}=\operatorname{diag}\{1, \gamma\}$ and $x, y$ be nonzero vectors in $\mathbb{C}^{2}$. If $\langle a| U_{1} \otimes A|x \otimes y\rangle=\langle c| U_{1} \otimes A|d\rangle$ for all $A \in \mathfrak{M}_{2}$ then either $|d\rangle=|z\rangle \otimes|y\rangle$ or $|c\rangle=|p\rangle \otimes|q\rangle$ for some vectors $p, q, z$ in $\mathbb{C}^{2}$.

Proof. By using the isomorphism $\mathbb{C}^{2} \otimes \mathbb{C}^{2} \ni u \otimes v \leftrightarrow\left[u_{1} v, u_{2} v\right]^{\top} \in \mathbb{C}^{2} \oplus \mathbb{C}^{2}$ the condition of the lemma can be rewritten as follows

$$
\left\langle\begin{array}{c|cc|c}
a_{1} & A & 0 & x_{1} y \\
a_{2} & 0 & \gamma A & x_{2} y
\end{array}\right\rangle=\left\langle\begin{array}{c|cc|c}
c_{1} & A & 0 & d_{1} \\
c_{2} & 0 & \gamma A & d_{2}
\end{array}\right\rangle, \quad \forall A \in \mathfrak{M}_{2},
$$

where $a_{1}, a_{2}$ are components of the vector $a$, etc. So, we have

$$
x_{1}\left\langle a_{1}\right| A|y\rangle+x_{2} \gamma\left\langle a_{2}\right| A|y\rangle=\left\langle c_{1}\right| A\left|d_{1}\right\rangle+\gamma\left\langle c_{2}\right| A\left|d_{2}\right\rangle \quad \forall A \in \mathfrak{M}_{2},
$$

which is equivalent to the equality $|y\rangle\left\langle\bar{x}_{1} a_{1}+\bar{x}_{2} \bar{\gamma} a_{2}\right|=\left|d_{1}\right\rangle\left\langle c_{1}\right|+\gamma\left|d_{2}\right\rangle\left\langle c_{2}\right|$. By Lemma 8 it follows that either $d_{1}\left\|d_{2}\right\| y$, which means that $|d\rangle=|z\rangle \otimes|y\rangle$, or $c_{1} \| c_{2}$, which means that $|c\rangle=|p\rangle \otimes|q\rangle$.

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[^1]:    ${ }^{1}$ In contrast to this paper $\gamma=\exp [\mathrm{i} \theta / 2]$ is used in [10.

[^2]:    ${ }^{2}$ We call a subspace $\mathcal{H}_{0}$ indistinguishable for an observable $\mathcal{M}$ if application of $\mathcal{M}$ to all states supported by $\mathcal{H}_{0}$ leads to the same outcomes probability distribution.

[^3]:    ${ }^{3}$ In the sense that all the vectors $\left|z_{i}\right\rangle, i=\overline{1,4}$, are not collinear to each other, but some two of them may be collinear.

