# Almost Disjunctive List-Decoding Codes 

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Abstract. A binary code is said to be a disjunctive list-decoding $s_{L}$-code, $s \geq 1, L \geq 1$, (briefly, LD $s_{L}$-code) if the code is identified by the incidence matrix of a family of finite sets in which the union of any $s$ sets can cover not more than $L-1$ other sets of the family. In this paper, we introduce a natural probabilistic generalization of LD $s_{L}$-code when the code is said to be an almost disjunctive LD $s_{L}$-code if the unions of almost all $s$ sets satisfy the given condition. We develop a random coding method based on the ensemble of binary constant-weight codes to obtain lower bounds on the capacity and error probability exponent of such codes. For the considered ensemble our lower bounds are asymptotically tight.

Index terms. Almost disjunctive codes, capacity, error probability exponent, random coding bounds, group testing, screening experiments, two-stage search designs.

## 1 Notations and Definitions

Let $N, t, s$, and $L$ be integers, where $1 \leq s<t, 1 \leq L \leq t-s$. Let $\triangleq$ denote the equality by definition, $|A|$ - the size of set $A$ and $[N] \triangleq\{1,2, \ldots, N\}$ - the set of integers from 1 to $N$. The standard symbol $\lfloor a\rfloor(\lceil a\rceil)$ will be used to denote the largest (least) integer $\leq a(\geq a)$. A binary $(N \times t)$-matrix

$$
\begin{equation*}
X=\left\|x_{i}(j)\right\|, \quad x_{i}(j)=0,1, \quad \boldsymbol{x}_{i} \triangleq\left(x_{i}(1), \ldots, x_{i}(t)\right), \quad \boldsymbol{x}(j) \triangleq\left(x_{1}(j), \ldots, x_{N}(j)\right), \tag{1}
\end{equation*}
$$

$i \in[N], j \in[t]$, with $N$ rows $\boldsymbol{x}_{1}, \ldots, \boldsymbol{x}_{N}$ and $t$ columns $\boldsymbol{x}(1), \ldots, \boldsymbol{x}(t)$ (codewords) is called a binary code of length $N$ and size $t=\left\lceil 2^{R N}\right\rceil$ (briefly, ( $N, R$ )-code), where a fixed parameter $R>0$ is called the rate of code $X$ [1]-[3]. For any code $X$ and any subset $\mathcal{S} \subset[t]$ of size $|\mathcal{S}|=s$, the symbol $\boldsymbol{x}(\mathcal{S}) \triangleq\{\boldsymbol{x}(j): j \in \mathcal{S}\}$ will denote the corresponding $s$-subset of codewords (columns) of the code $X$. The number of 1's in column $x(j)$, i.e., $|\boldsymbol{x}(j)| \triangleq \sum_{i=1}^{N} x_{i}(j)$, is called the weight of $x(j), j \in[t]$. A code $X$ is called a constant weight binary code of weight $w, 1<w<N$, if for any $j \in[t]$, the weight $|\boldsymbol{x}(j)|=w$. The standard symbol $\bigvee$ denotes the disjunctive (Boolean) sum of two binary numbers:

$$
0 \bigvee 0=0, \quad 0 \bigvee 1=1 \bigvee 0=1 \bigvee 1=1
$$

as well as the component-wise disjunctive sum of two binary columns. We say that a column $\mathbf{u}$ covers column $\mathbf{v}(\mathbf{u} \succeq \mathbf{v})$ if $\mathbf{u} \bigvee \mathbf{v}=\mathbf{u}$.

Definition 1. An $s$-subset of columns $\boldsymbol{x}(\mathcal{S}),|\mathcal{S}|=s$, of a code $X$ is said to be an $s_{L}$-bad subset of columns in the code $X$ if there exists a subset $\mathcal{L} \subset[t]$ of size $|\mathcal{L}|=L$, such that $\mathcal{S} \cap \mathcal{L}=\varnothing$ and the disjunctive sum

$$
\begin{equation*}
\bigvee_{i \in \mathcal{S}} x(i) \succeq \bigvee_{j \in \mathcal{L}} x(j) \tag{2}
\end{equation*}
$$

Otherwise, the $s$-subset $\boldsymbol{x}(\mathcal{S})$ is called $s_{L}$-good subset of columns in the code $X$. In other words, for any $s_{L}$-good subset of columns in a code $X$, the disjunctive sum of its $s$ columns can cover not more than $L-1$ columns of the code $X$ that are not components of the given $s$-subset.

Definition 2. Let $\epsilon, 0 \leq \epsilon<1$, be a fixed parameter. A code $X$ is said to be a disjunctive list-decoding $\left(s_{L}, \epsilon\right)$-code (or almost disjunctive list-decoding $s_{L}$-code) of strength $s$, list size $L$ and error probability $\epsilon, 0 \leq \epsilon<1$, (briefly, $L D\left(s_{L}, \epsilon\right)$-code), if the number $\mathbf{G}_{L}(s, X)$ of all $s_{L^{-}}$-good $s$-subsets of columns of the code $X$ is at least $(1-\epsilon) \cdot\binom{t}{s}$. In other words, the number $\mathbf{B}_{L}(s, X)$ of all $s_{L}$-bad $s$-subsets of columns for $\operatorname{LD}\left(s_{L}, \epsilon\right)$-code $X$ does not exceed $\epsilon\binom{t}{s}$, i.e.,

$$
\begin{equation*}
\mathbf{B}_{L}(s, X) \triangleq\binom{t}{s}-\mathbf{G}_{L}(s, X) \leq \epsilon \cdot\binom{t}{s} \Longleftrightarrow \frac{\mathbf{B}_{L}(s, X)}{\binom{t}{s}} \leq \epsilon \tag{3}
\end{equation*}
$$

The concept of LD $\left(s_{L}, \epsilon\right)$-code can be considered as a natural generalization of the classical superimposed $s$-code of Kautz-Singleton [4] corresponding to the case $L=1$ and $\epsilon=0$. For the case $L \geq 1$ and $\epsilon=0$, disjunctive list-decoding codes (LD $s_{L}$-codes) were investigated in works [5]-[13] and the last detailed survey of the most important results obtained for LD $s_{L}$-codes is given in the recent paper [14] (see, also, preprint [15]).

Definition 3. Let $t_{\epsilon}(N, s, L)$ be the maximal size of LD $\left(s_{L}, \epsilon\right)$-codes of length $N$ and let $N_{\epsilon}(t, s, L)$ be the minimal length of LD $\left(s_{L}, \epsilon\right)$-codes of size $t$. If $\epsilon=0$, then the number

$$
\begin{equation*}
R_{L}(s) \triangleq \varlimsup_{N \rightarrow \infty} \frac{\log _{2} t_{0}(N, s, L)}{N}=\varlimsup_{t \rightarrow \infty} \frac{\log _{2} t}{N_{0}(t, s, L)} \tag{4}
\end{equation*}
$$

is called [7] the rate of LD $s_{L^{-}}$-codes.
Observe [14] that at fixed $s \geq 2$, the number

$$
\begin{equation*}
R_{\infty}(s) \triangleq \lim _{L \rightarrow \infty} R_{L}(s), \quad s=2,3, \ldots \tag{5}
\end{equation*}
$$

can be interpreted as the maximal rate for two-stage group testing in the disjunctive search model of any $d, d \leq s$, defective elements based on LD $s_{L}$-codes. For the general two-stage group testing [10], the number $R_{\infty}(s)$ gives a lower bound on the corresponding rate.

Definition 4. Define the number

$$
\begin{equation*}
C_{L}(s) \triangleq \varlimsup_{\epsilon \rightarrow 0} \varlimsup_{N \rightarrow \infty} \frac{\log _{2} t_{\epsilon}(N, s, L)}{N}=\varlimsup_{\epsilon \rightarrow 0} \varlimsup_{t \rightarrow \infty} \frac{\log _{2} t}{N_{\epsilon}(t, s, L)} \geq R_{L}(s) \tag{6}
\end{equation*}
$$

called a capacity of almost disjunctive LD $s_{L}$-codes.
The definition (6) implies that if the parameter $N$ is sufficiently large, then for any fixed $\epsilon, \epsilon>0$, and any fixed rate $R>0$, there exists an $\operatorname{LD}\left(s_{L}, \epsilon\right)$-code $X$ of length $N$ and size $t=\left\lceil 2^{R N}\right\rceil$, i.e., $(N, R)$-code $X$, if and only if the rate $R<C_{L}(s)$. Obviously, $C_{L}(s) \leq 1 / s$ and the first open problem is: "how to improve this evident upper bound?"

Definition 5. Let $R, R_{L}(s) \leq R<C_{L}(s)$, be a fixed parameter. Taking into account the inequality (3) from Definition 2, we introduce the concept of error probability for almost disjunctive $L D s_{L}$-codes:

$$
\begin{equation*}
\epsilon_{L}(s, R, N) \triangleq \min _{X: t=\left\lceil 2^{R N}\right\rceil}\left\{\frac{\mathbf{B}_{L}(s, X)}{\binom{t}{s}}\right\}, \tag{7}
\end{equation*}
$$

where the minimum is taken over all $(N, R)$-codes $X$, and the function

$$
\begin{equation*}
\mathbf{E}_{L}(s, R) \triangleq \varlimsup_{N \rightarrow \infty} \frac{-\log _{2} \epsilon_{L}(s, R, N)}{N}, \quad R_{L}(s) \leq R<C_{L}(s) \tag{8}
\end{equation*}
$$

is said to be the exponent of error probability for almost disjunctive LD $s_{L}$-codes.
Immediately from definitions (4)-(8) it follows
Proposition 1. If one put the parameter $R=R_{L}(s)$, where the number $R_{L}(s)$ is the rate of $L D s_{L}$-codes defined by (4), then for the exponent of error probability (8), the equality

$$
\begin{equation*}
\mathbf{E}_{L}\left(s, R_{L}(s)\right)=s R_{L}(s), \quad s \geq 1, \quad L \geq 1 \tag{9}
\end{equation*}
$$

holds. In other words, the value $R=R_{L}(s)$ is the unique root of the equation

$$
\begin{equation*}
\mathbf{E}_{L}(s, R)=s R, \quad s \geq 1, \quad L \geq 1 . \tag{10}
\end{equation*}
$$

In addition, for any $s \geq 1$ and $L \geq 1$, the monotonicity inequalities hold:

$$
\begin{equation*}
R_{L}(s) \leq R_{L+1}(s), C_{L}(s) \leq C_{L+1}(s), \quad \mathbf{E}_{L}(s, R) \leq \mathbf{E}_{L+1}(s, R) \tag{11}
\end{equation*}
$$

One can easily understand that Proposition 1 yields
Proposition 2. Let there exist a lower bound $\underline{\mathbf{E}}_{L}(s, R)$ (upper bound $\overline{\mathbf{E}}_{L}(s, R)$ ) on the exponent of error probability $\mathbf{E}_{L}(s, R)$ for almost disjunctive $s_{L}$-codes, i.e.,

$$
\underline{\mathbf{E}}_{L}(s, R) \leq \mathbf{E}_{L}(s, R) \leq \overline{\mathbf{E}}_{L}(s, R), \quad R_{L}(s) \leq R<C_{L}(s) .
$$

Let $\underline{R}_{L}(s)\left(\bar{R}_{L}(s)\right)$ denote the unique root of the equation

$$
\begin{equation*}
\underline{\mathbf{E}}_{L}(s, R)=s R, \quad\left(\overline{\mathbf{E}}_{L}(s, R)=s R\right), \quad s \geq 1, \quad L \geq 1 . \tag{12}
\end{equation*}
$$

Then the number $\underline{R}_{L}(s)\left(\bar{R}_{L}(s)\right)$ is a lower (upper) bound on the rate of LD $s_{L}$-codes, i.e. the inequality

$$
\underline{R}_{L}(s) \leq R_{L}(s), \quad\left(R_{L}(s) \leq \bar{R}_{L}(s)\right), \quad s \geq 1, \quad L \geq 1 .
$$

holds.
In Definitions 2-5 for the case $L=1$, we use the terminology which is similar to a terminology for the concept of weakly separating designs introduced in [16]. Let $X$ be a code of length $N$ and size $t$ and let $\Omega_{\epsilon}(X, s, t)$ be a collection of $s$-subsets of columns of the code $X$ such that its size $\left|\Omega_{\epsilon}(X, s, t)\right| \geq(1-\epsilon) \cdot\binom{t}{s}$. The code $X$ is said $[16]$ to be a disjunctive $(s, \epsilon)$-design (or weakly separating $s$-design), if there exists a collection $\Omega_{\epsilon}(X, s, t)$ such that the disjunctive sums of any two s-subsets from the collection $\Omega_{\epsilon}(X, s, t)$ are different. Weakly separating $s$-design can be considered [17] (see, also [13]) as an important example of information-theoretical model for the multiple-access channel [3]. It was proved [16] that the capacity of weakly separating $s$-designs is equal to $1 / s$. For the case $L \geq 2$, the list-decoding weakly separating $s$-designs were suggested in the paper [18], where it was established that their capacity is equal to $1 / s$ as well.

## 2 Lower Bounds on $R_{L}(s), C_{L}(s)$ and $\mathbf{E}_{L}(s, R)$

The best known upper and lower bounds on the rate $R_{L}(s)$ of LD $s_{L}$-codes were presented in [14] (see, also, preprint [15]). For the classical case $L=1$, these bounds have the form:

$$
\begin{gather*}
R_{1}(s) \leq \bar{R}_{1}(s)=\frac{2 \log _{2} s}{s^{2}}(1+o(1)), \quad s \rightarrow \infty,  \tag{13}\\
R_{1}(s) \geq \underline{R}_{1}(s)=\frac{4 e^{-2} \log _{2} s}{s^{2}}(1+o(1))=\frac{0,542 \log _{2} s}{s^{2}}(1+o(1)), \quad s \rightarrow \infty . \tag{14}
\end{gather*}
$$

If $s \geq 1, L \geq 2$, then our lower random coding bound on $R_{L}(s)$ was established [14] as
Theorem 1. [14] (Random coding bound $\left.\underline{R}_{L}^{(1)}(s)\right)$. 1. The rate

$$
\begin{align*}
R_{L}(s) \geq \underline{R}_{L}^{(1)}(s) & \triangleq \frac{1}{s+L-1} \max _{0<Q<1} A_{L}(s, Q)=\frac{1}{s+L-1} A_{L}\left(s, Q_{L}^{(1)}(s)\right),  \tag{15}\\
A_{L}(s, Q) & \triangleq \log _{2} \frac{Q}{1-y}-s K(Q, 1-y)-L K\left(Q, \frac{1-y}{1-y^{s}}\right)  \tag{16}\\
K(a, b) & \triangleq a \cdot \log _{2} \frac{a}{b}+(1-a) \cdot \log _{2} \frac{1-a}{1-b}, \quad 0<a, b<1, \tag{17}
\end{align*}
$$

where parameter $y, 1-Q \leq y<1$, is defined as the unique root of the equation

$$
\begin{equation*}
y=1-Q+Q y^{s}\left[1-\left(\frac{y-y^{s}}{1-y^{s}}\right)^{L}\right], \quad 1-Q \leq y<1 . \tag{18}
\end{equation*}
$$

2. For fixed $L=2,3, \ldots$ and $s \rightarrow \infty$, the asymptotic behavior of the random coding bound $\underline{R}_{L}^{(1)}(s)$ has the form

$$
\begin{equation*}
\underline{R}_{L}^{(1)}(s)=\frac{L}{s^{2} \log _{2} e}(1+o(1))=\frac{L \ln 2}{s^{2}}(1+o(1)) . \tag{19}
\end{equation*}
$$

3. At fixed $s=1,2,3, \ldots$ and $L \rightarrow \infty$, for the maximal rate $R_{\infty}(s)$ of two-stage group testing defined by (5), the lower bound

$$
\begin{equation*}
R_{\infty}(s) \geq \underline{R}_{\infty}^{(1)}(s) \triangleq \lim _{L \rightarrow \infty} \underline{R}_{L}^{(1)}(s)=\log _{2}\left[\frac{(s-1)^{s-1}}{s^{s}}+1\right] . \tag{20}
\end{equation*}
$$

holds. If $s \rightarrow \infty$, then $\underline{R}_{\infty}^{(1)}(s)=\frac{\log _{2} e}{e \cdot s}(1+o(1))=\frac{0,5307}{s}(1+o(1))$.
In the given paper, we suggest a modification of the random coding method developed in [14] and obtain a lower bound on the capacity $C_{L}(s)$ along with a lower bound on the exponent of error probability $\mathbf{E}_{L}(s, R)$ for almost disjunctive $s_{L}$-codes. Let

$$
[x]^{+} \triangleq\left\{\begin{array}{ll}
x & \text { if } x \geq 0, \\
0 & \text { if } x<0,
\end{array} \quad \text { and } \quad h(a) \triangleq-a \log _{2} a-(1-a) \log _{2}(1-a), 0<a<1\right.
$$

be the standard notations for the positive part function and the binary entropy function.

Theorem 2. (Random coding lower bounds $\underline{C}(s)$ and $\underline{\mathbf{E}}_{L}(s, R)$ ). The following three claims hold. Claim 1. The capacity $C_{L}(s)$ and the exponent of error probability $\mathbf{E}_{L}(s, R)$ for almost disjunctive $L D s_{L}$-codes satisfy inequalities

$$
\begin{gather*}
C_{L}(s) \geq \underline{C}(s) \triangleq \max _{0<Q<1} C(s, Q)=C(s, Q(s)), \quad s \geq 1, \quad L \geq 1  \tag{21}\\
C(s, Q) \triangleq h(Q)-\left[1-(1-Q)^{s}\right] h\left(\frac{Q}{1-(1-Q)^{s}}\right), \quad s \geq 1, \quad 0<Q<1 \tag{22}
\end{gather*}
$$

and

$$
\begin{gather*}
\mathbf{E}_{L}(s, R) \geq \underline{\mathbf{E}}_{L}(s, R) \triangleq \max _{0<Q<1} E_{L}(s, R, Q), \quad s \geq 1, \quad L \geq 1  \tag{23}\\
E_{L}(s, R, Q) \triangleq \min _{Q \leq q \leq \min \{1, s Q\}}\left\{\mathcal{A}(s, Q, q)+L \cdot[h(Q)-q \cdot h(Q / q)-R]^{+}\right\} \tag{24}
\end{gather*}
$$

where the function $\mathcal{A}(s, Q, q), Q<q<\min \{1, s Q\}$, is defined in the parametric form:

$$
\begin{gather*}
\mathcal{A}(s, Q, q) \triangleq(1-q) \log _{2}(1-q)+q \log _{2}\left[\frac{Q y^{s}}{1-y}\right]+s Q \log _{2} \frac{1-y}{y}+\operatorname{sh}(Q)  \tag{25}\\
q=Q \frac{1-y^{s}}{1-y}, \quad 0<y<1 \tag{26}
\end{gather*}
$$

Claim 2. If $s \geq 1$ is fixed, then the random coding lower bound $\underline{C}(s)>\frac{\ln 2}{s}$ and at $s \rightarrow \infty$ the asymptotic behavior of $\underline{C}(s)$ and the asymptotic behavior of the optimal value $Q(s)$ in (21) are:

$$
\begin{equation*}
\underline{C}(s)=\frac{\ln 2}{s}(1+o(1)), \quad Q(s)=\frac{\ln 2}{s}(1+o(1)) \tag{27}
\end{equation*}
$$

Claim 3. For any $s \geq 1$ and $L \geq 1$, the lower bound $\underline{E}_{L}(s, R)$ defined by (23)-(26) is $a \cup$ convex function of the rate parameter $R>0$. If $0<R<\underline{C}(s)$, then $\underline{E}_{L}(s, R)>0$. If $R \geq \underline{C}(s)$, then $\underline{E}_{L}(s, R)=0$. In addition, there exist a number $R_{L}^{(c r)}(s), 0 \leq R_{L}^{(c r)}(s)<\underline{C}(s)$, such that

$$
\begin{equation*}
\underline{E}_{L}(s, R)=(s+L-1) \underline{R}_{L}^{(1)}(s)-L R, \quad \text { if } \quad 0 \leq R \leq R_{L}^{(c r)}(s) \tag{28}
\end{equation*}
$$

and

$$
\begin{equation*}
\underline{E}_{L}(s, R)>(s+L-1) \underline{R}_{L}^{(1)}(s)-L R, \quad \text { if } \quad R>R_{L}^{(c r)}(s) \tag{29}
\end{equation*}
$$

where the random coding bound $\underline{R}_{L}^{(1)}(s)$ is given by the formulas (15)-(18).
Table 1 gives some numerical values of the function

$$
\underline{R}_{L}(s) \triangleq \max \left\{\underline{R}_{1}(s), \underline{R}_{L}^{(1)}(s)\right\}, \quad 2 \leq s \leq 10, \quad 2 \leq L \leq 10
$$

along with the corresponding values $Q_{L}(s)$ of the optimal relative weight $Q_{L}^{(1)}(s)$ in the righthand side of $(15)$ if $\underline{R}_{L}(s)=\underline{R}_{L}^{(1)}(s)$, or we put $Q_{L}(s) \triangleq *$ if $\underline{R}_{L}(s)=\underline{R}_{1}(s)$, where the values $\underline{R}_{1}(s)$ were calculated in [14], i.e,

$$
Q_{L}(s) \triangleq \begin{cases}Q_{L}^{(1)}(s) & \text { if } \underline{R}_{L}(s)=\underline{R}_{L}^{(1)}(s) \text { for }(2 \leq s \leq 6, L=2) \text { or }(2 \leq s \leq 10,3 \leq L \leq 10) \\ * & \text { if } \underline{R}_{L}(s)=\underline{R}_{1}(s) \text { for }(7 \leq s \leq 10, L=2)\end{cases}
$$

The function $\underline{R}_{L}(s), L \geq 2, s \geq 2$, can be considered as the best presently known lower bound on the rate $R_{L}(s), L \geq 2, s \geq 2$, of LD $s_{L}$-codes.

Table 1:

| $s_{L}$ | $2_{2}$ | 23 | 24 | 25 | $2_{6}$ | $2_{7}$ | 28 | 29 | $2_{10}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $Q_{L}(s)$ | 0.244 | 0.233 | 0.226 | 0.221 | 0.218 | 0.215 | 0.212 | 0.211 | 0.209 |
| $\underline{R}_{L}(s)$ | 0.2358 | 0.2597 | 0.2729 | 0.2813 | 0.2871 | 0.2915 | 0.2948 | 0.2975 | 0.2997 |
| $\underline{R_{L}^{(c r)}(s)}$ | 0.3355 | 0.3279 | 0.3242 | 0.3226 | 0.3218 | 0.3216 | 0.3215 | 0.3215 | 0.3216 |
| $s_{L}$ | $3_{2}$ | $3_{3}$ | 34 | $3_{5}$ | $3_{6}$ | $3_{7}$ | 38 | 39 | $3_{10}$ |
| $Q_{L}(s)$ | 0.176 | 0.167 | 0.161 | 0.156 | 0.152 | 0.149 | 0.147 | 0.145 | 0.143 |
| $\underline{R}_{L}(s)$ | 0.1147 | 0.1346 | 0.1469 | 0.1552 | 0.1611 | 0.1656 | 0.1690 | 0.1718 | 0.1741 |
| $\underline{R_{L}^{(c r)}(s)}$ | 0.2177 | 0.2109 | 0.2065 | 0.2036 | 0.2017 | 0.2006 | 0.1998 | 0.1994 | 0.1992 |
| $s_{L}$ | $4_{2}$ | 43 | $4_{4}$ | 45 | $4_{6}$ | $4_{7}$ | 48 | 49 | 410 |
| $Q_{L}(s)$ | 0.139 | 0.133 | 0.128 | 0.123 | 0.120 | 0.117 | 0.115 | 0.113 | 0.111 |
| $\underline{R}_{L}(s)$ | 0.0684 | 0.0838 | 0.0941 | 0.1014 | 0.1068 | 0.1110 | 0.1143 | 0.1170 | 0.1192 |
| $\underline{R_{L}^{(c r)}(s)}$ | 0.1632 | 0.1580 | 0.1542 | 0.1514 | 0.1494 | 0.1479 | 0.1468 | 0.1460 | 0.1455 |
| $s_{L}$ | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 510 |
| $Q_{L}(s)$ | 0.115 | 0.110 | 0.106 | 0.103 | 0.100 | 0.098 | 0.096 | 0.094 | 0.092 |
| $\underline{R}_{L}(s)$ | 0.0456 | 0.0575 | 0.0660 | 0.0723 | 0.0771 | 0.0809 | 0.0840 | 0.0865 | 0.0886 |
| $\underline{R_{L}^{(c r)}(s)}$ | 0.1311 | 0.1271 | 0.1240 | 0.1216 | 0.1197 | 0.1183 | 0.1171 | 0.1162 | 0.1155 |
| $s_{L}$ | 62 | 63 | 64 | 65 | 66 | $6_{7}$ | 68 | 69 | $6_{10}$ |
| $Q_{L}(s)$ | 0.098 | 0.095 | 0.092 | 0.089 | 0.086 | 0.084 | 0.083 | 0.081 | 0.080 |
| $\underline{R}_{L}(s)$ | 0.0325 | 0.0420 | 0.0490 | 0.0544 | 0.0587 | 0.0621 | 0.0649 | 0.0672 | 0.0692 |
| $\underline{R_{L}^{(c r)}(s)}$ | 0.1098 | 0.1067 | 0.1041 | 0.1021 | 0.1004 | 0.0991 | 0.0980 | 0.0971 | 0.0963 |
| $s_{L}$ | 72 | 73 | 74 | 75 | $7_{6}$ | $7_{7}$ | 78 | 79 | 710 |
| $Q_{L}(s)$ | * | 0.083 | 0.080 | 0.078 | 0.076 | 0.074 | 0.073 | 0.072 | 0.070 |
| $\underline{R}_{L}(s)$ | 0.0260 | 0.0321 | 0.0380 | 0.0426 | 0.0463 | 0.0494 | 0.0519 | 0.0541 | 0.0559 |
| $\underline{R}^{(c r)}(s)$ | 0.0945 | 0.0920 | 0.0899 | 0.0882 | 0.0868 | 0.0855 | 0.0845 | 0.0837 | 0.0829 |
| $s_{L}$ | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 810 |
| $Q_{L}(s)$ | * | 0.074 | 0.072 | 0.070 | 0.068 | 0.067 | 0.065 | 0.064 | 0.063 |
| $\underline{R}_{L}(s)$ | 0.0213 | 0.0253 | 0.0303 | 0.0343 | 0.0376 | 0.0403 | 0.0426 | 0.0446 | 0.0463 |
| $\underline{R_{L}^{(c r)}(s)}$ | 0.0830 | 0.0810 | 0.0793 | 0.0778 | 0.0765 | 0.0754 | 0.0745 | 0.0737 | 0.0730 |
| $s_{L}$ | 92 | $9_{3}$ | 94 | 95 | $9_{6}$ | $9_{7}$ | 98 | $9_{9}$ | $9_{10}$ |
| $Q_{L}(s)$ | * | 0.067 | 0.065 | 0.063 | 0.062 | 0.061 | 0.059 | 0.058 | 0.057 |
| $\underline{R}_{L}(s)$ | 0.0178 | 0.0205 | 0.0248 | 0.0283 | 0.0312 | 0.0336 | 0.0357 | 0.0375 | 0.0391 |
| $\underline{R_{L}^{(c r)}(s)}$ | 0.0741 | 0.0724 | 0.0709 | 0.0696 | 0.0685 | 0.0676 | 0.0667 | 0.0660 | 0.0654 |
| $s_{L}$ | $10_{2}$ | $10_{3}$ | $10_{4}$ | $10_{5}$ | $10_{6}$ | $10_{7}$ | $10_{8}$ | 109 | $10_{10}$ |
| $Q_{L}(s)$ | * | 0.061 | 0.059 | 0.058 | 0.057 | 0.056 | 0.054 | 0.054 | 0.053 |
| $\underline{R}_{L}(s)$ | 0.0151 | 0.0169 | 0.0206 | 0.0237 | 0.0263 | 0.0285 | 0.0304 | 0.0320 | 0.0335 |
| $\underline{R_{L}^{(c r)}(s)}$ | 0.0668 | 0.0654 | 0.0642 | 0.0631 | 0.0621 | 0.0612 | 0.0605 | 0.0598 | 0.0592 |
| $s$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| $\underline{C}(s)$ | 0.3832 | 0.2455 | 0.1810 | 0.1434 | 0.1188 | 0.1014 | 0.0884 | 0.0784 | 0.0704 |
| $Q(s)$ | 0.2864 | 0.2028 | 0.1569 | 0.1280 | 0.1080 | 0.0935 | 0.0824 | 0.0736 | 0.0666 |
| $\underline{R_{1}^{(c r)}(s)}$ | 0.3510 | 0.2284 | 0.1705 | 0.1364 | 0.1137 | 0.0976 | 0.0855 | 0.0761 | 0.0685 |

## 3 On Constructions of Almost Disjunctive Codes

For $L=1$, constructions of LD $s_{1}$-codes (i.e classical disjunctive (superimposed) $s$-codes) based on the shortened Reed-Solomon codes were developed in [9]-[10]. The papers [9]-[10] significantly extend the optimal and suboptimal constructions of superimposed $s$-codes suggested in [4] and contain the detailed tables with parameters of the best known classical disjunctive (superimposed) s-codes. In addition, the table 3 from [10] along with the similar table presented in [11] gives a range of parameters $(t, N, s, \epsilon)$ corresponding to the best known $\operatorname{LD}\left(s_{1}, \epsilon\right)$-codes based on MDS codes. In the recent paper [19], it was proved that for the given parameters, the following parametric asymptotic equations

$$
\begin{equation*}
t=q^{\left\lfloor\frac{q}{\log _{2} q}\right\rfloor}, N=q(q+1), \epsilon=\epsilon(q) \rightarrow 0 \text { if } s<q \cdot \ln 2, q \text {-prime power, } q \rightarrow \infty, \tag{30}
\end{equation*}
$$

hold. Note that if $s \rightarrow \infty$ and $q \rightarrow \infty$, then the asymptotic behavior of the rate for LD $\left(s_{1}, \epsilon\right)$-codes with parameters (30) is

$$
\frac{\log _{2} t}{N}=\frac{1}{q}(1+o(1))=\frac{\ln 2}{s}(1+o(1))
$$

and coincides with the asymptotic behavior of the random coding bound $\underline{C}(s)$ defined by (27).

## 4 Proof of Theorem 2

Proof of claim 1. For an arbitrary code $X$, the number $\mathbf{B}_{L}(s, X)$ of $s_{L}$-bad subsets of columns in the code $X$ can be represented in the form:

$$
\mathbf{B}_{L}(s, X) \triangleq \sum_{\mathcal{S} \in[t], \mathcal{S} \mid=s} \psi_{L}(X, \mathcal{S}), \quad \psi_{L}(X, \mathcal{S}) \triangleq \begin{cases}1 & \text { if the set } \boldsymbol{x}(\mathcal{S}) \text { is } s_{L} \text {-bad in } X,  \tag{31}\\ 0 & \text { otherwise }\end{cases}
$$

Let $Q, 0<Q<1$, be a fixed parameter. Introduce the constant-weight ensemble $\{N, t, Q\}$ of binary $(N \times t)$-matrices $X$, where each column $\boldsymbol{x}(j), j \in[t]$, of $X$ is taken with replacement from the set containing $\binom{N}{w}$ binary columns of a given weight $w \triangleq\lfloor Q N\rfloor$. From (31) it follows that for the ensemble $\left\{N,\left\lfloor 2^{R N}\right\rfloor, Q\right\}$, the expectation $\overline{\mathbf{B}_{L}(s, X)}$ of the number $\mathbf{B}_{L}(s, X)$ is

$$
\overline{\mathbf{B}_{L}(s, X)}=\binom{t}{s} \operatorname{Pr}\left\{\boldsymbol{x}(S) \text { is } s_{L} \text {-bad in }(N, R) \text {-code } X\right\} .
$$

Therefore, the expectation of the error probability for almost disjunctive LD $s_{L}$-codes is

$$
\begin{equation*}
\mathcal{E}_{L}^{(N)}(s, R, Q) \triangleq\binom{t}{s}^{-1} \overline{\mathbf{B}_{L}(s, X)}=\operatorname{Pr}\left\{\boldsymbol{x}(S) \text { is } s_{L} \text {-bad in }(N, R) \text {-code } X\right\} . \tag{32}
\end{equation*}
$$

The evident random coding upper bound on the error probability (7) for almost disjunctive LD $s_{L}$-codes is formulated as the following inequality:

$$
\begin{equation*}
\epsilon_{L}(s, R, N) \triangleq \min _{\left.X: t=22^{R N}\right\rfloor}\left\{\frac{\mathbf{B}_{L}(s, X)}{\binom{t}{s}}\right\} \leq \mathcal{E}_{L}^{(N)}(s, R, Q), \quad 0<Q<1 \tag{33}
\end{equation*}
$$

The expectation $\mathcal{E}_{L}^{(N)}(s, R, Q)$ defined by (32) can be represented in the form

$$
\begin{equation*}
\mathcal{E}_{L}^{(N)}(s, R, Q)=\sum_{k=\lfloor Q N\rfloor}^{\min \{N, s\lfloor Q N\rfloor\}} \operatorname{Pr}\left\{\boldsymbol{x}(S) \text { is } s_{L} \text {-bad in } X /\left|\bigvee_{i \in \mathcal{S}} x(i)\right|=k\right\} \mathcal{P}^{(N)}(s, Q, k), \tag{34}
\end{equation*}
$$

where we applied the total probability formula and introduced the notation

$$
\begin{equation*}
\mathcal{P}^{(N)}(s, Q, k) \triangleq \operatorname{Pr}\left\{\left|\bigvee_{i \in \mathcal{S}} \boldsymbol{x}(i)\right|=k\right\}, \quad\lfloor Q N\rfloor \leq k \leq \min \{N, s\lfloor Q N\rfloor\} \tag{35}
\end{equation*}
$$

For the ensemble $\{N, t, Q\}$ and any $k,\lfloor Q N\rfloor \leq k \leq \min \{N, s\lfloor Q N\rfloor\}$, the conditional probability of event (2) is

$$
\begin{equation*}
\operatorname{Pr}\left\{\bigvee_{i \in \mathcal{S}} x(i) \succeq \bigvee_{j \in \mathcal{L}} x(j) /\left|\bigvee_{i \in \mathcal{S}} x(i)\right|=k\right\}=\left[\frac{\binom{k}{\lfloor Q\rfloor\rfloor}}{\binom{N}{\lfloor N\rfloor}}\right]^{L} \tag{36}
\end{equation*}
$$

In addition, with the help of the type (or composition) terminology:

$$
\{n(\mathbf{a})\}, \quad \mathbf{a} \triangleq\left(a_{1}, a_{2}, \ldots, a_{s}\right) \in\{0,1\}^{s}, \quad 0 \leq n(\mathbf{a}) \leq N, \quad \sum_{\mathbf{a}} n(\mathbf{a})=N,
$$

the probability of event (35) in the ensemble $\{N, t, Q\}$ can be written as follows:

$$
\begin{equation*}
\mathcal{P}^{(N)}(s, Q, k)=\binom{N}{\lfloor Q N\rfloor}^{-s} \cdot \sum_{(38)} \frac{N!}{\Pi_{\mathbf{a}} n(\mathbf{a})!}, \quad\lfloor Q N\rfloor \leq k \leq \min \{N, s\lfloor Q N\rfloor\}, \tag{37}
\end{equation*}
$$

and in the right-hand side of (37), the sum is taken over all types $\{n(\mathbf{a})\}$ provided that

$$
\begin{equation*}
n(\boldsymbol{0})=N-k, \quad \sum_{\mathbf{a}: a_{i}=1} n(\mathbf{a})=\lfloor Q N\rfloor \quad \text { for any } i \in[s] . \tag{38}
\end{equation*}
$$

Let the function

$$
\begin{equation*}
\mathcal{A}(s, Q, q) \triangleq \lim _{N \rightarrow \infty} \frac{-\log _{2} \mathcal{P}^{(N)}(s, Q,\lfloor q N\rfloor)}{N}, \quad Q \leq q \leq \min \{1, s Q\} \tag{39}
\end{equation*}
$$

denotes the exponent of the logarithmic asymptotic behavior for the probability of event (35) calculated by (37)-(38).

Further, the representation (34), the conditional probability (36) and the standard union bound

$$
\operatorname{Pr}\left\{\bigcup_{i} C_{i} / C\right\} \leq \min \left\{1 ; \sum_{i} \operatorname{Pr}\left\{C_{i} / C\right\}\right\}
$$

lead to the upper bound

$$
\begin{equation*}
\mathcal{E}_{L}^{(N)}(s, R, Q) \leq \sum_{k=\lfloor Q N\rfloor}^{\min \{N, s\lfloor Q N\rfloor\}} \mathcal{P}^{(N)}(s, Q, k) \min \left\{1 ;\binom{t-s}{L}\left[\frac{\binom{k}{\lfloor Q N\rfloor}}{(\lfloor Q N\rfloor}\right]^{L}\right\}, \tag{40}
\end{equation*}
$$

where the code size $t \triangleq\left\lfloor 2^{R N}\right\rfloor$. Inequality (40) and the random coding bound (33) imply that the error probability exponent (8) satisfies the inequality

$$
\begin{gather*}
\mathbf{E}_{L}(s, R) \geq \underline{\mathbf{E}}_{L}(s, R) \triangleq \max _{0<Q<1} E_{L}(s, R, Q)  \tag{41}\\
E_{L}(s, R, Q) \triangleq \min _{Q \leq q \leq \min \{1, s Q\}}\left\{\mathcal{A}(s, Q, q)+L \cdot[h(Q)-q \cdot h(Q / q)-R]^{+}\right\} . \tag{42}
\end{gather*}
$$

In Appendix we prove
Lemma 1. Let $\lfloor Q N\rfloor \leq k \leq \min \{N, s\lfloor Q N\rfloor\}$. For the conditional probability in the righthand side of (34), the lower bound

$$
\begin{equation*}
\operatorname{Pr}\left\{x(S) \text { is } s_{L^{-}} \text {-bad in } X /\left|\bigvee_{i \in \mathcal{S}} x(i)\right|=k\right\} \geq D(s, L) \times \min \left\{1 ;\binom{t-s}{L}\left[\frac{\binom{k}{Q N)}}{\binom{N}{Q N\rfloor}}\right]^{L}\right\}, \tag{43}
\end{equation*}
$$

holds, where $D(s, L)$ is some constant.
Lemma 1 establishes the asymptotic accuracy of the upper bound in (40), i.e., there exists

$$
\lim _{N \rightarrow \infty} \frac{-\log _{2} \mathcal{E}_{L}^{(N)}(s, R, Q)}{N}=E_{L}(s, R, Q), \quad R>0
$$

where the function $E_{L}(s, R, Q), R>0$, defined by (42) can be interpreted as the exponent of random coding bound on error probability for almost disjunctive $L D s_{L}$-codes in the ensemble $\left\{N,\left\lfloor 2^{R N}\right\rfloor, Q\right\}$ of constant weight codes.

The analytical properties of the function (39) are formulated below as Lemmas 2-4. They will be proved in Appendix.

Lemma 2. The function $\mathcal{A}(s, Q, q)$ of the parameter $q, Q<q<\min \{1, s Q\}$, defined by (39) can be represented in the parametric form (25)-(26). In addition, the function $\mathcal{A}(s, Q, q)$ is $\cup$-convex, monotonically decreases in the interval $\left(Q, 1-(1-Q)^{s}\right)$, monotonically increases in the interval $\left(1-(1-Q)^{s}, \min \{1, s Q\}\right)$ and its unique minimal value which is equal to 0 is attained at $q=1-(1-Q)^{s}$, i.e.,

$$
\min _{Q<q<\min \{1, s Q\}} \mathcal{A}(s, Q, q)=\mathcal{A}\left(s, Q, 1-(1-Q)^{s}\right)=0, \quad 0<Q<1
$$

Lemma 3. For any fixed $Q, 0<Q<1$, the function $q \cdot h(Q / q), Q<q<\min \{1, s Q\}$, is an $\cap$-convex and monotonically increases.

Lemma 4. For fixed $Q, 0<Q<1$, the function

$$
\begin{equation*}
\mathcal{A}(s, Q, q)+L \cdot[h(Q)-q \cdot h(Q / q)], \quad Q<q<\min \{1, s Q\}, \tag{44}
\end{equation*}
$$

is $\cup$-convex and its unique minimum is attained at some point $q=q_{L}^{(2)}(s, Q)>1-(1-Q)^{s}$ and is equal to the function $A_{L}(s, Q)$, defined by (16)-(18), i.e.,

$$
\min _{Q<q<\min \{1, s Q\}}\{\mathcal{A}(s, Q, q)+L \cdot[h(Q)-q \cdot h(Q / q)]\}=A_{L}(s, Q)
$$

Claim 1 is an evident consequence of Lemma 2.
Claim 3 is based on Lemmas 2-4.

Proof of Claim 2. First of all, let us rewrite the formula (22) in a more convenient form:
$C(s, Q)=\left(1-Q-(1-Q)^{s}\right) \log _{2}\left[1-\frac{Q(1-Q)^{s-1}}{\left.1-(1-Q)^{s}\right)}\right]-Q \log _{2}\left[1-(1-Q)^{s}\right]-(1-Q)^{s} \log _{2}[1-Q]$.
For $Q_{o}(s, a)=\frac{a}{s}(1+o(1)), s \rightarrow \infty$, the asymptotic behavior of (45) is the following:

$$
\begin{equation*}
C\left(s, Q_{o}(s, a)\right)=\frac{-a \log _{2}\left[1-e^{-a}\right]}{s}(1+o(1)), \quad s \rightarrow \infty \tag{46}
\end{equation*}
$$

Taking the derivative with respect to $a$ one can easily verify that for $a=\ln 2$ the maximum

$$
\max _{a>0}\left\{-a \log _{2}\left[1-e^{-a}\right]\right\}=\ln 2
$$

is attained. Thus,

$$
\begin{equation*}
\underline{C}(s) \geq \frac{\ln 2}{s}(1+o(1)), s \rightarrow \infty \tag{47}
\end{equation*}
$$

To complete the proof we need to achieve the opposite asymptotic inequality.
Let $0<Q(s)<1, s=2,3, \ldots$, be an arbitrary sequence, such that

$$
\max _{0<Q<1} C(s, Q)=C(s, Q(s))=\underline{C}(s)
$$

Suggest that $Q(s)>b$, for some fixed $b>0$. Then, one can obtain from (45) the inequality $C(s, Q(s)) \leq(1-b)^{s} O(1), s \rightarrow \infty$, and there is a contradiction with (47). Hence, without loss of generality, $Q(s) \rightarrow 0$, as $s \rightarrow \infty$.

Suggest that $0<Q=f(s) / s<1$ and $\lim _{s \rightarrow \infty} f(s)=\infty$. The assumption yields

$$
\lim _{s \rightarrow \infty}(1-Q)^{s} \leq \lim _{s \rightarrow \infty} e^{-f(s)}=0
$$

Using the previous property and the expansion of a logarithm one can derive from (45) the asymptotic inequality $C(s, Q(s)) \leq Q(1-Q)^{s} O(1), s \rightarrow \infty$. Therefore, the equality

$$
\lim _{s \rightarrow \infty} s Q(1-Q)^{s}=0
$$

establishes a contradiction with (47). Thus, without loss of generality, $s Q(s) \rightarrow a$, as $s \rightarrow \infty$, where the condition $0 \leq a<\infty$ holds.

Similarly, the assumption $a=0$ leads to the asymptotic inequality $C(s, Q(s)) \leq Q O(1)$, where is a contradiction with (47).

Thus, the asymptotics (27) holds. Claim 2 is proved.
Proof of Claim 3. Note that the $\cup$-convexity of $E_{L}(s, R, Q)$ for arbitrary $0<Q<1$ implies the $\cup$-convexity of $\underline{E}_{L}(s, R)$. Let us prove the $\cup$-convexity of $E_{L}(s, R, Q)$.

Fix arbitrary $0<Q<1$. For a fixed $R>0$, it follows from Lemmas 2-4 that the minimum in (42) is attained at some point $q \in\left[q^{(0)}(s, Q)=1-(1-Q)^{s}, q_{L}^{(2)}(s, Q)\right]$. Denote $\mathcal{B}(R, Q, q)=$ $h(Q)-q h(Q / q)-R$. If there exists a solution $q \in(0,1)$ of the equation $\mathcal{B}(R, Q, q)=0$, we will denote it as $q^{(1)}(R, Q)$. It's clear that the minimum in (42) is attained at the point $q=q_{L}^{(m i n)}(s, R, Q)$, defined as

$$
q_{L}^{(\text {min })}(s, R, Q)= \begin{cases}q_{L}^{(2)}(s, Q), & \text { if } \mathcal{B}\left(R, Q, q^{(2)}\right)>=0 \\ q^{(1)}(R, Q), & \text { if } \mathcal{B}\left(R, Q, q^{(0)}\right)>0 \text { and } \mathcal{B}\left(R, Q, q^{(2)}\right)<0 \\ q^{(0)}(s, Q), & \text { if } \mathcal{B}\left(R, Q, q^{(0)}\right)<=0\end{cases}
$$

Correspondingly, the substitution of $q^{(m i n)}$ into the expression (42) gives

$$
E_{L}(s, R, Q)= \begin{cases}A_{L}(s, Q)-L R, & \text { for } 0 \leq R \leq R_{L}^{(c r)}(s, Q)  \tag{48}\\ \mathcal{A}\left(s, Q, q^{(1)}\right), & \text { for } R_{L}^{(c r)}(s, Q) \leq R \leq C(s, Q) \\ 0, & \text { for } C(s, Q) \leq R\end{cases}
$$

where $A_{L}(s, Q)$ is defined by (16)-(18), $\mathcal{A}(s, Q, q)$ characterized by (25)-(26), $C(s, Q)$ determined by (22) and

$$
\begin{equation*}
R_{L}^{(c r)}(s, Q)=h(Q)-q^{(2)} h\left(Q / q^{(2)}\right) \tag{49}
\end{equation*}
$$

Note that $q^{(1)}(R, Q)$ is the implicit function of the parameter $R$ defined by the equation $\mathcal{B}(R, Q, q)=0$. Hence, one can calculate the following derivative in the domain of the function $q^{(1)}(R, Q)$ :

$$
\begin{equation*}
\left(q^{(1)}(R, Q)\right)_{R}^{\prime}=1 \div \log _{2} \frac{q-Q}{q} . \tag{50}
\end{equation*}
$$

Therefore, the use of (48) and (50) allows to compute the derivative of $E_{L}(s, R, Q)$ with respect to $R$ :

$$
\left(E_{L}(s, R, Q)\right)_{R}^{\prime}= \begin{cases}-L, & \text { for } 0 \leq R \leq R_{L}^{(c r)}(s, Q) \\ \log _{2} \frac{Q y^{s}}{1-Q-y+Q y^{s}} \div \log _{2} \frac{q-Q}{q}, & \text { for } R_{L}^{(c r)}(s, Q) \leq R \leq C(s, Q) \\ 0, & \text { for } C(s, Q) \leq R\end{cases}
$$

where in the second line $q$ denotes $q^{(1)}(R, Q)$ and $y$ is defined by (26). One can easily verify that the expression in the second line is nondecreasing function of the parameter $R$, moreover it equals $-L$ at $R=R_{L}^{(c r)}(s, Q)$ and 0 at $R=C(s, Q)$. Thus, the derivative of $E_{L}(s, R, Q)$ with respect to $R$ exists, is continuous and nondecreasing function, i.e. $E_{L}(s, R, Q)$ is $\cup$-convex.

In the case $R=0$, for any $0<Q<1$, it is clear that $h(Q)-q h(Q / q) \geq 0$, therefore the case $R=0$ satisfies (28).

In the case $R=\underline{C}(s)$, it is clear that $\underline{E}_{L}(s, R)=0$, therefore the case $R=\underline{C}(s)$ satisfies (29).

Thus, due to the U-convexity of $\underline{E}_{L}(s, R)$, there exists $R_{L}^{(c r)}(s)$, such that (28) holds for $0 \leq R \leq R_{L}^{(c r)}(s)$ and (29) holds for $R>R_{L}^{(c r)}(s)$.

Claim 3 is proved.

## 5 Appendix: Proofs of Lemmas 1-4

Proof of Lemma 1. Denote $p=\left[\frac{\binom{k}{(Q N)}}{\left(\begin{array}{l}N N\rfloor\end{array}\right)}\right]$. Let $A_{i}$ be an event, that $\boldsymbol{x}(S)$ covers $L$ fixed columns, $1 \leq i \leq\binom{ t-s}{L}$, then $\operatorname{Pr}\left(A_{i}\right)=p^{L}$.

$$
\operatorname{Pr}\left\{\boldsymbol{x}(S) \text { is } s_{L} \text {-bad in } X /\left|\bigvee_{i \in \mathcal{S}} \boldsymbol{x}(i)\right|=k\right\} \geq \sum_{i=1}^{\binom{t-s}{L}} \operatorname{Pr}\left(A_{i}\right)-\sum_{\left.1 \leq i<j \leq \begin{array}{c}
t-s) \\
L
\end{array}\right)} \operatorname{Pr}\left(A_{i} A_{j}\right)
$$

$\sum_{1 \leq i<j \leq\binom{(t-s)}{L}} \operatorname{Pr}\left(A_{i} A_{j}\right)=\frac{\binom{(-s)}{L}}{\left.\sum_{i=2}^{(t-s}{ }_{L}^{L}\right)} \operatorname{Pr}\left(A_{1} A_{i}\right)=$
$\frac{\binom{t-s}{L}}{\sum_{l=0}^{L-1}}\binom{L}{l}\binom{t-s-L}{L-l} \operatorname{Pr}\left(A_{1} A_{j} \mid\right.$ the cardinality of intersection $A_{1} A_{j}$ is equal to $\left.l\right)=$
$\frac{\binom{t-s}{2}}{\sum_{l=0}^{L-1}}\binom{L}{l}\binom{t-s-L}{L-l} p^{2 L-l}<\binom{t-s}{L} p^{L} \sum_{l=0}^{L-1}\binom{L}{L-l}(t p)^{L-l}<\binom{t-s}{L} p^{L}\left((1+t p)^{L}-1\right)$.
Let $t_{0}$ be a root of the equation $(1+t p)^{L}-1=0.5$, i.e. $t_{0}=\frac{(1.5)^{\frac{1}{L}}-1}{p}$.
If $t<t_{0}$, then $(1+p t)^{L}<1.5$ and

$$
\operatorname{Pr}\left\{\boldsymbol{x}(S) \text { is } s_{L^{-}} \text {-bad in } X /\left|\bigvee_{i \in \mathcal{S}} \boldsymbol{x}(i)\right|=k\right\} \geq \frac{1}{2}\binom{t-s}{L} p^{L} .
$$

If $t>t_{0}>s+L$ then
$\operatorname{Pr}\left\{\boldsymbol{x}(S)\right.$ is $s_{L}$-bad in $\left.X /\left|\bigvee_{i \in \mathcal{S}} \boldsymbol{x}(i)\right|=k\right\} \geq \frac{1}{2}\binom{t_{0}-s}{L} p^{L} \geq \frac{1}{2}\left(\frac{t_{0} p}{s+L+1}\right)^{L}=D_{1}(s, L)$.
If $t_{0} \leq s+L$, then $p \geq \frac{1.5 \frac{1}{L}-1}{s+L}$ and

$$
\operatorname{Pr}\left\{\boldsymbol{x}(S) \text { is } s_{L} \text {-bad in } X /\left|\bigvee_{i \in \mathcal{S}} x(i)\right|=k\right\} \geq D_{2}(s, L),
$$

where $D_{2}(s, L)$ is some constant.
Setting $D(s, L)=\min \left(D_{1}(s, L), D_{2}(s, L), 0.5\right)$ we obtain the inequality (43).
Lemma 1 is proved.
Proof of Lemma 2. Let $s \geq 2,0<Q<1, Q<q<\min \{1, s Q\}$ be fixed parameters. Let $k=\lfloor q N\rfloor$ and $N \rightarrow \infty$. For every type $\{n(\mathbf{a})\}$ we will consider corresponding distribution $\tau: \tau(\mathbf{a})=\frac{n(\mathbf{a})}{N}, \quad \forall \mathbf{a} \in\{0,1\}^{s}$.

Applying the Stirling approximation, we obtain the following logarithmic asymptotic behavior of the summand in the sum (37):

$$
\begin{gather*}
-\log _{2} \frac{N!}{\prod_{\mathbf{a}} n(\mathbf{a})!}\binom{N}{\lfloor Q N\rfloor}^{-s}=N F(\tau, Q, q)(1+o(1)), \quad \text { where } \\
F(\tau, Q, q)=\sum_{\mathbf{a}} \tau(\mathbf{a}) \log _{2} \tau(\mathbf{a})+s H(Q) \tag{51}
\end{gather*}
$$

Thus, one can reduce the calculation of $\mathcal{A}(s, Q, q)$ to the search of the following minimum:

$$
\begin{gather*}
\mathcal{A}(s, Q, q)=\min _{\tau \in(53):(54)} F(\tau, Q, q),  \tag{52}\\
\{\tau: \forall \mathbf{a} \quad 0<\tau(\mathbf{a})<1\},  \tag{53}\\
\sum_{\mathbf{a}} \tau(\mathbf{a})=1, \quad \tau(\boldsymbol{O})=1-q, \quad \sum_{\mathbf{a}: a_{i}=1} \tau(\mathbf{a})=Q \quad \forall i \in[s], \tag{54}
\end{gather*}
$$

where the restrictions (54) are induced by the definition of type and the properties (38).
To find the minimum, we use the standard Lagrange multipliers method. The Lagrangian is equal to

$$
\begin{aligned}
\Lambda \triangleq \sum_{\tau(\mathbf{a})} \tau(\mathbf{a}) \log _{2} \tau(\mathbf{a})+\operatorname{sh}(Q)+\lambda_{0}(\tau(\boldsymbol{0}) & +q-1)+ \\
& +\sum_{i=1}^{s} \lambda_{i}\left(\sum_{\mathbf{a}: a_{i}=1} \tau(\mathbf{a})-Q\right)+\lambda_{s+1}\left(\sum_{\mathbf{a}} \tau(\mathbf{a})-1\right) .
\end{aligned}
$$

Therefore, the necessary conditions for the extremal distribution are

$$
\left\{\begin{array}{l}
\frac{\partial \Lambda}{\partial \tau(\boldsymbol{O})}=\log _{2} \tau(\boldsymbol{O})+\log _{2} e+\lambda_{0}+\lambda_{s+1}=0,  \tag{55}\\
\frac{\partial \Lambda}{\partial \tau(\mathbf{a})}=\log _{2} \tau(\mathbf{a})+\log _{2} e+\lambda_{s+1}+\sum_{i=1}^{s} a_{i} \lambda_{i}=0 \quad \text { for any } \mathbf{a} \neq \boldsymbol{0} .
\end{array}\right.
$$

The matrix of second derivatives of the Lagrangian is obvious to be diagonal. Thus, this matrix is positive definite in the region (53) and the function $F(\tau, Q)$ defined by (51) is strictly U-convex in the region (53). The Karush-Kuhn-Tacker theorem (see, for example, [20]) states that each solution $\tau \in(53)$ satisfying system (55) and constraints (54) gives a local minimum of $F(\tau, Q)$. Thus, if there exists a solution of the system (55) and (54) in the region (53), then it is unique and gives a minimum in the minimization problem (52) - (54).

Note that the symmetry of problem yields the equality $v \triangleq \lambda_{1}=\lambda_{2}=\ldots \lambda_{s}$. Let $u \triangleq$ $\log _{2} e+\lambda_{s+1}$ and $w \triangleq \lambda_{0}$. One can rewrite (54) and (55) as follows:

$$
\left\{\begin{array}{l}
\text { 1) } \log _{2} \tau(\mathbf{a})+u+v \sum_{i=1}^{s} a_{i}=0 \quad \text { for any } \mathbf{a} \neq \boldsymbol{O},  \tag{56}\\
\text { 2) } \log _{2} \tau(\boldsymbol{O})+u+w=0, \\
\text { 3) } \tau(\boldsymbol{O})=1-q, \\
\text { 4) } \sum_{\mathbf{a}} \tau(\mathbf{a})=1, \\
\text { 5) } \sum_{\mathbf{a}: a_{i}=1} \tau(\mathbf{a})=Q \text { for any } i \in[s] .
\end{array}\right.
$$

Let $y \triangleq \frac{1}{1+2^{-v}}$ be a change of the variable $v$. The first equation of the system (56) means that

$$
\begin{equation*}
\text { for every } \mathbf{a} \neq \mathbf{0} \quad \tau(\mathbf{a})=\frac{1}{2^{u} y^{s}}(1-y)^{\sum a_{j}} y^{s-\sum a_{j}} . \tag{57}
\end{equation*}
$$

The substitution of (57) into the equation 5) of the previous system allows us to obtain

$$
\sum_{\mathbf{a}: a_{i}=1} \frac{1}{2^{u} y^{s}}(1-y)^{\sum a_{j}} y^{s-\sum a_{j}}=\frac{1-y}{2^{u} y^{s}},
$$

and therefore, the solution $u$ is determined by the equality

$$
\begin{equation*}
u=\log _{2} \frac{1-y}{Q y^{s}} . \tag{58}
\end{equation*}
$$

Substituting (57), (58) and the third equation of (56) into the equation 4) of the system (56) we achieve

$$
q=\sum_{\mathbf{a} \neq 0} \tau(\mathbf{a})=\frac{Q\left(1-y^{s}\right)}{1-y},
$$

i.e. the equation (26). Thus, the conditions (54) and (55) have the unique solution $\tau$ in the region (53):

$$
\begin{equation*}
\tau(\boldsymbol{O})=1-q, \quad \tau(\mathbf{a})=\frac{Q}{1-y}(1-y)^{\sum a_{j}} y^{s-\sum a_{j}} \quad \text { for any } \mathbf{a} \neq \boldsymbol{O} \tag{59}
\end{equation*}
$$

where the parameters $q$ and $y$ are related by the equation (26). To get the exact formula (25), the substitution of (59) into (51) is sufficient.

Let us prove the properties of the function (25). Note that the function $q(y)=Q \frac{1-y^{s}}{1-y}$ (26) monotonically increases in the interval $y \in(0,1)$ and correspondingly takes the values $Q$ and $s Q$ at the ends of the interval. That is why one can consider the function (25) as the function $\mathcal{T}(s, Q, y)=\mathcal{A}(s, Q, q(y))$ of the parameter $y$ in the interval $y \in\left(0, y_{1}\right)$, where $q\left(y_{1}\right)=\min \{1, s Q\}$. The derivative of the function $\mathcal{T}(s, Q, y)$ equals

$$
\begin{equation*}
\mathcal{T}^{\prime}(s, Q, y)=q^{\prime}(y) \log _{2} \frac{Q y^{s}}{1-Q-y+Q y^{s}} \tag{60}
\end{equation*}
$$

Thus, $\mathcal{T}(s, Q, y)$ decreases in the interval $y \in(0,1-Q)$, increases in the interval $y \in\left(1-Q, y_{1}\right)$, is $\cup$-convex, attains the minimal value 0 at $y_{0}=1-Q$ and $q\left(y_{0}\right)=1-(1-Q)^{s}$.

Lemma 2 is proved.
Proof of Lemma 3. Let $0<Q<1$ be a fixed value. The derivative of the function $f(Q, q)=q \cdot h(Q / q), Q<q<1$ equals

$$
\begin{equation*}
f_{q}^{\prime}(Q, q)=-\log _{2} \frac{q-Q}{q} \tag{61}
\end{equation*}
$$

Hence, the function $f(Q, q)$ increases in the interval $q \in(Q, 1)$, is $\cap$-convex and, for any halfinterval $q \in(Q, a], Q<a<1$, attains its unique maximal value at the point $q=a$.

Lemma 3 is proved.
Proof of Lemma 4. Let $0<Q<1$ be a fixed value. Due to the properties of (26), one can consider the function (44) as the function

$$
\mathcal{F}(s, L, Q, y)=\mathcal{A}(s, Q, q(y))+L[h(Q)-q(y) \cdot h(Q / q(y))]
$$

of the parameter $0<y<y_{1}$, where $q\left(y_{1}\right)=\min \{1, s Q\}$. Using (60) and (61) one can calculate the derivative of $\mathcal{F}(s, L, Q, y)$ :

$$
\mathcal{F}^{\prime}(s, L, Q, y)=\mathcal{T}^{\prime}(s, Q, y)-L q^{\prime}(y) f_{q}^{\prime}(Q, y)=q^{\prime}(y) \cdot \log _{2}\left[\frac{Q y^{s}}{1-Q-y+Q y^{s}}\left(\frac{y-y^{s}}{1-y^{s}}\right)^{L}\right]
$$

Thus, the equality $\mathcal{F}^{\prime}(s, L, Q, y)=0$ holds if and only if

$$
y=1-Q+Q y^{s}\left[1-\left(\frac{y-y^{s}}{1-y^{s}}\right)^{L}\right]
$$

i.e. the relation (18) is true. The function (44) is clear to be $\cap$-convex and to attain the minimum at the point $q=q\left(y_{2}\right)$, where $y_{2}$ is the solution of the equation (18).

Note that the following equality holds:

$$
1-q\left(y_{2}\right)=1-\frac{Q\left(1-y_{2}^{s}\right)}{1-y_{2}}=\frac{Q y_{2}^{s}}{1-y_{2}}\left(\frac{y_{2}-y_{2}^{s}}{1-y_{2}^{s}}\right)^{L}
$$

Thus

$$
\begin{aligned}
\mathcal{F}\left(s, L, Q, y_{2}\right)= & \left(1-Q \frac{1-y_{2}^{s}}{1-y_{2}}\right) \log _{2}\left[\frac{Q y_{2}^{s}}{1-y_{2}}\left(\frac{y_{2}-y_{2}^{s}}{1-y_{2}^{s}}\right)^{L}\right]+Q \frac{1-y_{2}^{s}}{1-y_{2}} \log _{2} \frac{Q y_{2}^{s}}{1-y_{2}}+ \\
& s Q \log _{2} \frac{1-y_{2}}{y_{2}}+\operatorname{sh}(Q)+L h(Q)+L Q \log _{2} \frac{1-y_{2}}{1-y_{2}^{s}}+L Q \frac{y_{2}-y_{2}^{s}}{1-y_{2}} \log _{2} \frac{y_{2}-y_{2}^{s}}{1-y_{2}^{s}} .
\end{aligned}
$$

The simplifying of the previous expression yields

$$
\min _{0<y<y_{1}} \mathcal{F}(s, L, Q, y)=A_{L}(s, Q),
$$

where the function $A_{L}(s, Q)$ is defined by (16)-(18).
Lemma 4 is proved.

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