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## ON DETECTION OF GAUSSIAN STOCHACTIC SEQUENCES ${ }^{1}$

The problem of minimax detection of Gaussian random signal vector in White Gaussian additive noise is considered. It is supposed that an unknown vector $\boldsymbol{\sigma}$ of the signal vector intensities belong to the given set $\mathcal{E}$. It is investigated when it is possible to replace the set $\mathcal{E}$ by a smaller set $\mathcal{E}_{0}$ without loss of quality (and, in particular, to replace it by a single point $\boldsymbol{\sigma}_{0}$ ).

## § 1. Inroduction

1. Simple hypotheses. There are two simple hypotheses $\mathcal{H}_{0}$ ("noise") and $\mathcal{H}_{1}$ ("noise + stochastic signal") on observations $\mathbf{y}=\left(y_{1}, \ldots, y_{n}\right) \in \mathbb{R}^{n}$ :

$$
\begin{gather*}
\mathcal{H}_{0}: \mathbf{y}=\boldsymbol{\xi} \\
\mathcal{H}_{1}: \mathbf{y}=\mathrm{s}+\boldsymbol{\xi} \tag{1}
\end{gather*}
$$

where $\boldsymbol{\xi}=\left(\xi_{1}, \ldots, \xi_{n}\right)$ - independent $\mathcal{N}(0,1)$-Gaussian random variables, and $\mathbf{s}=$ $\left(s_{1}, \ldots, s_{n}\right)$ - independent on $\boldsymbol{\xi}$, independent $\mathcal{N}\left(0, \sigma_{i}^{2}\right), i=1, \ldots, n$-Gaussian random variables (i.e. $\left.\mathbf{E}\left(s_{i}^{2}\right)=\sigma_{i}^{2}\right)$. Denote $\boldsymbol{\sigma}=\left(\sigma_{1}, \ldots, \sigma_{n}\right)$, where all $\sigma_{i} \geq 0$, and introduce functions (those notations will also be used below)

$$
\begin{equation*}
D(\boldsymbol{\sigma})=\sum_{i=1}^{n} \ln \left(1+\sigma_{i}^{2}\right), \quad T(\boldsymbol{\sigma})=\sum_{i=1}^{n} \frac{\sigma_{i}^{2}}{1+\sigma_{i}^{2}}, \quad B(\boldsymbol{\sigma})=2 \sum_{j=1}^{n} \frac{\sigma_{j}^{4}}{\left(1+\sigma_{j}^{2}\right)^{2}} . \tag{2}
\end{equation*}
$$

Then for conditional probability densities we have

$$
\begin{equation*}
p\left(\mathbf{y} \mid \mathcal{H}_{0}\right)=(2 \pi)^{-n / 2} e^{-\frac{1}{2} \sum_{i=1}^{n} y_{i}^{2}}, \quad p(\mathbf{y} \mid \boldsymbol{\sigma})=(2 \pi)^{-n / 2} e^{-\frac{1}{2} \sum_{i=1}^{n} y_{i}^{2} /\left(1+\sigma_{i}^{2}\right)-\frac{1}{2} D(\boldsymbol{\sigma})} \tag{3}
\end{equation*}
$$

Denote also

$$
\begin{equation*}
r(\mathbf{y}, \boldsymbol{\sigma})=\ln \frac{p(\mathbf{y} \mid \boldsymbol{\sigma})}{p\left(\mathbf{y} \mid \mathcal{H}_{0}\right)}=\frac{1}{2} \sum_{i=1}^{n} \frac{\sigma_{i}^{2} y_{i}^{2}}{1+\sigma_{i}^{2}}-\frac{1}{2} D(\boldsymbol{\sigma}) \tag{4}
\end{equation*}
$$

The optimal solution of that problem of testing the simple hypothesis $\mathcal{H}_{0}$ against the simple alternative $\mathcal{H}_{1}$ (Neyman-Pierson criteria) [1, 2] has the form

$$
\begin{equation*}
\mathbf{y} \in \mathcal{A}(A, \boldsymbol{\sigma}) \Rightarrow \mathcal{H}_{0}, \quad \mathbf{y} \notin \mathcal{A}(A, \boldsymbol{\sigma}) \Rightarrow \mathcal{H}_{1} \tag{5}
\end{equation*}
$$

[^0]where the set (ellipsoid) $\mathcal{A}(A, \boldsymbol{\sigma})$
\[

$$
\begin{equation*}
\mathcal{A}(A, \boldsymbol{\sigma})=\left\{\mathbf{y}: \sum_{i=1}^{n} \frac{\sigma_{i}^{2} y_{i}^{2}}{1+\sigma_{i}^{2}} \leq D(\boldsymbol{\sigma})+A\right\}, \quad \boldsymbol{\sigma}=\left(\sigma_{1}, \ldots, \sigma_{n}\right) \tag{6}
\end{equation*}
$$

\]

The level $A$ of that test is determined by a given 1 -st kind error probability ("false alarm probability") $\alpha=\alpha(A, \boldsymbol{\sigma})$ :

$$
\begin{equation*}
\left.\alpha(A, \boldsymbol{\sigma})=\mathbf{P}\left(\mathbf{y} \notin \mathcal{A} \mid \mathcal{H}_{0}\right)\right)=\mathbf{P}\left(\sum_{i=1}^{n} \frac{\sigma_{i}^{2} \xi_{i}^{2}}{1+\sigma_{i}^{2}}>D(\boldsymbol{\sigma})+A\right) \tag{7}
\end{equation*}
$$

If hypothesis $\mathcal{H}_{1}$ is true then $y_{i}=\xi_{i}+\sigma_{i} \eta_{i} \sim \sqrt{1+\sigma_{i}^{2}} \eta_{i}$, where $\left(\eta_{1}, \ldots, \eta_{n}\right)-$ independent $\mathcal{N}(0,1)$-Gaussian random variables. Therefore 2 -nd kind error probability (" miss probability") $\beta(A, \boldsymbol{\sigma})$ is defined by formula

$$
\begin{equation*}
\beta(A, \boldsymbol{\sigma})=\mathbf{P}\left(\mathbf{y} \in \mathcal{A} \mid \mathcal{H}_{1}\right)=\mathbf{P}\left(\sum_{i=1}^{n} \sigma_{i}^{2} \xi_{i}^{2}<D(\boldsymbol{\sigma})+A\right) \tag{8}
\end{equation*}
$$

For a given value $\alpha$ denote by $\beta(\alpha, \boldsymbol{\sigma})$ the minimum possible value $\beta(A, \boldsymbol{\sigma})$ for optimal choice of level $A$ (according to formulas (7)-(8)).

Since $\mathbf{E} \xi_{i}^{2}=1, i=1, \ldots, n$, then due to Large Numbers Law and (7)-(8) we get that for sufficiently small $\alpha, \beta$ the value $A$ should satisfy conditions

$$
\begin{equation*}
\sum_{i=1}^{n} \frac{\sigma_{i}^{2}}{1+\sigma_{i}^{2}}<D(\boldsymbol{\sigma})+A<\sum_{i=1}^{n} \sigma_{i}^{2} \tag{9}
\end{equation*}
$$

Below we assume satisfied both conditions (9). Note that with decreasing the level $A$ the error probability $\beta(A, \boldsymbol{\sigma})$ also decreases, but the error probability $\alpha(A, \boldsymbol{\sigma})$ increases. In particular, the case when the value $D(\boldsymbol{\sigma})+A$ is relatively close to the left side of the condition (9) will be interesting for us.
2. Simple hypothesis against composite alternative. Let a set $\mathcal{E}$ of nonnegative vectors $\boldsymbol{\sigma}=\left(\sigma_{1}, \ldots, \sigma_{n}\right)$ be given. Assume that on the vector $\boldsymbol{\sigma}$, describing the hypothesis $\mathcal{H}_{1}$ from (11) it is known only that $\boldsymbol{\sigma} \in \mathcal{E}$, but the vector $\boldsymbol{\sigma}$ itself is not known (i.e. the hypothesis $\mathcal{H}_{1}$ is composite).

Similarly to (5), for testing hypotheses $\mathcal{H}_{0}$ and $\mathcal{H}_{1}$ we choose a decision region $\mathcal{A} \in \mathbb{R}^{n}$ such that

$$
\mathbf{y} \in \mathcal{A} \Rightarrow \mathcal{H}_{0}, \quad \mathbf{y} \notin \mathcal{A} \Rightarrow \mathcal{H}_{1} .
$$

1-st kind and 2-nd kind error probabilities are defined by formulas, respectively,

$$
\alpha(\mathcal{A})=\mathbf{P}\left(\mathbf{y} \notin \mathcal{A} \mid \mathcal{H}_{0}\right)
$$

and

$$
\beta(\mathcal{A}, \mathcal{E})=\mathbf{P}\left(\mathbf{y} \in \mathcal{A} \mid \mathcal{H}_{1}\right)=\sup _{\boldsymbol{\sigma} \in \mathcal{E}} \mathbf{P}(\mathbf{y} \in \mathcal{A} \mid \boldsymbol{\sigma})
$$

In other words, the minimax problem of testing hypotheses $\mathcal{H}_{0}$ and $\mathcal{H}_{1}$ is considered.
Provided given 1 -st kind error probability $\alpha, 0<\alpha<1$, we are interested in minimal possible 2-nd kind error probability

$$
\begin{equation*}
\beta(\alpha, \mathcal{E})=\inf _{\mathcal{A}: \alpha(\mathcal{A}) \leq \alpha} \beta(\mathcal{A}, \mathcal{E}) \tag{10}
\end{equation*}
$$

and corresponding decision region $\mathcal{A}(\alpha)$.
Without loss of generality we assume the set $\mathcal{E}$ closed and Lebeques measurable on $\mathbb{R}^{n}$. Formally speaking, the optimal solution of the problem (10) of minimax testing of hypotheses $\mathcal{H}_{0}$ and $\mathcal{H}_{1}$ is described in Wald's general theory of statistical decisions [1]. For that solution we need to find the "least favorable" prior distribution $\pi_{\text {lf }}(d \mathcal{E})$ on $\mathcal{E}$, replace the composite hypothesis $\mathcal{H}_{1}$ by simple hypothesis $\mathcal{H}_{1}\left(\pi_{\mathrm{lf}}\right)$, and then to investigate characteristics of corresponding Neyman-Pierson criteria for testing simple hypotheses $\mathcal{H}_{0}$ and $\mathcal{H}_{1}\left(\pi_{\text {lf }}\right)$. Unfortunately, all that can be done only in some very special cases. Therefore it is natural to separate cases, when that "least favorable" prior distribution on $\mathcal{E}$ has the simplest form (for example, it is concentrated in one point from $\mathcal{E}$ ).

Clearly, for the value $\beta(\alpha, \mathcal{E})$ the lower bound holds

$$
\begin{equation*}
\beta(\alpha, \mathcal{E}) \geq \sup _{\boldsymbol{\sigma} \in \mathcal{E}} \beta(\alpha, \boldsymbol{\sigma}) \tag{11}
\end{equation*}
$$

The function $\beta(\alpha, \boldsymbol{\sigma}), \alpha \in[0,1], \boldsymbol{\sigma} \in \mathbb{R}_{+}^{n}$ is continuous on both arguments. Since the set $\mathcal{E} \in \mathbb{R}_{+}^{n}$ supposed to be closed then there exists $\boldsymbol{\sigma}_{0}=\boldsymbol{\sigma}_{0}(\mathcal{E}, \alpha) \in \mathcal{E}$, such that

$$
\beta\left(\alpha, \boldsymbol{\sigma}_{0}\right)=\sup _{\boldsymbol{\sigma} \in \mathcal{E}} \beta(\alpha, \boldsymbol{\sigma}) .
$$

First, we are interested for what kind of $\mathcal{E}$ the "least favorable" prior distribution is concentrated in the point $\boldsymbol{\sigma}_{0}$ and then the following equality holds

$$
\begin{equation*}
\beta(\alpha, \mathcal{E})=\beta\left(\alpha, \boldsymbol{\sigma}_{0}\right) \tag{12}
\end{equation*}
$$

If for the set $\mathcal{E}$ the equality (12) holds then without any loss of detection quality we may replace the composite hypothesis $\mathcal{H}_{1}=\{\mathcal{E}\}$ by the simple hypothesis $\mathcal{H}_{1}=\sigma_{0}$ and the optimal solution (5)-(6) for the simple hypothesis $\mathcal{H}_{1}=\boldsymbol{\sigma}_{0}$ remains optimal (in minimax sense) for the composite hypothesis $\mathcal{H}_{1}=\{\mathcal{E}\}$ as well (see similar question for shifts of measures [3]). Some sufficient conditions for having the equality (12) are given below in $\S 3$. Of course, those conditions set rather strong limitations on the set $\mathcal{E}$.

Earlier, it is shown in $\S 2$ that sometimes it is possible without any loss of detection quality to replace the set $\mathcal{E}$ by a smaller set $\mathcal{E}_{0}$ (i.e. to make a reduction of the set $\mathcal{E}$ ).

Usually in the problem considered the probability $\beta(\alpha, \mathcal{E})$ should be very small. For that reason often instead of the strong condition (12) its simpler asymptotic analogue is investigated, comparing exponents of error probabilities (see, for example, 4]). In that case we are interested in validity of a weaker condition:

$$
\begin{equation*}
\ln \beta(\alpha, \mathcal{E})=\ln \beta(\alpha, \boldsymbol{\sigma})+o(\ln \beta(\alpha, \boldsymbol{\sigma})), \quad|\ln \beta(\alpha, \boldsymbol{\sigma})| \rightarrow \infty \tag{13}
\end{equation*}
$$

It will be shown below that the condition (13) holds under a weaker restrictions on the set $\mathcal{E}$, than in the case of the condition (12).

Note that if for the set $\mathcal{E}$ asymptotic equality (13) holds, it does not mean that optimal solution (5)-(6) for simple hypothesis $\mathcal{H}_{1}=\boldsymbol{\sigma}_{0}$ remains optimal for composite hypothesis $\mathcal{H}_{1}=\{\mathcal{E}\}$ as well. Probably, it will be necessary to use another test. Some sufficient conditions for having equality (13) and corresponding test are described in § 4.

Since in the problem considered the probability $\beta(\alpha, \mathcal{E})$ usually should be very small, in the paper large deviations for the value $\beta(\alpha, \mathcal{E})$ (i.e. its logarithmic asymptotics as $n \rightarrow \infty)$ is also investigated. In $\S 4$ for that asymptotics upper bounds and in Appendix lower bounds are obtained (from which the exact logarithmic asymptotics of $\beta(\alpha, \mathcal{E})$ as $n \rightarrow \infty$ follows). In $\S 5$ similar upper bounds for the value $\alpha(A, \boldsymbol{\sigma})$ are derived. If the value $\alpha(A, \boldsymbol{\sigma})$ is not too small, then in order to have completeness in $\S 5.1$ it is investigated using the Central Limit Theorem and Berry-Esseen inequality, which give a more accurate estimates.

In $\S 6$ a special example is considered. Some useful estimates for large deviations of the distribution $\chi^{2}$, used in the paper, are given in Appendix.

All formulas in the paper are, essentially, non-asymptotic. All remaining terms can always be estimated.

Below, as usual, $\boldsymbol{\sigma} \leq \boldsymbol{\lambda}$ means $\sigma_{i} \leq \lambda_{i}, i=1, \ldots, n$.

## $\S 2$. Reduction of the set $\mathcal{E}$

We show that sometimes without any loss of detection quality it is possible to replace the set $\mathcal{E}$ by a smaller set $\mathcal{E}_{0}$. Define such set $\mathcal{E}_{0}=\mathcal{E}_{0}(\mathcal{E})$ as any set having the following property:

$$
\begin{equation*}
\text { for any } \boldsymbol{\sigma} \in \mathcal{E} \text { there exusts } \boldsymbol{\sigma}_{0} \in \mathcal{E}_{0} \text { with } \boldsymbol{\sigma}_{0} \leq \boldsymbol{\sigma} \tag{14}
\end{equation*}
$$

If the set $\mathcal{E}$ is closed (it is assumed in the paper), then $\mathcal{E}_{0} \subseteq \mathcal{E}$. Generally, the set $\mathcal{E}_{0}$ can be chosen non-uniquely.

It is shown below that for any Bayes criteria of testing a simple hypothesis $\mathcal{H}_{0}$ against a composite alternative $\mathcal{H}_{1}=\{\mathcal{E}\}$ the set $\mathcal{E}$ can be replaced by the set $\mathcal{E}_{0}$ without any loss of quality. It remains valid for likelihood ratio criteria as well. In one-dimensional case those properties are similar to the case of distributions with monotone likelihood ratio [2, Ch. 3.9].

The aim of the set $\mathcal{E}_{0}$ introduction is to decrease (if possible) the set $\mathcal{E}$ and so to simplify the test used.

1. Bayes criteria. Consider a Bayes criteria with a prior distribution $\pi(d \mathcal{E})$ on $\mathcal{E}$ and corresponding decision set $\mathcal{A} \in \mathbb{R}^{n}\left(\mathbf{y} \in \mathcal{A} \Rightarrow \mathcal{H}_{0}, \mathbf{y} \notin \mathcal{A} \Rightarrow \mathcal{H}_{1}\right)$ of the form

$$
\begin{equation*}
\mathcal{A}=\mathcal{A}(A)=\{\mathbf{y}: r(\mathbf{y}, \mathcal{E}, \pi) \leq A\} \tag{15}
\end{equation*}
$$

where (see (31), (4))

$$
p\left(\mathbf{y} \mid \mathcal{H}_{1}, \pi\right)=\int_{\boldsymbol{\sigma} \in \mathcal{E}} p(\mathbf{y} \mid \boldsymbol{\sigma}) \pi(d \mathcal{E})=(2 \pi)^{-n / 2} \int_{\boldsymbol{\sigma} \in \mathcal{E}} e^{-\frac{1}{2} \sum_{i=1}^{n} y_{i}^{2} /\left(1+\sigma_{i}^{2}\right)-\frac{1}{2} D(\boldsymbol{\sigma})} \pi(d \mathcal{E})
$$

and

$$
r(\mathbf{y}, \mathcal{E}, \pi)=\ln \frac{p\left(\mathbf{y} \mid \mathcal{H}_{1}, \pi\right)}{p\left(\mathbf{y} \mid \mathcal{H}_{0}\right)}=\ln \int_{\boldsymbol{\sigma} \in \mathcal{E}} e^{\frac{1}{2} \sum_{i=1}^{n} y_{i}^{2} \sigma_{i}^{2} /\left(1+\sigma_{i}^{2}\right)-\frac{1}{2} D(\boldsymbol{\sigma})} \pi(d \mathcal{E})
$$

Then $\mathcal{A}$ - convex set in $\mathbb{R}^{n}$, and if $\mathbf{y}=\left(y_{1}, \ldots, y_{n}\right) \in \mathcal{A}$, then all $\left( \pm y_{1}, \ldots, \pm y_{n}\right)$ belong to $\mathcal{A}$, i.e. the set $\mathcal{A}$ is symmetric with respect to any coordinate axis or plane. In particular, such $\mathcal{A}$ is also centrally symmetric set (i.e. if $\mathbf{y} \in \mathcal{A}$, then $(-\mathbf{y}) \in \mathcal{A}$ ).

Assume that for $\mathbf{y}=\mathbf{s}+\boldsymbol{\xi}, \boldsymbol{\sigma} \in \mathcal{E}$ from (1) a Bayes criteria with a prior distribution $\pi(d \mathcal{E})$ на $\mathcal{E}_{0}$ is used and $\mathcal{A} \in \mathbb{R}^{n}$ of the form (15) is the corresponding decision region. Assume also that for 2-nd kind error probability and some $\beta \geq 0$ we have

$$
\begin{equation*}
\beta\left(\mathcal{A}, \boldsymbol{\sigma}_{0}\right)=\mathbf{P}\left(\mathbf{y} \in \mathcal{A} \mid \boldsymbol{\sigma}_{0}\right)=\mathbf{P}\left\{r\left(\mathbf{y}, \mathcal{E}_{0}, \pi\right) \leq A \mid \boldsymbol{\sigma}_{0}\right\} \leq \beta, \quad \boldsymbol{\sigma}_{0} \in \mathcal{E}_{0} \tag{16}
\end{equation*}
$$

Show that the inequality (16) remains valid for any $\boldsymbol{\sigma} \in \mathcal{E}$, i.e.

$$
\begin{equation*}
\beta(\mathcal{A}, \boldsymbol{\sigma})=\mathbf{P}(\mathbf{y} \in \mathcal{A} \mid \boldsymbol{\sigma})=\mathbf{P}\left\{r\left(\mathbf{y}, \mathcal{E}_{0}, \pi\right) \leq A \mid \boldsymbol{\sigma}\right\} \leq \beta, \quad \boldsymbol{\sigma} \in \mathcal{E} \tag{17}
\end{equation*}
$$

In other words, for any Bayes criteria extension of the set $\mathcal{E}_{0}$ up to the set $\mathcal{E}$ does not increase 2 -nd kind error probability (1-st kind error probability $\alpha(\mathcal{A})$ does not change). In particular, since $\mathcal{E}_{0} \subseteq \mathcal{E}$, we get

$$
\begin{equation*}
\beta\left(\alpha, \mathcal{E}_{0}\right)=\beta(\alpha, \mathcal{E}), \quad 0 \leq \alpha \leq 1 \tag{18}
\end{equation*}
$$

We prove the relation (17). Let $\boldsymbol{\sigma} \in \mathcal{E}$, but $\boldsymbol{\sigma} \notin \mathcal{E}_{0}$. Then there exists $\boldsymbol{\sigma}_{0} \in \mathcal{E}_{0}$ with $\boldsymbol{\sigma}_{0}<\boldsymbol{\sigma}$. Let $\mathbf{s}_{0}$ - Gaussian "signal" in (11) in the case of $\boldsymbol{\sigma}_{0}$. Then in the case of $\boldsymbol{\sigma}$ such "signal" $\mathbf{s}$ has the form $\mathbf{s}=\mathbf{s}_{0}+\boldsymbol{\eta}$, where $\boldsymbol{\eta}$ - independent of $\mathbf{s}_{0}$ Gaussian random vector. The inequality (17) follows from the following auxiliary result (the set $\mathcal{A}$ satisfies its conditions).

L e mm a 1 . Let $\mathcal{B} \in \mathbb{R}^{n}-a$ convex set, such that if $\mathbf{y}=\left(y_{1}, \ldots, y_{n}\right) \in \mathcal{B}$, then all points of the form $\left( \pm y_{1}, \ldots, \pm y_{n}\right)$ belong to $\mathcal{B}$. Let also $\boldsymbol{\xi}, \boldsymbol{\eta}$-independent zero mean Gaussian vectors, consisting of independent (probably, with different distributions) components. Then

$$
\begin{equation*}
\mathbf{P}(\boldsymbol{\xi}+\boldsymbol{\eta} \in \mathcal{B}) \leq \mathbf{P}(\boldsymbol{\xi} \in \mathcal{B}) \tag{19}
\end{equation*}
$$

P r o o f. If $n=1$, then $\mathcal{B}=[-a, a], a>0$ and, clearly, the inequality (19) holds. Let $n=2$ and vectors $\left(\xi_{1}, \xi_{2}\right)$ and $\left(\xi_{1}+\eta_{1}, \xi_{2}+\eta_{2}\right)$ are compared. Compare first vectors $\left(\xi_{1}, \xi_{2}\right)$ and $\left(\xi_{1}+\eta_{1}, \xi_{2}\right)$. Denote

$$
\mathcal{B}_{x}=\left\{\mathbf{y} \in \mathcal{B}: y_{2}=x\right\} \in \mathbb{R}^{1}
$$

Due to assumptions of Lemma, for any $x$ we have $\mathcal{B}_{x}=[-a(x), a(x)], a(x)>0$. Therefore fox fixed $\xi_{2}$ the problem reduces to the case $n=1$ and

$$
\begin{equation*}
\mathbf{P}\left\{\xi_{1}+\eta_{1} \in \mathcal{B}_{\xi_{2}}\right\} \leq \mathbf{P}\left\{\xi_{1} \in \mathcal{B}_{\xi_{2}}\right\} \tag{20}
\end{equation*}
$$

and then

$$
\begin{equation*}
\mathbf{P}\left\{\left(\xi_{1}+\eta_{1}, \xi_{2}\right) \in \mathcal{B}\right\} \leq \mathbf{P}\left\{\left(\xi_{1}, \xi_{2}\right) \in \mathcal{B}\right\} . \tag{21}
\end{equation*}
$$

Compare now vectors $\left(\xi_{1}+\eta_{1}, \xi_{2}\right)$ and $\left(\xi_{1}+\eta_{1}, \xi_{2}+\eta_{2}\right)$. Similarly to (20) and (21) we get

$$
\mathbf{P}\left\{\xi_{2}+\eta_{2} \in \mathcal{B}_{\xi_{1}+\eta_{1}}\right\} \leq \mathbf{P}\left\{\xi_{2} \in \mathcal{B}_{\xi_{1}+\eta_{1}}\right\}
$$

and

$$
\begin{equation*}
\mathbf{P}\left\{\left(\xi_{1}+\eta_{1}, \xi_{2}+\eta_{2}\right) \in \mathcal{B}\right\} \leq \mathbf{P}\left\{\left(\xi_{1}+\eta_{1}, \xi_{2}\right) \in \mathcal{B}\right\} \tag{22}
\end{equation*}
$$

Then by (21) and (22) the inequality (19) follows for $n=2$. Similarly, the case $n=3$ reduces to the case $n=2$ and so on. It proves the inequality (19) for any $n$. $\triangle$
2. Likelihood ratio criteria. For any function $A(\boldsymbol{\sigma})$ the critical region $\mathcal{A}_{M L}(A, \mathcal{E})$ of that criteria is defined by the relation

$$
\begin{equation*}
\mathcal{A}_{M L}(A, \mathcal{E})=\left\{\mathbf{y}: \sup _{\boldsymbol{\sigma} \in \mathcal{E}}[2 r(\mathbf{y}, \boldsymbol{\sigma})-A(\boldsymbol{\sigma})] \leq 0\right\} \tag{23}
\end{equation*}
$$

and then $\mathbf{y} \in \mathcal{A}_{M L}(A, \mathcal{E}) \Rightarrow \mathcal{H}_{0}, \mathbf{y} \notin \mathcal{A}_{M L}(A, \mathcal{E}) \Rightarrow \mathcal{H}_{1}$.
Show that without any loss of quality we may replace the set $\mathcal{E}$ in (23) by smaller set $\mathcal{E}_{0}$ (see (14)), i.e. to use the criteria:

$$
\begin{equation*}
\mathcal{A}_{M L R}(A, \mathcal{E})=\left\{\mathbf{y}: \sup _{\boldsymbol{\sigma} \in \mathcal{E}_{0}}[2 r(\mathbf{y}, \boldsymbol{\sigma})-A(\boldsymbol{\sigma})] \leq 0\right\} \tag{24}
\end{equation*}
$$

keeping the same decision making method. In other words, for likelihood ratio criteria expansion of the set $\mathcal{E}_{0}$ up to the set $\mathcal{E}$ does not increase the 2-nd kind error probability (the 1 -st kind error probability $\alpha(\mathcal{A})$ does not change).

Indeed, if $\boldsymbol{\sigma} \in \mathcal{E}$, but $\boldsymbol{\sigma} \notin \mathcal{E}_{0}$, then there exists $\boldsymbol{\sigma}_{0} \in \mathcal{E}_{0}$ with $\boldsymbol{\sigma}_{0}<\boldsymbol{\sigma}$. Using the definition (24) and formulas (26) and (27) below, we have

$$
\begin{align*}
& \beta(A, \boldsymbol{\sigma})=\mathbf{P}\left\{\sup _{\boldsymbol{\lambda} \in \mathcal{E}_{0}}[2 r(\mathbf{y}, \boldsymbol{\lambda})-A(\boldsymbol{\lambda})] \leq 0 \mid \boldsymbol{\sigma}\right\}= \\
= & \mathbf{P}\left\{\sup _{\boldsymbol{\lambda} \in \mathcal{E}_{0}}\left[\sum_{i=1}^{n} \frac{\lambda_{i}^{2}\left(1+\sigma_{i}^{2}\right) \eta_{i}^{2}}{1+\lambda_{i}^{2}}-D(\boldsymbol{\lambda})-A(\boldsymbol{\lambda})\right] \leq 0\right\} \leq  \tag{25}\\
\leq & \mathbf{P}\left\{\sup _{\boldsymbol{\lambda} \in \mathcal{E}_{0}}\left[\sum_{i=1}^{n} \frac{\lambda_{i}^{2}\left(1+\sigma_{0 i}^{2}\right) \eta_{i}^{2}}{1+\lambda_{i}^{2}}-D(\boldsymbol{\lambda})-A(\boldsymbol{\lambda})\right] \leq 0\right\}= \\
= & \mathbf{P}\left\{\sup _{\boldsymbol{\lambda} \in \mathcal{E}_{0}}[2 r(\mathbf{y}, \boldsymbol{\lambda})-A(\boldsymbol{\lambda})] \leq 0 \mid \boldsymbol{\sigma}_{0}\right\}=\beta\left(A, \boldsymbol{\sigma}_{0}\right) .
\end{align*}
$$

Results (17) and (25) obtained can be formulated as follows.
Proposition 1. Consider the minimax problem of testing a simple hypothesis $\mathcal{H}_{0}$ against a composite alternative $\mathcal{H}_{1}=\left\{\mathcal{E}_{0}\right\}$ and let $\mathcal{E}_{0} \subseteq \mathcal{E}$. If for the set $\mathcal{E}$ the condition (14) is satisfied then for any Bayes criteria and the likelihood ratio criteria the 1 -st kind
and the 2-nd kind error probabilities do not change if the set $\mathcal{E}_{0}$ is replaced by the set $\mathcal{E}$. In particular, the equality (18) holds.

Remark 1. It seems that it would be more natural in Proposition 1 to start with a set $\mathcal{E}$ and to replace it by a set $\mathcal{E}_{0} \subseteq \mathcal{E}$. But in that case it would be necessary to describe "projections" of Bayes criteria from $\mathcal{E}$ on $\mathcal{E}_{0}$.

Remark 2. Similar to $\mathcal{E}_{0}$ "reduced" sets $\operatorname{red}_{1} S$ and $\operatorname{red}_{2} S$ have been introduced earlier in [3], where Gaussian measures differed from each other only by shifts. From analytical viewpoint, various convexity properties with respect to shifts of Gaussian measures were very useful in [3]. For example, due to them the set $\operatorname{red}_{1} S$ had very simple and natural form. Unfortunately, the author does not know similar convexity properties concerning variances of Gaussian measures and for that reason only certain monotonicity properties have been used (what is less productive).

## § 3. Exact equality (12)

1. The formula (12) has also another equivalent interpretation. Assume that initially we know that in the hypothesis $\mathcal{H}_{1}$ the "signal" is a certain $\sigma$ and therefore we use the optimal solution (5)-(6) for that $\boldsymbol{\sigma}$. Assume additionally that in fact the "signal" in the hypothesis $\mathcal{H}_{1}$ may also take another values $\boldsymbol{\lambda}$ from a set $\mathcal{E}$. For what $\mathcal{E}$ the solution (5)-(66) (oriented only on $\boldsymbol{\sigma}$ ) remains optimal for the set $\mathcal{E}$ as well ?

If $\boldsymbol{\sigma}$ is replaced by $\boldsymbol{\lambda}$ and decision (5)-(6) is used, then the 1 -st kind error probability $\alpha$ does not change. Therefore it is necessary to check only how the 2-nd kind error probability $\beta_{\boldsymbol{\sigma}}(A, \boldsymbol{\lambda})$ may change

$$
\begin{gather*}
\beta_{\boldsymbol{\sigma}}(A, \boldsymbol{\lambda})=\mathbf{P}(\mathbf{y} \in \mathcal{A} \mid \boldsymbol{\lambda})=\mathbf{P}\left(\left.\sum_{i=1}^{n} \frac{\sigma_{i}^{2}\left(\xi_{i}+s_{i}\right)^{2}}{1+\sigma_{i}^{2}}-D(\boldsymbol{\sigma})<A \right\rvert\, \boldsymbol{\lambda}\right)=  \tag{26}\\
=\mathbf{P}\left(\sum_{i=1}^{n} \nu_{i}^{2} \xi_{i}^{2}-D(\boldsymbol{\sigma})<A\right)
\end{gather*}
$$

since $\left(\xi_{i}+s_{i}\right)^{2}=\left(1+\lambda_{i}^{2}\right) \eta_{i}^{2}, i=1, \ldots, n$ and where

$$
\begin{equation*}
\nu_{i}^{2}=\frac{\sigma_{i}^{2}\left(1+\lambda_{i}^{2}\right)}{1+\sigma_{i}^{2}}=\sigma_{i}^{2}+\frac{\sigma_{i}^{2}\left(\lambda_{i}^{2}-\sigma_{i}^{2}\right)}{1+\sigma_{i}^{2}}, \quad i=1, \ldots, n \tag{27}
\end{equation*}
$$

and $\left\{\eta_{i}\right\}$ - independent $\mathcal{N}(0,1)$-Gaussian random variables.
If for any $\boldsymbol{\lambda} \in \mathcal{E}$ and $A$ the following inequality holds $\left(\boldsymbol{\nu}=\left(\nu_{1}, \ldots, \nu_{n}\right)\right.$ is defined in (27))

$$
\begin{equation*}
\beta_{\boldsymbol{\sigma}}(A, \boldsymbol{\lambda})=\mathbf{P}\left(\sum_{i=1}^{n} \nu_{i}^{2} \xi_{i}^{2}-D(\boldsymbol{\sigma})<A\right) \leq \mathbf{P}\left(\sum_{i=1}^{n} \sigma_{i}^{2} \xi_{i}^{2}-D(\boldsymbol{\sigma})<A\right)=\beta(A, \boldsymbol{\sigma}), \tag{28}
\end{equation*}
$$

then

$$
\beta(A, \mathcal{E}) \leq \sup _{\boldsymbol{\lambda} \in \mathcal{E}} \beta_{\boldsymbol{\sigma}}(A, \boldsymbol{\lambda}) \leq \beta(A, \boldsymbol{\sigma})
$$

and therefore the formula (12) is valid.
Some results showing validity of the inequality (28) for certain $\boldsymbol{\sigma}, \boldsymbol{\nu}, A$ can be found, for example, in [5, 6, 7].

In order to have $\beta_{\boldsymbol{\sigma}}(A, \boldsymbol{\lambda}) \leq \beta(A, \boldsymbol{\sigma})$ for any $A$ (see (28)), it is necessary, at least, to have (comparing of expectations)

$$
\sum_{i=1}^{n} \nu_{i}^{2}-\sum_{i=1}^{n} \sigma_{i}^{2}=\sum_{i=1}^{n} \frac{\sigma_{i}^{2}\left(\lambda_{i}^{2}-\sigma_{i}^{2}\right)}{1+\sigma_{i}^{2}} \geq 0
$$

Comparing (8), (26) and (27), we get simple
Proposition 2.1) If $\boldsymbol{\sigma} \leq \boldsymbol{\lambda}$, then $\beta(A, \boldsymbol{\lambda}) \leq \beta(A, \boldsymbol{\sigma})$ and $\beta_{\boldsymbol{\sigma}}(A, \boldsymbol{\lambda}) \leq \beta(A, \boldsymbol{\sigma})$ for any $A$.
2) If $\boldsymbol{\sigma} \leq \boldsymbol{\lambda}$ for any $\boldsymbol{\lambda} \in \mathcal{E}$, then $\beta(\alpha, \mathcal{E})=\beta(\alpha, \boldsymbol{\sigma})$ for any $\alpha$.
2. As an example consider the following result, which is the part of lemma 1 from [7].
$\mathrm{L} \mathrm{e} \mathrm{m} \mathrm{m} \mathrm{a} \mathrm{2} \mathrm{Assume} \mathrm{that} \mathrm{the} \mathrm{set} \mathrm{of} \mathrm{indices} I=.\{1,2, \ldots, n\}$ of vectors $\boldsymbol{\sigma}, \boldsymbol{\lambda}$ can be partitioned in $k \geq 1$ groups $I_{1}, \ldots, I_{k}$, such that $I=\bigcup_{j=1}^{k} I_{j}, I_{i} \cap I_{j}=\emptyset, i \neq j$, and the following conditions are fulfilled

$$
\sigma_{i} \leq \lambda_{0, j}, \quad i \in I_{j}, \quad j=1, \ldots, k
$$

where

$$
\lambda_{0, j}=\left(\prod_{i \in I_{j}} \lambda_{i}\right)^{1 /\left|I_{j}\right|}
$$

Then $\beta(A, \boldsymbol{\lambda}) \leq \beta(A, \boldsymbol{\sigma})$ for any $A$.
Example 1. Let for given $D>0$

$$
\mathcal{E}=\left\{\boldsymbol{\lambda} \geq \mathbf{0}: \prod_{i=1}^{n}\left(1+\lambda_{i}^{2}\right) \geq\left(1+D^{2}\right)^{n}\right\}
$$

Then from the formula (27) and Lemma 2 with $k=1$ it follows that the set $\mathcal{E}$ can be replaced (without loss of quality) by single point $\boldsymbol{\sigma}_{0}=(D, \ldots, D) \in \mathcal{E}$ (in the sense of exact equality (12)).

## $\S 4$. Asymptotic equality (13). Large deviations for $\beta(A, \boldsymbol{\sigma})$ and $\beta_{\boldsymbol{\sigma}}(A, \boldsymbol{\lambda})$

Consider conditions when the equality (13) holds. For that purpose we investigate the logarithmic asymptotics of probabilities $\beta(A, \boldsymbol{\sigma})$ and $\beta_{\boldsymbol{\sigma}}(A, \boldsymbol{\lambda})$ as $n \rightarrow \infty$.

1. Large deviations. Upper bounds. Since for $\xi \sim \mathcal{N}(0,1)$

$$
\begin{equation*}
\mathbf{E} e^{a(\xi+b)^{2}}=\frac{1}{\sqrt{1-2 a}} \exp \left\{\frac{2 a b^{2}}{1-2 a}\right\}, \quad a<1 / 2, \quad b \in \mathbb{R}^{1} \tag{29}
\end{equation*}
$$

then using exponential Chebychev inequality for any $u \geq 0$ we have

$$
\begin{equation*}
\beta(A, \boldsymbol{\sigma})=\mathbf{P}\left(\sum_{i=1}^{n} \sigma_{i}^{2} \xi_{i}^{2}<D(\boldsymbol{\sigma})+A\right) \leq e^{u[D(\boldsymbol{\sigma})+A] / 2} \mathbf{E} e^{-u \sum_{i=1}^{n} \sigma_{i}^{2} \xi_{i}^{2} / 2}=e^{-g_{\boldsymbol{\sigma}}(u)} \tag{30}
\end{equation*}
$$

where

$$
\begin{gather*}
2 g_{\boldsymbol{\sigma}}(u)=\sum_{i=1}^{n} \ln \left(1+u \sigma_{i}^{2}\right)-u[D(\boldsymbol{\sigma})+A], \quad 2 g_{\boldsymbol{\sigma}}^{\prime}(u)=\sum_{i=1}^{n} \frac{\sigma_{i}^{2}}{1+u \sigma_{i}^{2}}-D(\boldsymbol{\sigma})-A,  \tag{31}\\
g_{\boldsymbol{\sigma}}^{\prime \prime}(u)<0
\end{gather*}
$$

Since both conditions (9) supposed to be fulfilled, then $g_{\boldsymbol{\sigma}}^{\prime}(0)>0$ and $g_{\boldsymbol{\sigma}}^{\prime}(1)<0$. Therefore $\max _{u \geq 0} g_{\boldsymbol{\sigma}}(u)$ is attained for $0<u_{0}<1$, which is determined by the equation $g_{\boldsymbol{\sigma}}^{\prime}\left(u_{0}\right)=0$, i.e.

$$
\begin{equation*}
\sum_{i=1}^{n} \frac{\sigma_{i}^{2}}{1+u_{0} \sigma_{i}^{2}}=D(\boldsymbol{\sigma})+A \tag{32}
\end{equation*}
$$

Then from (30) and (31) we get

$$
\begin{equation*}
\beta(A, \boldsymbol{\sigma}) \leq e^{-g_{\boldsymbol{\sigma}}\left(u_{0}\right)} \tag{33}
\end{equation*}
$$

where

$$
\begin{equation*}
g_{\boldsymbol{\sigma}}\left(u_{0}\right)=\max _{u \geq 0} g_{\boldsymbol{\sigma}}(u) . \tag{34}
\end{equation*}
$$

Provided certain conditions (see Appendix, point 3) it is exact logarithmic asymptotics of the value $\beta(A, \boldsymbol{\sigma})$ as $n \rightarrow \infty$.

Similarly, from (26) and (27) for any $v \geq 0$ we have

$$
\beta_{\boldsymbol{\sigma}}(A, \boldsymbol{\lambda})=\mathbf{P}\left(\sum_{i=1}^{n} \nu_{i}^{2} \xi_{i}^{2}<D(\boldsymbol{\sigma})+A\right) \leq e^{v[D(\boldsymbol{\sigma})+A] / 2} \prod_{i=1}^{n} \frac{1}{\sqrt{1+v \nu_{i}^{2}}}=e^{-g_{\boldsymbol{\sigma}}(v, \boldsymbol{\lambda})},
$$

where values $\left\{\nu_{i}^{2}\right\}$ are defined in (27) and

$$
\begin{gathered}
2 g_{\boldsymbol{\sigma}}(v, \boldsymbol{\lambda})=\sum_{i=1}^{n} \ln \left(1+v \nu_{i}^{2}\right)-v[D(\boldsymbol{\sigma})+A] \\
2 g_{\boldsymbol{\sigma}}^{\prime}(v, \boldsymbol{\lambda})=\sum_{i=1}^{n} \frac{\nu_{i}^{2}}{1+v \nu_{i}^{2}}-D(\boldsymbol{\sigma})-A, \quad g_{\boldsymbol{\sigma}}^{\prime \prime}(v, \boldsymbol{\lambda})<0
\end{gathered}
$$

Then

$$
\begin{equation*}
\beta_{\boldsymbol{\sigma}}(A, \boldsymbol{\lambda}) \leq e^{-g_{\boldsymbol{\sigma}}\left(v_{0}, \boldsymbol{\lambda}\right)} \tag{35}
\end{equation*}
$$

where

$$
\begin{equation*}
g_{\boldsymbol{\sigma}}\left(v_{0}, \boldsymbol{\lambda}\right)=\max _{v \geq 0} g_{\boldsymbol{\sigma}}(v, \boldsymbol{\lambda}) . \tag{36}
\end{equation*}
$$

There is a sense to consider only $\boldsymbol{\lambda}$ such that $g_{\boldsymbol{\sigma}}^{\prime}(0, \boldsymbol{\lambda})=\sum_{i=1}^{n} \nu_{i}^{2}-D(\boldsymbol{\sigma})-A>0$ (otherwise $\left.v_{0}=0\right)$. Then $\max _{v \geq 0} g_{\boldsymbol{\sigma}}(v, \boldsymbol{\lambda})$ is attained for $v_{0}>0$, which is determined by the equation

$$
\sum_{i=1}^{n} \frac{\nu_{i}^{2}}{1+v_{0} \nu_{i}^{2}}=\sum_{i=1}^{n} \frac{\sigma_{i}^{2}\left(1+\lambda_{i}^{2}\right)}{1+\sigma_{i}^{2}+v_{0} \sigma_{i}^{2}\left(1+\lambda_{i}^{2}\right)}=D(\boldsymbol{\sigma})+A .
$$

If estimates (33)-(34) and (35)-(36) give the right logarithmic asymptotics (as $n \rightarrow$ $\infty)$ of values $\beta(A, \boldsymbol{\sigma})$ and $\beta_{\boldsymbol{\sigma}}(A, \boldsymbol{\lambda})$, then $g_{\boldsymbol{\sigma}}\left(u_{0}\right)-g_{\boldsymbol{\sigma}}\left(v_{0}, \boldsymbol{\lambda}\right) \geq 0$. Then the condition (13) is equivalent to the question: if $\boldsymbol{\sigma}$ is given, then for what $\boldsymbol{\lambda}$ the following condition holds

$$
\begin{equation*}
g_{\boldsymbol{\sigma}}\left(u_{0}\right)-g_{\boldsymbol{\sigma}}\left(v_{0}, \boldsymbol{\lambda}\right)=o\left(g_{\boldsymbol{\sigma}}\left(u_{0}\right)\right) \text { as } g_{\boldsymbol{\sigma}}\left(u_{0}\right) \rightarrow \infty ? \tag{37}
\end{equation*}
$$

If the condition (37) is fulfilled and we replace $\boldsymbol{\sigma}$ by $\boldsymbol{\lambda}$, using the decision (5)-(6) (oriented on $\boldsymbol{\sigma}$ ), then the 1 -st kind error probability $\alpha$ does not change and the 2-nd kind error probability $\beta_{\boldsymbol{\sigma}}(A, \boldsymbol{\lambda})$ changes slightly.

Generally, the condition (37) is rather complicated for checking (since we must find the value $v_{0}$ for each $\boldsymbol{\lambda}$ ). Sufficient for having (37) is a simpler condition

$$
\begin{equation*}
g_{\boldsymbol{\sigma}}\left(u_{0}\right)-\max \left\{g_{\boldsymbol{\sigma}}\left(u_{0}, \boldsymbol{\lambda}\right), g_{\boldsymbol{\sigma}}(1, \boldsymbol{\lambda})\right\}=o\left(g_{\boldsymbol{\sigma}}\left(u_{0}\right)\right), \quad g_{\boldsymbol{\sigma}}\left(u_{0}\right) \rightarrow \infty \tag{38}
\end{equation*}
$$

or, in particular,

$$
\begin{equation*}
g_{\boldsymbol{\sigma}}\left(u_{0}, \boldsymbol{\lambda}\right)-g_{\boldsymbol{\sigma}}\left(u_{0}\right)=\sum_{i=1}^{n} \ln \left[1+\frac{u_{0} \sigma_{i}^{2}\left(\lambda_{i}^{2}-\sigma_{i}^{2}\right)}{\left(1+\sigma_{i}^{2}\right)\left(1+u_{0} \sigma_{i}^{2}\right)}\right]=o\left(g_{\boldsymbol{\sigma}}\left(u_{0}\right)\right) \tag{39}
\end{equation*}
$$

Note that the condition (38) (or (39)) is only sufficient, but not necessary. It may give satisfactory results, if $\boldsymbol{\lambda}$ is not very different from $\boldsymbol{\sigma}$. If $\boldsymbol{\lambda}$ is very different from $\boldsymbol{\sigma}$, then essential loss of accuracy is possible (see below example 3, where the condition (38) is not fulfilled, but the condition (37) is satisfied). We give another similar example (omitting some details).

Ex a m ple 2. Choose $\boldsymbol{\sigma}$ and $\boldsymbol{\lambda}$, such that $u_{0} \neq v_{0}$, and in the formula (37) the equality holds, i.e.

$$
g_{\boldsymbol{\sigma}}\left(u_{0}\right)=\max _{u \geq 0} g_{\boldsymbol{\sigma}}(u)=g_{\boldsymbol{\lambda}}\left(v_{0}\right)=\max _{v \geq 0} g_{\boldsymbol{\lambda}}(v)
$$

Now if the condition (38) is satisfied, then similar condition

$$
\begin{equation*}
g_{\boldsymbol{\lambda}}\left(v_{0}\right) \leq g_{\boldsymbol{\sigma}}\left(v_{0}\right) \tag{40}
\end{equation*}
$$

can not be fulfilled. It means that when changing mutually $\boldsymbol{\sigma}$ and $\boldsymbol{\lambda}$ the condition (40) stops being necessary.
2. Case $u_{0} \approx 1$. Consider an important particular case when the set $\mathcal{E}$ can be replaced by a point $\boldsymbol{\sigma} \in \mathcal{E}$, and the sufficient condition (39) takes a simple form. Let $\alpha$ be not very
small and we need only that $\alpha(A, \mathcal{E})$ satisfies the inequality $\alpha(A, \mathcal{E}) \leq 6 B_{n}^{-1 / 2}$, where $B_{n}=\inf _{\boldsymbol{\sigma} \in \mathcal{E}} B(\boldsymbol{\sigma})$, and $B(\boldsymbol{\sigma})$ is defined in (2). In order to do so, keeping in mind some $\boldsymbol{\sigma} \in \mathcal{E}$, we set (see (2) and (43))

$$
A=T(\boldsymbol{\sigma})-D(\boldsymbol{\sigma})+\varepsilon,
$$

where

$$
\varepsilon=\sqrt{B(\boldsymbol{\sigma}) \ln B(\boldsymbol{\sigma})}
$$

Denoting $u_{0}=1-\delta, \delta \geq 0$, we show that the value $\delta$ is small for large $B(\boldsymbol{\sigma})$. Indeed, the equation (32) takes the form

$$
\sum_{i=1}^{n} \frac{\sigma_{i}^{2}}{1+u_{0} \sigma_{i}^{2}}=D(\boldsymbol{\sigma})+A=T(\boldsymbol{\sigma})+\varepsilon=\sum_{i=1}^{n} \frac{\sigma_{i}^{2}}{1+\sigma_{i}^{2}}+\varepsilon
$$

from which it follows

$$
\sum_{i=1}^{n} \frac{\delta \sigma_{i}^{4}}{\left(1+\sigma_{i}^{2}-\delta \sigma_{i}^{2}\right)\left(1+\sigma_{i}^{2}\right)}=\varepsilon \geq \sum_{i=1}^{n} \frac{\delta \sigma_{i}^{4}}{\left(1+\sigma_{i}^{2}\right)^{2}}=\frac{\delta B}{2}
$$

Therefore

$$
0 \leq 1-u_{0}=\delta \leq \frac{2 \varepsilon}{B}=2 \sqrt{\frac{\ln B}{B}}
$$

Since $g_{\boldsymbol{\sigma}}(1)=-A / 2, g_{\boldsymbol{\sigma}}^{\prime}(1)=-\varepsilon / 2$ and $g_{\boldsymbol{\sigma}}^{\prime \prime}(u)<0$, then

$$
-A \leq 2 g_{\boldsymbol{\sigma}}\left(u_{0}\right)=2 g_{\boldsymbol{\sigma}}(1-\delta) \leq 2 g_{\boldsymbol{\sigma}}(1)-2 \delta g_{\boldsymbol{\sigma}}^{\prime}(1)=-A+\delta \varepsilon \leq-A+2 \ln B
$$

Therefore, in the sufficient condition (39) we may set $u_{0}=1$ and then it takes the form

$$
\begin{equation*}
g_{\boldsymbol{\sigma}}(1, \boldsymbol{\lambda})-g_{\boldsymbol{\sigma}}(1)=\sum_{i=1}^{n} \ln \left[1+\frac{\sigma_{i}^{2}\left(\lambda_{i}^{2}-\sigma_{i}^{2}\right)}{\left(1+\sigma_{i}^{2}\right)^{2}}\right]=o\left(g_{\boldsymbol{\sigma}}(1)\right), \quad g_{\boldsymbol{\sigma}}(1) \rightarrow \infty \tag{41}
\end{equation*}
$$

The results obtained can be formulated as follows
Proposition 3.1) If there exists $\boldsymbol{\sigma} \in \mathcal{E}$ such that for any $\boldsymbol{\lambda} \in \mathcal{E}$ the condition (39) is satisfied then the property (13) holds and the set $\mathcal{E}$ can be replaced by the point $\boldsymbol{\sigma} \in \mathcal{E}$ without any loss of detection quality.
2) If only $\alpha(A, \mathcal{E}) \leq 6 B_{n}^{-1 / 2}$ is desirable and there exists $\boldsymbol{\sigma} \in \mathcal{E}$ such that for any $\boldsymbol{\lambda} \in \mathcal{E}$ the condition (41) is satisfied then the property (13) holds and the set $\mathcal{E}$ can be replaced by the point $\boldsymbol{\sigma} \in \mathcal{E}$ without any loss of detection quality.

In the case of stationary sequences similar to (41) condition appeared from different arguments in [4, Theorem 1, formula (6)]. Authors of 4] called the analog of the condition (41) "surprising" since, in particular, it does not demand the set $\mathcal{E}$ to be convex. But, as was already mentioned (see Remark 2), in the considered problems with unknown
correlations such convexity is not so important. The condition (41) itself is a corollary of a quite natural sufficient condition (38).

It is shown in Appendix that under certain assumptions upper bounds for $\beta(A, \boldsymbol{\sigma})$ and $\beta_{\boldsymbol{\sigma}}(A, \boldsymbol{\lambda})$ used above give exact logarithmic asymptotics for them as $n \rightarrow \infty$ (and then it is sufficient to compare functions $g_{\boldsymbol{\sigma}}(u)$ and $\left.g_{\boldsymbol{\sigma}}(v, \boldsymbol{\lambda})\right)$.

## $\S$ 5. Relation of $A$ and $\alpha(A, \boldsymbol{\sigma})$

1. Central Limit Theorem. If the given value $\alpha(A, \boldsymbol{\sigma})$ is not too small, then it is possible to evaluate it rather accurately using the Central Limit Theorem and Berry-Esseen inequality. Let $X_{1}, \ldots, X_{n}$ - independent random variables, $\mathbf{E} X_{j}=0$, $\mathbf{E}\left|X_{j}\right|^{3}<\infty, j=1, \ldots, n$. Denote

$$
\begin{aligned}
b_{j}^{2}=\mathbf{E} X_{j}^{2}, \quad B_{n}= & \sum_{j=1}^{n} b_{j}^{2}, \quad F_{n}(x)=\mathbf{P}\left(B_{n}^{-1 / 2} \sum_{j=1}^{n} X_{j}<x\right) \\
& L_{n}=B_{n}^{-3 / 2} \sum_{j=1}^{n} \mathbf{E}\left|X_{j}\right|^{3}
\end{aligned}
$$

Then by Berry-Esseen inequality [9, Ch. V, §2, Theorem 3]

$$
\sup _{x}\left|F_{n}(x)-\Phi(x)\right| \leq L_{n} .
$$

In our case

$$
\begin{gathered}
X_{j}=\frac{\sigma_{j}^{2}\left(\xi_{j}^{2}-1\right)}{1+\sigma_{j}^{2}}, \quad b_{j}^{2}=\frac{2 \sigma_{j}^{4}}{\left(1+\sigma_{j}^{2}\right)^{2}}, \quad B_{n}=\sum_{j=1}^{n} b_{j}^{2} \\
\mathbf{E}\left|X_{j}\right|^{3} \leq \frac{10 \sigma_{j}^{6}}{\left(1+\sigma_{j}^{2}\right)^{3}} \leq \frac{10 \sigma_{j}^{4}}{\left(1+\sigma_{j}^{2}\right)^{2}}=5 b_{j}^{2}, \quad L_{n} \leq 5 B_{n}^{-1 / 2}
\end{gathered}
$$

Therefore

$$
\alpha(A, \boldsymbol{\sigma})=\mathbf{P}\left(\sum_{i=1}^{n} \frac{\sigma_{i}^{2} \xi_{i}^{2}}{1+\sigma_{i}^{2}}>D(\boldsymbol{\sigma})+A\right)=\mathbf{P}\left(\sum_{i=1}^{n} X_{i}>D(\boldsymbol{\sigma})+A-T(\boldsymbol{\sigma})\right)
$$

and then we get

$$
|\alpha(A, \boldsymbol{\sigma})-Q(x)| \leq 5 B_{n}^{-1 / 2}, \quad x=B_{n}^{-1 / 2}[D(\boldsymbol{\sigma})+A-T(\boldsymbol{\sigma})]
$$

where

$$
Q(x)=\frac{1}{\sqrt{2 \pi}} \int_{x}^{\infty} e^{-u^{2} / 2} d u \leq \min \left\{\frac{1}{2}, \frac{1}{x \sqrt{2 \pi}}\right\} e^{-x^{2} / 2}, \quad x>0
$$

In particular,

$$
\begin{align*}
\alpha(A, \boldsymbol{\sigma}) & \leq \frac{5}{\sqrt{B}}+\min \left\{\frac{1}{2}, \frac{1}{z} \sqrt{\frac{B}{2 \pi}}\right\} e^{-z^{2} /(2 B)},  \tag{42}\\
z & =D(\boldsymbol{\sigma})+A-T(\boldsymbol{\sigma})>0
\end{align*}
$$

where $B=B(\boldsymbol{\sigma})$ is defined in (22). It follows from (42)
Proposition 4. If $A \geq T(\boldsymbol{\sigma})-D(\boldsymbol{\sigma})+\sqrt{B(\ln B-\ln \ln B)}$, then

$$
\begin{equation*}
\alpha(A, \boldsymbol{\sigma}) \leq \frac{6}{\sqrt{B(\boldsymbol{\sigma})}}, \quad B=B(\boldsymbol{\sigma}) \tag{43}
\end{equation*}
$$

The estimate (43) quite accurately shows dependence of the value $\alpha(A, \boldsymbol{\sigma})$ on $B(\boldsymbol{\sigma})$ (for large $B$ ), if the given value $\alpha>5 B^{-1 / 2}$. Usually, $B(\boldsymbol{\sigma}) \sim n$.
2. Large deviations. Upper bound. Since

$$
\begin{equation*}
\alpha(A, \boldsymbol{\sigma})=\mathbf{P}\left(\sum_{i=1}^{n} r_{i}^{2} \xi_{i}^{2}>D(\boldsymbol{\sigma})+A\right), \quad r_{i}^{2}=\frac{\sigma_{i}^{2}}{1+\sigma_{i}^{2}} \tag{44}
\end{equation*}
$$

then for any $t \geq 0$ similarly to (30), (31) we have

$$
\alpha(A, \boldsymbol{\sigma}) \leq e^{-t[D(\boldsymbol{\sigma})+A] / 2} \mathbf{E} e^{t \sum_{i=1}^{n} r_{i}^{2} \xi_{i}^{2} / 2}=e^{-f_{\boldsymbol{\sigma}}(t)}
$$

where

$$
\begin{gather*}
2 f_{\boldsymbol{\sigma}}(t)=t[D(\boldsymbol{\sigma})+A]+\sum_{i=1}^{n} \ln \left(1-t r_{i}^{2}\right), \quad 2 f_{\boldsymbol{\sigma}}^{\prime}(t)=D(\boldsymbol{\sigma})+A-\sum_{i=1}^{n} \frac{r_{i}^{2}}{1-t r_{i}^{2}}  \tag{45}\\
f_{\boldsymbol{\sigma}}^{\prime \prime}(t)<0
\end{gather*}
$$

Since both conditions (9) supposed to be satisfied then $f_{\boldsymbol{\sigma}}^{\prime}(0)>0$ and $f_{\boldsymbol{\sigma}}^{\prime}(1)<0$. Therefore $\max _{t \geq 0} f_{\boldsymbol{\sigma}}(t)$ is attained for $0<t_{0}<1$, which is determined by the equation

$$
\sum_{i=1}^{n} \frac{r_{i}^{2}}{1-t_{0} r_{i}^{2}}=\sum_{i=1}^{n} \frac{\sigma_{i}^{2}}{1+\left(1-t_{0}\right) \sigma_{i}^{2}}=D(\boldsymbol{\sigma})+A
$$

Then

$$
\begin{equation*}
\alpha(A, \boldsymbol{\sigma}) \leq e^{-f_{\boldsymbol{\sigma}}\left(t_{0}\right)} \tag{46}
\end{equation*}
$$

For $t=1$ we have $f_{\boldsymbol{\sigma}}(1)=A / 2$, from which the estimate follows

$$
\begin{equation*}
\alpha(A, \boldsymbol{\sigma}) \leq e^{-f_{\boldsymbol{\sigma}}(1)}=e^{-A / 2} \tag{47}
\end{equation*}
$$

Simple estimate (47) is sufficiently accurate, if $t_{0}$ is close to 1 (i.e. if all $\left\{\sigma_{i}^{2}\right\}$ are small).

## §6. One more example

Consider more complicated
Example 3. Let for a given $R>0$

$$
\mathcal{E}=\left\{\boldsymbol{\sigma} \geq \mathbf{0}: \sum_{i=1}^{n} \sigma_{i}^{2} \geq n R^{2}\right\}
$$

Then

$$
\mathcal{E}_{0}=\left\{\boldsymbol{\sigma} \geq \mathbf{0}: \sum_{i=1}^{n} \sigma_{i}^{2}=n R^{2}\right\} .
$$

Denote $\boldsymbol{\sigma}_{0}=(R, \ldots, R)$. Then

$$
D\left(\boldsymbol{\sigma}_{0}\right)=n \ln \left(1+R^{2}\right), \quad \min _{\boldsymbol{\sigma} \in \mathcal{E}} D(\boldsymbol{\sigma})=\ln \left(1+n R^{2}\right)
$$

and that minimum is attained for $\boldsymbol{\sigma}$, which has only one nonzero (equal to $R \sqrt{n}$ ) coordinate. Denote $\boldsymbol{\sigma}_{i}, i=1, \ldots, n$ all those vectors. For example, $\boldsymbol{\sigma}_{1}=(R \sqrt{n}, 0, \ldots, 0)$. Denote also

$$
\mathcal{E}_{1}=\left\{\boldsymbol{\sigma}_{i}, i=1, \ldots, n\right\} .
$$

We show that without any loss of quality (in the sense of asymptotic equality (13)) all set $\mathcal{E}$ can be replaced by the set $\mathcal{E}_{1}$ and get the same results as for one point $\boldsymbol{\sigma}_{1}$. Notice that it does not follow from the sufficient condition (38).

In order to show possibility of such reduction of the set $\mathcal{E}$ we use the likelihood ratio criteria with the set $\mathcal{E}_{1}($ see (24))

$$
\mathcal{A}\left(A, \mathcal{E}, \mathcal{E}_{1}\right)=\left\{\mathbf{y}: 2 \sup _{\boldsymbol{\lambda} \in \mathcal{E}_{1}} r(\mathbf{y}, \boldsymbol{\lambda}) \leq A\right\}
$$

Now, if $D\left(\boldsymbol{\sigma}_{1}\right)+A=\ln \left(1+n R^{2}\right)+A \geq 0$, then

$$
\begin{gather*}
\beta\left(A, \boldsymbol{\sigma}_{1}\right)=\mathbf{P}\left\{2 \max _{\boldsymbol{\lambda} \in \mathcal{E}_{1}} r(\mathbf{y}, \boldsymbol{\lambda}) \leq A \mid \boldsymbol{\sigma}_{1}\right\}= \\
=\mathbf{P}\left\{2 r\left(\mathbf{y}, \boldsymbol{\sigma}_{1}\right) \leq A \mid \boldsymbol{\sigma}_{1}\right\} \prod_{i=2}^{n} \mathbf{P}\left\{2 r\left(\mathbf{y}, \boldsymbol{\sigma}_{i}\right) \leq A \mid \boldsymbol{\sigma}_{1}\right\}=  \tag{48}\\
=\mathbf{P}\left\{\xi_{1}^{2} \leq \frac{D\left(\boldsymbol{\sigma}_{1}\right)+A}{n R^{2}}\right\} \prod_{i=2}^{n} \mathbf{P}\left\{\frac{n R^{2} \xi_{i}^{2}}{1+n R^{2}} \leq D\left(\boldsymbol{\sigma}_{1}\right)+A\right\} \leq \frac{\sqrt{D\left(\boldsymbol{\sigma}_{1}\right)+A}}{R \sqrt{n}}
\end{gather*}
$$

The estimate (48) gives the correct asymptotics in $n$, since for $n \rightarrow \infty$ and small $\alpha(A)$

$$
\begin{aligned}
\prod_{i=2}^{n} \mathbf{P}\left\{\frac{n R^{2} \xi_{i}^{2}}{1+n R^{2}} \leq\right. & \left.D\left(\boldsymbol{\sigma}_{1}\right)+A\right\} \sim \mathbf{P}^{n}\left\{\left|\xi_{1}\right| \leq \sqrt{D\left(\boldsymbol{\sigma}_{1}\right)+A}\right\} \approx \\
& \approx\left[1-\frac{\alpha(A)}{n}\right]^{n} \approx e^{-\alpha(A)}
\end{aligned}
$$

We also have (see estimates (62))

$$
\begin{aligned}
& \alpha(A)=\mathbf{P}\left\{2 \max _{\boldsymbol{\lambda} \in \mathcal{E}_{1}} r(\mathbf{y}, \boldsymbol{\lambda}) \geq A \mid \mathcal{H}_{0}\right\} \leq \mathbf{P}\left\{\max _{i=1, \ldots, n} \xi_{i}^{2} \geq D\left(\boldsymbol{\sigma}_{1}\right)+A\right\} \leq \\
& \leq n \mathbf{P}\left\{\xi_{1}^{2} \geq D\left(\boldsymbol{\sigma}_{1}\right)+A\right\} \leq \frac{n}{\sqrt{D\left(\boldsymbol{\sigma}_{1}\right)+A}} \exp \left\{-\frac{\left[D\left(\boldsymbol{\sigma}_{1}\right)+A\right]}{2}\right\} .
\end{aligned}
$$

To simplify formulas we set $A$ as follows

$$
A=2 \ln n-D\left(\boldsymbol{\sigma}_{1}\right)=2 \ln n-\ln \left(1+n R^{2}\right)
$$

Then

$$
\begin{equation*}
\alpha\left(\boldsymbol{\sigma}_{1}\right) \leq \frac{1}{\sqrt{2 \ln n}} \quad \text { and } \quad \beta\left(\boldsymbol{\sigma}_{1}\right) \leq \frac{\sqrt{2 \ln n}}{R \sqrt{n}} \tag{49}
\end{equation*}
$$

Consider the value $\beta(\boldsymbol{\lambda})$ for $\boldsymbol{\lambda} \in \mathcal{E}_{0}$. Denote

$$
z^{2}=\frac{2\left(1+n R^{2}\right) \ln n}{n R^{2}}
$$

Without loss of generality, assume $\lambda_{1} \leq \lambda_{2} \leq \ldots \leq \lambda_{n}$, and introduce an auxiliary level $\lambda_{0}, 0 \leq \lambda_{0} \leq z$ (level $\lambda_{0}$ will be defined below). Then we have

$$
\begin{equation*}
\ln \beta\left(A, \mathcal{E}_{1}, \boldsymbol{\lambda}\right)=\ln \mathbf{P}\left\{\max _{i=1, \ldots, n}\left[\left(1+\lambda_{i}^{2}\right) \xi_{i}^{2}\right] \leq z^{2}\right\}=B_{1}+B_{2}+B_{3} \tag{50}
\end{equation*}
$$

where

$$
\begin{gather*}
B_{1}=\ln \mathbf{P}\left\{\max _{\lambda_{i} \leq \lambda_{0}}\left[\left(1+\lambda_{i}^{2}\right) \xi_{i}^{2}\right] \leq z^{2}\right\}<0, \\
B_{2}=\ln \mathbf{P}\left\{\max _{\lambda_{0}^{2}<\lambda_{i}^{2} \leq z^{2}-1}\left[\left(1+\lambda_{i}^{2}\right) \xi_{i}^{2}\right] \leq z^{2}\right\}<0,  \tag{51}\\
B_{3}=\ln \mathbf{P}\left\{\max _{\lambda_{i}^{2}>z^{2}-1}\left[\left(1+\lambda_{i}^{2}\right) \xi_{i}^{2}\right] \leq z^{2}\right\}<0 .
\end{gather*}
$$

We estimate sequentially values $B_{1}, B_{2}, B_{3}$ from (51). For that purpose denote

$$
\begin{gather*}
n_{1}=\#\left\{\lambda_{i}: \lambda_{i} \leq \lambda_{0}\right\}, \quad n_{2}=\#\left\{\lambda_{i}: \lambda_{0}^{2}<\lambda_{i}^{2} \leq z^{2}-1\right\}, \\
n_{3}=\#\left\{\lambda_{i}: \lambda_{i}^{2}>z^{2}-1\right\},  \tag{52}\\
s_{1} n_{1}=\sum_{\left\{\lambda_{i} \leq \lambda_{0}\right\}} \lambda_{i}^{2}, \quad s_{2} n_{2}=\sum_{\left\{\lambda_{0}^{2}<\lambda_{i}^{2} \leq z^{2}-1\right\}} \lambda_{i}^{2}, \quad s_{3} n_{3}=\sum_{\left\{\lambda_{i}^{2}>z^{2}-1\right\}} \lambda_{i}^{2} .
\end{gather*}
$$

Using notations (52) and the inequality $\ln (1+z) \leq z$, for the value $B_{1}$ we have

$$
\begin{gather*}
B_{1}=\sum_{i=1}^{n_{1}} \ln \left[1-\mathbf{P}\left\{\left(1+\lambda_{i}^{2}\right) \xi_{i}^{2} \geq z^{2}\right\}\right] \leq  \tag{53}\\
\leq-\sum_{i=1}^{n_{1}} \mathbf{P}\left\{\left(1+\lambda_{i}^{2}\right) \xi_{i}^{2} \geq z^{2}\right\}=-2 \sum_{i=1}^{n_{1}} \mathbf{P}\left(\sqrt{1+\lambda_{i}^{2}} \xi \geq z\right)
\end{gather*}
$$

Consider first the variational task of minimization of the sum of any two terms from the right-hand side of (53)

$$
\mathbf{P}\left(\sqrt{1+\lambda_{i}^{2}} \xi \geq z\right)+\mathbf{P}\left(\sqrt{1+\lambda_{j}^{2}} \xi \geq z\right)
$$

over variables $\lambda_{i}, \lambda_{j}$ with a given sum $\lambda_{i}^{2}+\lambda_{j}^{2}=r^{2}-2 \geq 0$. Denoting by $u^{2}=1+\lambda_{i}^{2}$, $v^{2}=1+\lambda_{j}^{2}, u^{2}+v^{2}=r^{2}$, and by $f(u, z, r)$ that sum, we have

$$
f(u, z, r)=\mathbf{P}\left\{\xi \geq \frac{z}{u}\right\}+\mathbf{P}\left\{\xi \geq \frac{z}{v}\right\}=2\left[1-\Phi\left(\frac{z}{u}\right)-\Phi\left(\frac{z}{v}\right)\right], \quad v=\sqrt{r^{2}-u^{2}} \geq 1
$$

We are interested in $\min _{1 \leq u \leq \sqrt{r^{2}-1}} f(u, z, r)$. We have

$$
\begin{gathered}
f^{\prime}(u)=f_{u}^{\prime}+f_{v}^{\prime} v_{u}^{\prime}, \quad f_{u}^{\prime}=\frac{2 z e^{-z^{2} /\left(2 u^{2}\right)}}{\sqrt{2 \pi} u^{2}}, \quad f_{v}^{\prime}=\frac{2 z e^{-z^{2} /\left(2 v^{2}\right)}}{\sqrt{2 \pi} v^{2}}, \quad v_{u}^{\prime}=-\frac{u}{v}, \\
f^{\prime}(u)=\frac{2 z}{\sqrt{2 \pi}}\left\{\frac{e^{-z^{2} /\left(2 u^{2}\right)}}{u^{2}}-\frac{u e^{-z^{2} /\left(2 v^{2}\right)}}{v^{3}}\right\}=\frac{2 u}{\sqrt{2 \pi} z^{2}}\left[\exp \left\{t\left(\frac{z}{u}\right)\right\}-\exp \left\{t\left(\frac{z}{v}\right)\right\}\right],
\end{gathered}
$$

where we denoted

$$
t(x)=3 \ln x-\frac{x^{2}}{2}, \quad t^{\prime}(x)=\frac{3}{x}-x, \quad x>0
$$

The function $t(x)$ monotonically decreases for $x>\sqrt{3}$ and monotonically increases for $0<x<\sqrt{3}$. Without loss of generality we may assume that $z / v \leq z / u$, i.e. $2 u^{2} \leq r^{2}$. Therefore if $z / v=z / \sqrt{r^{2}-u^{2}} \geq z / \sqrt{r^{2}-1} \geq \sqrt{3}$, then $f^{\prime}(u) \leq 0, u \leq r / \sqrt{2}$, and then the minimum of the function $f(u, z, r)$ in $u$ is attained for $u=v$ (i.e. for $\lambda_{i}=\lambda_{j}$ ). To fulfill those conditions it is sufficient to set $r^{2}=z^{2} / 3+1$. As a result, we get that if among $\left\{\lambda_{i}\right\}$ there is a pair $\lambda_{i}, \lambda_{j}$ such that $\lambda_{i} \neq \lambda_{j}$ and $\lambda_{i}^{2}+\lambda_{j}^{2} \leq r^{2}-2=z^{2} / 3-1$, then $f(u, z, r)$ decreases if we replace each $\lambda_{i}^{2}, \lambda_{j}^{2}$ by their half-sum $\left(\lambda_{i}^{2}+\lambda_{j}^{2}\right) / 2$. Therefore we define the level $\lambda_{0}$ as follows

$$
\begin{equation*}
\lambda_{0}^{2}=z^{2} / 6-1 / 2 \tag{54}
\end{equation*}
$$

Continuing that process of maximization of the right-hand side of (53) we get that its maximum is attained when

$$
\lambda_{1}^{2}=\ldots=\lambda_{n_{1}}^{2} \leq \lambda_{0}^{2}
$$

For remaining $n-n_{1}$ components $\left\{\lambda_{i}\right\}$ we have

$$
\lambda_{i}>\lambda_{0}, \quad i=n_{1}+1, \ldots, n
$$

Therefore for the value $B_{1}$ from (53) we get for large $n$ and some $C>0$ (see estimates (62))

$$
\begin{equation*}
B_{1} \leq-2 n_{1} \mathbf{P}\left(\sqrt{1+\lambda_{1}^{2}} \xi \geq z\right) \leq-\frac{C n_{1}}{z} e^{-\frac{z^{2}}{2\left(1+\lambda_{1}^{2}\right)}} \leq-\frac{C n^{\frac{\lambda_{1}^{2}}{1+\lambda_{1}^{2}}}}{\sqrt{\ln n}} \tag{55}
\end{equation*}
$$

since $\left(n-n_{1}\right) \lambda_{0}^{2} \leq R^{2} n$ and then $n_{1} \geq n\left(1-R^{2} / \lambda_{0}^{2}\right)$.
For the value $B_{2}$ from (51) we get

$$
\begin{equation*}
B_{2} \leq n_{2} \ln \mathbf{P}\left\{\left(1+\lambda_{0}^{2}\right) \xi^{2} \leq z^{2}\right\} \leq n_{2} \ln \mathbf{P}\{|\xi| \leq \sqrt{6}\} \leq-\frac{n_{2}}{100} \tag{56}
\end{equation*}
$$

Now we estimate the value $B_{3}$ from (51). We have

$$
\begin{equation*}
B_{3}=\sum_{\lambda_{i}^{2}>z^{2}-1} \ln \left(2 \mathbf{P}\left\{0 \leq \xi_{i} \leq \frac{z}{\sqrt{1+\lambda_{i}^{2}}}\right\}\right) \leq-\frac{n_{3}}{2} \ln \frac{\pi}{2}-\frac{1}{2} I_{3}, \tag{57}
\end{equation*}
$$

where

$$
I_{3}=\sum_{\left\{\lambda_{i}^{2} \geq z^{2}-1\right\}} \ln \frac{1+\lambda_{i}^{2}}{z^{2}}
$$

Consider the value $I_{3}$ for given $s_{3}$ and $n_{3}$. Since the function $I_{3}$ is $\bigcap$-concave in $\left\{\lambda_{i}^{2}\right\}$, its minimum is attained at an extreme point, i.e. when one of coordinates $\lambda_{j}^{2}$ equals $s_{3} n_{3}-\left(n_{3}-1\right)\left(z^{2}-1\right)$, and all remaining coordinates $\lambda_{i}^{2}$ equal $z^{2}-1$. Hence

$$
\begin{equation*}
I_{3} \geq \ln \frac{1+s_{3} n_{3}-\left(n_{3}-1\right)\left(z^{2}-1\right)}{z^{2}}=\ln \frac{z^{2}+n_{3}\left(s_{3}-z^{2}+1\right)}{z^{2}} \tag{58}
\end{equation*}
$$

Therefore from (50) and (55)-(58) we get for large $n$

$$
\begin{equation*}
\ln \beta\left(A, \mathcal{E}_{1}, \boldsymbol{\lambda}\right) \leq-\frac{C n^{\frac{\lambda_{1}^{2}}{1+\lambda_{1}^{2}}}}{\sqrt{\ln n}}-\frac{n_{2}}{100}-\frac{n_{3}}{5}-\frac{1}{2} \ln \frac{z^{2}+n_{3}\left(s_{3}-z^{2}+1\right)}{z^{2}} \tag{59}
\end{equation*}
$$

It remains to show that the right-hand side of (59) satisfies the inequality

$$
\begin{align*}
\min _{n_{2}, n_{3}, \lambda_{1}}\left\{\frac{C n^{\frac{\lambda_{1}^{2}}{1+\lambda_{1}^{2}}}}{\sqrt{\ln n}}\right. & \left.+\frac{n_{2}}{100}+\frac{n_{3}}{5}+\frac{1}{2} \ln \frac{z^{2}+n_{3}\left(s_{3}-z^{2}+1\right)}{z^{2}}\right\} \geq  \tag{60}\\
& \geq \frac{1}{2} \ln \left(R^{2} n\right)+o\left(\ln \left(R^{2} n\right)\right),
\end{align*}
$$

where minimum is taken provided $n_{2} \lambda_{0}^{2}+n_{3} s_{3} \geq R^{2} n+o\left(R^{2} n\right)$.
We may assume that (see (49) and (55))

$$
n_{2}<50 \ln \left(R^{2} n\right), \quad n_{3}<3 \ln \left(R^{2} n\right) \quad \text { и } \quad \lambda_{1}^{2}<\frac{2 \ln \ln n}{\ln n}+\frac{2 \ln R}{\ln ^{2} n}
$$

(otherwise the inequality (60) holds). In other words, almost all power $R^{2} n$ is distributed on last $n_{3}$ components. Hence

$$
n_{3} s_{3}=R^{2} n-n_{1} \lambda_{1}^{2}-n_{2} z^{2}=R^{2} n+o(n)
$$

Then the inequality (60) holds and therefore for any $\boldsymbol{\lambda} \in \mathcal{E}$ we get as $n \rightarrow \infty$

$$
\begin{equation*}
\ln \beta\left(A, \mathcal{E}_{1}, \boldsymbol{\lambda}\right) \leq-\frac{1}{2} \ln \left(R^{2} n\right)+o\left(\ln \left(R^{2} n\right)\right)=(1+o(1)) \ln \beta\left(\boldsymbol{\sigma}_{1}\right) \tag{61}
\end{equation*}
$$

The relation (61) means that the likelihood ratio criteria with the set $\mathcal{E}_{1}$ allows to get for the whole set $\mathcal{E}$ the same results as for the single point $\boldsymbol{\sigma}_{1}$.

APPENDIX

1. Tails of $\mathcal{N}(0,1)$. Let $\xi \sim \mathcal{N}(0,1)$. Then the following estimates are known

$$
\begin{equation*}
\frac{z e^{-z^{2} / 2}}{\left(z^{2}+1\right) \sqrt{2 \pi}} \leq \mathbf{P}\{\xi \geq z\} \leq \frac{e^{-z^{2} / 2}}{z \sqrt{2 \pi}}, \quad z>0 \tag{62}
\end{equation*}
$$

where the lower bound is derived via integration by parts.
2. Distribution $\chi^{2}$. Large deviations. Consider the value

$$
\begin{equation*}
\beta(A, n)=\mathbf{P}\left(\sum_{i=1}^{n} \xi_{i}^{2}<A\right) \tag{63}
\end{equation*}
$$

L e m m a 3. For $A \leq n$ and $n \geq 1$ the following estimates hold

$$
\begin{equation*}
-\frac{1}{2} \ln (\pi n)-\frac{1}{3 n} \leq \ln \beta(A, n)+\frac{1}{2}\left(n \ln \frac{n}{e A}+A\right) \leq 0 \tag{64}
\end{equation*}
$$

Proof. The right one of inequalities (64) follows from exponential Chebychev inequality (see (29) и (30)). To prove the left one of inequalities (64) denote

$$
\begin{equation*}
\mathcal{B}_{n}(r)=\left\{\mathbf{y}: \sum_{i=1}^{n} y_{i}^{2} \leq r^{2}\right\} . \tag{65}
\end{equation*}
$$

Then

$$
\begin{gathered}
\left|\mathcal{B}_{n}(r)\right|=\frac{\pi^{n / 2} r^{n}}{\Gamma(n / 2+1)}, \\
\ln \Gamma(z)=z \ln \frac{z}{e}-\frac{1}{2} \ln z+\frac{1}{2} \ln (2 \pi)+\frac{\theta}{6 z}, \quad z>0, \quad 0 \leq \theta \leq 1
\end{gathered}
$$

Therefore

$$
\beta(A, n)=\frac{1}{(2 \pi)^{n / 2}} \int_{0}^{\sqrt{A}} e^{-r^{2} / 2} d\left|\mathcal{B}_{n}(r)\right|=\frac{1}{\Gamma(n / 2)} \int_{0}^{A / 2} v^{n / 2-1} e^{-v} d v
$$

Integrating by parts, we have ( $a=n / 2-1, B=A / 2,0<\theta<1$ )

$$
\int_{0}^{B} v^{a} e^{-v} d v=\frac{B^{a+1} e^{-B}}{a+1}+\frac{1}{(a+1)} \int_{0}^{B} v^{a+1} e^{-v} d v=\frac{B^{a+1} e^{-B}}{a+1}+\frac{\theta B}{(a+1)} \int_{0}^{B} v^{a} e^{-v} d v
$$

Therefore

$$
\int_{0}^{B} v^{a} e^{-v} d v=\frac{B^{a+1} e^{-B}}{a+1-\theta B}, \quad 0<\theta<1
$$

Then

$$
\beta(A, n)=\frac{1}{\Gamma(n / 2)} \int_{0}^{A / 2} v^{n / 2-1} e^{-v} d v=\frac{2}{(n-\theta A) \Gamma(n / 2)}\left(\frac{A}{2}\right)^{n / 2} e^{-A / 2}
$$

Hence $\left(0 \leq \theta, \theta_{1} \leq 1\right)$

$$
\ln \beta(A, n)=-\frac{n}{2} \ln \frac{n}{e A}-\frac{A}{2}+\frac{1}{2} \ln \frac{n}{4 \pi}-\frac{\theta_{1}}{3 n}+\ln \frac{2}{n-\theta A},
$$

from which the left one of inequalities (64) follows.
Consider the value

$$
\begin{equation*}
\alpha(A, n)=\mathbf{P}\left(\sum_{i=1}^{n} \xi_{i}^{2}>A\right) . \tag{66}
\end{equation*}
$$

L em m a 4 . For $A \geq n$ and $n \geq 2$ the following estimates hold

$$
\begin{equation*}
-\frac{1}{3 n}-\frac{1}{2} \ln \frac{\pi A^{2}}{n} \leq \ln \alpha(A, n)+\frac{1}{2}\left(n \ln \frac{n}{e A}+A\right) \leq 0 \tag{67}
\end{equation*}
$$

Proof. The right one of inequalities (67) follows from exponential Chebychev inequality (see (44)-(46)). To prove the left one of inequalities (67), we have, using the notation (65)

$$
\alpha(A, n)=\frac{1}{(2 \pi)^{n / 2}} \int_{\sqrt{A}}^{\infty} e^{-r^{2} / 2} d\left|\mathcal{B}_{n}(r)\right|=\frac{1}{\Gamma(n / 2)} \int_{A / 2}^{\infty} v^{n / 2-1} e^{-v} d v
$$

Integrating by parts, we have for the last integral ( $a=n / 2-1, B=A / 2$ )

$$
\int_{B}^{\infty} v^{a} e^{-v} d v=B^{a} e^{-B}+\frac{\theta a}{B} \int_{B}^{\infty} v^{a} e^{-v} d v, \quad 0<\theta<1
$$

Therefore

$$
\int_{B}^{\infty} v^{a} e^{-v} d v=\frac{B^{a} e^{-B}}{1-\theta a / B}, \quad 0<\theta<1
$$

and then

$$
\alpha(A, n)=\frac{2}{A \Gamma(n / 2)}\left(\frac{A}{2}\right)^{n / 2} e^{-A / 2} \frac{1}{1-\theta(n-2) / A}, \quad 0<\theta<1
$$

As a result, for $A \geq n$ and $n \geq 2$ we get $\left(0<\theta, \theta_{1}<1\right)$

$$
\ln \alpha(A, n)=-\frac{n}{2} \ln \frac{n}{e A}-\frac{A}{2}-\frac{1}{2} \ln \frac{\pi}{n}-\frac{\theta_{1}}{3 n}-\ln [A-\theta(n-2)],
$$

from which the left one of inequalities (67) follows.
3. Large deviations for $\beta(A, \boldsymbol{\sigma})$. Lower bound. To estimate $\beta(A, \boldsymbol{\sigma})$ from below, we use the approach, similar to [8, proof of Theorem 1]. Let $\sigma_{1} \geq \sigma_{2} \geq \ldots \sigma_{n}$. We partition the segment $[1, n]$ onto $K$ equal parts of length $\Delta=(n-1) / K$ by points $n_{k}=1+\Delta k, 1 \leq k \leq K$, and represent $A$ as a sum $A=A_{1}+\ldots+A_{K}$. Then

$$
\begin{equation*}
\beta(A, \boldsymbol{\sigma})=\mathbf{P}\left(\sum_{i=1}^{n} \sigma_{i}^{2} \xi_{i}^{2}<A\right) \geq \max \prod_{k=1}^{K} \mathbf{P}\left(\sigma_{n_{k-1}+1}^{2} \sum_{i=n_{k-1}+1}^{n_{k}} \xi_{i}^{2}<A_{k}\right) \tag{68}
\end{equation*}
$$

where maximum is taken over all $K$ and $\left\{A_{k}\right\}$. To evaluate probabilities in the right-hand side of (68), we use the estimate (64). Denoting

$$
b_{k}=\sigma_{n_{k-1}+1}^{2}, \quad k=1, \ldots, K
$$

and assuming $A_{k} \leq b_{k} \Delta, k=1, \ldots, K$ (see (63)), we have from (64)

$$
\begin{align*}
& 2 \ln \beta(A, \boldsymbol{\sigma}) \geq 2 \max \sum_{k=1}^{K} \ln \mathbf{P}\left(\sum_{i=n_{k-1}+1}^{n_{k}} \xi_{i}^{2}<\frac{A_{k}}{b_{k}}\right) \geq  \tag{69}\\
\geq & -\min _{\left\{A_{k}\right\}} \sum_{k=1}^{K}\left(\Delta \ln \frac{b_{k}}{A_{k}}+\frac{A_{k}}{b_{k}}\right)-(n-1) \ln \frac{\Delta}{e}-K \ln (\pi \Delta) .
\end{align*}
$$

Minimum in the right-hand side of (69) provided $A=A_{1}+\ldots+A_{K}$ is attained for

$$
A_{k}=\frac{\Delta b_{k}}{1+u_{1} b_{k}}, \quad k=1, \ldots, K
$$

where $u_{1}$ is determined by the equation similar to (32)

$$
\begin{equation*}
\sum_{k=1}^{K} \frac{\Delta b_{k}}{1+u_{1} b_{k}}=A \tag{70}
\end{equation*}
$$

Since $\Delta b_{k} \geq A_{k}$ then $u_{1} \geq 0$. Moreover,

$$
\sum_{k=1}^{K}\left(\Delta \ln \frac{b_{k}}{A_{k}}+\frac{A_{k}}{b_{k}}\right)=\Delta \sum_{k=1}^{K} \ln \left(1+u_{1} b_{k}\right)-(n-1) \ln \frac{\Delta}{e}-u_{1} A
$$

Since for any $u \geq 0$

$$
\Delta \sum_{k=1}^{K} \ln \left(1+u b_{k+1}\right) \leq \sum_{i=1}^{n} \ln \left(1+u \sigma_{i}^{2}\right) \leq \Delta \sum_{k=1}^{K} \ln \left(1+u b_{k}\right)
$$

then using (31), we have

$$
\begin{gather*}
2 \ln \beta(A, \boldsymbol{\sigma})+K \ln \frac{\pi n}{K} \geq-\Delta \sum_{k=1}^{K} \ln \left(1+u_{1} b_{k}\right)+u_{1} A= \\
=-2 g_{\boldsymbol{\sigma}}\left(u_{0}\right)-\Delta \sum_{k=1}^{K} \ln \left(1+u_{1} b_{k}\right)+\sum_{i=1}^{n} \ln \left(1+u_{0} \sigma_{i}^{2}\right)+\left(u_{1}-u_{0}\right) A \geq  \tag{71}\\
\geq-2 g_{\boldsymbol{\sigma}}\left(u_{0}\right)-\Delta \sum_{k=1}^{K} \ln \frac{1+u_{1} b_{k}}{1+u_{1} b_{k+1}}+\left(u_{1}-u_{0}\right) A \geq \\
\geq-2 g_{\boldsymbol{\sigma}}\left(u_{0}\right)-\Delta \ln \frac{1+u_{1} b_{1}}{1+u_{1} b_{n}}=-2 g_{\boldsymbol{\sigma}}\left(u_{0}\right)-\Delta \ln \frac{b_{1}}{b_{n}},
\end{gather*}
$$

where the inequality $u_{1} \geq u_{0}$ was used. Indeed, from the formula (70) we have

$$
\left(u_{1}\right)_{b_{k}}^{\prime}=\frac{1}{b_{k}^{2}} \geq 0, \quad k=1, \ldots, K
$$

and since $\sigma_{1} \geq \ldots \geq \sigma_{n}$, we get that $u_{1} \geq u_{0}$. Therefore denoting

$$
\delta_{\boldsymbol{\sigma}}=\ln \frac{\max _{i} \sigma_{i}^{2}}{\min _{i} \sigma_{i}^{2}} \geq 0
$$

from (71) we get

$$
\ln \beta(A, \boldsymbol{\sigma})+g_{\boldsymbol{\sigma}}\left(u_{0}\right) \geq-\frac{1}{2} \min _{K \geq 1}\left\{\frac{n \delta_{\boldsymbol{\sigma}}}{K}+K \ln (\pi n)\right\}=-\left[n \delta_{\boldsymbol{\sigma}} \ln (\pi n)\right]^{1 / 2}
$$

provided that maximizing $K=K_{0} \geq 1$, where

$$
K_{0}^{2}=\frac{n \delta_{\boldsymbol{\sigma}}}{\ln (\pi n)}
$$

If $K_{0}<1$ (i.e. $\sigma_{1}^{2} / \sigma_{n}^{2}$ is close to 1 ), then setting $K=1$ we get

$$
\ln \beta(A, \boldsymbol{\sigma})+g_{\boldsymbol{\sigma}}\left(u_{0}\right) \geq-\ln (\pi n)
$$

Both cases $K_{0} \geq 1, K_{0}<1$, and the formula (33) can be combined as follows

$$
\begin{equation*}
-\sqrt{\delta_{\boldsymbol{\sigma}} n \ln (\pi n)}-\ln (\pi n) \leq \ln \beta(A, \boldsymbol{\sigma})+g_{\boldsymbol{\sigma}}\left(u_{0}\right) \leq 0 \tag{72}
\end{equation*}
$$

Notice that usually $g_{\boldsymbol{\sigma}}\left(u_{0}\right) \sim n$. Then (72) gives the right logarithmic asymptotics for $\beta(A, \boldsymbol{\sigma})$, if $\delta_{\boldsymbol{\sigma}}=o(n / \ln n), n \rightarrow \infty$.

As a result, we get
Proposition 5. 1) For the value $\ln \beta(A, \boldsymbol{\sigma})$ upper and lower bounds (72) hold.
2) If $g_{\boldsymbol{\sigma}}\left(u_{0}\right) \leq g_{\boldsymbol{\lambda}}\left(u_{0}\right)$ (for example, the sufficient condition (39) is fulfilled), then

$$
\begin{equation*}
\ln \beta(A, \boldsymbol{\lambda}) \leq \ln \beta(A, \boldsymbol{\sigma})+\sqrt{\delta_{\boldsymbol{\sigma}} n \ln (\pi n)}+\ln (\pi n) . \tag{73}
\end{equation*}
$$

The formula (73) follows from (72):

$$
\ln \beta(A, \boldsymbol{\lambda}) \leq-g_{\boldsymbol{\lambda}}\left(u_{0}\right) \leq-g_{\boldsymbol{\sigma}}\left(u_{0}\right) \leq \ln \beta(A, \boldsymbol{\sigma})+\sqrt{\delta_{\boldsymbol{\sigma}} n \ln (\pi n)}+\ln (\pi n) .
$$

Similarly the lower bound for $\alpha(A, \boldsymbol{\sigma})$ can be derived.

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