# Polar Codes with Higher-Order Memory 

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#### Abstract

We introduce the design of a set of code sequences $\left\{\mathcal{C}_{n}^{(m)}: n \geqslant 1, m \geqslant 1\right\}$, with memory order $m$ and code-length $N=O\left(\phi^{n}\right)$, where $\phi \in(1,2]$ is the largest real root of the polynomial equation $F(m, \rho)=\rho^{m}-\rho^{m-1}-1$ and $\phi$ is decreasing in $m$. $\left\{\mathrm{e}_{n}^{(m)}\right\}$ is based on the channel polarization idea, where $\left\{\mathrm{e}_{n}^{(1)}\right\}$ coincides with the polar codes presented by Arkan in [1] and can be encoded and decoded with complexity $O(N \log N)$. $\left\{\mathrm{e}_{n}^{(m)}\right\}$ achieves the symmetric capacity, $I(W)$, of an arbitrary binary-input, discrete-output memoryless channel, $W$, for any fixed $m$ and its encoding and decoding complexities decrease with growing $m$. We obtain an achievable bound on the probability of block-decoding error, $P_{e}$, of $\left\{\mathrm{e}_{n}^{(m)}\right\}$ and showed that $P_{e}=$ $O\left(2^{-N^{\beta}}\right)$ is achievable for $\beta<\frac{\phi-1}{1+m(\phi-1)}$.


Index Terms-Channel polarization, polar codes, capacityachieving codes, method of types, successive cancellation decoding

## I. Introduction and Overview

Channel polarization [1] is a method to achieve the symmetric capacity, $I(W)$, of an arbitrary binary-input, discreteoutput memoryless channel (B-DMC), $W$. By applying channel combining and splitting operations [2], one transforms $N$ uses of $W$ into another set of synthesized binary-input channels. As $N$ increases, the symmetric capacities of the synthesized binary-input channels polarize as $I(W)$ fraction of them gets close to 1 and $1-I(W)$ fraction of them gets close to 0 . The resulting code sequences, called polar codes, have encoding and decoding complexities $O(N \log N)$, and their block error probabilities scale as $2^{-N^{\beta}}$ where $\beta<1 / 2$ is the exponent of the code [3].

Let $W: \mathcal{X} \rightarrow \mathcal{Y}$ denote a B-DMC with binary-input $x \in \mathcal{X}=\{0,1\}$ and arbitrary discrete-output $y \in \mathcal{Y}$. Considering Arikan's polar codes, let us write $W_{n}$ to denote the vector channel, $W_{n}: \mathcal{X}^{N} \rightarrow \mathcal{Y}^{N}, N=2^{n}, n \geqslant 1$, obtained at channel combining level $n$. The vector channel, $W_{n}$, is obtained from $W_{n-1}$ in a recursive manner where one first injects an independent realization of $W_{n-1}$, denoted as $\hat{W}_{n-1}$, and then combines the input of $W_{n-1}$ and $\hat{W}_{n-1}$ to obtain $W_{n}$, where the recursion starts with $W_{0}=W$. The injection of $\hat{W}_{n-1}$, in a way, creates $N / 2$ diversity paths for the $N / 2$ inputs of $W_{n-1}$, and this allows polarization which one sees in the synthesized binary-input channels obtained by splitting $W_{n}$. Consequently, at each combining level the code-

[^0]length doubles with respect to the previous step scaling as $N=2^{n}$.
With higher-order memory in channel polarization, let us write $N=N(n, m)$ to denote the code-length at channel combining level $n$ and memory parameter $m, m \geqslant 1$, which we assume to be fixed. The vector channel, $W_{n}$, is obtained by combining the inputs of $W_{n-1}$ with $\hat{W}_{n-m}$, where one chooses $W_{0}=W_{-1}=\ldots=W_{1-m}=W$ to initiate the recursion. The number of binary-inputs in $W_{n-1}$ and $\hat{W}_{n-m}$ are $N(n-1)$ and $N(n-m)$, respectively. In turn, with the controlled memory parameter, $m$, and at channel combining level $n$, one only injects $N(n-m)$ new diversity paths with $\hat{W}_{n-m}$, for the $N(n-1)$ inputs of $W_{n-1}$, to obtain $W_{n}$. Because $N(n-m)$ gets smaller compared to $N(n-1)$ as $m$ increases, it is possible to slow the speed at which one inject new channels to provide polarization. At first glance, it seems that increasing $m$ will decrease the polarization effect obtained after each combining and splitting stage, however it will also allow the code-length to increase less rapidly in $n$. In order to see this consider the code-length obeying the recursion
\[

$$
\begin{equation*}
N=N(n-1)+N(n-m), \quad n \geqslant 1, m \geqslant 1, \tag{1}
\end{equation*}
$$

\]

with initial conditions

$$
\begin{equation*}
N(0)=N(-1)=\ldots=N(1-m)=1, \quad m \geqslant 1 . \tag{2}
\end{equation*}
$$

As will be explained in the sequel, the code-length takes the form

$$
\begin{equation*}
N=O\left(\phi^{n}\right), \quad n \geqslant 1 \tag{3}
\end{equation*}
$$

where $\phi \in(1,2]$ is the largest real root of the $m$-th order polynomial equation

$$
\begin{equation*}
F(m, \rho)=\rho^{m}-\rho^{m-1}-1, \tag{4}
\end{equation*}
$$

and $\phi$ decreases with increasing $m$. Therefore, if we increase $m$, it will take more channel combining and splitting stages to reach a pre-defined code-length, where the ratio of injected diversity paths to existing paths in each combining stage will also decrease. The aim of this paper is to understand the effects of this trade-off on the polarization performance one can obtain at a fixed code-length $N$.

The original construction of polar codes by Arikan is closely related to the recursive construction of Reed-Muller codes based on the $2 \times 2$ kernel $\mathbf{F}_{2}=\left[\begin{array}{ll}1 & 0 \\ 1 & 1\end{array}\right]$. For these codes the encoding matrix, $\mathbf{G}_{N}$, is of the form $\mathbf{G}_{N}=\mathbf{F}_{2}^{\otimes n}$, where $\otimes$
denotes the Kronecker power, suitably defined in [1]. In [4] Korada et al. generalize the channel polarization idea where $\ell \geqslant 2$ independent uses of $W_{n-1}$ are arbitrarily combined to obtain $W_{n}$ and code-length scales as $N=\ell^{n}$. Although the channel combining mechanism is generalized to combining arbitrary numbers of $W_{n-1}$ to obtain $W_{n}$, this setup has also first order memory in the channel combining. The authors express the combining mechanism by an $\ell \times \ell$ polarization kernel $\mathbf{K}_{\ell}$. With an arbitrary $\mathbf{K}_{\ell}$, the encoding matrix takes the form $\mathbf{G}_{N}=\mathbf{K}_{\ell}^{\otimes n}$. The asymptotic polarization performance is characterized by the distance properties of the rows of $\mathbf{K}_{\ell}$. The encoding and decoding complexities of these polar codes increases with $l$ scaling as $O(l N \log N)$ and $O\left(\frac{2^{l}}{l} N \log N\right)$, respectively. Our work differs from [4] in the sense that by introducing higher-order memory we modify the channel combining process. Moreover the encoding matrix of polar codes with memory $m>1$ can not be obtained by applying Kronecker power to an arbitrary polarization kernel. As a result, one needs new mathematical tools to investigate $\beta$.

The contributions of this paper are as follows: $i$ ) We present a novel polar code family, $\left\{\mathcal{C}_{n}^{(m)}: n \geqslant, m \geqslant 1\right\}$, with codelength $N=O\left(\phi^{n}\right), \phi \in(1,2]$, and arbitrary but fixed memory parameter $m$. We show that $\left\{\mathcal{C}_{n}^{(m)}\right\}$ achieves the symmetric capacity of arbitrary BDMCs for any choice of $m$ which complements Arıkan's conjecture that channel polarization is in fact a general phenomenon. ii) By developing a new mathematical framework, we obtain an asymptotic bound on the achievable exponent, $\beta$, of $\left\{\mathrm{e}_{n}^{(m)}\right\}$. iii) We show that the encoding and decoding complexities of $\left\{\mathcal{C}_{n}^{(m)}\right\}$ decrease with increasing $m .\left\{\mathcal{C}_{n}^{(m)}\right\}$ is the first example of a polar code family that has lower complexity compared to the original codes presented by Arıkan.

The outline of the paper is a as follows. Section $I$ provides the necessary material for the analysis in the sequel. In Section [III we explain the design, encoding and the decoding of $\left\{\mathcal{C}_{n}^{(m)}\right\}$. In Section IV we develop a probabilistic framework to investigate $\left\{\mathcal{C}_{n}^{(m)}\right\}$. After showing that $\left\{\mathrm{C}_{n}^{(m)}\right\}$ achieves the symmetric capacity of arbitrary B-DMCs we obtain an achievable bound on its block-decoding error probability. In Section V we analyze impact of higher-order memory on the encoding and decoding complexities of $\left\{\mathcal{C}_{n}^{(m)}\right\}$. Section VI concludes the paper and provides some future research directions.

Notation: We use uppercase letter $A, B$ for random variables and lower cases $a, b$ for their realizations taking values from sets $\mathcal{A}, \mathcal{B}$, where the sets have sizes $|\mathcal{A}|$ and $|\mathcal{B}|$ respectively. $\operatorname{Pr}(a)$ denotes the probability of the event $A=a$. We write $\mathbf{a}_{n}=\left(a_{1}, a_{2}, \ldots, a_{n}\right)$ to denote a vector and $\left(\mathbf{a}_{n}, \mathbf{b}_{n}\right)$ to denote the concatenation of $\mathbf{a}_{n}$ and $\mathbf{b}_{n}$. We use standard Landau notation $o(n), O(N)$ to denote the limiting values of functions. Note: Proofs, unless stated otherwise, are provided in the Appendix.

## II. Preliminaries

Let $W(y \mid x), x \in \mathcal{X}, y \in \mathcal{Y}$ denote the transition probabilities of $W$. Throughout the paper we assume that $x$ is uniformly distributed in $\mathcal{X}$, and use base- 2 logarithm. The symmetric capacity, $I(W)$, of $W$ is

$$
\begin{equation*}
I(W) \triangleq \sum_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} \frac{1}{2} W(y \mid x) \log \frac{W(y \mid x)}{\frac{1}{2} W(y \mid 0)+\frac{1}{2} W(y \mid 1)} \tag{5}
\end{equation*}
$$

The Bhattacharyya parameter, $Z(W)$, of $W$ provides an upper bound on the probability of error for maximum likelihood (ML) decoding over $W$ and is defined as

$$
\begin{equation*}
Z(W) \triangleq \sum_{y \in \mathcal{Y}} \sqrt{W(y \mid 0) W(y \mid 1)} \tag{6}
\end{equation*}
$$

The symmetric cut-off rate, $J(W)$, of $W$ is [1]

$$
\begin{equation*}
J(W) \triangleq \log \frac{2}{1+Z(W)} \tag{7}
\end{equation*}
$$

As Arıkan shows in [1, Prop. 1] $Z(W)=1$ implies $I(W)=0$ and $Z(W)=0$ implies $I(W)=1$. By using this fact and from (7) we see that if $J(W)=0$ then $I(W)=0$ holds and $J(W)=1$ indicates $I(W)=1$.

Let $W^{\prime}$ and $W^{\prime \prime}$ be two B-DMCs with inputs $x_{1}, x_{2} \in \mathcal{X}$ and outputs $y_{1} \in \mathcal{Y}_{1}$ and $y_{2} \in \mathcal{Y}_{2}$, respectively. Channel polarization is based on a single-step channel transformation where one first combines the inputs of $W^{\prime}$ and $W^{\prime \prime}$ to obtain a vector channel

$$
\begin{equation*}
W\left(y_{1}, y_{2} \mid x_{1}, x_{2}\right)=W^{\prime}\left(y_{1} \mid x_{1} \oplus x_{2}\right) W^{\prime \prime}\left(y_{2} \mid x_{2}\right) \tag{8}
\end{equation*}
$$

Next, by choosing a channel ordering, one splits the vector channel to obtain two new binary-input channels, $W^{-}: \mathcal{X} \rightarrow$ $\mathcal{Y}_{1} \times \mathcal{Y}_{2}$ and $W^{+}: \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{Y}_{1} \times \mathcal{Y}_{2}$, with transition probabilities

$$
\begin{align*}
W^{-}\left(y_{1}, y_{2} \mid x_{1}\right) & =\sum_{x_{2}} \frac{1}{2} W^{\prime}\left(y_{1} \mid x_{1} \oplus x_{2}\right) W^{\prime \prime}\left(y_{2} \mid x_{2}\right),  \tag{9}\\
W^{+}\left(y_{1}, y_{2}, x_{1} \mid x_{2}\right) & =\frac{1}{2} W^{\prime}\left(y_{1} \mid x_{1} \oplus x_{2}\right) W^{\prime \prime}\left(y_{2} \mid x_{2}\right), \tag{10}
\end{align*}
$$

We use the following short-hand notations for the transforms in (9) and (10), respectively.

$$
\begin{align*}
& W^{-}=W^{\prime} \boxminus W^{\prime \prime}  \tag{11}\\
& W^{+}=W^{\prime} \boxplus W^{\prime \prime} \tag{12}
\end{align*}
$$

The polarization transforms preserve the symmetric capacity as

$$
\begin{equation*}
I\left(W^{-}\right)+I\left(W^{+}\right)=I\left(W^{\prime}\right)+I\left(W^{\prime \prime}\right) \tag{13}
\end{equation*}
$$

and they help polarization by creating disparities in $I\left(W^{+}\right)$ and $I\left(W^{-}\right)$such that

$$
\begin{align*}
I\left(W^{+}\right) & \geqslant \max \left\{I\left(W^{\prime}\right), I\left(W^{\prime \prime}\right)\right\}  \tag{14}\\
I\left(W^{-}\right) & \leqslant \min \left\{I\left(W^{\prime}\right), I\left(W^{\prime \prime}\right)\right\} \tag{15}
\end{align*}
$$

where the above inequalities are strict as long as $I\left(W^{\prime}\right) \in$ $(0,1)$ and $I\left(W^{\prime \prime}\right) \in(0,1)$. This polarization effect quantitatively observed in the Bhattacharyya parameters as they take the form

$$
\begin{gather*}
Z\left(W^{+}\right)=Z\left(W^{\prime}\right) Z\left(W^{\prime \prime}\right)  \tag{16}\\
Z\left(W^{-}\right) \leqslant Z\left(W^{\prime}\right)+Z\left(W^{\prime \prime}\right)-Z\left(W^{\prime}\right) Z\left(W^{\prime \prime}\right) \tag{17}
\end{gather*}
$$

where the equality in 17 is achieved if $Z\left(W^{\prime}\right) \in\{0,1\}$ or $Z\left(W^{\prime \prime}\right) \in\{0,1\}$, or if $W^{\prime}$ and $W^{\prime \prime}$ are binary erasure channels (BECs).

Equations (13)-(17) are proved in [1] when $W^{\prime}$ is identical to $W^{\prime \prime}$. Their generalizations for the case $W^{\prime}$ and $W^{\prime \prime}$ are different channels are straightforward and omitted. The proposition below will be crucial in the sequel.

## Proposition 1.

$$
J\left(W^{-}\right)+J\left(W^{+}\right) \geqslant J\left(W^{\prime}\right)+J\left(W^{\prime \prime}\right)
$$

where equality is achieved only if $J\left(W^{\prime}\right) \in\{0,1\}$ or $J\left(W^{\prime \prime}\right) \in$ $\{0,1\}$.

The above proposition indicates that one can obtain coding gain by applying channel combining and splitting operations as long as the symmetric cut-off rate of $W^{\prime}$ and $W^{\prime \prime}$ is in $(0,1)$, where the coding gain manifests itself as an increase in the sum cut-off rate of channels $W^{-}$and $W^{-}$compared to $W^{\prime}$ and $W^{+}$. In this paper we use the parameters $J(W)$ and $I(W)$ together to show that $\left\{\mathrm{e}_{n}^{(m)}\right\}$ achieves $I(W)$ of an arbitrary $W$, whereas the parameter $Z(W)$ will be used to characterize polarization performance of $\left\{\mathrm{e}_{n}^{(m)}\right\}$.

## III. Polarization with Higher-Order Memory

We develop a method to design a family of code sequences $\left\{\mathcal{C}_{n}^{(m)} ; n \geqslant 1, m \geqslant 1\right\}$ with code-length $N=N(n, m)=$ $O\left(\phi^{n}\right), \phi \in(1,2]$, and fixed memory order $m$. $\left\{\mathfrak{C}_{n}^{(m)}\right\}$ is based on the channel polarization idea of Arrkan in [1]. This section is devoted to explaining the design, encoding and decoding of $\left\{\mathcal{C}_{n}^{(m)}\right\}$, while preparing some grounds for investigating its characteristics in the following sections.

## A. Channel Combining

Consider an arbitrary B-DMC, $W$, where its $N$ independent uses take the form $W\left(\mathbf{y}_{N} \mid \mathbf{x}_{N}\right)=\prod_{i=1}^{N} W\left(y_{i} \mid x_{i}\right), \mathbf{x}_{N} \in \mathcal{X}^{N}$, $\mathbf{y}_{N} \in \mathcal{Y}^{N}$. Let $\mathbf{u}_{N} \in \mathcal{X}^{N}$ be the binary information vector that needs to be transmitted over $N$ uses of $W$. Channel combining phase creates a vector channel $W_{n}: \mathcal{X}^{N} \rightarrow \mathcal{Y}^{N}$ of the form

$$
W_{n}\left(\mathbf{y}_{N} \mid \mathbf{u}_{N}\right)=\prod_{i=1}^{N} W\left(y_{i} \mid x_{i}\right)
$$

where $\mathbf{x}_{N}=\mathbf{u}_{N} \mathbf{G}_{N} . \mathbf{G}_{N}$ is an $N \times N$ encoding matrix where encoding takes place in GF(2).

Let $\mathbb{N}_{n}=\{1,2, \ldots, N\}, N=O\left(\phi^{n}\right)$, denote the set of the indices at the channel combining level $n$. There are $N$ binary-input channels in $W_{n}$ to transmit information. We index those channels as $W_{n}^{(i)}, i \in \mathbb{N}_{n}$, and demonstrate the channel combining operations in Fig 1 Inspecting this figure


Fig. 1: Recursive construction of the vector channel $W_{n}$ from $W_{n-1}$ and $\hat{W}_{n-m}$, where $W_{n}^{(i)}, i \in \mathbb{N}_{n}$, denotes the binary-input channels in $W_{n}$. The arrows on the left show the directions of flow for the binary-inputs of $W_{n}^{(i)}$ and $\oplus$ is the XOR operation. The arrows on the right show the outputs of successive uses of $W$. The XOR operations that take place on the dotted arrows within $W_{n-1}$ and $\hat{W}_{n-1}$ are not shown as they obey the same recursion.
observe that we index the topmost binary-input channel of $W_{n}$ as $W_{n}^{(1)}$ and index $i$ of $W_{n}^{(i)}$ increases as one move downwards. The vector channel $W_{n}$ is obtained by combining $W_{n-1}$ with $\hat{W}_{n-m}$. To accomplish this combining we apply XOR operations on the binary-inputs of $W_{n}$ and transmit the resultant bits through the inputs of $W_{n-1}$ and $\hat{W}_{n-m}$. By continuing the same recursion within $W_{n-1}$ and $\hat{W}_{n-m}$, the encoded bits are transmitted through independent uses of $W$ channels because we start the combining recursion by choosing $W_{0}=W_{-1}=\ldots=W_{1-m}=W$. If we use the binary-input channels $W_{n}^{(1)}, W_{n}^{(2)}, \ldots, W_{n}^{(N)}$ to transmit the symbols $u_{1}, u_{2}, \ldots, u_{N}$, respectively, the encoding matrix $\mathbf{G}_{N}$ can be expressed as

$$
\mathbf{G}_{N}=\left[\begin{array}{c|c}
\mathbf{G}_{N(n-1)} & \mathbf{G}_{N(n-m)}  \tag{18}\\
\hline \mathbf{0}_{2} \\
\hline \mathbf{0}_{1} & \mathbf{G}_{N(n-m)}
\end{array}\right], \quad n \geqslant 1
$$

where $\mathbf{G}_{N(0)}=\mathbf{G}_{N(-1)}=\ldots=\mathbf{G}_{N(1-m)}=[1]$, and $\mathbf{0}_{1}$ and $\mathbf{0}_{2}$ are $N(n-m) \times N(n-1)$ and $(N(n-1)-N(n-$ $m)) \times N(n-m)$ all zero matrices, respectively. Observe that when $m=1, \mathbf{0}_{2}$ matrix vanishes and $\mathbf{G}_{N}$ can be represented as $\mathbf{G}_{n}=\left(\mathbf{F}_{2}^{\mathbf{T}}\right)^{\otimes n}$, where $\mathbf{F}_{2}=\left[\begin{array}{cc}1 & 0 \\ 1 & 1\end{array}\right]$ is the Kernel used by Arıkan in [1]. However, when $m>1, \mathbf{G}_{N}$ can not be represented via Kronecker power.

## B. Channel Ordering

After performing channel combining operation we have to define an order to split the vector $W_{n}: \mathcal{X}^{N} \rightarrow \mathcal{Y}^{N}$ and obtain $N$ binary-input channels. This ordering is carried out with the help of a permutation $\pi_{n}: \mathbb{N}_{n} \rightarrow \mathbb{N}_{n}$. The $W_{n}^{(i)}$ channels in $W_{n}$ are split in increasing $\pi_{n}(i)$ values (from 1 to $N$ ) so that each $W_{n}^{(i)}$ channel is of the form $W_{n}^{(i)}: \mathcal{X} \rightarrow \mathcal{Y}^{N} \times \mathcal{X}^{\pi(i)-1}$. In order to explain this operation we associate a unique state vector $\mathbf{s}_{n}^{(i)}$ with each $W_{n}^{(i)}$ channel, which has the form

$$
\mathbf{s}_{n}^{(i)}=\left(s_{1}^{(i)}, s_{2}^{(i)}, \ldots, s_{n}^{(i)}\right)
$$

where

$$
\mathbf{s}_{k}^{(i)} \in\{+,-, \star\}, \quad k=1,2, \ldots, n
$$

$s_{k}^{(i)}$ terms will be referred as a "state" and we use,,$+- \star$ symbols to track down the channel transformations that $W_{n}^{(i)}$ channels undergo as $n=1,2, \ldots$. States + , - will correspond to the polarization transforms $\boxplus$ and $\square$, as defined in $\sqrt[91]{ }$ and (10), respectively; whereas state $\star$ will correspond to a nonpolarizing transform. We let

$$
\begin{equation*}
\mathcal{S}_{n}=\left\{\mathbf{s}_{n}^{(i)}: i \in \mathbb{N}_{n}\right\} \tag{19}
\end{equation*}
$$

to be the set of all possible state vectors at level $n$. Since each $\mathbf{s}_{n}^{(i)} \in \mathcal{S}_{n}$ is unique (as we will show shortly) we have $\left|\mathcal{S}_{n}\right|=N$ and $\mathcal{S}_{n} \subset\{+,-, \star\}^{n}$. The vectors, $\mathbf{s}_{n}^{(i)} \in \mathcal{S}_{n}$, are assigned recursively from $\mathbf{s}_{n-1}^{(j)} \in \mathcal{S}_{n-1}$, with a state assigning procedure $\varphi_{n}: \mathcal{S}_{n-1} \rightarrow \mathcal{S}_{n}$. The operation of $\varphi_{n}$ is explained in the following definition.
Definition 1. State Vector Assigning Procedure: Let $s_{n-1}^{(j)} \in$ $\mathcal{S}_{n-1}$ be the state vector of $W_{n-1}^{(j)}$. The state vectors $\boldsymbol{s}_{n}^{(i)} \in \mathcal{S}_{n}$, associated with $W_{n}^{(i)}$ take the form

$$
\begin{array}{rlrl}
\boldsymbol{s}_{n}^{(j)} & =\left(\boldsymbol{s}_{n-1}^{(j)},+\right), & & \\
\boldsymbol{s}_{n}^{(j+N(n-1))} & =\left(\boldsymbol{s}_{n-1}^{(j)},-\right),  \tag{21}\\
\boldsymbol{s}_{n}^{(j)} & =\left(\boldsymbol{s}_{n-1}^{(j)}, \star\right), & & j \in \mathbb{N}_{n-1} \backslash \mathbb{N}_{n-m} .
\end{array}
$$

Investigating the above definition, as also demonstrated in Fig. 2. we observe that $\varphi_{n}$ appends a new state, $\{+,-, \star\}$, to $\mathbf{s}_{n-1}^{(j)} \in \mathcal{S}_{n-1}$ in order to construct $\mathbf{s}_{n}^{(i)} \in \mathcal{S}_{n}$. For $j \in \mathbb{N}_{n-m}$, $\varphi_{n}$ appends + and - to $\mathbf{s}_{n-1}^{(j)}$ to obtain $\mathbf{s}_{n}^{(j)}$ and $\mathbf{s}_{n}^{\left(j+N_{n-1}\right)}$, respectively. For $j \in \mathbb{N}_{n-1} \backslash \mathbb{N}_{n-m}, \varphi_{n}$ appends $\star$ to $\mathbf{s}_{n-1}^{(j)}$ in order to construct $\mathbf{s}_{n}^{(j)}$. Because of the inherent memory in the combining procedure, it is difficult to obtain closed form expressions for $\mathbf{s}_{n}^{(i)}$, for any $i$ and $m$. Nevertheless, with the above definition one can recursively obtain $\mathbf{s}_{n}^{(i)}$, by applying


Fig. 2: State labeling procedure $\varphi_{n}: \mathcal{S}_{n-1} \rightarrow \mathcal{S}_{n}$. State vectors $\mathbf{s}_{n}^{(i)} \in \mathcal{S}_{n}$, are obtained by appending a new state $\{+,-, \star\}$, to the vectors $\mathbf{s}_{n-1}^{(j)} \in \mathcal{S}_{n-1}$.
$\varphi_{1}, \varphi_{2}, \ldots, \varphi_{n}$. With the following proposition, we give the formal structure of the possible state vector, $\mathbf{s}_{n}^{(i)}$, and thus the set $\mathcal{S}_{n}$.

Proposition 2. Let $\boldsymbol{s}_{n}, \boldsymbol{s}_{n} \in \mathcal{S}_{n}$, be a valid state vector one can obtain after applying $\varphi_{1}, \varphi_{2}, \ldots, \varphi_{n}$. Only the transitions between $s_{k}$ and $s_{k+1}, k=1,2, \ldots, n$, that are shown in the state transition diagram of Fig. 3 are possible, where the imposed initial condition is $s_{1} \in\{+,-\}$.

The above proposition is a direct consequence of the channel combining and state vector assigning procedure, $\varphi_{n}$, and it can be verified by induction through stages $\varphi_{1}, \varphi_{2}, \ldots, \varphi_{n}$.
Proposition 3. The state vector $s_{n}^{(i)} \in \mathcal{S}_{n}, i \in \mathbb{N}_{n}$, assigned to each $W_{n}^{(i)} \in \mathcal{W}_{n}$ is unique.

The above proposition will be crucial for the ongoing analysis as it states that each $W_{n}^{(i)}$ is uniquely addressable by $\mathbf{s}_{n}^{(i)}$. We will use this fact to obtain the ordering $\pi_{n}$. Before accomplishing this, we obtain binary vectors $\mathbf{b}_{n}^{(i)}=$ $\left(b_{1}^{(i)}, b_{2}^{(i)}, \ldots, b_{n}^{(i)}\right), b_{k}^{(i)} \in \mathcal{X}, k=1,2, \ldots, n$, from $\mathbf{s}_{n}^{(i)}$, which will allows us to sort and provide an order. The mapping between $\mathbf{s}_{n}^{(i)}$ and $\mathbf{b}_{n}^{(i)}$ is obtained as

$$
b_{k}^{(i)}=\left\{\begin{array}{lll}
0 & \text { if } & s_{k}^{(i)} \in\{-, \star\},  \tag{22}\\
1 & \text { if } & s_{k}^{(i)}=+,
\end{array} \quad k=1,2, \ldots, n\right.
$$

We notice that although both $s_{k}^{(i)}=-$ and $s_{k}^{(i)}=\star$ are mapped as $b_{k}^{(i)}=0$, the $\mathbf{b}_{n}^{(i)}$ vectors will also be unique for each $i$ because every state - in $\mathbf{s}_{n}^{(i)}$ is followed by $m-1$


Fig. 3: Possible state transitions observed between $s_{k}$ and $s_{k+1}, k=1,2, \ldots, n$.
occurrences of state $\star$, and the distinction between different $\mathbf{s}_{n}^{(i)}$ is hidden in the location of + states in $\mathbf{s}_{n}^{(i)}$. The following definition uses this uniqueness property to obtain the ordering, $\pi_{n}$. It is an adaptation of the bit-reversed order of Arıkan in [1] to the proposed coding scheme.

Definition 2. Bit-Reversed Order: Let $\left(\boldsymbol{b}_{n}^{(i)}\right)_{2}$ denote value of $\boldsymbol{b}_{n}^{(i)}$ in Mod-2 as $\left(b_{1}^{(i)}, b_{2}^{(i)}, \ldots, b_{n}^{(i)}\right)_{2}$ where $b_{1}^{(i)}$ is the most significant bit. The uniqueness of $\boldsymbol{b}_{n}^{(i)}$ for each $i$ ensures the existence of a permutation $\pi_{n}: \mathbb{N}_{n} \rightarrow \mathbb{N}_{n}$, so that for some $i, j \in \mathbb{N}_{n}$, we have $\pi_{n}(i)<\pi_{n}(j)$ if $\left(\boldsymbol{b}_{n}^{(i)}\right)_{2}<\left(\boldsymbol{b}_{n}^{(j)}\right)_{2}$.

Therefore the bit-reversed order $\pi_{n}$ is obtained in terms of increasing $\left(\mathbf{b}_{n}^{(i)}\right)_{2}$ values.

Notice that the binary input channels $\hat{W}_{n-m}^{(j)}, j \in \mathbb{N}_{n-m}$, of Fig. 1 have no effect in the recursive state assigning procedure, $\varphi_{n}$, and thus in the bit-reversed order. Their sole purpose is to provide auxiliary channels for the combining process. In fact, the $N(n-m)$ inputs of $\hat{W}_{n-m}$ can be combined with the $N(n-1)$ inputs of $\hat{W}_{n-1}$ in $\frac{N(n-1)!}{N(n-m)!}$ different ways. However, we deliberately align the inputs of $W_{n-1}$ and $\hat{W}_{n-m}$ so that the first $N(n-m)$ inputs of $W_{n-1}$ are combined, respectively, with the the first $N(n-m)$ inputs of $\hat{W}_{n-m}$ as shown in Fig. 1 . This alignment in the combining process will be crucial in the next section when we investigate the evolution of binary-input channels in a probabilistic setting, because the channel pairs, $W_{n-1}^{(j)}$ and $\hat{W}_{n-m}^{(j)}$, share the same state history as explained in the following proposition.
Proposition 4. Let $\boldsymbol{s}_{n-1}^{(j)}=\left(s_{1}, s_{2}, \ldots, s_{n-1}\right) \in \mathcal{S}_{n-1}$ be the state vector of $W_{n-1}^{(j)}$. Channel $\hat{W}_{(n-m)}^{(j)}$ shares the same state history with $W_{(n-1)}^{(j)}$, through combining stages $1,2, \ldots, n-m$, in the sense that its state vector is $\boldsymbol{s}_{n-m}^{(j)}=$ $\left(s_{1}, s_{2}, \ldots, s_{n-m}\right) \in \mathcal{S}_{n-m}$.

## C. Channel Splitting

We assume a genie-aided decoding mechanism where the $W_{n}^{(i)}$ channels are decoded successively in increasing $\pi_{n}(i)$ values, from 1 to $N$, and the genie provides the true values of already decoded bits. The decoder has no knowledge of the future bits that it will decode. With these assumptions $W_{n}^{(i)}$ is the effective bit-channel that this genie-aided decoder faces
while trying to decode its next bit. Let us define $u_{n}^{(i)} \in \mathcal{X}$ as

$$
u_{n}^{(i)}=\text { binary input of the channel } W_{n}^{(i)}
$$

and for $i, j \in \mathbb{N}_{n}$ let

$$
\begin{align*}
& \mathbf{u}_{n, b}^{(i)} \triangleq\left(u_{n}^{(j)}: \pi_{n}(j)<\pi_{n}(i)\right), \\
& \mathbf{u}_{n, a}^{(i)} \triangleq\left(u_{n}^{(j)}: \pi_{n}(j)>\pi_{n}(i)\right) . \tag{23}
\end{align*}
$$

$\mathbf{u}_{n, b}^{(i)}$ and $\mathbf{u}_{n, a}^{(i)}$ are the information vectors that are decoded, by the genie-aided decoder, before and after $u_{n}^{(i)}$, respectively. The length of $\mathbf{u}_{n, b}^{(i)}$ is $\pi_{n}(i)-1$ and the length of $\mathbf{u}_{n, a}^{(i)}$ is $N-\pi_{n}(i)$ so that $\mathbf{u}_{n, b}^{(i)} \in \mathcal{X}^{\pi_{n}(i)-1}$ and $\mathbf{u}_{n, a}^{(i)} \in \mathcal{X}^{N_{n}-\pi_{n}(i)}$. The following definition formalizes the transition probabilities of the $W_{n}^{(i)}$ channels.

$$
\begin{equation*}
W_{n}^{(i)} \triangleq \sum_{\substack{(i) \\ \mathbf{u}_{n, a}}} \operatorname{Pr}\left(\mathbf{y}_{N}, \mathbf{u}_{n, a}^{(i)}, \mathbf{u}_{n, b}^{(i)} \mid u_{n}^{(i)}\right) \tag{24}
\end{equation*}
$$

The above definition indicates that $W_{n}^{(i)}$ is the posterior probability of an arbitrary B-DMC obtained at channel combining and splitting level $n$. The genie-aided decoder has no knowledge of $\mathbf{u}_{n, a}^{(i)}$, therefore it averages the joint probability of all outputs and all inputs over $\mathbf{u}_{n, a}^{(i)}$ and takes $\mathbf{y}_{N}$ and $\mathbf{u}_{n, b}^{(i)}$ as the effective output (observation) of the combined channels. Hence each $W_{n}^{(i)}$ has input $u_{n}^{(i)} \in \mathcal{X}$ and output $\left(\mathbf{y}_{N}, \mathbf{u}_{n, b}^{(i)}\right) \in \mathcal{Y}^{N} \times \mathcal{X}^{\pi_{n}(i)-1}$.
Proposition 5. The transition probabilities of $W_{n}^{(i)}$ channels take the following forms

$$
\begin{align*}
W_{n}^{(j)} & =\hat{W}_{n-m}^{(j)} \boxplus W_{n-1}^{(j)}, \\
W_{n}^{\left(j+N_{n-1}\right)} & =\hat{W}_{n-m}^{(j)} \boxplus W_{n-1}^{(j)},  \tag{25}\\
W_{n}^{(j)} & =\gamma(n) W_{n-1}^{(j)}, \quad j \in \mathbb{N}_{n-m} \tag{26}
\end{align*} \quad j \in \mathbb{N}_{n-1} \backslash \mathbb{N}_{n-m}
$$

where $\gamma(n)=\operatorname{Pr}\left(y_{N(n-1)+1}, y_{N(n-1)+2}, \ldots, y_{N}\right)$ and $W_{0}=$ $W_{-1}=\ldots=W_{1-m}=W$.

The above proposition is illustrated in Fig. 4 In order to provide a proof for the above proposition and explain the underlying idea behind the bit-reversed order we make the following analysis. Investigating Fig. 4, we see that the overall effect of XOR operations, after channel splitting, is to provide diversity paths for the $N(n-m)$ inputs of $W_{n-1}$ in the sense that for $j \in \mathbb{N}_{n-m}$ we have $W_{n}^{(j)}=\hat{W}_{n-m}^{(j)} \boxplus W_{n-1}^{(j)}$. Therefore the input of $W_{n}^{(j)}$ is transmitted through both $\hat{W}_{n-m}^{(j)}$ and $\hat{W}_{n-m}^{(j)}$. Notice that in order to provide this diversity, the inputs of $W_{n}^{\left(j+N_{n-1}\right)}$ must be decoded, by the genie-aided decoder, before the inputs of $W_{n}^{(j)}$ indicating $\pi_{n}(j)>\pi_{n}(j+N(n-1))$ must hold. Thanks to the bit-reversed order, as explained in Definition. 2, this requirement can be easily accomplished. To see this consider the state vectors $\mathbf{s}_{n-1}^{(j)}$ of $W_{n-1}^{(j)}$ to which one appends + and - in order to construct $\mathbf{s}_{n}^{(j)}$ and $\mathbf{s}_{n}^{(j+N(n-1))}$, respectively. After this operation, the mapping between $\mathbf{s}_{n}^{(i)}$


Fig. 4: Transition probabilities of $W_{n}^{(i)}$ channels after combining and splitting $W_{n-1}$ and $\hat{W}_{n-m}$.
and $\mathbf{b}_{n}^{(i)}$, as given by 22, indicates that $\mathbf{b}_{n}^{(j)}=\left(\mathbf{b}_{n-1}^{(j)}, 1\right)$ and $\mathbf{b}_{n}^{(j+N(n-1))}=\left(\mathbf{b}_{n-1}^{(j)}, 0\right)$ holds. Therefore

$$
\left(\mathbf{b}_{n}^{(j)}\right)_{2}>\left(\mathbf{b}_{n}^{(j+N(n-1))}\right)_{2}, \quad n=1,2, \ldots
$$

and by Definition 2, $\pi_{n}(j)>\pi_{n}(j+N(n-1))$ holds for all $n \geqslant 1$. On the other hand, in order to decode $W_{n}^{\left(j+N_{n-1}\right)}$ correctly, the inputs of $W_{n-1}^{(j)}$ and $\hat{W}_{n-m}^{(j)}$ must be decoded correctly indicating we must have $W_{n}^{(j+N(n-1))}=\hat{W}_{n-m}^{(j)}$ ■ $W_{n-1}^{(j)}$. The above analysis, by induction through combining and splitting stages $1,2, \ldots, n$ proves (25). In order to prove (26), we inspect that for $j \in \mathbb{N}_{n-1} \backslash \mathbb{N}_{n-m}$ the channel $W_{n}^{(j)}$ is as good as $W_{n-1}^{(j)}$ in the sense that the genie-aided decoder can always decode $W_{n-1}^{(j)}$ instead of $W_{n}^{(j)}$. Inspecting Fig. 4 we notice that the binary-input of $W_{n}^{(j)}$ is not transmitted through the inputs of $\hat{W}_{n-m}$. Therefore, the combining of $\hat{W}_{n-m}$ with $W_{n-1}$ does not provide any new information regarding the input of $W_{n}^{(j)}$. This, in turn, indicates that $W_{n}^{(j)}$ is the same as $W_{n-1}^{(j)}$ except for a scaling factor $\gamma(n)$, as in 26.

## D. Effects of Channel Combining and Splitting on the Symmetric Capacity

Let us define $I_{n}^{(i)}=I\left(W_{n}^{(i)}\right)$ and analyze the implications of Proposition 5 Equation (25) states that the channel pairs,
$\hat{W}_{n-m}^{(j)}$ and $W_{n-1}^{(j)}, j \in \mathbb{N}_{n-m}$, undergo a polarization transform, $\square$ and $\boxplus$, from which two new channels, $W_{n}^{(j)}$ and $W_{n}^{\left(j+N_{n-1}\right)}$, emerge. In the light of (14) we have

$$
\begin{equation*}
I_{n}^{(j)} \geqslant \max \left\{I_{n-1}^{(j)}, I_{n-m}^{(j)}\right\}, \quad j \in \mathbb{N}_{n-m} . \tag{27}
\end{equation*}
$$

Therefore, the injection of $\hat{W}_{n-m}^{(j)}$ allows $W_{n}^{(j)}$ to be superior channel compared to $\hat{W}_{n-m}^{(j)}$ and $W_{n-1}^{(j)}$. This comes with the expense that now $W_{n}^{(j+N(n-1)}$ is an inferior channel compared to $\hat{W}_{n-m}^{(j)}$ and $W_{n-1}^{(j)}$ because, from (15), one has

$$
\begin{equation*}
I_{n}^{(j+N(n-1))} \leqslant \min \left\{I_{n-1}^{(j)}, I_{n-m}^{(j)}\right\}, \quad j \in \mathbb{N}_{n-m} \tag{28}
\end{equation*}
$$

Although $I_{n}^{(j)}$ and $I_{n}^{(j+N(n-1))}$ move away from $I_{n-1}^{(j)}$ and $I_{n-m}^{(j)}$, the transformations preserve the symmetric capacity because, as indicated by (13), we have

$$
\begin{equation*}
I_{n}^{(j)}+I_{n}^{(j+N n-1)}=I_{n-1}^{(j)}+I_{n-m}^{(j)}, \quad j \in \mathbb{N}_{n-m} \tag{29}
\end{equation*}
$$

The remaining channels $W_{n}^{(j)}, j \in \mathbb{N}_{n-1} \backslash \mathbb{N}_{n-m}$, in Equation (26, do not see any polarization transforms as their transition probabilities are scaled by $\operatorname{Pr}\left(y_{N(n-1)+1}, \ldots, y_{N}\right)$ with respect to $W_{n-1}^{(j)}$. This scaling, in turn, results in

$$
\begin{equation*}
I_{n}^{(j)}=I_{n-1}^{(j)}, \quad j \in \mathbb{N}_{n-1} \backslash \mathbb{N}_{n-m} \tag{30}
\end{equation*}
$$

All in all, the combining and splitting of $W_{n-1}$ and $W_{n-m}$ preserves the sum symmetric capacity as

$$
\begin{equation*}
\sum_{i \in \mathbb{N}_{n}} I_{n}^{(i)}=\sum_{j \in \mathbb{N}_{n-1}} I_{n-1}^{(j)}+\sum_{k \in \mathbb{N}_{n-m}} I_{n-m}^{(k)} \tag{31}
\end{equation*}
$$

## E. Decoding

We will take successive cancellation decoding (SCD) of [1] as the default decoding method for $\left\{\mathrm{C}_{n}^{(m)}\right\}$. The genieaided decoder that we have explained in Section $\Pi$ III $B$ and the definition of $W_{n}^{(i)}$ as given by (24) already provide us a guideline for SCD. The only difference is, during the calculation of 24, SCD uses its own estimates for the vector $\mathbf{u}_{n, b}^{(i)}$, which we denote as $\hat{\mathbf{u}}_{n, b}^{(i)}$.

Likelihood ratios (LRs) should be preferred in SCD so that one can eliminate the $P\left(y_{N_{n-1}+1}, y_{N_{n-1}+1}, \ldots, y_{N_{n}}\right)$ term in (26). The LR for the channel $W_{n}^{(i)}$ is defined as

$$
L_{n}^{(i)} \triangleq \frac{\sum_{\mathbf{u}_{n, a}^{(i)}} \operatorname{Pr}\left(\mathbf{y}_{N}, \mathbf{u}_{n, a}^{(i)}, \hat{\mathbf{u}}_{n, b}^{(i)} \mid 0\right)}{\sum_{\mathbf{u}_{n, a}^{(i)}} \operatorname{Pr}\left(\mathbf{y}_{N}, \mathbf{u}_{n, a}^{(i)}, \hat{\mathbf{u}}_{n, b}^{(i)} \mid 1\right)}
$$

By using the LR relations given in [1] for $\boxplus$ and $\square$ transformations and from Proposition 5 we obtain

$$
\begin{align*}
& L_{n}^{(j)}=L_{n-1}^{(j)}\left(L_{n-m}^{(j)}\right)^{1-2 \hat{u}_{n}^{\left(j+N_{n-1}\right)}}, \\
& L_{n}^{\left(j+N_{n-1}\right)}=\frac{L_{n-1}^{(j)} L_{n-m}^{(j)}+1}{L_{n-1}^{(j)}+L_{n-m}^{(j)}},  \tag{32}\\
& L_{n}^{(j)}=L_{n-1}^{(j)}, \quad j \in \mathbb{N}_{n-m},  \tag{33}\\
&
\end{align*}
$$

Therefore, while decoding $W_{n}^{(i)}$ one only needs to calculate $2 N(n-m)$ LRs as given by 32 while the remaining
$N-N(n-m)$ LRs for (33) are the same as the previous level. This fact can be exploited to avoid unnecessary decoding complexity in hardware implementation.

## F. Code-Length

Recall that the code-length $N=N(n, m)$ obeys the recursion in (1) with initial conditions of (2). It is easy to show that $N$ can be calculated as

$$
\begin{equation*}
N=\sum_{i=1}^{m} c_{i}\left(\rho_{i}\right)^{n} \tag{34}
\end{equation*}
$$

where each $\rho_{i}, i=1,2, \ldots, m$, is a root of the $m$ th order polynomial equation

$$
\begin{equation*}
F(m, \rho)=\rho^{m}-\rho^{m-1}-1 \tag{35}
\end{equation*}
$$

and constants, $c_{i}$, are calculated by using the initial conditions in (2) together with (34).
Proposition 6. For $m \geqslant 1$, let $\phi \in(1,2]$ be a real root of $F(m, \rho)$.
i) $\phi$ is unique, i.e., there is only one real root in $\in(1,2]$.
ii) If $\rho_{i} \neq \phi$ we have $\sqrt{\rho_{i} \rho_{i}^{*}} / \phi<1$ indicating $\phi$ is the the largest magnitude root of $F(m, \rho)$.
iii) $\phi$ is decreasing in increasing $m$.

Part $i i$ of the above proposition indicates that, as $n$ gets large, the summation in will be dominated by $\phi^{n}$ term therefore the code-length will scale as $N=\kappa \phi^{n}=O\left(\phi^{n}\right)$ where $\kappa>0$ is the constant scaler of $\phi^{n}$ in (34). Part iii of Proposition 6 implies that as $m$ increases the code-length increases less rapidly in $n$ which we have mentioned in the beginning of the paper.

## G. Code Construction

The following proposition is a generalization of [1] Prop. 5] and it's proof is omitted.
Proposition 7. If $W$ is a BEC, then $W_{n}^{(i)}$ channels obeying the transition probabilities as given by Proposition 5 are also BECs.

In order to use $\left\{\mathfrak{C}_{n}^{(m)}\right\}$ one has to fix a code parameter vector $(W, N, K, \mathcal{A})$, where $W$ is the underlying B-DMC, $N$ is the code-length, $K$ is the dimensionality of the code, and $\mathcal{A} \subseteq \mathbb{N}_{n}$ is the set of information carrying symbols. We have $|\mathcal{A}|=K$ and $K / N=R$, where $R \in[0,1]$ is the rate of the code.

Let $P_{e, n}^{(i)}, i \in \mathbb{N}_{n}$, denote the bit-error probability of $W_{n}^{(i)}$ with SCD. Code construction problem is choosing the set $\mathcal{A}$ so that $\sum_{i \in \mathcal{A}} P_{e, n}^{(i)}$ is minimum. This problem can be analytically solved only when $W$ is a BEC [1] since for this case the $W_{n}^{(i)}$ channels are also BECs (Proposition 7) and the Bhattacaryya parameters of $W_{n}^{(i)}$, which we denote as $Z_{n}^{(i)}$, obey $P_{e, n}^{(i)}=$ $Z_{n}^{(i)}$. In this case, in the light of 16-17) and Proposition 5 ,
$Z_{n}^{(i)}$ terms can be recursively calculated as

$$
\begin{aligned}
Z_{n}^{(j)} & =Z_{n-1}^{(j)} Z_{n-m}^{(j)}, \\
Z_{n}^{\left(j+N_{n-1}\right)} & =Z_{n-1}^{(j)}+Z_{n-m}^{(j)}-Z_{n-1}^{(j)}+Z_{n-m}^{(j)}, \quad j \in \mathbb{N}_{n-m}, \\
Z_{n}^{(j)} & =Z_{n-1}^{(j)} \quad j \in \mathbb{N}_{n-1} \backslash \mathbb{N}_{n-1} .
\end{aligned}
$$

The case when $W$ is not a BEC is a well-studied problem, where one approximates a suitable reliability measure for $W_{n}^{(i)}$ channels and uses this measure to choose the set $\mathcal{A}$. We refer the reader to [5] for an overview.

## IV. Channel Polarization

Channel polarization should be investigated by observing the evolution of the set $\left\{W_{n}^{(i)}: i \in \mathbb{N}_{n}\right\}$ as $n$ increases. To track this evolution we use the state vectors $\mathbf{s}_{n}^{(i)} \in \mathcal{S}_{n}$ assigned to $W_{n}^{(i)}$ because each $W_{n}^{(i)}$ is uniquely addressable by its $\mathbf{s}_{n}^{(i)}$.

## A. Probabilistic Model for Channel Evolution

We define a random process $\left\{S_{n}\right\}$ and a random vector $\mathbf{S}_{n}=\left(S_{1}, S_{2}, \ldots, S_{n}\right)$ obtained from the process $\left\{S_{n}\right\}$ where the state vectors, $\mathbf{s}_{n}=\left(s_{1}, s_{2}, \ldots, s_{n}\right), \mathbf{s}_{n} \in \mathcal{S}_{n}$, of Section II. are the realizations of $\mathbf{S}_{n}$. The process $\left\{S_{n}\right\}$ can be regarded as a tree process where $\mathbf{s}_{n}$ form the branches of the tree where we illustrate it in Fig. 5 for the case $m=2$. Since $\left|\mathcal{S}_{n}\right|=$ $N=N(n)$, there are $N(n)$ different branches at tree level $n$. The process $\left\{S_{n}\right\}$ starts with the initial conditions $S_{1} \in$ $\{+,-\}$. At tree level $n, N(n)$ new branches emerge from $N(n-1)$ branches of level $n-1$. We assume that each branch is observed with identical probability

$$
\begin{equation*}
\operatorname{Pr}\left(\mathbf{S}_{n}=\mathbf{s}_{n}\right)=\frac{1}{N(n)} \tag{36}
\end{equation*}
$$

This, in turn, implies that each valid state transition of Fig. 3 between $s_{n-1}$ and $s_{n}$, has probability $N(n-1) / N(n)$. Investigating this figure, consider the case $m=1$, which coincides with Arıkan's setup in [1], where there are two possible states as $S_{n} \in\{+,-\}$ and $\left|\mathcal{S}_{n}\right|=N(n)=2^{n}$. Since transitions between $S_{n-1}$ and $S_{n}$ are valid if $S_{n} \in\{+,-\}$ and $S_{n-1} \in\{+,-\}$, each possible transition has probability $N(n-1) / N(n)=1 / 2$. Consequently, the process $\left\{S_{n}\right\}$ is composed of independent realizations of $\operatorname{Bernoulli}(1 / 2)$ random variables as $\operatorname{Pr}\left(S_{n}=+\right)=\operatorname{Pr}\left(S_{n}=-\right)=1 / 2$. On the other hand, when $m>1$, there exists a memory in the state transition model as depicted in Fig. 3. Therefore, the process $\left\{S_{n}\right\}$ can be modeled as a Markov process with order $m-1$ in the sense that

$$
\operatorname{Pr}\left(S_{n} \mid \mathbf{S}_{n-1}\right)=\operatorname{Pr}\left(S_{n} \mid S_{n-1}, S_{n-2}, \ldots, S_{n-(m-1)}\right)
$$

Throughout the paper we find it easier to work with the random vector $\mathbf{S}_{n}$ keeping in mind the Markovian property of the process $\left\{S_{n}\right\}$.

We define a random channel process $\left\{K_{n}\right\}$, driven by $\left\{S_{n}\right\}$, as $K_{n}=W_{S_{1}, S_{2}, \ldots, S_{n}}$. The realizations of $K_{n}$ are $k_{n}=W_{s_{1}, s_{2}, \ldots, s_{n}}$ and they correspond to the binary-input channels, $W_{n}^{(i)}$, with state vectors $\mathbf{s}_{n}=\left(s_{1}, s_{2}, \ldots, s_{n}\right) \in \mathcal{S}_{n}$.


Fig. 5: Illustration of the evolution of $\left\{S_{n}\right\}$ as a tree for the case $m=2$, where each branch is a state vector $\mathbf{s}_{n} \in \mathcal{S}_{n}$.

In order to obtain a characterization for the process $\left\{K_{n}\right\}$ we fix $\left(s_{1}, s_{2}, \ldots, s_{n-1}\right)$ to be the state vector associated with $W_{n-1}^{(j)}, j \in \mathbb{N}_{n-m}$ and let $k_{n-1}=W_{n-1}^{(j)}$. In the light of Proposition 4 , we know that the state vector of $\hat{W}_{n-m}^{(j)}$ is $\left(s_{1}, s_{2}, \ldots, s_{n-m}\right)$ indicating $k_{n-m}=\hat{W}_{n-m}^{(j)}$. Investigating the operation of $\varphi_{n}: \mathcal{S}_{n-1} \rightarrow \mathcal{S}_{n}$ in Fig. 2. we observe that the state vectors of $W_{n}^{(j)}$ and $W_{n}^{\left(j+N_{n-1}\right)}$ are $\left(s_{1}, s_{2}, \ldots, s_{n-1},+\right)$ and $\left(s_{1}, s_{2}, \ldots, s_{n-1},-\right)$, respectively. From Proposition 5 we notice that $W_{n}^{(j)}=\hat{W}_{n-m}^{(j)} \boxplus W_{n-1}^{(j)}$ and $W_{n}^{(j+N(n-1))}=W_{n-m}^{(j)} \boxminus W_{n-1}^{(j)}$ holds. These observations, in turn, indicate $k_{n}=k_{n-1} \boxplus k_{n-m}$ holds when $s_{n}=+$, and $k_{n}=k_{n-1} \boxminus k_{n-m}$ holds when $s_{n}=-$. Next, we fix $\left(s_{1}, s_{2}, \ldots, s_{n-1}\right)$ to be the state vector associated with $W_{n-1}^{(j)}, j \in \mathbb{N}_{n-1} \backslash \mathbb{N}_{n-m}$ and hence $k_{n-1}=W_{n-1}^{(j)}$. From the operation of $\varphi_{n}: \mathcal{S}_{n-1} \rightarrow \mathcal{S}_{n}$ we know that the state vector of $W_{n}^{(j)}$ is $\left(s_{1}, s, \ldots, s_{n-1}, \boldsymbol{\star}\right)$ and Proposition 5 tells us $W_{n}^{(j)}=\gamma(n) W_{n-1}^{(j)}$. Combining these facts tells us $k_{n}=\gamma(n) k_{n-1}$ holds if $s_{n}=\star$. The above analysis relates $k_{n}$ to $k_{n-1}$ and $k_{n-m}$ for all $s_{n} \in\{+,-, \star\}$, which we formally present with the below recursion.

$$
K_{n}= \begin{cases}K_{n-m} \boxplus K_{n-1} & \text { if } \quad S_{n}=+,  \tag{37}\\ K_{n-m} \boxminus K_{n-1} & \text { if } \quad S_{n}=- \\ \gamma(n) K_{n-1} & \text { otherwise },\end{cases}
$$

where $K_{n}=W$ for $n<1$.

## B. Polarization:

We define the processes $\left\{I_{n}: n \geqslant 1\right\}$ and $\left\{J_{n}: n \geqslant 1\right\}$ where $I_{n}=I\left(K_{n}\right) \in[0,1]$ and $J_{n}=J\left(K_{n}\right) \in[0,1]$. In [1] Arkan shows that $I_{n}$ converges to a random variable $I_{\infty}$ as $\operatorname{Pr}\left(I_{\infty}=1\right)=I(W)$ and $\operatorname{Pr}\left(I_{\infty}=0\right)=1-I(W)$. This result indicates that the synthesized binary-input channels, $W_{n}^{(i)}$, either become error-free or useless. We will show that the same holds for polar codes with higher-order memory as well. This result is presented with the following theorem.
Theorem 1. For any fixed $m \geqslant 1$ and for some $\delta \in(0,1)$ as $n$ tends to infinity, the probability of $I_{n} \in(1-\delta, 1]$ goes to $I(W)$ and the probability of having $I_{n} \in[0, \delta)$ goes to $1-I(W)$.

Proof: We investigate the polarization of $\left\{J_{n}\right\}$ towards 0 and 1 as it will imply the polarization of $\left\{I_{n}\right\}$ as well. We write $E\left[J_{n}\right]=\sum_{\mathbf{s}_{n}} \operatorname{Pr}\left(\mathbf{S}_{n}=\mathbf{s}_{n}\right) J_{n}=\frac{1}{N(n)} \sum_{\mathbf{s}_{n}} J_{n}$ to denote the expected value of $J_{n}$ and $\left\{E\left[J_{n}\right]: n \geqslant 1\right\}$ to denote the deterministic sequences obtained from $E\left[J_{n}\right]$. The following lemma will be crucial for the proof

## Lemma 1.

$$
\begin{equation*}
E\left[J_{n}\right] \geqslant \mu E\left[J_{n-1}\right]+(1-\mu) E\left[J_{n-m}\right], \tag{38}
\end{equation*}
$$

where $\mu=N(n-1) / N(n)$ and the above equality is achieved only if $J_{n-1} \in\{0,1\}$ or $J_{n-m} \in\{0,1\}$ holds for all $S_{n} \in$ $\{+,-\}$

We apply a decimation operation on the sequence $\left\{E\left[J_{n}\right]\right\}$ and obtain a subsequence $\left\{E\left[\hat{J}_{k}\right]: k=1,2, \ldots,\lfloor n / m]\right\}$, where the decimation operation is performed as

$$
\begin{equation*}
E\left[\hat{J}_{k}\right]=\min _{i \in\{0,1, \ldots, m-1\}}\left\{E\left[J_{k m-i}\right]\right\} \tag{39}
\end{equation*}
$$

The elements of $\left\{E\left[\hat{J}_{k}\right]\right\}$ are obtained by choosing the minimum of $m$ consecutive and non-overlapping elements of $\left\{E\left[J_{n}\right]\right\}$.
Lemma 2. The sequence $\left\{E\left[\hat{J}_{k}\right]\right\}$ is monotonically increasing in the sense that

$$
E\left[\hat{J}_{k}\right] \geqslant E\left[\hat{J}_{k-1}\right] .
$$

We know that $E\left[\hat{J}_{k}\right]$ is bounded in $[0,1]$ and since $\left\{E\left[\hat{J}_{k}\right]\right\}$ is monotonically increasing, from the monotone convergence theorem [6] p. 21.] we conclude that there exists a unique limit for $\left\{E\left[\hat{J}_{k}\right]\right\}$ in the sense that

$$
\begin{equation*}
\lim _{k \rightarrow \infty} E\left[\hat{J}_{k}\right]=\sup \left\{E\left[\hat{J}_{k}\right]\right\} . \tag{40}
\end{equation*}
$$

Next, we let $n=k m-i$ in Lemma 1 to obtain

$$
\begin{equation*}
E\left[J_{k m-i}\right] \geqslant \mu E\left[J_{k m-(i+1)}\right]+(1-\mu) E\left[J_{(k-1) m-i}\right] . \tag{41}
\end{equation*}
$$

We fix $i$ such that $E\left[J_{k m-i}\right]=E\left[\hat{J}_{k}\right]$ is satisfied. For any choice of $i$ observe that $E\left[J_{(k-1) m-i)}\right] \geqslant E\left[\hat{J}_{k-1}\right]$ and $E\left[J_{k m-(i+1)}\right] \geqslant \min \left\{E\left[\hat{J}_{k}\right], E\left[\hat{J}_{k-1}\right]\right\} \geqslant E\left[\hat{J}_{k-1}\right]$ hold. Using these results in (41) gives

$$
\begin{equation*}
E\left[\hat{J}_{k}\right] \geqslant \mu E\left[\hat{J}_{k-1}\right]+(1-\mu) E\left[\hat{J}_{k-1}\right] \geqslant E\left[\hat{J}_{k-1}\right] \tag{42}
\end{equation*}
$$

Therefore, the monotonic increase in $E\left[\hat{J}_{k}\right]$ will continue until the inequality in Lemma 1 is achieved with equality. This fact, together with the convergence of $E\left[\hat{J}_{k}\right]$, indicates that conditioned on the event $\left\{S_{n}: S_{n} \in\{+,-\}\right\}$ either $\lim _{n \rightarrow \infty} J_{n-1} \in\{0,1\}$ or $\lim _{n \rightarrow \infty} J_{n-m} \in\{0,1\}$ holds, indicating

$$
\begin{equation*}
\lim _{n \rightarrow \infty} J_{n} \in\{0,1\}, \quad S_{n} \in\{+,-\} . \tag{43}
\end{equation*}
$$

Investigating the operation of $\varphi_{n}: \mathcal{S}_{n-1} \rightarrow \mathcal{S}_{n}$ in $\operatorname{Fig} 2$ we see that

$$
\begin{equation*}
\operatorname{Pr}\left(S_{n} \in\{+,-\}\right)=\frac{2 N(n-m)}{N(n)} \geqslant 0, \tag{44}
\end{equation*}
$$

which implies that the event $\left\{S_{n}: S_{n-1} \in\{+,-\}\right\}$ occurs infinitely many times as $n \rightarrow \infty$ and $\sum_{n \rightarrow \infty} \operatorname{Pr}\left(S_{n-1} \in\{+,-\}\right)$ diverges. Consequently, and by using the first Borel Contelli lemma [7] p. 36] we conclude that

$$
\lim _{n \rightarrow \infty} \operatorname{Pr}\left(J_{n} \in\{0,1\}\right)=1
$$

One to one correspondence between $J_{n}$ and $I_{n}$ implies

$$
\lim _{n \rightarrow \infty} \operatorname{Pr}\left(I_{n} \in\{0,1\}\right)=1
$$

and having $E\left[I_{n}\right]=I(W)$ results in

$$
\lim _{n \rightarrow \infty} \operatorname{Pr}\left(I_{n}=1\right)=I(W)
$$

and

$$
\lim _{n \rightarrow \infty} \operatorname{Pr}\left(I_{n}=0\right)=1-I(W)
$$

which completes the proof.

## C. A Typicality Result

In this section we use the Method of Types to investigate the state vectors, $\mathbf{s}_{n}$, obtained from the realizations of the process $\left\{S_{n}\right\}$. We let $s \in\{+,-, \star\}$ and write $P_{\mathbf{s}_{n}}^{(s)}, P_{\mathbf{s}_{n}}^{(s)} \in[0,1]$, to denote the type (frequency) of $s$ in $\mathbf{s}_{n}$ as

$$
P_{\mathbf{s}_{n}}^{(s)}=\#\left(\mathbf{s}_{n} \mid s\right) / n
$$

where $\#\left(\mathbf{s}_{n} \mid s\right)$ denotes the number times the symbol $s$ occurs in $\mathbf{s}_{n}$. Investigating the state transition diagram of Fig. 3 we inspect that, as $n$ gets large, $P_{\mathbf{s}_{n}}^{(\boldsymbol{\star})}=(m-1) P_{\mathbf{s}_{n}}^{(-)}$holds because each - state in $\mathbf{s}_{n}$ is followed by $m-1$ occurrences of state $\star$. As the remaining states in $\mathbf{s}_{n}$ will be + , we must have $P_{\mathbf{s}_{n}}^{(+)}=1-m P_{\mathbf{s}_{n}}^{(-)}$indicating $P_{\mathbf{s}_{n}}^{(+)} \in[0,1], P_{\mathbf{s}_{n}}^{(-)} \in\left[0, \frac{1}{m}\right]$, and $P_{\mathbf{s}_{n}}^{(\star)} \in\left[0, \frac{m-1}{m}\right]$. As it tuns out, depending on $P_{\mathbf{s}^{n}}^{(s)}$, not all realizations of $\left\{S_{n}\right\}$ are observed with the same probability. This is explained with the following theorem.

Theorem 2. As $n$ gets large, except for a vanishing fraction of $s_{n} \in \mathcal{S}_{n}$, and for some $\epsilon \in(0,1)$ we have

$$
\begin{aligned}
& \left|P_{s^{n}}^{(-)}-p^{-}\right| \leqslant \epsilon \\
& \left|P_{s^{n}}^{(+)}-p^{+}\right| \leqslant \epsilon \\
& \left|P_{s^{n}}^{(\star)}-p^{\star}\right| \leqslant \epsilon
\end{aligned}
$$

where $p^{-}=\frac{\phi-1}{1+m(\phi-1)}, p^{\star}=(m-1) p^{-}$and $p^{+}=1-m p^{-}$.
Therefore we can consider $p^{+}, p^{-}$and $p^{\star}$ as the frequencies of states,+- , and $\star$, in $\mathbf{s}_{n}$, respectively, that one typically observes as $n$ gets large.

Proof of Theorem 2: The proof is based on the Method of Types [8]. We let $q \in[0,1 / m]$ and define

$$
\begin{equation*}
\mathcal{T}_{n}^{(q)}=\left\{\mathbf{s}^{n}: P_{\mathbf{s}^{n}}^{(-)}=q\right\} \tag{45}
\end{equation*}
$$

$\mathcal{T}_{n}^{(q)}$ is a type class and it consists of $\mathbf{s}_{n}$ having $n q \in[0, n / m]$ occurrences of state - . For all $m \geqslant 1$, there are at most $n+1$ different such type classes. However, the number of all possible $\mathbf{s}_{n},\left|\mathcal{S}_{n}\right|$, increases exponentially in $n$ as $\left|\mathcal{S}_{n}\right|=N=$
$O\left(\phi^{n}\right)$. The Method of Types ensures the existence of a type class with exponentially many elements. Our aim is to find this type class. Recalling that each $\mathbf{s}_{n}$ is observed with probability $1 / N$, the probability of observing a given $\mathbf{s}_{n}$ in $\mathcal{T}_{n}^{(q)}$ is

$$
\operatorname{Pr}\left(\mathbf{s}_{n} \in \mathcal{T}_{q}^{n}\right)=\frac{\left|\mathcal{T}_{q}^{n}\right|}{N}
$$

## Lemma 3.

$$
\begin{equation*}
\left|\mathcal{T}_{q}^{n}\right|<2^{n(G(m, q)+o(1))} \tag{46}
\end{equation*}
$$

where

$$
G(m, q)=(1-(m-1) q)) H\left(\frac{q}{1-(m-1) q}\right)
$$

and $H$ is the binary entropy function.
Investigating $G(m, q)$ we observe that it is a concave function of $q \in[0,1 / m]$. We establish a similarity between $\frac{\partial G(m, q)}{\partial q}$ and $F(m, \rho)$ in (35). The following proposition is a direct consequence of this result.

Lemma 4. The function $G(m, q)$ attains its maximum when $q=p^{-}$and its maximum value is

$$
G\left(m, p^{-}\right)=\log \phi
$$

Consequently, for every $\mathcal{T}_{n}^{(q)}$ with $\left|q-p^{-}\right|>0$ there exists a $D\left(q, p^{-}\right)>0$ such that

$$
\begin{aligned}
D\left(q, p^{-}\right) & \triangleq G\left(m, p^{-}\right)-G(m, q) \\
& =\log \phi-G(m, q)
\end{aligned}
$$

Using the above fact in (46) results in

$$
\left|\mathcal{T}_{n}^{(q)}\right| \leqslant \phi^{n} 2^{n\left(-D\left(q, p^{-}\right)+o(1)\right)}
$$

From the above result and the fact that $N=O\left(\phi^{n}\right)$ we obtain

$$
\begin{equation*}
\operatorname{Pr}\left(\mathbf{s}_{n} \in \mathcal{T}_{n}^{(q)}\right) \leqslant 2^{-n\left(D\left(q, p^{-}\right)+o(1)\right)} \tag{47}
\end{equation*}
$$

The above result shows that depending on $D\left(q, p^{-}\right)$, and in turn $q$, the probabilities of some type classes decay exponentially in $n$. The following proposition results from this fact.
Proposition 8. As $n$ tends to infinity $D\left(q, p^{-}\right)$converges to 0 with probability 1.

The above proposition implies the convergence of $q$ to $p^{-}$ as well, because $D\left(q, p^{-}\right)$is 0 only if $q=p^{-}$. Therefore among all $T_{n}^{(q)}$, one observes the ones with $\left|q-p^{-}\right| \leqslant \epsilon$ with probability 1.

## D. Rate of Polarization

We define the Bhattacharyya process $\left\{Z_{n}\right\}$ where $Z_{n}=$ $Z\left(K_{n}\right)$ is the Bhattacharyya parameter of the random channel $K_{n}$. By using the channel evolution model in 37, this process can be expressed as

$$
Z_{n} \begin{cases}=Z_{n-1} Z_{n-m} & \text { if } S_{n}=+  \tag{48}\\ \leqslant Z_{n-1}+Z_{n-m}-Z_{n-1} Z_{n-m} & \text { if } S_{n}=- \\ =Z_{n-1} & \text { otherwise }\end{cases}
$$

where $Z_{n}=Z(W)$ for $n<1$.
Theorem 3. For any $\epsilon \in(0,1)$ there exists an $n$ such that for $\beta<p^{+}$we have

$$
\begin{equation*}
\operatorname{Pr}\left(Z_{n} \leqslant 2^{-\phi^{n \beta}}\right) \geqslant I(W)-\epsilon \tag{49}
\end{equation*}
$$

Proof: We consider another process $\left\{\hat{Z}_{n}\right\}$, driven by $\left\{S_{n}\right\}$, so that for $i=1,2, \ldots, n_{0}, n_{0}<n$, we have $\hat{Z}_{i}=Z_{i}$ and for $i>n_{0}, \hat{Z}_{i}$ obeys

$$
\hat{Z}_{i}= \begin{cases}\hat{Z}_{i-1} \hat{Z}_{i-m} & \text { if } S_{n}=+,  \tag{50}\\ \hat{Z}_{i-1}+\hat{Z}_{i-m}-\hat{Z}_{i-1} \hat{Z}_{i-m} & \text { if } S_{n}=-, \\ \hat{Z}_{i-1} & \text { otherwise }\end{cases}
$$

Comparing (48) and (50) we observe that $Z_{n}$ is stochastically dominated by $\hat{Z}_{n}$ in the sense that for some $f_{n} \in(0,1)$, $\operatorname{Pr}\left(Z_{n} \leqslant f_{n}\right) \geqslant \operatorname{Pr}\left(\hat{Z}_{n} \leqslant f_{n}\right)$. For the proof it will suffice to show that $\operatorname{Pr}\left(\hat{Z}_{n} \leqslant f_{n}\right) \geqslant I(W)-\epsilon$ holds for $f_{n}=2^{-\phi^{n \beta}}$ and $\beta<p_{+}$.

In [9, Lemma 1] authors derive an upper bound on $\hat{Z}_{n}$, for the case $m=1$, by using the frequency of state + in the realizations of $\left\{S_{n_{0}+1}, S_{n_{0}+2}, \ldots, S_{n}\right\}$ and the fact that $Z_{n_{0}}$ gets arbitrarily close to 0 , with probability $I(W)$, when $n_{0}$ is large enough. Following lemma is a generalization of this approach for arbitrary $m \geqslant 1$.
Lemma 5. For some $\zeta \in(0,1)$ and $\gamma \in(0,1)$ define the events

$$
\begin{aligned}
C_{n_{0}}(\zeta) & =\left\{Z_{n_{0}} \leqslant \zeta\right\} \\
D_{n_{0}}^{n}(\gamma) & =\left\{\#\left(\left(S_{n_{0}+1}, \ldots, S_{n}\right) \mid+\right) \geqslant \gamma\left(n-n_{0}\right)\right\}
\end{aligned}
$$

We have

$$
\hat{Z}_{n} \leqslant 2^{-\phi^{(\gamma-\epsilon)\left(n-n_{0}\right)}}, \quad C_{n_{0}}(\zeta) \cap D_{n_{0}}^{n}(\gamma)
$$

From the convergence of $Z_{n}$ to $Z_{\infty}$ with probability $\operatorname{Pr}\left(Z_{\infty}=0\right)=I(W)$ we know that for any $\epsilon \in(0,1)$ there exist a fixed $n_{0}$ such that

$$
\operatorname{Pr}\left(C_{n_{0}}(\zeta)\right) \geqslant I(W)-\epsilon
$$

Next, from Theorem 2, we infer that when $m \ll n-n_{0}$

$$
\begin{equation*}
\operatorname{Pr}\left(D_{n_{0}}^{n}(\gamma)\right) \geqslant 1-\epsilon, \quad \gamma \geqslant p^{+}-\epsilon \tag{51}
\end{equation*}
$$

holds. This results from the fact that the probability of observing + in $\left\{S_{n_{0}+1}, \ldots, S_{n_{0}}\right\}$ approaches to $p^{+}$when $n-n_{0}$ is much larger than the memory, $m$, of the process $\left\{S_{n}\right\}$.

Choosing $n_{0}=n \epsilon$ and using the above results in lemma 5 gives

$$
\begin{aligned}
\operatorname{Pr}\left(\hat{Z}_{n} \leqslant 2^{-\phi^{n\left(p^{+}-2 \epsilon\right)(1-\epsilon)}}\right) & \geqslant(1-\epsilon)(I(W)-\epsilon) \\
& \geqslant I(W)-\epsilon
\end{aligned}
$$

Since $\epsilon \in(0,1)$ can be chosen arbtirarily close to 0 , the above result indicates that

$$
\operatorname{Pr}\left(\hat{Z}_{n} \leqslant 2^{-\phi^{n \beta}}\right) \geqslant I(W)-\epsilon
$$

holds for $\beta<p^{+}$.

Let us analyze the implications of Theorem 3 on the blockdecoding error probability, $P_{e}$, of $\left\{\mathrm{C}_{n}^{(m)}\right\}$. It states that for $I(W)-\epsilon$ fraction of $W_{n}^{(i)}$ the corresponding Bhattacharyya parameters will be bounded as $Z_{n}^{(i)} \leqslant 2^{-\phi^{n \beta}}$ for $\beta<p^{+}$. We have $P_{e} \leqslant \sum_{i=1}^{N} Z_{n}^{(i)} \leqslant N 2^{-\phi^{n \beta}}=O\left(2^{-\phi^{n \beta}}\right)$. Since the code-length of $\left\{\mathrm{C}_{n}^{(m)}\right\}$ scales as $N=O\left(\phi^{n}\right)$ we also see that $P_{e}=O\left(2^{-N^{\beta}}\right)$ holds for $\beta<p^{+}$.

The term $p^{+}$is plotted in Fig. 6 as a $m$ increases from 1 to 50 . Investigating this figure we see that $p^{+}$equals to 0.5 when $m=1$ which coincides with the bound for the exponent of polar codes presented by Arıkan and Telatar in [3]. As $m$ increases from 1 to $50, p^{+}$and thus the achievable exponent decreases. The decrease is more steep for small values of $m$ and it becomes more monotone as $m$ increases.

In order to fully characterize the asymptotic performance of $\left\{\mathcal{C}_{n}^{(m)}\right\}$ one needs to provide a converse bound on $\beta$ which may be a difficult task. We believe that for the case $m>1$, the achievable $\beta$ for $\left\{\mathcal{C}_{n}^{(m)}\right\}$ may show a dependency on the rate, $R \in[0,1]$, chosen for the code; a phenomenon that does not exist when $m=1$ (see [10]). In order explain our conjecture, consider the process $\left\{\hat{Z}_{n}\right\}$ in 50 which we use to obtain an achievable bound on $\beta$ as $\beta<p^{+}$. Our proof is based on the observation that once the realizations of $\hat{Z}_{n_{0}}$ are sufficiently close to 0 , which happens with probability $I(W)$, the scaling of $Z_{n}$ is mostly determined by the number of occurrences of state + in $\left\{S_{n_{0}+1}, S_{n_{0}+2}, \ldots, S_{n}\right\}$. From Theorem 2 we know that one typically observes $\left(n-n_{0}\right) p^{+}$ occurrences of + in $\left\{S_{n_{0}+1}, S_{n_{0}+2}, \ldots, S_{n}\right\}$, therefore the value of $\log Z_{n}$ decreases $\left(n-n_{0}\right) p^{+}$times with the same speed as the code-length, $\log \hat{Z}_{n}=\log \hat{Z}_{n-1}+\log \hat{Z}_{n-m}$, scaling as $\log Z_{n}=-\phi^{\left(n-n_{0}\right) p^{+}}=-\phi^{n(1-\epsilon) p^{+}}$. This result in the achievable exponent $\beta<p^{+}$. However, when $m>1$ the value of $\log \hat{Z}_{n}$ may also decrease with a faster rate compared to that of the code-length. To see this, consider the case $\left.\left(S_{n-1}, S_{n-2}, \ldots, S_{n-(m-1)}\right)=(\star, \star, \ldots, \star)\right\}$ and $S_{n}=+$, where we have $\hat{Z}_{n-1}=\hat{Z}_{n-2}=\ldots=\hat{Z}_{n-(m-1)}$ and $\log \hat{Z}_{n}=\log \hat{Z}_{n-1}+\log \hat{Z}_{n-m}=\log \hat{Z}_{n-1}^{2}$. Therefore, there may be times where $\log Z_{n}$ decreases with a faster rate as $\log \hat{Z}_{n}=\log Z_{n-1}^{2}$ instead of $\log \hat{Z}_{n}=\log \hat{Z}_{n-1}+\log \hat{Z}_{n-m}$ and this may result in a higher achievable $\beta$. In order to quantify this we need to know not only the number of times state + occurs in $\left\{S_{n}\right\}$, but also the number of times a state + in $\left\{S_{n}\right\}$ is preceded by $\star$ states. Therefore, we need to refine Theorem 2 in terms of the number of transitions between states ,+- and $\star$, as well. This might be a difficult but important problem whose solution will provide a full characterization of the asymptotic polarization performance of $\left\{\mathcal{C}_{n}^{(m)}\right\}$ and we leave it as a future work.

## V. COMPLEXITY AND SPARSITY

## A. Encoding and Decoding Complexity

We consider a single core processor with random access memory and investigate the time complexity of encoding and decoding of $\left\{\mathcal{C}_{n}^{(m)}\right\}$. Let $\chi_{n}^{E}$ denote the complexity for encoding the information vector $\mathbf{u}_{N}$ to encoded bits $\mathbf{x}_{N}$.


Fig. 6: Achievable exponent, $\beta<p^{+}$, as scaled with $m$.

We take complexity of each XOR operation as 1 unit. By inspection of Fig 1, we have

$$
\begin{equation*}
\chi_{n}^{E}=\chi_{n-1}^{E}+\chi_{n-m}^{E}+N_{n-m} \quad n, m \geqslant 1 \tag{52}
\end{equation*}
$$

where $\chi_{1}^{E}=1$ and $\chi_{0}^{E}=\chi_{-1}^{E}=\ldots=\chi_{1-m}^{E}=0$.
Similarly, let $\chi_{n}^{D}$ denote the complexity for decoding the inputs of $W_{n}^{(i)}$ channels, where SCD is the decoding method. We take the complexity of computing the LR. relations in (32) as 1 unit. We observe that one does not make any operations to calculate the LR in 33). By inspection of Fig 1 , we have

$$
\begin{equation*}
\chi_{n}^{D}=\chi_{n-1}^{D}+\chi_{n-m}^{D}+2 N_{n-m} \quad n, m \geqslant 1 \tag{53}
\end{equation*}
$$

where $\chi_{0}^{D}=\chi_{-1}^{D}=\ldots=\chi_{1-m}^{D}=0$.
The recursions in (52) and (53) are cumbersome to deal with. To observe the scaling behavior of $\chi_{n}^{E}$ and $\chi_{n}^{D}$ in $m$, we define

$$
\begin{equation*}
\eta^{E} \triangleq \frac{\chi_{n}^{E}}{N \log N}, \quad \eta^{D} \triangleq \frac{\chi_{n}^{D}}{N \log N} \tag{54}
\end{equation*}
$$

and demonstrate the scaling of $\eta^{E}$ and $\eta^{D}$ in Fig 7, where we have numerically calculated $\chi_{n}^{E}$ and $\chi_{n}^{D}$ as in (52) and (53) by choosing $N=O\left(\phi^{n}\right)$ to be the code-length closest to $10^{4}$ and $10^{6}$. From Fig. 7 we observe that, there exist a decrease in $\eta_{n}^{E}$ and $\eta_{n}^{D}$ as $m$ increases, where the decrease is more steep for small values of $m$ and it becomes more monotone as $m$ increases. This decrease in complexity, although not being orders of magnitude, is promising in showing the existence of polar codes requiring lower complexity. For example, from Fig. 7 we observe that $\eta_{n}^{D}$ is around $1 / 2$ when $m=12$. This indicates that the decoding complexity of $\left\{\mathfrak{C}_{n}^{(12)}\right\}$ is reduced by half compared to $\left\{\mathrm{C}_{n}^{(1)}\right\}$ which is the polar code presented by Arıkan in [1].

## B. Sparsity

As we have explained in Section II, there exist a sparsity in the channel combining process in the sense that at each combining level, the vector channel $W_{n}$ is obtained by combining $W_{n-1}$ and $\hat{W}_{n-m}$ which are obtained from $N(n-1)$ and $N(n-m)$ uses of underlying B-DMC, $W$, respectively. From Proposition 5 we observe that the overall


Fig. 7: Scaling of encoding and decoding complexities as $m$ increases where $N$ is chosen to be the code-length closest to $1 \times 10^{4}, 1 \times 10^{6}$.
effect of channel combining and splitting is that, at each level $n$, there exist $N(n-m)$ bit-channel pairs that participate in $\boxplus$ and $\square$ transforms. As $m$ increases $N(n-m)$ decreases with respect to $N(n-1)$ implying the fraction of bit-channels participating in $\boxplus$ and $\square$ transforms also decreases. On the other hand, as $m$ increases, the code-length increases less rapidly in $n$ because $N=O\left(\phi^{n}\right)$ and $\phi$ is decreasing in $m$, thus one can fit more channel combining and splitting levels within fixed code-length. A natural question is to understand the overall effect of increasing $m$ on the total number of $\boxplus$ and $\boxminus$ transforms that one can obtain when the number of uses of $W$ channels is fixed. The importance of $\chi_{n}^{D}$ in (53) comes to play at this point because it gives us the total number of $\boxplus$ and $\square$ transformation that are recursively applied to independent uses of $W$ channels to obtain the bit-channels in $W_{n}$. Consequently, one can view $\eta_{D}$ as a packing ratio in the sense that one can pack $\eta_{n}^{D} N \log N$ recursive applications of $\boxplus$ and $\boxminus$ transformation to $N$ independent uses of $W$. Inspecting the scaling of $\eta_{D}$ in Fig. 7 we observe that this packing ratio is 1 when $m=1$ and it decreases with increasing $m$, and this decrease manifests itself as a reduction in the decoding complexity of $\left\{\mathrm{C}_{n}^{(m)}\right\}$.

## VI. Conclusion and Future Work

We have introduced a method to design a class of code sequences $\left\{\mathrm{C}_{n}^{(m)} ; n \geqslant 1, m \geqslant 1\right\}$ with code-length $N=$ $O\left(\phi^{n}\right), \phi \in(1,2]$, and memory order $m$. The design of $\left\{\mathcal{C}_{n}^{(m)}\right\}$ is based on the channel polarization idea of Arıkan [1] and $\left\{\mathrm{C}_{n}^{(m)}\right\}$ coincides with the polar codes presented by Arıkan when $m=1$. We showed that $\left\{\mathcal{C}_{n}^{(m)}\right\}$ achieves the symmetric capacity of arbitrary BDMCs for arbitrary but fixed $m$. We have obtained an achievable bound on the asymptotic polarization of performance of $\left\{\mathfrak{C}_{n}^{(m)}\right\}$ as scaled with $m$ and showed that the encoding and decoding complexities of $\left\{\mathcal{C}_{n}^{(m)}\right\}$ decrease with increasing $m$. Our introduction of $\left\{\mathrm{C}_{n}^{(m)}\right\}$ complements Arıkan's conjecture that channel polarization is a general phenomenon and it shows the existence of polar codes requiring lower complexity. Future work will include a rate
dependent analysis and a converse result on the asymptotic polarization performance of $\left\{\mathcal{C}_{n}^{(m)}\right\}$.

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## VII. Appendix

## A. Proof of Proposition 11

We have $J\left(W^{-}\right)=\frac{2}{1+Z\left(W^{-}\right)}$and $J\left(W^{+}\right)=\frac{2}{1+Z\left(W^{+}\right)}$. By using (17) and (16) we obtain

$$
\begin{align*}
& J\left(W^{+}\right)+J\left(W^{-}\right) \geqslant \log \frac{2}{1+Z\left(W^{\prime}\right) Z\left(W^{\prime \prime}\right)}+ \\
& \log \frac{2}{1+Z\left(W^{\prime}\right)+Z\left(W^{\prime \prime}\right)-Z\left(W^{\prime}\right) Z\left(W^{\prime \prime}\right)}  \tag{55}\\
& =\log \frac{2}{1+Z\left(W^{\prime}\right)+Z\left(W^{\prime \prime}\right)+w\left(W^{\prime}, W^{\prime \prime}\right) Z\left(W^{\prime}\right) Z\left(W^{\prime \prime}\right)}
\end{align*}
$$

where $w\left(W^{\prime}, W^{\prime \prime}\right)=Z\left(W^{\prime}\right)+Z\left(W^{\prime \prime}\right)-Z\left(W^{\prime}\right) Z\left(W^{\prime \prime}\right) \leqslant 1$ indicating

$$
\begin{align*}
& J\left(W^{+}\right)+J\left(W^{-}\right) \geqslant \log \frac{2}{1+Z\left(W^{\prime}\right)}+\log \frac{2}{1+Z\left(W^{\prime \prime}\right)}  \tag{56}\\
& \quad=J\left(W^{\prime}\right)+J\left(W^{\prime \prime}\right)
\end{align*}
$$

In order to have $J\left(W^{+}\right)+J\left(W^{-}\right)=J\left(W^{\prime}\right)+J\left(W^{\prime \prime}\right)$, the equalities in (55) and (56) must be achieved. From (17) we know that the equality in 55) is achieved only if $Z\left(W^{\prime}\right) \in$ $\{0,1\}$ or $Z\left(W^{\prime \prime}\right) \in\{0,1\}$ or if $W^{\prime}$ and $W^{\prime \prime}$ are BECs. When $\left(Z\left(W^{\prime}\right), Z\left(W^{\prime \prime}\right)\right) \in(0,1)^{2}$ we have $w\left(W^{\prime}, W^{\prime \prime}\right)<1$ and the inequality in (56) is always strict, whether or not $W^{\prime}$ and $W^{\prime \prime}$ being BECs. Consider the case $Z\left(W^{\prime}\right)=1$ or $Z\left(W^{\prime \prime}\right)=1$, then we have $w\left(W^{\prime}, W^{\prime \prime}\right)=1$ and the equalities in (55) and (56) are achieved. When $Z\left(W^{\prime}\right)=0$ we have $J\left(W^{\prime}\right)=1$, $w\left(W^{\prime}, W^{\prime \prime}\right)=0$ and $J\left(W^{+}\right)+J\left(W^{-}\right)=J\left(W^{\prime}\right)+J\left(W^{\prime \prime}\right)$, and the case $J\left(W^{\prime}\right)=1$ follows from the symmetry in (55) and (56). Hence the equalities in 55 and (56) are both achieved only if $Z\left(W^{\prime}\right) \in\{0,1\}$ or $Z\left(W^{\prime \prime}\right) \in\{0,1\}$, or alternatively only if $J\left(W^{\prime}\right) \in\{0,1\}$ or $J\left(W^{\prime \prime}\right) \in\{0,1\}$.

## B. Proof of Proposition 3

From the operation of $\varphi_{n}$ in Defn. 1 we obtain $\mathcal{S}_{1}=\{+,-\}$ such that $\mathbf{s}_{1}^{(1)}=(+)$ and $\mathbf{s}_{1}^{(2)}=(-)$, indicating $\mathbf{s}_{1}^{(1)}$ and $\mathbf{s}_{1}^{(2)}$ are unique. Proof is by induction, assume that $s_{n-1}^{(j)} \in \mathcal{S}_{n-1}$ are unique. Let $j \in \mathbb{N}_{n-m}$ and consider $\mathbf{s}_{n-1}^{(j)}$ to whom by appending + and - one obtains $\mathbf{s}_{n}^{(j)}$ and $\mathbf{s}_{n}^{(j+N(n-1))}$, respectively, indicating $\mathbf{s}_{n}^{(j+N(n-1))}$ and $\mathbf{s}_{n}^{(j)}$ are different from each other. Next, let $j \in \mathbb{N}_{n-1} \backslash \mathbb{N}_{n-m}$ then $\mathbf{s}_{n}^{(j)}$ are obtained by appending $\star$ to $\mathbf{s}_{n-1}^{(j)}$ which, by assumption, are unique. Combining the result we see that for all $j \in \mathbb{N}_{n}$ the vectors $s_{n}^{(j)} \in \mathcal{S}_{n}$ are different from each other.

## C. Proof of Proposition 4

Investigating Fig 2 consider the operation of $\varphi_{n-1}$ where $s_{n-2}^{(k)}=\left(s_{1}, s_{2}, \ldots, s_{n-2}\right), k \in \mathbb{N}_{n-2}$, holds at level $n-1$. Next, consider the operation of $\varphi_{n-2}$ where one has $s_{n-3}^{(k)}=$ $\left(s_{1}, s_{2}, \ldots, s_{n-3}\right)$ for $k \in \mathbb{N}_{n-3}$. In turn and by induction through $\varphi_{n-2}, \varphi_{n-3}, \ldots, \varphi_{n-(m-1)}$ we conclude that $s_{n-m}^{(j)}=$ $\left(s_{1}, s_{2}, \ldots, s_{n-m}\right), j \in \mathbb{N}_{n-m}$.

## D. Proof of Proposition 6

i) For $m>1$ we have $F(m, 1)=-1<0$ and $F(m, 2)=$ $2^{m-1}-1 \geqslant 0$ so that there exists at least one real root in $(1,2]$. Proof is by contradiction, let $\rho_{1}, \rho_{2} \in(1,2]$ be two real roots of $F(m, \rho)$ then from (35) we have

$$
\begin{align*}
& \rho_{1}^{m-1}\left(\rho_{1}-1\right)=1  \tag{57}\\
& \rho_{2}^{m-1}\left(\rho_{2}-1\right)=1 \tag{58}
\end{align*}
$$

Let $\rho_{1}<\rho_{2}$, then $\rho_{2}^{m-1}>\rho_{1}^{m-1}$ and $\rho_{2}-1>\rho_{1}-1>0$ implying $\rho_{2}^{m-1}\left(\rho_{2}-1\right)>1$ if $\rho_{1}^{m-1}\left(\rho_{1}-1\right)=1$ which contradicts 58), carrying a similar analysis for $\rho_{1}<\rho_{2}$ also contradicts (58), which indicates $\rho_{1}=\rho_{2}=\phi$.
ii) Assume that $\rho$ is a complex root of $F(m, \rho)$, with $\sqrt{\rho \rho^{*}}=\sigma>1$ where $*$ denotes the conjugate operation. Since the coefficients of $F(m, \rho)$ are real, its complex roots must be in conjugate pairs. From 35)

$$
\begin{array}{r}
\rho^{m-1}(\rho-1)=1 \\
\rho^{* m-1}\left(\rho^{*}-1\right)=1
\end{array}
$$

Multiplying the above equations we obtain

$$
\begin{align*}
\sigma^{2(m-1)}\left(\sigma^{2}-2 \operatorname{Re}(\rho)+1\right) & =1 \\
\sigma^{2(m-1)}\left(\sigma^{2}-2 \sigma \alpha+1\right) & =1 \tag{59}
\end{align*}
$$

where $0 \leqslant \alpha<1$. In turn for any $\rho, \sigma$ must be a root of

$$
\begin{equation*}
g(\sigma, \alpha)=\sigma^{2(m-1)}\left(\sigma^{2}-2 \sigma \alpha+1\right)-1 \tag{60}
\end{equation*}
$$

Observe that when $\sigma$ is fixed $g(\sigma, \alpha)$ is decreasing in $\alpha$. We also have

$$
\begin{aligned}
\frac{\partial g(\sigma, \alpha)}{\partial \sigma}= & 2(m-1) \sigma^{2(m-1)-1}\left(\sigma^{2}-2 \sigma \alpha+1\right) \\
& +\sigma^{2(m-1)}(2 \sigma-2 \alpha)
\end{aligned}
$$

From (59) observe that $\left(\sigma^{2}-2 \sigma \alpha+1\right)>0$, and since $(2 \sigma-$ $2 \alpha)>0$ for $\sigma>1$ we have $\frac{\partial g(\sigma, \alpha)}{\partial \sigma}>0$. This indicates that
$g(\sigma, \alpha)$ is increasing with $\sigma$. But $\phi$ is a root of $g(\sigma, \alpha)$ with $\alpha=1$ and thus $g(\phi, 1)=0$. Since $g(\sigma, \alpha)$ is decreasing in $\alpha$ we have $g(\phi, \alpha) \geqslant 0$ and $g(\sigma, \alpha)=0$ is only achieved if $\sigma<\phi$ because $g(\sigma, \alpha)$ is increasing with $\sigma$.
iii) Observe that for some $\rho \in(1,2]$ we have $\frac{\partial F(m, \rho)}{\partial \rho}>0$ so that $F(m, \rho)$ is increasing in $\rho$ and when $\rho$ is fixed $F(m, \rho)$ is also increasing in $m$. Assume that $\rho_{1}, \rho_{2} \in(1,2]$ are real roots of $F\left(m_{1}, \rho\right)$ and $F\left(m_{2}, \rho\right)$, respectively, where $m_{1}, m_{2} \geqslant 1$. Then $f\left(m_{1}, \rho_{1}\right)<f\left(m_{2}, \rho_{1}\right)$ holds if $m_{2}>m_{1}$ and $f\left(m_{1}, \rho_{1}\right)=f\left(m_{2}, \rho_{2}\right)=0$ is satisfied only if $\rho_{1}<\rho_{2}$.

## E. Proof of Lemma 1

Let $J_{n}^{(i)}=J\left(W_{n}^{(i)}\right)$ denote symmetric cut-off rate of $W_{n}^{(i)}$. From Proposition 5 we know that for $j \in \mathbb{N}_{n-m}$ we have $W_{n}^{(j)}=W_{n-1}^{(j)} \boxplus W_{n-m}^{(j)}$ and $W_{n}^{(j+N(n-1))}=W_{n-1}^{(j)} \boxminus W_{n-m}^{(j)}$. Proposition 1 indicates that these transforms increase the sum cut-off rate as $J_{n}^{(j)}+J_{n}^{(j+N(n-1))} \geqslant J_{n-1}^{(j)}+J_{n-1}^{(j)}$ where the equality is achieved only if $J_{n-1}^{(j)} \in\{0,1\}$ or $J_{n-m}^{(j)} \in\{0,1\}$ holds. For $j \in \mathbb{N}_{n-1} \backslash \mathbb{N}_{n-m}$, from Proposition 5, we have $J_{n}^{(j)}=\gamma(n) J_{n-1}^{(j)}$ which implies $J_{n}^{(j)}=J_{n-1}^{(j)}$. Combining the above results gives

$$
\sum_{i \in \mathbb{N}_{n}} J_{n}^{(i)} \geqslant \sum_{j \in \mathbb{N}_{n-1}} J_{n}^{(j)}+\sum_{k \in \mathbb{N}_{n-m}} J_{n}^{(k)}
$$

where the equality is achieved only of if $J_{n-1}^{(j)} \in\{0,1\}$ or $J_{n-m}^{(j)} \in\{0,1\}$ holds for all $j \in \mathbb{N}_{n-m}$. In the probabilistic domain of Sectior $[\mathrm{IV}$ the above result is equivalent to

$$
\sum_{\mathbf{s}_{n} \in \mathcal{S}_{n}} J_{n} \geqslant \sum_{\mathbf{s}_{n-1} \in \mathcal{S}_{n-1}} J_{n-1}+\sum_{\mathbf{s}_{n-m} \in \mathcal{S}_{n-m}} J_{n-m}
$$

where the equality is achieved only of if $J_{n-1} \in\{0,1\}$ or $J_{n-m} \in\{0,1\}$ holds for all $S_{n} \in\{+,-\}$. Dividing both sides of the above inequality by $1 / N(n)$ and using $E\left[J_{n}\right]=\frac{1}{N(n)} \sum_{\mathbf{s}_{n} \in \mathcal{S}_{n}} J_{n}$ we obtain

$$
E\left[J_{n}\right] \geqslant \frac{N(n-1)}{N(n)} E\left[J_{n-1}\right]+\frac{N(n-m)}{N(n)} E\left[J_{n-m}\right]
$$

Noticing $\frac{N(n-1)}{N(n)}=\mu(n)$ and $\frac{N(n-m)}{N(n)}=1-\mu(n)$ completes the proof.

## F. Proof of Lemma 2

From (38) we have

$$
\begin{array}{r}
E\left[J_{n}\right] \geqslant \mu E\left[J_{n-1}\right]+(1-\mu) E\left[J_{n-m}\right] \\
\geqslant \min \left\{E\left[J_{n-1}\right], E\left[J_{n-m}\right]\right\} \tag{61}
\end{array}
$$

Let us define the set

$$
\mathcal{E}_{k}^{(m)} \triangleq\left\{E_{k m}, E_{k m-1}, \ldots, E_{k m-(m-1)}\right\}
$$

By definition in 39 we have we have $E\left[\hat{J}_{k}\right]=\min \mathcal{E}_{k}^{(m)}$. Proof is by induction. We use (61) to upper bound the elements of $\mathcal{E}_{k}^{(m)}$ with respect to $\min \mathcal{E}_{k-1}^{(m)}=E\left[\hat{J}_{k-1}\right]$. Let $n=k m-$ $(m-1)$ and use (61) to obtain

$$
\begin{aligned}
E_{k m-(m-1)} & \geqslant \min \left\{E_{(k-1) m}, E_{(k-1) m-(m-1)}\right\} \\
& \geqslant \min \mathcal{E}_{k-1}^{(m)}
\end{aligned}
$$

For $i=2,3, \ldots, m-1$ assume

$$
E_{k m-(m-i)} \geqslant \min \mathcal{E}_{k-1}^{(m)}
$$

holds. Next, let $n=k m-(m-(i+1))$ in 61) to write

$$
E_{k m-(m-(i+1))} \geqslant \min \left\{E_{k m-(m-i)}, E_{(k-1) m-(m-(i+1))}\right\}
$$

By assumption $E_{k m-(m-i)} \geqslant \min \mathcal{E}_{k-1}^{(m)}$ and by definition $E_{(k-1) m-(m-(i+1))} \geqslant \min \mathcal{E}_{k-1}^{(m)}$ holds, indicating

$$
E_{k m-(m-(i+1))} \geqslant \min \mathcal{E}_{k-1}^{(m)}
$$

Combining the above results tells us for $i=1,2, \ldots, m$ we have $E_{k m-(m-i)} \geqslant \min \mathcal{E}_{k-1}^{(m)}=E\left[\hat{J}_{k-1}\right]$ which indicates $E\left[\hat{J}_{k}\right] \geqslant E\left[\hat{J}_{k-1}\right]$.

## G. Proof of Lemma 3

In order to bound $\left|\mathcal{T}_{n}^{(q)}\right|$ we decompose $\mathcal{T}_{n}^{(q)}$ it into two different sets

$$
\begin{aligned}
& \mathcal{T}_{n}^{(a, q)} \triangleq\left\{\mathbf{s}^{n}: P_{\mathbf{s}^{n}}^{(-)}=q, s_{n}=+\right\} \\
& \mathcal{T}_{n}^{(b, q)} \triangleq\left\{\mathbf{s}^{n}: P_{\mathbf{s}^{n}}^{(-)}=q, s_{n} \neq+\right\}
\end{aligned}
$$

and we have $T_{n}^{(q)}=\mathcal{T}_{n}^{(a, q)} \cup \mathcal{T}_{n}^{(b, q)}$. Recall that each state - in $\mathbf{s}_{n}$ is followed by $m-1$ occurrences of state $\star$. In turn, $\mathcal{T}_{n}^{(a, q)}$ consists of $\mathbf{s}_{n}$ having $k=n q, 0 \leqslant k \leqslant n / m$, occurrences of the vector $\mathbf{a}=(-, \underbrace{\star, \star, \ldots, \star}_{m-1 \text { times }})$ and $n-k m$ occurrences of state + . By combinatorial analysis we have

$$
\left|\mathcal{T}_{n}^{(a, q)}\right|=\binom{n-(m-1) k}{k}
$$

$\mathcal{T}_{n}^{(b, q)}$ consists of $k-1$ occurrences of the vector $\mathbf{a}$, an occurrence of $\mathbf{b}=(-, \underbrace{0,0, \ldots, 0}_{p \text { times }}), 1 \leqslant p<m-1$, and $n-m k-(p+1)$ occurrences of state + . The vector $\mathbf{b}$ can only occur in the last $p+1$ entries in $\mathbf{s}_{n}$ and it will be completed to a vector a if we had prolonged the channel combining operation $m-1-p \leqslant m$ more levels. Therefore

$$
\left|\mathcal{T}_{n}^{(b, q)}\right| \leqslant\binom{ n+m-(m-1) k}{k}
$$

For some $c \in \mathbb{Z}$ and $d \in \mathbb{Z}$ with $c<d$ we have $\binom{d}{c}=$ $\frac{d}{d-c}\binom{d-1}{c} \leqslant d\binom{d-1}{c}$, using this fact we obtain

$$
\begin{aligned}
&\binom{n+m-(m-1) k}{k} \leqslant(n+m)\binom{n+(m-1)-(m-1) k}{k} \\
&<(n+m)^{2}\binom{n+(m-2)-(m-1) k}{k} \\
& \vdots \\
&<(n+m)^{m}\binom{n-(m-1) k}{k}
\end{aligned}
$$

Then we have

$$
\begin{align*}
\left|\mathcal{T}_{n}^{(q)}\right| & =\left|\mathcal{T}_{n}^{(a, q)}\right|+\left|\mathcal{T}_{n}^{(b, q)}\right|, \\
& <\left(1+(n+m)^{m}\right)\binom{n-(m-1) k}{k}, \\
& <(1+(n+m))^{m}\binom{n-(m-1) k}{k}, \\
& =2^{n B(m, n)}\binom{n-(m-1) k}{k} \tag{62}
\end{align*}
$$

where $B(m, n)=\frac{m \log (1+n+m)}{n}=o(1)$. Next, we use the upper bound $\binom{n}{k} \leqslant 2^{n} n(k / n)$ in [8] to upper bound $\binom{n-(m-1) k}{k}$ as

$$
\begin{align*}
\binom{n-(m-1) k}{k} & \leqslant 2^{n(1-(m-1)(k / n)) H\left(\frac{(k / n)}{1-(m-1)(k / n)}\right.}, \\
& =2^{n G(m, q)} . \tag{63}
\end{align*}
$$

Combining 62 and 63)we obtain the desired bound as $\left|T_{n}^{(q)}\right|<2^{n(G(m, q)+B(m, n))}=2^{n(G(m, q)+o(1))}$.

## H. Proof of Lemma 4

We have

$$
G(m, q)=(1-(m-1) q) H\left(\frac{(q)}{1-(m-1) q}\right)
$$

We know that, for $q \in[0,1 / m], H\left(\frac{q}{1-(m-1) q}\right)$ is concave in $q$ and $(1-(m-1) q)$ is linear in $q$ indicating $G(m, q)$ is concave in $q$. Let $q *$ denote the maximizer of $G(m, q)$. The maximum of $H\left(\frac{q}{1-(m-1) q}\right)$ occurs when $\frac{q}{1-(m-1) q}=\frac{1}{2}$ or equivalently when $q=\frac{1}{m+1}$ and since $(1-(m-1) q)$ is decreasing in $q$, we have $q * \in\left[0, \frac{1}{m+1}\right]$. We next evaluate $\frac{\partial G(m, q)}{\partial q}$

$$
\begin{array}{r}
\frac{\partial G(m, q)}{\partial q}=(m-1) \log (1-(m-1) q) \\
+\log q-m \log (1-m q)
\end{array}
$$

setting $\left.\frac{\partial G(m, q)}{\partial q}\right|_{q=q *}=0$ gives
$(m-1) \log \left(1-(m-1) q^{*}\right)+\log q^{*}=m \log \left(1-m q^{*}\right)$.
Re-arranging the above equation we obtain

$$
\begin{align*}
& m \log \frac{\left(1-(m-1) q^{*}\right)}{1-m q^{*}}+\log \frac{q^{*}}{1-m q^{*}} \\
& \quad=\log \frac{\left(1-(m-1) q^{*}\right)}{1-m q^{*}} \tag{65}
\end{align*}
$$

Let us use the following substitutions

$$
\eta=\frac{1-(m-1) q^{*}}{1-m q^{*}}, \quad \eta-1=\frac{q^{*}}{1-m q^{*}} .
$$

For $q * \in\left[0, \frac{1}{m+1}\right]$ we have $\eta \in[1,2]$. Using the above substitutions in 65) we obtain

$$
m \log \eta+\log (\eta-1)=\log \eta
$$

or alternatively

$$
\eta^{m}(\eta-1)=\eta
$$

Dividing both sides of the above relation by $\eta$ and re-arranging the terms we obtain

$$
\begin{equation*}
\eta^{m}-\eta^{m-1}-1=0 \tag{66}
\end{equation*}
$$

But the above polynomial is same as 35. Consequently from part $i$ of Proposition. 6we conclude that $\eta=\phi$ which indicates that $\frac{1-(m-1) q^{*}}{1-m q^{*}}=\phi$ and hence $q^{*}=\frac{1}{1+m(\phi-1)}=p^{-}$. Next we evaluate the maximum of $G(m, q)$ attained at $q=q^{*}$.

$$
\begin{array}{r}
G\left(m, q^{*}\right)=-q^{*} \log \frac{q^{*}}{1-(m-1) q^{*}}+ \\
\left(m q^{*}-1\right) \log \frac{1-m q^{*}}{1-(m-1) q^{*}} \tag{67}
\end{array}
$$

Re-arranging (64) we observe that

$$
\log \frac{q^{*}}{1-(m-1) q^{*}}=m \log \frac{1-m q^{*}}{1-(m-1) q^{*}}
$$

Using the above relation in 67) gives

$$
G\left(m, q^{*}\right)=\log \frac{1-(m-1) q^{*}}{1-m q^{*}}=\log \phi
$$

## I. Proof of Proposition 8

We define a typical set $\mathcal{T}_{n}^{(q, \epsilon)}$ as

$$
\mathcal{T}_{n}^{(q, \epsilon)}=\left\{\mathbf{s}_{n}: P_{s^{n}}^{(-)}=q, D\left(q, p^{-}\right) \leqslant \epsilon\right\}
$$

The probability that $\mathcal{T}_{n}^{(q)}$ is not typical is

$$
\begin{align*}
1-\operatorname{Pr}\left(\mathcal{T}_{n}^{(q, \epsilon)}\right) & =\sum_{\operatorname{Pr}\left(D\left(q, p^{-}\right)>\epsilon\right)} \operatorname{Pr}\left(\mathcal{T}_{n}^{(q)}\right) \\
& \stackrel{a}{\leqslant} \sum_{\operatorname{Pr}\left(D\left(q, p^{-}\right)>\epsilon\right)} 2^{-n\left(D\left(q, s_{-}\right)+o(1)\right)} \\
& \leqslant \sum_{\operatorname{Pr}\left(D\left(q, p^{-}\right)>\epsilon\right)} 2^{-n(\epsilon+o(1))} \\
& \leqslant(n+1) 2^{-n(\epsilon+o(1))} \\
& =2^{-n(\epsilon+o(1))} \tag{68}
\end{align*}
$$

In the above derivation (a) follows from (47) and (b) follows from the fact that there exist at most $n+1$ different type classes having $\operatorname{Pr}\left(D\left(q, s_{-}\right)>\epsilon\right)$. The above result indicates that $\sum_{n \rightarrow \infty} \operatorname{Pr}\left(D\left(q, s_{-}\right) \geqslant \epsilon\right)$ converges, thus the expected number of the occurrences of the event $D\left(q, s_{-}\right)>\epsilon$ for all $n$ is finite. By using the first Borel Cantelli Lemma [7, p. 59] we conclude that $D\left(q, s_{-}\right)$converges to 0 with probability 1 .

## J. Proof of Lemma 5

Conditioned on the event $D_{n_{0}}^{n}(\gamma)=\#\left(\left(s_{n_{0}+1}, \ldots, s_{n}\right) \mid+\right.$ $) \geqslant \gamma\left(n-n_{0}\right)$ there exists at least $\gamma\left(n-n_{0}\right)$ occurrences of state + in $\left\{S_{n_{0}+1}, S_{n_{0}+2}, \ldots, S_{n}\right\}$. Investigating (50), we have $\hat{Z}_{n} \leqslant \hat{Z}_{n-1}$ when $S_{n}=+$ and $Z_{n} \geqslant Z_{n-1}$ when $S_{n} \neq$ + . Moreover, $Z_{n}$ is increasing in $Z_{n-1}$ when $S_{n}$ is fixed. Consequently, if we fix $\hat{Z}_{m}$, the largest value of $\hat{Z}_{n}$ will occur if $\left\{S_{n_{0}+1}, S_{n_{0}+2}, \ldots S_{n}\right\}$ has the following realization

$$
\{\overbrace{\mathbf{a}, \mathbf{a}, \ldots, \mathbf{a},}^{(1-\gamma)\left(n-n_{0}\right) / m \text { times }} \underbrace{+,+, \ldots,+}_{\gamma\left(n-n_{0}\right) \text { times }}\} .
$$

where $\mathbf{a}=(-, \underbrace{\star, \star, \ldots, \star}_{m-1 \text { times }})$. In order to upper bound $\hat{Z}_{n}$ we assume that the above realization has occured for $\left\{S_{n_{0}+1}, S_{n_{0}+2}, \ldots S_{n}\right\}$. During consecutive runs of + , the value of $\log \hat{Z}_{n}$ increases with the same recursion as the codelength in (1) as $\log \hat{Z}_{n}=\log \hat{Z}_{n-1}+\log \hat{Z}_{n-m}$. This recursion happens $\gamma(n-m)$ times and since the code-legth obeying the same recursion scales as $\phi^{\gamma(n-m)}, \phi \in(1,2]$, we have

$$
\begin{equation*}
\log \hat{Z}_{n}=\phi^{\gamma\left(n-n_{0}\right)} \log \hat{Z}_{k} \tag{69}
\end{equation*}
$$

where $k=n_{0}+(1-\gamma)(n-m)$. During consecutive runs of a the value of $\hat{Z}_{i}$ does not change with respect to $\hat{Z}_{i-1}$ when $S_{i}=\star$ and it increases as $\hat{Z}_{i}=\hat{Z}_{i-1}+\hat{Z}_{i-m}-\hat{Z}_{n-1} \hat{Z}_{i-m}$ when $S_{i}=-$. By construction of $\left\{S_{n_{0}+1}, S_{n_{0}+2}, \ldots S_{n}\right\}$ each state - is preceed by $m-1$ occurances of $\star$ therefore if $S_{i}=-$ we have $\left(S_{i-1}, S_{i-2}, \ldots, S_{i-(m-1)}\right)=(\star, \star, \ldots, \star)$ indicating $\hat{Z}_{i-1}=\hat{Z}_{i-2}=\ldots=\hat{Z}_{i-(m-1)}$. Therefore during each occurance of state - in a we see the recursion $\hat{Z}_{i-1}+\hat{Z}_{i-m}-\hat{Z}_{i-1} \hat{Z}_{i-m}=2 \hat{Z}_{i-1}-\hat{Z}_{i}^{(i)}$ or equvalently $1-\hat{Z}_{i}=\left(1-\hat{Z}_{i}^{(i)}\right)^{2}$. This recursion occurs $(1-\gamma)\left(n-n_{0}\right)$ times resulting in $1-\hat{Z}_{k}=\left(1-\hat{Z}_{n_{0}}\right)^{2(1-\gamma)\left(n-n_{0}\right)}$ and $\hat{Z}_{k}=1-\left(1-\hat{Z}_{n_{0}}\right)^{2(1-\gamma)\left(n-n_{0}\right)}$. Next, employ the inequality $\log x \leqslant x-1, x \in[0,1]$, by letting $x=\hat{Z}_{k}$ to obtain

$$
\begin{equation*}
\log \hat{Z}_{k} \leqslant-\left(1-\hat{Z}_{n_{0}}\right)^{2(1-\gamma)\left(n-n_{0}\right)} \tag{70}
\end{equation*}
$$

Using (70) in 69 gives

$$
\begin{aligned}
\log \hat{Z}_{n} & =-\phi^{\gamma\left(n-n_{0}\right)}\left(1-Z_{n_{0}}\right)^{2(1-\gamma)\left(n-n_{0}\right)} \\
& \leqslant-\phi^{\gamma\left(n-n_{0}\right)}\left(1-Z_{n_{0}}\right)^{2\left(n-n_{0}\right)} \\
& =-\phi^{(\gamma-\epsilon)\left(n-n_{0}\right)}\left(\left(1-Z_{n_{0}}\right)^{2} \phi^{\epsilon}\right)^{\left(n-n_{0}\right)}
\end{aligned}
$$

Choose $\zeta \in(0,1)$ so that $\zeta \leqslant 1-\phi^{\frac{-\epsilon}{2}}$ holds. Conditioned on $C_{n_{0}}(\zeta)=\left\{Z_{n_{0}} \leqslant \zeta\right\}$ we have $\left(1-Z_{n_{0}}\right)^{2} \phi^{\epsilon} \geqslant 1$, resulting in

$$
\log _{2} \hat{Z}_{n} \leqslant-\phi^{(\gamma-\epsilon)(n-m)}, \quad C_{n_{0}}(\zeta) \cap D_{n_{0}}^{n}(\gamma)
$$

which proves the lemma.


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