# Polar Codes with Higher-Order Memory

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Abstract—We introduce the design of a set of code sequences  $\{\mathbb{C}_n^{(m)} : n \ge 1, m \ge 1\}$ , with memory order m and code-length  $N = O(\phi^n)$ , where  $\phi \in (1, 2]$  is the largest real root of the polynomial equation  $F(m, \rho) = \rho^m - \rho^{m-1} - 1$  and  $\phi$  is decreasing in m.  $\{\mathbb{C}_n^{(m)}\}$  is based on the channel polarization idea, where  $\{\mathbb{C}_n^{(1)}\}$  coincides with the polar codes presented by Arıkan in [1] and can be encoded and decoded with complexity  $O(N \log N)$ .  $\{\mathbb{C}_n^{(m)}\}$  achieves the symmetric capacity, I(W), of an arbitrary binary-input, discrete-output memoryless channel, W, for any fixed m and its encoding and decoding complexities decrease with growing m. We obtain an achievable bound on the probability of block-decoding error,  $P_e$ , of  $\{\mathbb{C}_n^{(m)}\}$  and showed that  $P_e = O(2^{-N^{\beta}})$  is achievable for  $\beta < \frac{\phi-1}{1+m(\phi-1)}$ .

Index Terms—Channel polarization, polar codes, capacityachieving codes, method of types, successive cancellation decoding

#### I. INTRODUCTION AND OVERVIEW

Channel polarization [1] is a method to achieve the symmetric capacity, I(W), of an arbitrary binary-input, discreteoutput memoryless channel (B-DMC), W. By applying channel combining and splitting operations [2], one transforms N uses of W into another set of synthesized binary-input channels. As N increases, the symmetric capacities of the synthesized binary-input channels polarize as I(W) fraction of them gets close to 1 and 1 - I(W) fraction of them gets close to 0. The resulting code sequences, called polar codes, have encoding and decoding complexities  $O(N \log N)$ , and their block error probabilities scale as  $2^{-N^{\beta}}$  where  $\beta < 1/2$ is the exponent of the code [3].

Let  $W : \mathcal{X} \to \mathcal{Y}$  denote a B-DMC with binary-input  $x \in \mathcal{X} = \{0, 1\}$  and arbitrary discrete-output  $y \in \mathcal{Y}$ . Considering Arikan's polar codes, let us write  $W_n$  to denote the vector channel,  $W_n : \mathcal{X}^N \to \mathcal{Y}^N$ ,  $N = 2^n$ ,  $n \ge 1$ , obtained at channel combining level n. The vector channel,  $W_n$ , is obtained from  $W_{n-1}$  in a recursive manner where one first injects an independent realization of  $W_{n-1}$ , denoted as  $\hat{W}_{n-1}$ , and then combines the input of  $W_{n-1}$  and  $\hat{W}_{n-1}$  to obtain  $W_n$ , where the recursion starts with  $W_0 = W$ . The injection of  $\hat{W}_{n-1}$ , in a way, creates N/2 diversity paths for the N/2 inputs of  $W_{n-1}$ , and this allows polarization which one sees in the synthesized binary-input channels obtained by splitting  $W_n$ . Consequently, at each combining level the code-

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With higher-order memory in channel polarization, let us write N = N(n, m) to denote the code-length at channel combining level n and memory parameter  $m, m \ge 1$ , which we assume to be fixed. The vector channel,  $W_n$ , is obtained by combining the inputs of  $W_{n-1}$  with  $W_{n-m}$ , where one chooses  $W_0 = W_{-1} = \ldots = W_{1-m} = W$  to initiate the recursion. The number of binary-inputs in  $W_{n-1}$  and  $W_{n-m}$ are N(n-1) and N(n-m), respectively. In turn, with the controlled memory parameter, m, and at channel combining level n, one only injects N(n-m) new diversity paths with  $W_{n-m}$ , for the N(n-1) inputs of  $W_{n-1}$ , to obtain  $W_n$ . Because N(n-m) gets smaller compared to N(n-1) as m increases, it is possible to slow the speed at which one inject new channels to provide polarization. At first glance, it seems that increasing m will decrease the polarization effect obtained after each combining and splitting stage, however it will also allow the code-length to increase less rapidly in n. In order to see this consider the code-length obeying the recursion

$$N = N(n-1) + N(n-m), \quad n \ge 1, m \ge 1,$$
(1)

with initial conditions

$$N(0) = N(-1) = \dots = N(1-m) = 1, \quad m \ge 1.$$
 (2)

As will be explained in the sequel, the code-length takes the form

$$N = O(\phi^n), \quad n \ge 1 \tag{3}$$

where  $\phi \in (1, 2]$  is the largest real root of the *m*-th order polynomial equation

$$F(m,\rho) = \rho^m - \rho^{m-1} - 1,$$
(4)

and  $\phi$  decreases with increasing *m*. Therefore, if we increase *m*, it will take more channel combining and splitting stages to reach a pre-defined code-length, where the ratio of injected diversity paths to existing paths in each combining stage will also decrease. The aim of this paper is to understand the effects of this trade-off on the polarization performance one can obtain at a fixed code-length *N*.

The original construction of polar codes by Arıkan is closely related to the recursive construction of Reed-Muller codes based on the 2 × 2 kernel  $\mathbf{F}_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ . For these codes the encoding matrix,  $\mathbf{G}_N$ , is of the form  $\mathbf{G}_N = \mathbf{F}_2^{\otimes n}$ , where  $\otimes$  denotes the Kronecker power, suitably defined in [1]. In [4] Korada et al. generalize the channel polarization idea where  $\ell \ge 2$  independent uses of  $W_{n-1}$  are arbitrarily combined to obtain  $W_n$  and code-length scales as  $N = \ell^n$ . Although the channel combining mechanism is generalized to combining arbitrary numbers of  $W_{n-1}$  to obtain  $W_n$ , this setup has also first order memory in the channel combining. The authors express the combining mechanism by an  $\ell \times \ell$  polarization kernel  $\mathbf{K}_{\ell}$ . With an arbitrary  $\mathbf{K}_{\ell}$ , the encoding matrix takes the form  $\mathbf{G}_N = \mathbf{K}_{\ell}^{\otimes n}$ . The asymptotic polarization performance is characterized by the distance properties of the rows of  $K_{\ell}$ . The encoding and decoding complexities of these polar codes increases with l scaling as  $O(lN \log N)$  and  $O(\frac{2^{l}}{l} N \log N)$ , respectively. Our work differs from [4] in the sense that by introducing higher-order memory we modify the channel combining process. Moreover the encoding matrix of polar codes with memory m > 1 can not be obtained by applying Kronecker power to an arbitrary polarization kernel. As a result, one needs new mathematical tools to investigate  $\beta$ .

The contributions of this paper are as follows: *i*) We present a novel polar code family,  $\{\mathbb{C}_n^{(m)} : n \ge m \ge 1\}$ , with codelength  $N = O(\phi^n), \phi \in (1, 2]$ , and arbitrary but fixed memory parameter *m*. We show that  $\{\mathbb{C}_n^{(m)}\}$  achieves the symmetric capacity of arbitrary BDMCs for any choice of *m* which complements Arıkan's conjecture that channel polarization is in fact a general phenomenon. *ii*) By developing a new mathematical framework, we obtain an asymptotic bound on the achievable exponent,  $\beta$ , of  $\{\mathbb{C}_n^{(m)}\}$ . *iii*) We show that the encoding and decoding complexities of  $\{\mathbb{C}_n^{(m)}\}$  decrease with increasing *m*.  $\{\mathbb{C}_n^{(m)}\}$  is the first example of a polar code family that has lower complexity compared to the original codes presented by Arıkan.

The outline of the paper is a as follows. Section II provides the necessary material for the analysis in the sequel. In Section III we explain the design, encoding and the decoding of  $\{\mathbb{C}_n^{(m)}\}$ . In Section IV we develop a probabilistic framework to investigate  $\{\mathbb{C}_n^{(m)}\}$ . After showing that  $\{\mathbb{C}_n^{(m)}\}$  achieves the symmetric capacity of arbitrary B-DMCs we obtain an achievable bound on its block-decoding error probability. In Section V we analyze impact of higher-order memory on the encoding and decoding complexities of  $\{\mathbb{C}_n^{(m)}\}$ . Section VI concludes the paper and provides some future research directions.

**Notation:** We use uppercase letter A, B for random variables and lower cases a, b for their realizations taking values from sets A, B, where the sets have sizes  $|\mathcal{A}|$  and  $|\mathcal{B}|$  respectively.  $\Pr(a)$  denotes the probability of the event A = a. We write  $\mathbf{a}_n = (a_1, a_2, \dots, a_n)$  to denote a vector and  $(\mathbf{a}_n, \mathbf{b}_n)$  to denote the concatenation of  $\mathbf{a}_n$  and  $\mathbf{b}_n$ . We use standard Landau notation o(n), O(N) to denote the limiting values of functions. Note: Proofs, unless stated otherwise, are provided in the Appendix.

#### **II. PRELIMINARIES**

Let W(y|x),  $x \in \mathcal{X}$ ,  $y \in \mathcal{Y}$  denote the transition probabilities of W. Throughout the paper we assume that x is uniformly distributed in  $\mathcal{X}$ , and use base-2 logarithm. The symmetric capacity, I(W), of W is

$$I(W) \stackrel{\Delta}{=} \sum_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} \frac{1}{2} W(y|x) \log \frac{W(y|x)}{\frac{1}{2} W(y|0) + \frac{1}{2} W(y|1)}.$$
 (5)

The Bhattacharyya parameter, Z(W), of W provides an upper bound on the probability of error for maximum likelihood (ML) decoding over W and is defined as

$$Z(W) \stackrel{\Delta}{=} \sum_{y \in \mathcal{Y}} \sqrt{W(y|0)W(y|1)}.$$
 (6)

The symmetric cut-off rate, J(W), of W is [1]

$$J(W) \stackrel{\Delta}{=} \log \frac{2}{1 + Z(W)}.$$
(7)

As Arikan shows in [1, Prop. 1] Z(W) = 1 implies I(W) = 0and Z(W) = 0 implies I(W) = 1. By using this fact and from (7) we see that if J(W) = 0 then I(W) = 0 holds and J(W) = 1 indicates I(W) = 1.

Let W' and W'' be two B-DMCs with inputs  $x_1, x_2 \in \mathcal{X}$ and outputs  $y_1 \in \mathcal{Y}_1$  and  $y_2 \in \mathcal{Y}_2$ , respectively. Channel polarization is based on a single-step channel transformation where one first combines the inputs of W' and W'' to obtain a vector channel

$$W(y_1, y_2 | x_1, x_2) = W'(y_1 | x_1 \oplus x_2) W''(y_2 | x_2).$$
(8)

Next, by choosing a channel ordering, one splits the vector channel to obtain two new binary-input channels,  $W^- : \mathcal{X} \to \mathcal{Y}_1 \times \mathcal{Y}_2$  and  $W^+ : \mathcal{X} \to \mathcal{X} \times \mathcal{Y}_1 \times \mathcal{Y}_2$ , with transition probabilities

$$W^{-}(y_{1}, y_{2}|x_{1}) = \sum_{x_{2}} \frac{1}{2} W'(y_{1}|x_{1} \oplus x_{2}) W''(y_{2}|x_{2}), \quad (9)$$

$$W^{+}(y_{1}, y_{2}, x_{1}|x_{2}) = \frac{1}{2}W'(y_{1}|x_{1} \oplus x_{2})W''(y_{2}|x_{2}), \quad (10)$$

We use the following short-hand notations for the transforms in (9) and (10), respectively.

$$W^{-} = W' \boxminus W'', \tag{11}$$

$$W^+ = W' \boxplus W''. \tag{12}$$

The polarization transforms preserve the symmetric capacity as

$$I(W^{-}) + I(W^{+}) = I(W') + I(W''),$$
(13)

and they help polarization by creating disparities in  $I(W^+)$  and  $I(W^-)$  such that

$$I(W^+) \ge \max\{I(W'), I(W'')\},$$
 (14)

$$I(W^{-}) \leq \min\{I(W'), I(W'')\},$$
 (15)

where the above inequalities are strict as long as  $I(W') \in (0,1)$  and  $I(W'') \in (0,1)$ . This polarization effect quantitatively observed in the Bhattacharyya parameters as they take the form

$$Z(W^{+}) = Z(W')Z(W''),$$
(16)

$$Z(W^{-}) \leq Z(W') + Z(W'') - Z(W')Z(W''), \qquad (17)$$

where the equality in (17) is achieved if  $Z(W') \in \{0, 1\}$  or  $Z(W'') \in \{0, 1\}$ , or if W' and W'' are binary erasure channels (BECs).

Equations (13)-(17) are proved in [1] when W' is identical to W''. Their generalizations for the case W' and W'' are different channels are straightforward and omitted. The proposition below will be crucial in the sequel.

## **Proposition 1.**

$$J(W^{-}) + J(W^{+}) \ge J(W') + J(W''),$$

where equality is achieved only if  $J(W') \in \{0,1\}$  or  $J(W'') \in \{0,1\}$ .

The above proposition indicates that one can obtain coding gain by applying channel combining and splitting operations as long as the symmetric cut-off rate of W' and W'' is in (0,1), where the coding gain manifests itself as an increase in the sum cut-off rate of channels  $W^-$  and  $W^-$  compared to W' and  $W^+$ . In this paper we use the parameters J(W)and I(W) together to show that  $\{\mathcal{C}_n^{(m)}\}$  achieves I(W) of an arbitrary W, whereas the parameter Z(W) will be used to characterize polarization performance of  $\{\mathcal{C}_n^{(m)}\}$ .

## III. POLARIZATION WITH HIGHER-ORDER MEMORY

We develop a method to design a family of code sequences  $\{\mathcal{C}_n^{(m)}; n \ge 1, m \ge 1\}$  with code-length  $N = N(n,m) = O(\phi^n), \phi \in (1,2]$ , and fixed memory order m.  $\{\mathcal{C}_n^{(m)}\}$  is based on the channel polarization idea of Arıkan in [1]. This section is devoted to explaining the design, encoding and decoding of  $\{\mathcal{C}_n^{(m)}\}$ , while preparing some grounds for investigating its characteristics in the following sections.

## A. Channel Combining

Consider an arbitrary B-DMC, W, where its N independent uses take the form  $W(\mathbf{y}_N|\mathbf{x}_N) = \prod_{i=1}^N W(y_i|x_i), \mathbf{x}_N \in \mathcal{X}^N$ ,  $\mathbf{y}_N \in \mathcal{Y}^N$ . Let  $\mathbf{u}_N \in \mathcal{X}^N$  be the binary information vector that needs to be transmitted over N uses of W. Channel combining phase creates a vector channel  $W_n : \mathcal{X}^N \to \mathcal{Y}^N$  of the form

$$W_n(\mathbf{y}_N|\mathbf{u}_N) = \prod_{i=1}^N W(y_i|x_i),$$

where  $\mathbf{x}_N = \mathbf{u}_N \mathbf{G}_N$ .  $\mathbf{G}_N$  is an  $N \times N$  encoding matrix where encoding takes place in GF(2).

Let  $\mathbb{N}_n = \{1, 2, \dots, N\}$ ,  $N = O(\phi^n)$ , denote the set of the indices at the channel combining level n. There are N binary-input channels in  $W_n$  to transmit information. We index those channels as  $W_n^{(i)}$ ,  $i \in \mathbb{N}_n$ , and demonstrate the channel combining operations in Fig 1. Inspecting this figure



Fig. 1: Recursive construction of the vector channel  $W_n$ from  $W_{n-1}$  and  $\hat{W}_{n-m}$ , where  $W_n^{(i)}$ ,  $i \in \mathbb{N}_n$ , denotes the binary-input channels in  $W_n$ . The arrows on the left show the directions of flow for the binary-inputs of  $W_n^{(i)}$  and  $\oplus$  is the XOR operation. The arrows on the right show the outputs of successive uses of W. The XOR operations that take place on the dotted arrows within  $W_{n-1}$  and  $\hat{W}_{n-1}$  are not shown as they obey the same recursion.

observe that we index the topmost binary-input channel of  $W_n$  as  $W_n^{(1)}$  and index *i* of  $W_n^{(i)}$  increases as one move downwards. The vector channel  $W_n$  is obtained by combining  $W_{n-1}$  with  $\hat{W}_{n-m}$ . To accomplish this combining we apply XOR operations on the binary-inputs of  $W_n$  and transmit the resultant bits through the inputs of  $W_{n-1}$  and  $\hat{W}_{n-m}$ . By continuing the same recursion within  $W_{n-1}$  and  $\hat{W}_{n-m}$ , the encoded bits are transmitted through independent uses of W channels because we start the combining recursion by choosing  $W_0 = W_{-1} = \ldots = W_{1-m} = W$ . If we use the binary-input channels  $W_n^{(1)}, W_n^{(2)}, \ldots, W_n^{(N)}$  to transmit the symbols  $u_1, u_2, \ldots, u_N$ , respectively, the encoding matrix  $\mathbf{G}_N$ can be expressed as

$$\mathbf{G}_{N} = \begin{bmatrix} \mathbf{G}_{N(n-1)} & \mathbf{G}_{N(n-m)} \\ \mathbf{0}_{2} & \mathbf{0}_{1} & \mathbf{G}_{N(n-m)} \end{bmatrix}, \quad n \ge 1$$
(18)

where  $\mathbf{G}_{N(0)} = \mathbf{G}_{N(-1)} = \ldots = \mathbf{G}_{N(1-m)} = [1]$ , and  $\mathbf{0}_1$ and  $\mathbf{0}_2$  are  $N(n-m) \times N(n-1)$  and  $(N(n-1) - N(n-m)) \times N(n-m)$  all zero matrices, respectively. Observe that when  $m = 1, \mathbf{0}_2$  matrix vanishes and  $\mathbf{G}_N$  can be represented as  $\mathbf{G}_n = (\mathbf{F}_2^\mathsf{T})^{\otimes n}$ , where  $\mathbf{F}_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$  is the Kernel used by Arıkan in [1]. However, when m > 1,  $\mathbf{G}_N$  can not be represented via Kronecker power.

## B. Channel Ordering

After performing channel combining operation we have to define an order to split the vector  $W_n : \mathcal{X}^N \to \mathcal{Y}^N$  and obtain N binary-input channels. This ordering is carried out with the help of a permutation  $\pi_n : \mathbb{N}_n \to \mathbb{N}_n$ . The  $W_n^{(i)}$  channels in  $W_n$  are split in increasing  $\pi_n(i)$  values (from 1 to N) so that each  $W_n^{(i)}$  channel is of the form  $W_n^{(i)} : \mathcal{X} \to \mathcal{Y}^N \times \mathcal{X}^{\pi(i)-1}$ . In order to explain this operation we associate a unique state vector  $\mathbf{s}_n^{(i)}$  with each  $W_n^{(i)}$  channel, which has the form

$$\mathbf{s}_{n}^{(i)} = (s_{1}^{(i)}, s_{2}^{(i)}, \dots, s_{n}^{(i)}),$$

where

$$\mathbf{s}_{k}^{(i)} \in \{+, -, \star\}, \quad k = 1, 2, \dots, n$$

 $s_k^{(i)}$  terms will be referred as a "state" and we use  $+, -, \bigstar$  symbols to track down the channel transformations that  $W_n^{(i)}$  channels undergo as  $n = 1, 2, \ldots$ . States +, - will correspond to the polarization transforms  $\boxplus$  and  $\boxminus$ , as defined in (9) and (10), respectively; whereas state  $\bigstar$  will correspond to a non-polarizing transform. We let

$$\mathcal{S}_n = \{ \mathbf{s}_n^{(i)} : i \in \mathbb{N}_n \}$$
(19)

to be the set of all possible state vectors at level *n*. Since each  $\mathbf{s}_n^{(i)} \in S_n$  is unique (as we will show shortly) we have  $|S_n| = N$  and  $S_n \subset \{+, -, \star\}^n$ . The vectors,  $\mathbf{s}_n^{(i)} \in S_n$ , are assigned recursively from  $\mathbf{s}_{n-1}^{(j)} \in S_{n-1}$ , with a state assigning procedure  $\varphi_n : S_{n-1} \to S_n$ . The operation of  $\varphi_n$  is explained in the following definition.

**Definition 1.** State Vector Assigning Procedure: Let  $s_{n-1}^{(j)} \in S_{n-1}$  be the state vector of  $W_{n-1}^{(j)}$ . The state vectors  $s_n^{(i)} \in S_n$ , associated with  $W_n^{(i)}$  take the form

$$\boldsymbol{s}_{n}^{(j)} = (\boldsymbol{s}_{n-1}^{(j)}, \boldsymbol{\star}), \qquad j \in \mathbb{N}_{n-1} \setminus \mathbb{N}_{n-m}.$$
(21)

Investigating the above definition, as also demonstrated in Fig. 2, we observe that  $\varphi_n$  appends a new state,  $\{+, -, \star\}$ , to  $\mathbf{s}_{n-1}^{(j)} \in S_{n-1}$  in order to construct  $\mathbf{s}_n^{(i)} \in S_n$ . For  $j \in \mathbb{N}_{n-m}$ ,  $\varphi_n$  appends + and - to  $\mathbf{s}_{n-1}^{(j)}$  to obtain  $\mathbf{s}_n^{(j)}$  and  $\mathbf{s}_n^{(j+N_{n-1})}$ , respectively. For  $j \in \mathbb{N}_{n-1} \setminus \mathbb{N}_{n-m}$ ,  $\varphi_n$  appends  $\star$  to  $\mathbf{s}_{n-1}^{(j)}$  in order to construct  $\mathbf{s}_n^{(j)}$ . Because of the inherent memory in the combining procedure, it is difficult to obtain closed form expressions for  $\mathbf{s}_n^{(i)}$ , for any i and m. Nevertheless, with the above definition one can recursively obtain  $\mathbf{s}_n^{(i)}$ , by applying



Fig. 2: State labeling procedure  $\varphi_n : S_{n-1} \to S_n$ . State vectors  $\mathbf{s}_n^{(i)} \in S_n$ , are obtained by appending a new state  $\{+, -, \mathbf{\star}\}$ , to the vectors  $\mathbf{s}_{n-1}^{(j)} \in S_{n-1}$ .

 $\varphi_1, \varphi_2, \ldots, \varphi_n$ . With the following proposition, we give the formal structure of the possible state vector,  $\mathbf{s}_n^{(i)}$ , and thus the set  $S_n$ .

**Proposition 2.** Let  $s_n$ ,  $s_n \in S_n$ , be a valid state vector one can obtain after applying  $\varphi_1, \varphi_2, \ldots, \varphi_n$ . Only the transitions between  $s_k$  and  $s_{k+1}$ ,  $k = 1, 2, \ldots, n$ , that are shown in the state transition diagram of Fig. 3 are possible, where the imposed initial condition is  $s_1 \in \{+, -\}$ .

The above proposition is a direct consequence of the channel combining and state vector assigning procedure,  $\varphi_n$ , and it can be verified by induction through stages  $\varphi_1, \varphi_2, \dots, \varphi_n$ .

**Proposition 3.** The state vector  $\mathbf{s}_n^{(i)} \in S_n$ ,  $i \in \mathbb{N}_n$ , assigned to each  $W_n^{(i)} \in \mathcal{W}_n$  is unique.

The above proposition will be crucial for the ongoing analysis as it states that each  $W_n^{(i)}$  is uniquely addressable by  $\mathbf{s}_n^{(i)}$ . We will use this fact to obtain the ordering  $\pi_n$ . Before accomplishing this, we obtain binary vectors  $\mathbf{b}_n^{(i)} = (b_1^{(i)}, b_2^{(i)}, \ldots, b_n^{(i)}), b_k^{(i)} \in \mathcal{X}, k = 1, 2, \ldots, n$ , from  $\mathbf{s}_n^{(i)}$ , which will allows us to sort and provide an order. The mapping between  $\mathbf{s}_n^{(i)}$  and  $\mathbf{b}_n^{(i)}$  is obtained as

$$b_k^{(i)} = \begin{cases} 0 & \text{if } s_k^{(i)} \in \{-, \star\}, \\ 1 & \text{if } s_k^{(i)} = +, \end{cases} \qquad k = 1, 2, \dots, n.$$
 (22)

We notice that although both  $s_k^{(i)} = -$  and  $s_k^{(i)} = \star$  are mapped as  $b_k^{(i)} = 0$ , the  $\mathbf{b}_n^{(i)}$  vectors will also be unique for each *i* because every state - in  $\mathbf{s}_n^{(i)}$  is followed by m - 1



Fig. 3: Possible state transitions observed between  $s_k$  and  $s_{k+1}$ , k = 1, 2, ..., n.

occurrences of state  $\star$ , and the distinction between different  $\mathbf{s}_n^{(i)}$  is hidden in the location of + states in  $\mathbf{s}_n^{(i)}$ . The following definition uses this uniqueness property to obtain the ordering,  $\pi_n$ . It is an adaptation of the bit-reversed order of Arıkan in [1] to the proposed coding scheme.

**Definition 2.** Bit-Reversed Order: Let  $(\boldsymbol{b}_n^{(i)})_2$  denote value of  $\boldsymbol{b}_n^{(i)}$  in Mod-2 as  $(b_1^{(i)}, b_2^{(i)}, \dots, b_n^{(i)})_2$  where  $b_1^{(i)}$  is the most significant bit. The uniqueness of  $\boldsymbol{b}_n^{(i)}$  for each i ensures the existence of a permutation  $\pi_n : \mathbb{N}_n \to \mathbb{N}_n$ , so that for some  $i, j \in \mathbb{N}_n$ , we have  $\pi_n(i) < \pi_n(j)$  if  $(\boldsymbol{b}_n^{(i)})_2 < (\boldsymbol{b}_n^{(j)})_2$ .

Therefore the bit-reversed order  $\pi_n$  is obtained in terms of increasing  $(\mathbf{b}_n^{(i)})_2$  values.

Notice that the binary input channels  $\hat{W}_{n-m}^{(j)}$ ,  $j \in \mathbb{N}_{n-m}$ , of Fig. 1 have no effect in the recursive state assigning procedure,  $\varphi_n$ , and thus in the bit-reversed order. Their sole purpose is to provide auxiliary channels for the combining process. In fact, the N(n-m) inputs of  $\hat{W}_{n-m}$  can be combined with the N(n-1) inputs of  $\hat{W}_{n-1}$  in  $\frac{N(n-1)!}{N(n-m)!}$  different ways. However, we deliberately align the inputs of  $W_{n-1}$  and  $\hat{W}_{n-m}$  so that the first N(n-m) inputs of  $W_{n-1}$  are combined, respectively, with the the first N(n-m) inputs of  $\hat{W}_{n-m}$  as shown in Fig. 1. This alignment in the combining process will be crucial in the next section when we investigate the evolution of binary-input channels in a probabilistic setting, because the channel pairs,  $W_{n-1}^{(j)}$  and  $\hat{W}_{n-m}^{(j)}$ , share the same state history as explained in the following proposition.

**Proposition 4.** Let  $\mathbf{s}_{n-1}^{(j)} = (s_1, s_2, \dots, s_{n-1}) \in S_{n-1}$ be the state vector of  $W_{n-1}^{(j)}$ . Channel  $\hat{W}_{(n-m)}^{(j)}$  shares the same state history with  $W_{(n-1)}^{(j)}$ , through combining stages  $1, 2, \dots, n-m$ , in the sense that its state vector is  $\mathbf{s}_{n-m}^{(j)} = (s_1, s_2, \dots, s_{n-m}) \in S_{n-m}$ .

## C. Channel Splitting

We assume a genie-aided decoding mechanism where the  $W_n^{(i)}$  channels are decoded successively in increasing  $\pi_n(i)$  values, from 1 to N, and the genie provides the true values of already decoded bits. The decoder has no knowledge of the future bits that it will decode. With these assumptions  $W_n^{(i)}$  is the effective bit-channel that this genie-aided decoder faces

while trying to decode its next bit. Let us define  $u_n^{(i)} \in \mathcal{X}$  as

$$u_n^{(i)}$$
 = binary input of the channel  $W_n^{(i)}$ ,

and for  $i, j \in \mathbb{N}_n$  let

$$\mathbf{u}_{n,b}^{(i)} \stackrel{\Delta}{=} (u_n^{(j)} : \pi_n(j) < \pi_n(i)), 
\mathbf{u}_{n,a}^{(i)} \stackrel{\Delta}{=} (u_n^{(j)} : \pi_n(j) > \pi_n(i)).$$
(23)

 $\mathbf{u}_{n,b}^{(i)}$  and  $\mathbf{u}_{n,a}^{(i)}$  are the information vectors that are decoded, by the genie-aided decoder, before and after  $u_n^{(i)}$ , respectively. The length of  $\mathbf{u}_{n,b}^{(i)}$  is  $\pi_n(i) - 1$  and the length of  $\mathbf{u}_{n,a}^{(i)}$  is  $N - \pi_n(i)$  so that  $\mathbf{u}_{n,b}^{(i)} \in \mathcal{X}^{\pi_n(i)-1}$  and  $\mathbf{u}_{n,a}^{(i)} \in \mathcal{X}^{N_n - \pi_n(i)}$ . The following definition formalizes the transition probabilities of the  $W_n^{(i)}$  channels.

$$W_n^{(i)} \stackrel{\Delta}{=} \sum_{\mathbf{u}_{n,a}^{(i)}} \Pr\left(\mathbf{y}_N, \mathbf{u}_{n,a}^{(i)}, \mathbf{u}_{n,b}^{(i)} | u_n^{(i)}\right).$$
(24)

The above definition indicates that  $W_n^{(i)}$  is the posterior probability of an arbitrary B-DMC obtained at channel combining and splitting level n. The genie-aided decoder has no knowledge of  $\mathbf{u}_{n,a}^{(i)}$ , therefore it averages the joint probability of all outputs and all inputs over  $\mathbf{u}_{n,a}^{(i)}$  and takes  $\mathbf{y}_N$  and  $\mathbf{u}_{n,b}^{(i)}$  as the effective output (observation) of the combined channels. Hence each  $W_n^{(i)}$  has input  $u_n^{(i)} \in \mathcal{X}$  and output  $(\mathbf{y}_N, \mathbf{u}_{n,b}^{(i)}) \in \mathcal{Y}^N \times \mathcal{X}^{\pi_n(i)-1}$ .

**Proposition 5.** The transition probabilities of  $W_n^{(i)}$  channels take the following forms

$$W_{n}^{(j)} = \hat{W}_{n-m}^{(j)} \boxplus W_{n-1}^{(j)}, \qquad j \in \mathbb{N}_{n-m}, \quad W_{n}^{(j+N_{n-1})} = \hat{W}_{n-m}^{(j)} \boxminus W_{n-1}^{(j)}, \qquad j \in \mathbb{N}_{n-m}, \quad (25)$$

$$W_n^{(j)} = \gamma(n) W_{n-1}^{(j)}, \qquad j \in \mathbb{N}_{n-1} \setminus \mathbb{N}_{n-m}, \quad (26)$$

where  $\gamma(n) = \Pr(y_{N(n-1)+1}, y_{N(n-1)+2}, \dots, y_N)$  and  $W_0 = W_{-1} = \dots = W_{1-m} = W$ .

The above proposition is illustrated in Fig. 4. In order to provide a proof for the above proposition and explain the underlying idea behind the bit-reversed order we make the following analysis. Investigating Fig. 4, we see that the overall effect of XOR operations, after channel splitting, is to provide diversity paths for the N(n-m) inputs of  $W_{n-1}$  in the sense that for  $j \in \mathbb{N}_{n-m}$  we have  $W_n^{(j)} = \hat{W}_{n-m}^{(j)} \boxplus W_{n-1}^{(j)}$ . Therefore the input of  $W_n^{(j)}$  is transmitted through both  $\hat{W}_{n-m}^{(j)}$  and  $\hat{W}_{n-m}^{(j)}$ . Notice that in order to provide this diversity, the inputs of  $W_n^{(j)}$  indicating  $\pi_n(j) > \pi_n(j+N(n-1))$  must be decoded, by the genie-aided decoder, before the inputs of  $W_n^{(j)}$  indicating  $\pi_n(j) > \pi_n(j+N(n-1))$  must hold. Thanks to the bit-reversed order, as explained in Definition. 2, this requirement can be easily accomplished. To see this consider the state vectors  $\mathbf{s}_{n-1}^{(j)}$  of  $W_{n-1}^{(j)}$  to which one appends + and - in order to construct  $\mathbf{s}_n^{(j)}$  and  $\mathbf{s}_n^{(j+N(n-1))}$ , respectively. After this operation, the mapping between  $\mathbf{s}_n^{(i)}$ 



Fig. 4: Transition probabilities of  $W_n^{(i)}$  channels after combining and splitting  $W_{n-1}$  and  $\hat{W}_{n-m}$ .

and  $\mathbf{b}_n^{(i)}$ , as given by (22), indicates that  $\mathbf{b}_n^{(j)} = (\mathbf{b}_{n-1}^{(j)}, 1)$  and  $\mathbf{b}_n^{(j+N(n-1))} = (\mathbf{b}_{n-1}^{(j)}, 0)$  holds. Therefore

$$(\mathbf{b}_n^{(j)})_2 > (\mathbf{b}_n^{(j+N(n-1))})_2, \quad n = 1, 2, \dots$$

and by Definition 2,  $\pi_n(j) > \pi_n(j + N(n-1))$  holds for all  $n \ge 1$ . On the other hand, in order to decode  $W_n^{(j+N_{n-1})}$ correctly, the inputs of  $W_{n-1}^{(j)}$  and  $\hat{W}_{n-m}^{(j)}$  must be decoded correctly indicating we must have  $W_n^{(j+N(n-1))} = \hat{W}_{n-m}^{(j)} \square$  $W_{n-1}^{(j)}$ . The above analysis, by induction through combining and splitting stages  $1, 2, \ldots, n$  proves (25). In order to prove (26), we inspect that for  $j \in \mathbb{N}_{n-1} \setminus \mathbb{N}_{n-m}$  the channel  $W_n^{(j)}$  is as good as  $W_{n-1}^{(j)}$  in the sense that the genie-aided decoder can always decode  $W_{n-1}^{(j)}$  instead of  $W_n^{(j)}$ . Inspecting Fig. 4 we notice that the binary-input of  $W_n^{(j)}$  is not transmitted through the inputs of  $\hat{W}_{n-m}$ . Therefore, the combining of  $\hat{W}_{n-m}$  with  $W_{n-1}$  does not provide any new information regarding the input of  $W_n^{(j)}$ . This, in turn, indicates that  $W_n^{(j)}$  is the same as  $W_{n-1}^{(j)}$  except for a scaling factor  $\gamma(n)$ , as in (26).

# D. Effects of Channel Combining and Splitting on the Symmetric Capacity

Let us define  $I_n^{(i)} = I(W_n^{(i)})$  and analyze the implications of Proposition 5. Equation (25) states that the channel pairs,

 $\hat{W}_{n-m}^{(j)}$  and  $W_{n-1}^{(j)}$ ,  $j \in \mathbb{N}_{n-m}$ , undergo a polarization transform,  $\boxminus$  and  $\boxminus$ , from which two new channels,  $W_n^{(j)}$  and  $W_n^{(j+N_{n-1})}$ , emerge. In the light of (14) we have

$$I_n^{(j)} \ge \max\{I_{n-1}^{(j)}, I_{n-m}^{(j)}\}, \quad j \in \mathbb{N}_{n-m}.$$
 (27)

Therefore, the injection of  $\hat{W}_{n-m}^{(j)}$  allows  $W_n^{(j)}$  to be superior channel compared to  $\hat{W}_{n-m}^{(j)}$  and  $W_{n-1}^{(j)}$ . This comes with the expense that now  $W_n^{(j+N(n-1))}$  is an inferior channel compared to  $\hat{W}_{n-m}^{(j)}$  and  $W_{n-1}^{(j)}$  because, from (15), one has

$$I_{n}^{(j+N(n-1))} \leq \min\{I_{n-1}^{(j)}, I_{n-m}^{(j)}\}, \quad j \in \mathbb{N}_{n-m}.$$
 (28)

Although  $I_n^{(j)}$  and  $I_n^{(j+N(n-1))}$  move away from  $I_{n-1}^{(j)}$  and  $I_{n-m}^{(j)}$ , the transformations preserve the symmetric capacity because, as indicated by (13), we have

$$I_n^{(j)} + I_n^{(j+Nn-1)} = I_{n-1}^{(j)} + I_{n-m}^{(j)}, \quad j \in \mathbb{N}_{n-m}.$$
 (29)

The remaining channels  $W_n^{(j)}$ ,  $j \in \mathbb{N}_{n-1} \setminus \mathbb{N}_{n-m}$ , in Equation (26), do not see any polarization transforms as their transition probabilities are scaled by  $\Pr(y_{N(n-1)+1}, \ldots, y_N)$  with respect to  $W_{n-1}^{(j)}$ . This scaling, in turn, results in

$$I_n^{(j)} = I_{n-1}^{(j)}, \quad j \in \mathbb{N}_{n-1} \backslash \mathbb{N}_{n-m}.$$
(30)

All in all, the combining and splitting of  $W_{n-1}$  and  $W_{n-m}$  preserves the sum symmetric capacity as

$$\sum_{i \in \mathbb{N}_n} I_n^{(i)} = \sum_{j \in \mathbb{N}_{n-1}} I_{n-1}^{(j)} + \sum_{k \in \mathbb{N}_{n-m}} I_{n-m}^{(k)}, \qquad (31)$$

## E. Decoding

We will take successive cancellation decoding (SCD) of [1] as the default decoding method for  $\{\mathcal{C}_n^{(m)}\}$ . The genieaided decoder that we have explained in Section III.B and the definition of  $W_n^{(i)}$  as given by (24) already provide us a guideline for SCD. The only difference is, during the calculation of (24), SCD uses its own estimates for the vector  $\mathbf{u}_{n,b}^{(i)}$ , which we denote as  $\hat{\mathbf{u}}_{n,b}^{(i)}$ .

Likelihood ratios (LRs) should be preferred in SCD so that one can eliminate the  $P(y_{N_{n-1}+1}, y_{N_{n-1}+1}, \ldots, y_{N_n})$  term in (26). The LR for the channel  $W_n^{(i)}$  is defined as

$$L_n^{(i)} \triangleq \frac{\sum_{\mathbf{u}_{n,a}^{(i)}} \Pr\left(\mathbf{y}_N, \mathbf{u}_{n,a}^{(i)}, \hat{\mathbf{u}}_{n,b}^{(i)}|0\right)}{\sum_{\mathbf{u}_{n,a}^{(i)}} \Pr\left(\mathbf{y}_N, \mathbf{u}_{n,a}^{(i)}, \hat{\mathbf{u}}_{n,b}^{(i)}|1\right)}$$

By using the LR relations given in [1] for  $\boxplus$  and  $\boxminus$  transformations and from Proposition 5 we obtain

$$L_{n}^{(j)} = L_{n-1}^{(j)} (L_{n-m}^{(j)})^{1-2\hat{u}_{n}^{(j+N_{n-1})}},$$

$$L_{n}^{(j+N_{n-1})} = \frac{L_{n-1}^{(j)} L_{n-m}^{(j)} + 1}{L_{n-1}^{(j)} + L_{n-m}^{(j)}},$$

$$L_{n}^{(j)} = L_{n-1}^{(j)}, \qquad j \in \mathbb{N}_{n-1} \setminus \mathbb{N}_{n-1}.$$
(32)
(32)

Therefore, while decoding  $W_n^{(i)}$  one only needs to calculate 2N(n-m) LRs as given by (32) while the remaining

N - N(n - m) LRs for (33) are the same as the previous level. This fact can be exploited to avoid unnecessary decoding complexity in hardware implementation.

## F. Code-Length

Recall that the code-length N = N(n,m) obeys the recursion in (1) with initial conditions of (2). It is easy to show that N can be calculated as

$$N = \sum_{i=1}^{m} c_i (\rho_i)^n,$$
 (34)

where each  $\rho_i$ , i = 1, 2, ..., m, is a root of the *m*th order polynomial equation

$$F(m,\rho) = \rho^m - \rho^{m-1} - 1,$$
(35)

and constants,  $c_i$ , are calculated by using the initial conditions in (2) together with (34).

**Proposition 6.** For  $m \ge 1$ , let  $\phi \in (1,2]$  be a real root of  $F(m, \rho)$ .

- i)  $\phi$  is unique, i.e., there is only one real root in  $\in (1, 2]$ .
- ii) If  $\rho_i \neq \phi$  we have  $\sqrt{\rho_i \rho_i^*}/\phi < 1$  indicating  $\phi$  is the the largest magnitude root of  $F(m, \rho)$ .
- iii)  $\phi$  is decreasing in increasing m.

Part *ii* of the above proposition indicates that, as *n* gets large, the summation in (34) will be dominated by  $\phi^n$  term therefore the code-length will scale as  $N = \kappa \phi^n = O(\phi^n)$  where  $\kappa > 0$  is the constant scaler of  $\phi^n$  in (34). Part *iii* of Proposition 6 implies that as *m* increases the code-length increases less rapidly in *n* which we have mentioned in the beginning of the paper.

## G. Code Construction

The following proposition is a generalization of [1, Prop. 5] and it's proof is omitted.

**Proposition 7.** If W is a BEC, then  $W_n^{(i)}$  channels obeying the transition probabilities as given by Proposition 5 are also BECs.

In order to use  $\{\mathcal{C}_n^{(m)}\}\)$  one has to fix a code parameter vector  $(W, N, K, \mathcal{A})$ , where W is the underlying B-DMC, N is the code-length, K is the dimensionality of the code, and  $\mathcal{A} \subseteq \mathbb{N}_n$  is the set of information carrying symbols. We have  $|\mathcal{A}| = K$  and K/N = R, where  $R \in [0, 1]$  is the rate of the code.

Let  $P_{e,n}^{(i)}$ ,  $i \in \mathbb{N}_n$ , denote the bit-error probability of  $W_n^{(i)}$  with SCD. Code construction problem is choosing the set  $\mathcal{A}$  so that  $\sum_{i \in \mathcal{A}} P_{e,n}^{(i)}$  is minimum. This problem can be analytically solved only when W is a BEC [1] since for this case the  $W_n^{(i)}$  channels are also BECs (Proposition 7) and the Bhattacaryya parameters of  $W_n^{(i)}$ , which we denote as  $Z_n^{(i)}$ , obey  $P_{e,n}^{(i)} = Z_n^{(i)}$ . In this case, in the light of (16)-(17) and Proposition 5,

 $Z_n^{(i)}$  terms can be recursively calculated as

$$Z_{n}^{(j)} = Z_{n-1}^{(j)} Z_{n-m}^{(j)}, \qquad j \in \mathbb{N}_{n-m}, Z_{n}^{(j+N_{n-1})} = Z_{n-1}^{(j)} + Z_{n-m}^{(j)} - Z_{n-1}^{(j)} + Z_{n-m}^{(j)}, \qquad j \in \mathbb{N}_{n-m}, Z_{n}^{(j)} = Z_{n-1}^{(j)} \qquad j \in \mathbb{N}_{n-1} \setminus \mathbb{N}_{n-1}.$$

The case when W is not a BEC is a well-studied problem, where one approximates a suitable reliability measure for  $W_n^{(i)}$ channels and uses this measure to choose the set  $\mathcal{A}$ . We refer the reader to [5] for an overview.

## IV. CHANNEL POLARIZATION

Channel polarization should be investigated by observing the evolution of the set  $\{W_n^{(i)} : i \in \mathbb{N}_n\}$  as *n* increases. To track this evolution we use the state vectors  $\mathbf{s}_n^{(i)} \in \mathcal{S}_n$  assigned to  $W_n^{(i)}$  because each  $W_n^{(i)}$  is uniquely addressable by its  $\mathbf{s}_n^{(i)}$ .

# A. Probabilistic Model for Channel Evolution

We define a random process  $\{S_n\}$  and a random vector  $\mathbf{S}_n = (S_1, S_2, \ldots, S_n)$  obtained from the process  $\{S_n\}$  where the state vectors,  $\mathbf{s}_n = (s_1, s_2, \ldots, s_n)$ ,  $\mathbf{s}_n \in S_n$ , of Section II, are the realizations of  $\mathbf{S}_n$ . The process  $\{S_n\}$  can be regarded as a tree process where  $\mathbf{s}_n$  form the branches of the tree where we illustrate it in Fig. 5 for the case m = 2. Since  $|S_n| =$ N = N(n), there are N(n) different branches at tree level n. The process  $\{S_n\}$  starts with the initial conditions  $S_1 \in$  $\{+, -\}$ . At tree level n, N(n) new branches emerge from N(n-1) branches of level n-1. We assume that each branch is observed with identical probability

$$\Pr(\mathbf{S}_n = \mathbf{s}_n) = \frac{1}{N(n)}.$$
(36)

This, in turn, implies that each valid state transition of Fig. 3, between  $s_{n-1}$  and  $s_n$ , has probability N(n-1)/N(n). Investigating this figure, consider the case m = 1, which coincides with Arıkan's setup in [1], where there are two possible states as  $S_n \in \{+, -\}$  and  $|S_n| = N(n) = 2^n$ . Since transitions between  $S_{n-1}$  and  $S_n$  are valid if  $S_n \in \{+, -\}$  and  $S_{n-1} \in \{+, -\}$ , each possible transition has probability N(n-1)/N(n) = 1/2. Consequently, the process  $\{S_n\}$  is composed of independent realizations of Bernoulli(1/2) random variables as  $\Pr(S_n = +) = \Pr(S_n = -) = 1/2$ . On the other hand, when m > 1, there exists a memory in the state transition model as depicted in Fig. 3. Therefore, the process  $\{S_n\}$  can be modeled as a Markov process with order m - 1 in the sense that

$$\Pr(S_n | \mathbf{S}_{n-1}) = \Pr(S_n | S_{n-1}, S_{n-2}, \dots, S_{n-(m-1)}).$$

Throughout the paper we find it easier to work with the random vector  $\mathbf{S}_n$  keeping in mind the Markovian property of the process  $\{S_n\}$ .

We define a random channel process  $\{K_n\}$ , driven by  $\{S_n\}$ , as  $K_n = W_{S_1,S_2,...,S_n}$ . The realizations of  $K_n$  are  $k_n = W_{s_1,s_2,...,s_n}$  and they correspond to the binary-input channels,  $W_n^{(i)}$ , with state vectors  $\mathbf{s}_n = (s_1, s_2, ..., s_n) \in S_n$ .



Fig. 5: Illustration of the evolution of  $\{S_n\}$  as a tree for the case m = 2, where each branch is a state vector  $\mathbf{s}_n \in S_n$ .

In order to obtain a characterization for the process  $\{K_n\}$  we fix  $(s_1, s_2, \ldots, s_{n-1})$  to be the state vector associated with  $W_{n-1}^{(j)}$ ,  $j \in \mathbb{N}_{n-m}$  and let  $k_{n-1} = W_{n-1}^{(j)}$ . In the light of Proposition 4, we know that the state vector of  $\hat{W}_{n-m}^{(j)}$  is  $(s_1, s_2, \ldots, s_{n-m})$  indicating  $k_{n-m} = \hat{W}_{n-m}^{(j)}$ . Investigating the operation of  $\varphi_n : S_{n-1} \to S_n$  in Fig. 2, we observe that the state vectors of  $W_n^{(j)}$  and  $W_n^{(j+N_{n-1})}$  are  $(s_1, s_2, \ldots, s_{n-1}, +)$  and  $(s_1, s_2, \ldots, s_{n-1}, -)$ , respectively. From Proposition 5 we notice that  $W_n^{(j)} = \hat{W}_{n-m}^{(j)} \boxplus W_{n-1}^{(j)}$  and  $W_n^{(j)}$  metry  $k_{n-1} = \hat{W}_{n-1}^{(j)}$  body. These observations, in turn, indicate  $k_n = k_{n-1} \boxplus k_{n-m}$  holds. These observations, in turn, indicate  $k_n = k_{n-1} \boxplus k_{n-m}$  holds when  $s_n = -$ . Next, we fix  $(s_1, s_2, \ldots, s_{n-1})$  to be the state vector associated with  $W_{n-1}^{(j)}$ ,  $j \in \mathbb{N}_{n-1} \setminus \mathbb{N}_{n-m}$  and hence  $k_{n-1} = W_{n-1}^{(j)}$ . From the operation of  $\varphi_n$  :  $S_{n-1} \to S_n$  we know that the state vector of  $W_n^{(j)}$  is  $(s_1, s_2, \ldots, s_{n-1}, \star)$  and Proposition 5 tells us  $W_n^{(j)} = \gamma(n)W_{n-1}^{(j)}$ . Combining these facts tells us  $k_n = \gamma(n)k_{n-1}$  holds if  $s_n = \star$ . The above analysis relates  $k_n$  to  $k_{n-1}$  and  $k_{n-m}$  for all  $s_n \in \{+, -, \star\}$ , which we formally present with the below recursion.

$$K_n = \begin{cases} K_{n-m} \boxplus K_{n-1} & \text{if } S_n = +, \\ K_{n-m} \boxminus K_{n-1} & \text{if } S_n = -, \\ \gamma(n)K_{n-1} & \text{otherwise}, \end{cases}$$
(37)

where  $K_n = W$  for n < 1.

## B. Polarization:

We define the processes  $\{I_n : n \ge 1\}$  and  $\{J_n : n \ge 1\}$ where  $I_n = I(K_n) \in [0, 1]$  and  $J_n = J(K_n) \in [0, 1]$ . In [1] Arıkan shows that  $I_n$  converges to a random variable  $I_{\infty}$  as  $\Pr(I_{\infty} = 1) = I(W)$  and  $\Pr(I_{\infty} = 0) = 1 - I(W)$ . This result indicates that the synthesized binary-input channels,  $W_n^{(i)}$ , either become error-free or useless. We will show that the same holds for polar codes with higher-order memory as well. This result is presented with the following theorem.

**Theorem 1.** For any fixed  $m \ge 1$  and for some  $\delta \in (0,1)$ as *n* tends to infinity, the probability of  $I_n \in (1 - \delta, 1]$  goes to I(W) and the probability of having  $I_n \in [0, \delta)$  goes to 1 - I(W). *Proof:* We investigate the polarization of  $\{J_n\}$  towards 0 and 1 as it will imply the polarization of  $\{I_n\}$  as well. We write  $E[J_n] = \sum_{\mathbf{s}_n} \Pr(\mathbf{S}_n = \mathbf{s}_n) J_n = \frac{1}{N(n)} \sum_{\mathbf{s}_n} J_n$  to denote the expected value of  $J_n$  and  $\{E[J_n] : n \ge 1\}$  to denote the deterministic sequences obtained from  $E[J_n]$ . The following lemma will be crucial for the proof

## Lemma 1.

$$E[J_n] \ge \mu E[J_{n-1}] + (1-\mu) E[J_{n-m}], \qquad (38)$$

where  $\mu = N(n-1)/N(n)$  and the above equality is achieved only if  $J_{n-1} \in \{0,1\}$  or  $J_{n-m} \in \{0,1\}$  holds for all  $S_n \in \{+,-\}$ 

We apply a decimation operation on the sequence  $\{E[J_n]\}\$ and obtain a subsequence  $\{E[\hat{J}_k] : k = 1, 2, \dots, \lfloor n/m \rfloor\}$ , where the decimation operation is performed as

$$E[\hat{J}_k] = \min_{i \in \{0, 1, \dots, m-1\}} \left\{ E[J_{km-i}] \right\}.$$
 (39)

The elements of  $\{E[\hat{J}_k]\}\$  are obtained by choosing the minimum of m consecutive and non-overlapping elements of  $\{E[J_n]\}\$ .

**Lemma 2.** The sequence  $\{E[\hat{J}_k]\}$  is monotonically increasing in the sense that

$$E[\hat{J}_k] \ge E[\hat{J}_{k-1}].$$

We know that  $E[\hat{J}_k]$  is bounded in [0, 1] and since  $\{E[\hat{J}_k]\}$  is monotonically increasing, from the monotone convergence theorem [6, p. 21.] we conclude that there exists a unique limit for  $\{E[\hat{J}_k]\}$  in the sense that

$$\lim_{k \to \infty} E[\hat{J}_k] = \sup\{E[\hat{J}_k]\}.$$
(40)

Next, we let n = km - i in Lemma 1 to obtain

$$E[J_{km-i}] \ge \mu E[J_{km-(i+1)}] + (1-\mu)E[J_{(k-1)m-i}].$$
 (41)

We fix *i* such that  $E[J_{km-i}] = E[\hat{J}_k]$  is satisfied. For any choice of *i* observe that  $E[J_{(k-1)m-i}] \ge E[\hat{J}_{k-1}]$  and  $E[J_{km-(i+1)}] \ge \min\{E[\hat{J}_k], E[\hat{J}_{k-1}]\} \ge E[\hat{J}_{k-1}]$  hold. Using these results in (41) gives

$$E[\hat{J}_k] \ge \mu E[\hat{J}_{k-1}] + (1-\mu)E[\hat{J}_{k-1}] \ge E[\hat{J}_{k-1}]$$
(42)

Therefore, the monotonic increase in  $E[\hat{J}_k]$  will continue until the inequality in Lemma 1 is achieved with equality. This fact, together with the convergence of  $E[\hat{J}_k]$ , indicates that conditioned on the event  $\{S_n : S_n \in \{+, -\}\}$  either  $\lim_{n\to\infty} J_{n-1} \in \{0, 1\}$  or  $\lim_{n\to\infty} J_{n-m} \in \{0, 1\}$  holds, indicating

$$\lim_{n \to \infty} J_n \in \{0, 1\}, \quad S_n \in \{+, -\}.$$
(43)

Investigating the operation of  $\varphi_n : S_{n-1} \to S_n$  in Fig.2 we see that

$$\Pr(S_n \in \{+, -\}) = \frac{2N(n-m)}{N(n)} \ge 0,$$
(44)

which implies that the event  $\{S_n : S_{n-1} \in \{+, -\}\}$  occurs infinitely many times as  $n \to \infty$  and  $\sum_{n \to \infty} \Pr(S_{n-1} \in \{+, -\})$  diverges. Consequently, and by using the first Borel Contelli lemma [7, p. 36] we conclude that

$$\lim_{n \to \infty} \Pr(J_n \in \{0, 1\}) = 1$$

One to one correspondence between  $J_n$  and  $I_n$  implies

$$\lim_{n \to \infty} \Pr(I_n \in \{0, 1\}) = 1,$$

and having  $E[I_n] = I(W)$  results in

$$\lim_{n \to \infty} \Pr(I_n = 1) = I(W)$$

and

$$\lim_{m \to \infty} \Pr(I_n = 0) = 1 - I(W).$$

which completes the proof.

## C. A Typicality Result

In this section we use the Method of Types to investigate the state vectors,  $\mathbf{s}_n$ , obtained from the realizations of the process  $\{S_n\}$ . We let  $s \in \{+, -, \star\}$  and write  $P_{\mathbf{s}_n}^{(s)}$ ,  $P_{\mathbf{s}_n}^{(s)} \in [0, 1]$ , to denote the type (frequency) of s in  $\mathbf{s}_n$  as

$$P_{\mathbf{s}_n}^{(s)} = \#(\mathbf{s}_n|s)/n$$

where  $\#(\mathbf{s}_n|s)$  denotes the number times the symbol *s* occurs in  $\mathbf{s}_n$ . Investigating the state transition diagram of Fig. 3 we inspect that, as *n* gets large,  $P_{\mathbf{s}_n}^{(\bigstar)} = (m-1)P_{\mathbf{s}_n}^{(-)}$  holds because each – state in  $\mathbf{s}_n$  is followed by m-1 occurrences of state  $\bigstar$ . As the remaining states in  $\mathbf{s}_n$  will be +, we must have  $P_{\mathbf{s}_n}^{(+)} = 1 - mP_{\mathbf{s}_n}^{(-)}$  indicating  $P_{\mathbf{s}_n}^{(+)} \in [0, 1], P_{\mathbf{s}_n}^{(-)} \in [0, \frac{1}{m}]$ , and  $P_{\mathbf{s}_n}^{(\bigstar)} \in [0, \frac{m-1}{m}]$ . As it tuns out, depending on  $P_{\mathbf{s}_n}^{(s)}$ , not all realizations of  $\{S_n\}$  are observed with the same probability. This is explained with the following theorem.

**Theorem 2.** As *n* gets large, except for a vanishing fraction of  $\mathbf{s}_n \in S_n$ , and for some  $\epsilon \in (0, 1)$  we have

$$\begin{aligned} |P_{s^n}^{(-)} - p^-| &\leq \epsilon, \\ |P_{s^n}^{(+)} - p^+| &\leq \epsilon, \\ |P_{s^n}^{(\bigstar)} - p^\bigstar| &\leq \epsilon, \end{aligned}$$

where  $p^- = \frac{\phi - 1}{1 + m(\phi - 1)}$ ,  $p^{\bigstar} = (m - 1)p^-$  and  $p^+ = 1 - mp^-$ .

Therefore we can consider  $p^+$ ,  $p^-$  and  $p^*$  as the frequencies of states +, -, and \*, in  $s_n$ , respectively, that one typically observes as n gets large.

**Proof of Theorem 2 :** The proof is based on the Method of Types [8]. We let  $q \in [0, 1/m]$  and define

$$\mathcal{T}_{n}^{(q)} = \{ \mathbf{s}^{n} : P_{\mathbf{s}^{n}}^{(-)} = q \}.$$
(45)

 $\mathcal{T}_n^{(q)}$  is a type class and it consists of  $\mathbf{s}_n$  having  $nq \in [0, n/m]$  occurrences of state -. For all  $m \ge 1$ , there are at most n+1 different such type classes. However, the number of all possible  $\mathbf{s}_n$ ,  $|\mathcal{S}_n|$ , increases exponentially in n as  $|\mathcal{S}_n| = N =$ 

 $O(\phi^n)$ . The Method of Types ensures the existence of a type class with exponentially many elements. Our aim is to find this type class. Recalling that each  $\mathbf{s}_n$  is observed with probability 1/N, the probability of observing a given  $\mathbf{s}_n$  in  $\mathcal{T}_n^{(q)}$  is

$$\Pr\left(\mathbf{s}_n \in \mathcal{T}_q^n\right) = \frac{|\mathcal{T}_q^n|}{N}.$$

Lemma 3.

$$|\mathcal{T}_{q}^{n}| < 2^{n(G(m,q)+o(1))}.$$
(46)

where

$$G(m,q) = (1 - (m-1)q))H\left(\frac{q}{1 - (m-1)q}\right),\,$$

and H is the binary entropy function.

Investigating G(m,q) we observe that it is a concave function of  $q \in [0, 1/m]$ . We establish a similarity between  $\frac{\partial G(m,q)}{\partial q}$  and  $F(m, \rho)$  in (35). The following proposition is a direct consequence of this result.

**Lemma 4.** The function G(m,q) attains its maximum when  $q = p^-$  and its maximum value is

$$G(m, p^-) = \log \phi.$$

Consequently, for every  $\mathcal{T}_n^{(q)}$  with  $|q-p^-| > 0$  there exists a  $D(q, p^-) > 0$  such that

$$D(q, p^{-}) \stackrel{\Delta}{=} G(m, p^{-}) - G(m, q),$$
  
= log  $\phi - G(m, q).$ 

Using the above fact in (46) results in

$$|\mathcal{T}_n^{(q)}| \leqslant \phi^n 2^{n\left(-D(q,p^-)+o(1)\right)}.$$

From the above result and the fact that  $N = O(\phi^n)$  we obtain

$$\Pr(\mathbf{s}_n \in \mathcal{T}_n^{(q)}) \leqslant 2^{-n\left(D(q, p^-) + o(1)\right)},\tag{47}$$

The above result shows that depending on  $D(q, p^-)$ , and in turn q, the probabilities of some type classes decay exponentially in n. The following proposition results from this fact.

**Proposition 8.** As *n* tends to infinity  $D(q, p^-)$  converges to 0 with probability 1.

The above proposition implies the convergence of q to  $p^$ as well, because  $D(q, p^-)$  is 0 only if  $q = p^-$ . Therefore among all  $T_n^{(q)}$ , one observes the ones with  $|q - p^-| \leq \epsilon$  with probability 1.

## D. Rate of Polarization

We define the Bhattacharyya process  $\{Z_n\}$  where  $Z_n = Z(K_n)$  is the Bhattacharyya parameter of the random channel  $K_n$ . By using the channel evolution model in (37), this process can be expressed as

$$Z_{n} \begin{cases} = Z_{n-1}Z_{n-m} & \text{if } S_{n} = +, \\ \leqslant Z_{n-1} + Z_{n-m} - Z_{n-1}Z_{n-m} & \text{if } S_{n} = -, \\ = Z_{n-1} & \text{otherwise}, \end{cases}$$
(48)

where  $Z_n = Z(W)$  for n < 1.

**Theorem 3.** For any  $\epsilon \in (0, 1)$  there exists an n such that for  $\beta < p^+$  we have

$$\Pr\left(Z_n \leqslant 2^{-\phi^{n\beta}}\right) \geqslant I(W) - \epsilon, \tag{49}$$

*Proof:* We consider another process  $\{\hat{Z}_n\}$ , driven by  $\{S_n\}$ , so that for  $i = 1, 2, ..., n_0$ ,  $n_0 < n$ , we have  $\hat{Z}_i = Z_i$  and for  $i > n_0$ ,  $\hat{Z}_i$  obeys

$$\hat{Z}_{i} = \begin{cases}
\hat{Z}_{i-1}\hat{Z}_{i-m} & \text{if } S_{n} = +, \\
\hat{Z}_{i-1} + \hat{Z}_{i-m} - \hat{Z}_{i-1}\hat{Z}_{i-m} & \text{if } S_{n} = -, \\
\hat{Z}_{i-1} & \text{otherwise.}
\end{cases}$$
(50)

Comparing (48) and (50) we observe that  $Z_n$  is stochastically dominated by  $\hat{Z}_n$  in the sense that for some  $f_n \in (0,1)$ ,  $\Pr(Z_n \leq f_n) \ge \Pr(\hat{Z}_n \leq f_n)$ . For the proof it will suffice to show that  $\Pr(\hat{Z}_n \leq f_n) \ge I(W) - \epsilon$  holds for  $f_n = 2^{-\phi^{n\beta}}$ and  $\beta < p_+$ .

In [9, Lemma 1] authors derive an upper bound on  $Z_n$ , for the case m = 1, by using the frequency of state + in the realizations of  $\{S_{n_0+1}, S_{n_0+2}, \ldots, S_n\}$  and the fact that  $Z_{n_0}$ gets arbitrarily close to 0, with probability I(W), when  $n_0$ is large enough. Following lemma is a generalization of this approach for arbitrary  $m \ge 1$ .

**Lemma 5.** For some  $\zeta \in (0, 1)$  and  $\gamma \in (0, 1)$  define the events

$$C_{n_0}(\zeta) = \{Z_{n_0} \leq \zeta\},\$$
  
$$D_{n_0}^n(\gamma) = \{\#((S_{n_0+1}, \dots, S_n)|+) \ge \gamma(n-n_0)\}.$$

We have

$$\hat{Z}_n \leqslant 2^{-\phi^{(\gamma-\epsilon)(n-n_0)}}, \quad C_{n_0}(\zeta) \cap D_{n_0}^n(\gamma).$$

From the convergence of  $Z_n$  to  $Z_\infty$  with probability  $Pr(Z_\infty = 0) = I(W)$  we know that for any  $\epsilon \in (0, 1)$  there exist a fixed  $n_0$  such that

$$\Pr(C_{n_0}(\zeta)) \ge I(W) - \epsilon.$$

Next, from Theorem 2, we infer that when  $m \ll n - n_0$ 

$$\Pr(D_{n_0}^n(\gamma)) \ge 1 - \epsilon, \quad \gamma \ge p^+ - \epsilon \tag{51}$$

holds. This results from the fact that the probability of observing + in  $\{S_{n_0+1}, \ldots, S_{n_0}\}$  approaches to  $p^+$  when  $n - n_0$  is much larger than the memory, m, of the process  $\{S_n\}$ .

Choosing  $n_0 = n\epsilon$  and using the above results in lemma 5 gives

$$\Pr\left(\hat{Z}_n \leqslant 2^{-\phi^{n(p^+ - 2\epsilon)(1-\epsilon)}}\right) \ge (1-\epsilon)(I(W) - \epsilon)$$
$$\ge I(W) - \epsilon$$

Since  $\epsilon \in (0, 1)$  can be chosen arbitrarily close to 0, the above result indicates that

$$\Pr\left(\hat{Z}_n \leqslant 2^{-\phi^{n\beta}}\right) \ge I(W) - \epsilon$$

holds for  $\beta < p^+$ .

Let us analyze the implications of Theorem 3 on the block-decoding error probability,  $P_e$ , of  $\{\mathcal{C}_n^{(m)}\}$ . It states that for  $I(W) - \epsilon$  fraction of  $W_n^{(i)}$  the corresponding Bhattacharyya parameters will be bounded as  $Z_n^{(i)} \leq 2^{-\phi^{n\beta}}$  for  $\beta < p^+$ . We have  $P_e \leq \sum_{i=1}^N Z_n^{(i)} \leq N2^{-\phi^{n\beta}} = O(2^{-\phi^{n\beta}})$ . Since the code-length of  $\{\mathcal{C}_n^{(m)}\}$  scales as  $N = O(\phi^n)$  we also see that  $P_e = O(2^{-N^{\beta}})$  holds for  $\beta < p^+$ .

The term  $p^+$  is plotted in Fig. 6 as a m increases from 1 to 50. Investigating this figure we see that  $p^+$  equals to 0.5 when m = 1 which coincides with the bound for the exponent of polar codes presented by Arıkan and Telatar in [3]. As m increases from 1 to 50,  $p^+$  and thus the achievable exponent decreases. The decrease is more steep for small values of m and it becomes more monotone as m increases.

In order to fully characterize the asymptotic performance of  $\{\mathcal{C}_n^{(m)}\}\$  one needs to provide a converse bound on  $\beta$  which may be a difficult task. We believe that for the case m > 1, the achievable  $\beta$  for  $\{\mathcal{C}_n^{(m)}\}$  may show a dependency on the rate,  $R \in [0,1]$ , chosen for the code; a phenomenon that does not exist when m = 1 (see [10]). In order explain our conjecture, consider the process  $\{\hat{Z}_n\}$  in (50) which we use to obtain an achievable bound on  $\beta$  as  $\beta < p^+$ . Our proof is based on the observation that once the realizations of  $Z_{n_0}$ are sufficiently close to 0, which happens with probability I(W), the scaling of  $Z_n$  is mostly determined by the number of occurrences of state + in  $\{S_{n_0+1}, S_{n_0+2}, \ldots, S_n\}$ . From Theorem 2 we know that one typically observes  $(n - n_0)p^+$ occurrences of + in  $\{S_{n_0+1}, S_{n_0+2}, \ldots, S_n\}$ , therefore the value of  $\log Z_n$  decreases  $(n - n_0)p^+$  times with the same speed as the code-length,  $\log \hat{Z}_n = \log \hat{Z}_{n-1} + \log \hat{Z}_{n-m}$ , scaling as  $\log Z_n = -\phi^{(n-n_0)p^+} = -\phi^{n(1-\epsilon)p^+}$ . This result in the achievable exponent  $\beta < p^+$ . However, when m > 1the value of  $\log \hat{Z}_n$  may also decrease with a faster rate compared to that of the code-length. To see this, consider the case  $(S_{n-1}, S_{n-2}, ..., S_{n-(m-1)}) = (\star, \star, ..., \star)$  and  $S_n = +$ , where we have  $\hat{Z}_{n-1} = \hat{Z}_{n-2} = \ldots = \hat{Z}_{n-(m-1)}$ and  $\log \hat{Z}_n = \log \hat{Z}_{n-1} + \log \hat{Z}_{n-m} = \log \hat{Z}_{n-1}^2$ . Therefore, there may be times where  $\log Z_n$  decreases with a faster rate as  $\log \hat{Z}_n = \log Z_{n-1}^2$  instead of  $\log \hat{Z}_n = \log \hat{Z}_{n-1} + \log \hat{Z}_{n-m}$ and this may result in a higher achievable  $\beta$ . In order to quantify this we need to know not only the number of times state + occurs in  $\{S_n\}$ , but also the number of times a state + in  $\{S_n\}$  is preceded by  $\star$  states. Therefore, we need to refine Theorem 2 in terms of the number of transitions between states +, - and  $\star$ , as well. This might be a difficult but important problem whose solution will provide a full characterization of the asymptotic polarization performance of  $\{\mathbb{C}_n^{(m)}\}$  and we leave it as a future work.

## V. COMPLEXITY AND SPARSITY

## A. Encoding and Decoding Complexity

We consider a single core processor with random access memory and investigate the time complexity of encoding and decoding of  $\{\mathcal{C}_n^{(m)}\}$ . Let  $\chi_n^E$  denote the complexity for encoding the information vector  $\mathbf{u}_N$  to encoded bits  $\mathbf{x}_N$ .



Fig. 6: Achievable exponent,  $\beta < p^+$ , as scaled with m.

We take complexity of each XOR operation as 1 unit. By inspection of Fig 1, we have

$$\chi_{n}^{E} = \chi_{n-1}^{E} + \chi_{n-m}^{E} + N_{n-m} \quad n, m \ge 1,$$
 (52)

where  $\chi_1^E = 1$  and  $\chi_0^E = \chi_{-1}^E = \ldots = \chi_{1-m}^E = 0$ . Similarly, let  $\chi_n^D$  denote the complexity for decoding the

inputs of  $W_n^{(i)}$  channels, where SCD is the decoding method. We take the complexity of computing the LR. relations in (32) as 1 unit. We observe that one does not make any operations to calculate the LR in (33). By inspection of Fig 1, we have

$$\chi_n^D = \chi_{n-1}^D + \chi_{n-m}^D + 2N_{n-m} \quad n, m \ge 1,$$
 (53)

where  $\chi_0^D = \chi_{-1}^D = \ldots = \chi_{1-m}^D = 0$ . The recursions in (52) and (53) are cumbersome to deal with. To observe the scaling behavior of  $\chi_n^E$  and  $\chi_n^D$  in m, we define

$$\eta^E \stackrel{\Delta}{=} \frac{\chi_n^E}{N \log N}, \quad \eta^D \stackrel{\Delta}{=} \frac{\chi_n^D}{N \log N}, \tag{54}$$

and demonstrate the scaling of  $\eta^E$  and  $\eta^D$  in Fig .7, where we have numerically calculated  $\chi^E_n$  and  $\chi^D_n$  as in (52) and (53) by choosing  $N = O(\phi^n)$  to be the code-length closest to  $10^4$ and  $10^6$ . From Fig. 7 we observe that, there exist a decrease in  $\eta_n^E$  and  $\eta_n^D$  as *m* increases, where the decrease is more steep for small values of m and it becomes more monotone as *m* increases. This decrease in complexity, although not being orders of magnitude, is promising in showing the existence of polar codes requiring lower complexity. For example, from Fig. 7 we observe that  $\eta_n^D$  is around 1/2 when m = 12. This indicates that the decoding complexity of  $\{\mathbb{C}_n^{(12)}\}$  is reduced by half compared to  $\{\mathcal{C}_n^{(1)}\}$  which is the polar code presented by Arıkan in [1].

## B. Sparsity

As we have explained in Section II, there exist a sparsity in the channel combining process in the sense that at each combining level, the vector channel  $W_n$  is obtained by combining  $W_{n-1}$  and  $W_{n-m}$  which are obtained from N(n-1) and N(n-m) uses of underlying B-DMC, W, respectively. From Proposition 5 we observe that the overall



Fig. 7: Scaling of encoding and decoding complexities as mincreases where N is chosen to be the code-length closest to  $1 \times 10^4, 1 \times 10^6.$ 

effect of channel combining and splitting is that, at each level n, there exist N(n-m) bit-channel pairs that participate in  $\boxplus$  and  $\boxminus$  transforms. As *m* increases N(n-m) decreases with respect to N(n-1) implying the fraction of bit-channels participating in  $\square$  and  $\square$  transforms also decreases. On the other hand, as m increases, the code-length increases less rapidly in n because  $N = O(\phi^n)$  and  $\phi$  is decreasing in m, thus one can fit more channel combining and splitting levels within fixed code-length. A natural question is to understand the overall effect of increasing m on the total number of  $\boxplus$ and  $\square$  transforms that one can obtain when the number of uses of W channels is fixed. The importance of  $\chi_n^D$  in (53) comes to play at this point because it gives us the total number of  $\boxplus$  and  $\square$  transformation that are recursively applied to independent uses of W channels to obtain the bit-channels in  $W_n$ . Consequently, one can view  $\eta_D$  as a packing ratio in the sense that one can pack  $\eta_n^D N \log N$  recursive applications of  $\boxplus$  and  $\boxminus$  transformation to N independent uses of W. Inspecting the scaling of  $\eta_D$  in Fig. 7 we observe that this packing ratio is 1 when m = 1 and it decreases with increasing m, and this decrease manifests itself as a reduction in the decoding complexity of  $\{\mathcal{C}_n^{(m)}\}$ .

## VI. CONCLUSION AND FUTURE WORK

We have introduced a method to design a class of code sequences  $\{\mathcal{C}_n^{(m)}; n \ge 1, m \ge 1\}$  with code-length N = $O(\phi^n), \phi \in (1,2]$ , and memory order m. The design of  $\{\mathbb{C}_n^{(m)}\}$  is based on the channel polarization idea of Arıkan [1] and  $\{\mathcal{C}_n^{(m)}\}$  coincides with the polar codes presented by Arikan when m = 1. We showed that  $\{\mathcal{C}_n^{(m)}\}$  achieves the symmetric capacity of arbitrary BDMCs for arbitrary but fixed m. We have obtained an achievable bound on the asymptotic polarization of performance of  $\{\mathcal{C}_n^{(m)}\}\$  as scaled with m and showed that the encoding and decoding complexities of  $\{\mathbb{C}_n^{(m)}\}$ decrease with increasing m. Our introduction of  $\{\mathcal{C}_n^{(m)}\}$  complements Arıkan's conjecture that channel polarization is a general phenomenon and it shows the existence of polar codes requiring lower complexity. Future work will include a rate dependent analysis and a converse result on the asymptotic polarization performance of  $\{\mathcal{C}_n^{(m)}\}$ .

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#### VII. APPENDIX

#### A. Proof of Proposition 1

We have  $J(W^-) = \frac{2}{1+Z(W^-)}$  and  $J(W^+) = \frac{2}{1+Z(W^+)}$ . By using (17) and (16) we obtain

$$J(W^{+}) + J(W^{-}) \ge \log \frac{2}{1 + Z(W')Z(W'')} + \log \frac{2}{1 + Z(W') + Z(W'') - Z(W')Z(W'')}$$
(55)  
$$= \log \frac{2}{1 + Z(W') + Z(W'') + w(W', W'')Z(W')Z(W'')}$$

where  $w(W',W'')=Z(W')+Z(W'')-Z(W')Z(W'')\leqslant 1$  indicating

$$J(W^{+}) + J(W^{-}) \ge \log \frac{2}{1 + Z(W')} + \log \frac{2}{1 + Z(W'')}$$
(56)  
=  $J(W') + J(W'').$ 

In order to have  $J(W^+) + J(W^-) = J(W') + J(W'')$ , the equalities in (55) and (56) must be achieved. From (17) we know that the equality in (55) is achieved only if  $Z(W') \in \{0, 1\}$  or  $Z(W'') \in \{0, 1\}$  or if W' and W'' are BECs. When  $(Z(W'), Z(W'')) \in (0, 1)^2$  we have w(W', W'') < 1 and the inequality in (56) is always strict, whether or not W' and W'' being BECs. Consider the case Z(W') = 1 or Z(W'') = 1, then we have w(W', W'') = 1 and the equalities in (55) and (56) are achieved. When Z(W') = 0 we have J(W') = 1, w(W', W'') = 0 and  $J(W^+) + J(W^-) = J(W') + J(W'')$ , and the case J(W') = 1 follows from the symmetry in (55) and (56). Hence the equalities in (55) and (56) are both achieved only if  $Z(W') \in \{0, 1\}$  or  $Z(W'') \in \{0, 1\}$ , or alternatively only if  $J(W') \in \{0, 1\}$  or  $J(W'') \in \{0, 1\}$ .

## B. Proof of Proposition 3

From the operation of  $\varphi_n$  in Defn. 1 we obtain  $S_1 = \{+, -\}$ such that  $\mathbf{s}_1^{(1)} = (+)$  and  $\mathbf{s}_1^{(2)} = (-)$ , indicating  $\mathbf{s}_1^{(1)}$  and  $\mathbf{s}_1^{(2)}$ are unique. Proof is by induction, assume that  $s_{n-1}^{(j)} \in S_{n-1}$ are unique. Let  $j \in \mathbb{N}_{n-m}$  and consider  $\mathbf{s}_{n-1}^{(j)}$  to whom by appending + and - one obtains  $\mathbf{s}_n^{(j)}$  and  $\mathbf{s}_n^{(j+N(n-1))}$ , respectively, indicating  $\mathbf{s}_n^{(j+N(n-1))}$  and  $\mathbf{s}_n^{(j)}$  are different from each other. Next, let  $j \in \mathbb{N}_{n-1} \setminus \mathbb{N}_{n-m}$  then  $\mathbf{s}_n^{(j)}$  are obtained by appending  $\star$  to  $\mathbf{s}_{n-1}^{(j)}$  which, by assumption, are unique. Combining the result we see that for all  $j \in \mathbb{N}_n$  the vectors  $s_n^{(j)} \in S_n$  are different from each other.

# C. Proof of Proposition 4

Investigating Fig 2 consider the operation of  $\varphi_{n-1}$  where  $s_{n-2}^{(k)} = (s_1, s_2, \ldots, s_{n-2}), k \in \mathbb{N}_{n-2}$ , holds at level n-1. Next, consider the operation of  $\varphi_{n-2}$  where one has  $s_{n-3}^{(k)} = (s_1, s_2, \ldots, s_{n-3})$  for  $k \in \mathbb{N}_{n-3}$ . In turn and by induction through  $\varphi_{n-2}, \varphi_{n-3}, \ldots, \varphi_{n-(m-1)}$  we conclude that  $s_{n-m}^{(j)} = (s_1, s_2, \ldots, s_{n-m}), j \in \mathbb{N}_{n-m}$ .

## D. Proof of Proposition 6

i) For m > 1 we have F(m, 1) = -1 < 0 and  $F(m, 2) = 2^{m-1} - 1 \ge 0$  so that there exists at least one real root in (1,2]. Proof is by contradiction, let  $\rho_1, \rho_2 \in (1,2]$  be two real roots of  $F(m, \rho)$  then from (35) we have

$$\rho_1^{m-1}(\rho_1 - 1) = 1, \tag{57}$$

$$\rho_2^{m-1}(\rho_2 - 1) = 1. \tag{58}$$

Let  $\rho_1 < \rho_2$ , then  $\rho_2^{m-1} > \rho_1^{m-1}$  and  $\rho_2 - 1 > \rho_1 - 1 > 0$ implying  $\rho_2^{m-1}(\rho_2 - 1) > 1$  if  $\rho_1^{m-1}(\rho_1 - 1) = 1$  which contradicts (58), carrying a similar analysis for  $\rho_1 < \rho_2$  also contradicts (58), which indicates  $\rho_1 = \rho_2 = \phi$ .

ii) Assume that  $\rho$  is a complex root of  $F(m, \rho)$ , with  $\sqrt{\rho\rho^*} = \sigma > 1$  where \* denotes the conjugate operation. Since the coefficients of  $F(m, \rho)$  are real, its complex roots must be in conjugate pairs. From (35)

$$\rho^{m-1}(\rho - 1) = 1,$$
  
$$\rho^{*^{m-1}}(\rho^* - 1) = 1.$$

Multiplying the above equations we obtain

$$\sigma^{2(m-1)}(\sigma^2 - 2Re(\rho) + 1) = 1,$$
  

$$\sigma^{2(m-1)}(\sigma^2 - 2\sigma\alpha + 1) = 1,$$
(59)

where  $0 \le \alpha < 1$ . In turn for any  $\rho$ ,  $\sigma$  must be a root of

$$g(\sigma, \alpha) = \sigma^{2(m-1)}(\sigma^2 - 2\sigma\alpha + 1) - 1,$$
 (60)

Observe that when  $\sigma$  is fixed  $g(\sigma,\alpha)$  is decreasing in  $\alpha.$  We also have

$$\frac{\partial g(\sigma, \alpha)}{\partial \sigma} = 2(m-1)\sigma^{2(m-1)-1}(\sigma^2 - 2\sigma\alpha + 1) + \sigma^{2(m-1)}(2\sigma - 2\alpha)$$

From (59) observe that  $(\sigma^2 - 2\sigma\alpha + 1) > 0$ , and since  $(2\sigma - 2\alpha) > 0$  for  $\sigma > 1$  we have  $\frac{\partial g(\sigma, \alpha)}{\partial \sigma} > 0$ . This indicates that

 $q(\sigma, \alpha)$  is increasing with  $\sigma$ . But  $\phi$  is a root of  $q(\sigma, \alpha)$  with  $\alpha = 1$  and thus  $g(\phi, 1) = 0$ . Since  $g(\sigma, \alpha)$  is decreasing in  $\alpha$  we have  $q(\phi, \alpha) \ge 0$  and  $q(\sigma, \alpha) = 0$  is only achieved if  $\sigma < \phi$  because  $g(\sigma, \alpha)$  is increasing with  $\sigma$ .

*iii*) Observe that for some  $\rho \in (1, 2]$  we have  $\frac{\partial F(m, \rho)}{\partial \rho} > 0$  so that  $F(m, \rho)$  is increasing in  $\rho$  and when  $\rho$  is fixed  $F(m,\rho)$  is also increasing in m. Assume that  $\rho_1, \rho_2 \in (1,2]$ are real roots of  $F(m_1, \rho)$  and  $F(m_2, \rho)$ , respectively, where  $m_1, m_2 \ge 1$ . Then  $f(m_1, \rho_1) < f(m_2, \rho_1)$  holds if  $m_2 > m_1$ and  $f(m_1, \rho_1) = f(m_2, \rho_2) = 0$  is satisfied only if  $\rho_1 < \rho_2$ .

# E. Proof of Lemma 1

L. Proof of Lemma 1 Let  $J_n^{(i)} = J(W_n^{(i)})$  denote symmetric cut-off rate of  $W_n^{(i)}$ . From Proposition 5 we know that for  $j \in \mathbb{N}_{n-m}$  we have  $W_n^{(j)} = W_{n-1}^{(j)} \boxplus W_{n-m}^{(j)}$  and  $W_n^{(j+N(n-1))} = W_{n-1}^{(j)} \boxplus W_{n-m}^{(j)}$ . Proposition 1 indicates that these transforms increase the sum cut-off rate as  $J_n^{(j)} + J_n^{(j+N(n-1))} \ge J_{n-1}^{(j)} + J_{n-1}^{(j)}$  where the equality is achieved only if  $J_{n-1}^{(j)} \in \{0,1\}$  or  $J_{n-m}^{(j)} \in \{0,1\}$ holds. For  $j \in \mathbb{N}_{n-1} \setminus \mathbb{N}_{n-m}$ , from Proposition 5, we have  $J_n^{(j)} = \gamma(n) J_{n-1}^{(j)}$  which implies  $J_n^{(j)} = J_{n-1}^{(j)}$ . Combining the above results gives above results gives

$$\sum_{i \in \mathbb{N}_n} J_n^{(i)} \ge \sum_{j \in \mathbb{N}_{n-1}} J_n^{(j)} + \sum_{k \in \mathbb{N}_{n-m}} J_n^{(k)},$$

where the equality is achieved only of if  $J_{n-1}^{(j)} \in \{0,1\}$  or  $J_{n-m}^{(j)} \in \{0,1\}$  holds for all  $j \in \mathbb{N}_{n-m}$ . In the probabilistic domain of SectionIV the above result is equivalent to

$$\sum_{\mathbf{s}_n \in \mathcal{S}_n} J_n \geqslant \sum_{\mathbf{s}_{n-1} \in \mathcal{S}_{n-1}} J_{n-1} + \sum_{\mathbf{s}_{n-m} \in \mathcal{S}_{n-m}} J_{n-m},$$

where the equality is achieved only of if  $J_{n-1} \in \{0,1\}$ or  $J_{n-m} \in \{0,1\}$  holds for all  $S_n \in \{+,-\}$ . Dividing both sides of the above inequality by 1/N(n) and using  $E[J_n] = \frac{1}{N(n)} \sum_{\mathbf{s}_n \in S_n} J_n$  we obtain

$$E[J_n] \ge \frac{N(n-1)}{N(n)} E[J_{n-1}] + \frac{N(n-m)}{N(n)} E[J_{n-m}].$$

Noticing  $\frac{N(n-1)}{N(n)} = \mu(n)$  and  $\frac{N(n-m)}{N(n)} = 1 - \mu(n)$  completes the proof.

#### F. Proof of Lemma 2

From (38) we have

$$E[J_n] \ge \mu E[J_{n-1}] + (1-\mu) E[J_{n-m}],$$
  
$$\ge \min\{E[J_{n-1}], E[J_{n-m}]\}, \qquad (61)$$

Let us define the set

$$\mathcal{E}_k^{(m)} \stackrel{\Delta}{=} \{ E_{km}, E_{km-1}, \dots, E_{km-(m-1)} \}.$$

By definition in (39) we have we have  $E[\hat{J}_k] = \min \mathcal{E}_k^{(m)}$ . Proof is by induction. We use (61) to upper bound the elements of  $\mathcal{E}_{k}^{(m)}$  with respect to  $\min \hat{\mathcal{E}}_{k-1}^{(m)} = E[\hat{J}_{k-1}]$ . Let n = km - (m-1) and use (61) to obtain

$$E_{km-(m-1)} \ge \min\{E_{(k-1)m}, E_{(k-1)m-(m-1)}\},\\ \ge \min \mathcal{E}_{k-1}^{(m)}$$

For i = 2, 3, ..., m - 1 assume

$$E_{km-(m-i)} \ge \min \mathcal{E}_{k-1}^{(m)}$$

holds. Next, let n = km - (m - (i + 1)) in (61) to write

$$E_{km-(m-(i+1))} \ge \min\{E_{km-(m-i)}, E_{(k-1)m-(m-(i+1))}\}$$

By assumption  $E_{km-(m-i)} \ge \min \mathcal{E}_{k-1}^{(m)}$  and by definition  $E_{(k-1)m-(m-(i+1))} \ge \min \mathcal{E}_{k-1}^{(m)}$  holds, indicating

$$E_{km-(m-(i+1))} \ge \min \mathcal{E}_{k-1}^{(m)}.$$

Combining the above results tells us for i = 1, 2, ..., m we have  $E_{km-(m-i)} \ge \min \mathcal{E}_{k-1}^{(m)} = E[\hat{J}_{k-1}]$  which indicates  $E[\hat{J}_k] \ge E[\hat{J}_{k-1}].$ 

# G. Proof of Lemma 3

In order to bound  $|\mathcal{T}_n^{(q)}|$  we decompose  $\mathcal{T}_n^{(q)}$  it into two different sets

$$\mathcal{T}_{n}^{(a,q)} \stackrel{\Delta}{=} \left\{ \mathbf{s}^{n} : P_{\mathbf{s}^{n}}^{(-)} = q, s_{n} = + \right\}$$
$$\mathcal{T}_{n}^{(b,q)} \stackrel{\Delta}{=} \left\{ \mathbf{s}^{n} : P_{\mathbf{s}^{n}}^{(-)} = q, s_{n} \neq + \right\}$$

and we have  $T_n^{(q)} = \mathcal{T}_n^{(a,q)} \cup \mathcal{T}_n^{(b,q)}$ . Recall that each state – in  $\mathbf{s}_n$  is followed by m-1 occurrences of state  $\star$ . In turn,  $\mathcal{T}_n^{(a,q)}$ consists of  $\mathbf{s}_n$  having k = nq,  $0 \le k \le n/m$ , occurrences of the vector  $\mathbf{a} = (-, \underbrace{\star}, \star, \ldots, \star)$  and n - km occurrences of m-1 times

state +. By combinatorial analysis we have

$$|\mathcal{T}_n^{(a,q)}| = \binom{n - (m-1)k}{k}.$$

 $\mathcal{T}_n^{(b,q)}$  consists of k-1 occurrences of the vector **a**, an occurrence of **b** =  $(-, \underbrace{0, 0, \ldots, 0}), 1 \leq p < m-1$ , and p times

n-mk-(p+1) occurrences of state +. The vector **b** can only occur in the last p+1 entries in  $s_n$  and it will be completed to a vector **a** if we had prolonged the channel combining operation  $m-1-p \leq m$  more levels. Therefore

$$|\mathcal{T}_n^{(b,q)}| \leq \binom{n+m-(m-1)k}{k}.$$

For some  $c \in \mathbb{Z}$  and  $d \in \mathbb{Z}$  with c < d we have  $\binom{d}{c} = \frac{d}{d-c}\binom{d-1}{c} \leqslant d\binom{d-1}{c}$ , using this fact we obtain

$$\binom{n+m-(m-1)k}{k} \leqslant (n+m)\binom{n+(m-1)-(m-1)k}{k}, \\ < (n+m)^2\binom{n+(m-2)-(m-1)k}{k}, \\ \vdots \\ < (n+m)^m\binom{n-(m-1)k}{k}$$

Then we have

$$\begin{aligned} |\mathcal{T}_{n}^{(q)}| &= |\mathcal{T}_{n}^{(a,q)}| + |\mathcal{T}_{n}^{(b,q)}|, \\ &< (1 + (n+m)^{m}) \binom{n - (m-1)k}{k}, \\ &< (1 + (n+m))^{m} \binom{n - (m-1)k}{k}, \\ &= 2^{nB(m,n)} \binom{n - (m-1)k}{k}, \end{aligned}$$
(62)

where  $B(m,n) = \frac{m \log(1+n+m)}{n} = o(1)$ . Next, we use the upper bound  $\binom{n}{k} \leq 2^{nH(k/n)}$  in [8] to upper bound  $\binom{n-(m-1)k}{k}$  as

$$\binom{n - (m - 1)k}{k} \leqslant 2^{n(1 - (m - 1)(k/n))H(\frac{(k/n)}{1 - (m - 1)(k/n)}},$$
$$= 2^{nG(m,q)}.$$
(63)

Combining (62) and (63)we obtain the desired bound as  $|T_n^{(q)}| < 2^{n(G(m,q)+B(m,n))} = 2^{n(G(m,q)+o(1))}$ .

#### H. Proof of Lemma 4

We have

$$G(m,q) = (1 - (m-1)q)H\left(\frac{(q)}{1 - (m-1)q}\right).$$

We know that, for  $q \in [0, 1/m]$ ,  $H(\frac{q}{1-(m-1)q})$  is concave in q and (1-(m-1)q) is linear in q indicating G(m,q) is concave in q. Let q\* denote the maximizer of G(m,q). The maximum of  $H(\frac{q}{1-(m-1)q})$  occurs when  $\frac{q}{1-(m-1)q} = \frac{1}{2}$  or equivalently when  $q = \frac{1}{m+1}$  and since (1-(m-1)q) is decreasing in q, we have  $q* \in [0, \frac{1}{m+1}]$ . We next evaluate  $\frac{\partial G(m,q)}{\partial q}$ 

$$\frac{\partial G(m,q)}{\partial q} = (m-1)\log(1-(m-1)q) + \log q - m\log(1-mq).$$

setting  $\frac{\partial G(m,q)}{\partial q}|_{q=q*} = 0$  gives

$$(m-1)\log(1-(m-1)q^*) + \log q^* = m\log(1-mq^*).$$
 (64)

Re-arranging the above equation we obtain

$$m \log \frac{(1 - (m - 1)q^*)}{1 - mq^*} + \log \frac{q^*}{1 - mq^*}$$
  
=  $\log \frac{(1 - (m - 1)q^*)}{1 - mq^*}.$  (65)

Let us use the following substitutions

$$\eta = \frac{1 - (m - 1)q^*}{1 - mq^*}, \quad \eta - 1 = \frac{q^*}{1 - mq^*}.$$

For  $q \in [0, \frac{1}{m+1}]$  we have  $\eta \in [1, 2]$ . Using the above substitutions in (65) we obtain

$$m\log\eta + \log(\eta - 1) = \log\eta,$$

or alternatively

$$\eta^m(\eta - 1) = \eta.$$

Dividing both sides of the above relation by  $\eta$  and re-arranging the terms we obtain

$$\eta^m - \eta^{m-1} - 1 = 0. \tag{66}$$

But the above polynomial is same as 35. Consequently from part *i* of Proposition. 6 we conclude that  $\eta = \phi$  which indicates that  $\frac{1-(m-1)q^*}{1-mq^*} = \phi$  and hence  $q^* = \frac{1}{1+m(\phi-1)} = p^-$ . Next we evaluate the maximum of G(m,q) attained at  $q = q^*$ .

$$G(m, q^*) = -q^* \log \frac{q^*}{1 - (m-1)q^*} + (mq^* - 1) \log \frac{1 - mq^*}{1 - (m-1)q^*}$$
(67)

Re-arranging (64) we observe that

$$\log \frac{q^*}{1 - (m - 1)q^*} = m \log \frac{1 - mq^*}{1 - (m - 1)q^*}$$

Using the above relation in (67) gives

$$G(m, q^*) = \log \frac{1 - (m - 1)q^*}{1 - mq^*} = \log \phi.$$

I. Proof of Proposition 8

We define a typical set  $\mathcal{T}_n^{(q,\epsilon)}$  as

$$\mathcal{T}_n^{(q,\epsilon)} = \{ \mathbf{s}_n : P_{s^n}^{(-)} = q, D(q, p^-) \leqslant \epsilon \}.$$

The probability that  $\mathcal{T}_n^{(q)}$  is not typical is

$$1 - \Pr(\mathcal{T}_{n}^{(q,\epsilon)}) = \sum_{\Pr(D(q,p^{-})>\epsilon)} \Pr(\mathcal{T}_{n}^{(q)}),$$

$$\stackrel{a}{\leq} \sum_{\Pr(D(q,p^{-})>\epsilon)} 2^{-n(D(q,s_{-})+o(1))},$$

$$\leq \sum_{\Pr(D(q,p^{-})>\epsilon)} 2^{-n(\epsilon+o(1))},$$

$$\stackrel{b}{\leq} (n+1)2^{-n(\epsilon+o(1))},$$

$$= 2^{-n(\epsilon+o(1))}, \qquad (68)$$

In the above derivation (a) follows from (47) and (b) follows from the fact that there exist at most n + 1 different type classes having  $\Pr(D(q, s_-) > \epsilon)$ . The above result indicates that  $\sum_{n\to\infty} \Pr(D(q, s_-) \ge \epsilon)$  converges, thus the expected number of the occurrences of the event  $D(q, s_-) > \epsilon$  for all n is finite. By using the first Borel Cantelli Lemma [7, p. 59] we conclude that  $D(q, s_-)$  converges to 0 with probability 1.

## J. Proof of Lemma 5

Conditioned on the event  $D_{n_0}^n(\gamma) = \#((s_{n_0+1},\ldots,s_n)| + ) \ge \gamma(n-n_0)$  there exists at least  $\gamma(n-n_0)$  occurrences of state + in  $\{S_{n_0+1}, S_{n_0+2}, \ldots, S_n\}$ . Investigating (50), we have  $\hat{Z}_n \le \hat{Z}_{n-1}$  when  $S_n = +$  and  $Z_n \ge Z_{n-1}$  when  $S_n \neq +$ . Moreover,  $Z_n$  is increasing in  $Z_{n-1}$  when  $S_n$  is fixed. Consequently, if we fix  $\hat{Z}_m$ , the largest value of  $\hat{Z}_n$  will occur if  $\{S_{n_0+1}, S_{n_0+2}, \ldots, S_n\}$  has the following realization

$$(1-\gamma)(n-n_0)/m$$
 times  
{ $\overline{\mathbf{a}, \mathbf{a}, \dots, \mathbf{a}}, \underbrace{+, +, \dots, +}_{\gamma(n-n_0) \text{ times}}$ }.

where  $\mathbf{a} = (-, \underbrace{\bigstar, \bigstar, \dots, \bigstar}_{m-1 \text{ times}})$ . In order to upper bound

 $\hat{Z}_n$  we assume that the above realization has occured for  $\{S_{n_0+1}, S_{n_0+2}, \ldots, S_n\}$ . During consecutive runs of +, the value of  $\log \hat{Z}_n$  increases with the same recursion as the codelength in (1) as  $\log \hat{Z}_n = \log \hat{Z}_{n-1} + \log \hat{Z}_{n-m}$ . This recursion happens  $\gamma(n-m)$  times and since the code-legth obeying the same recursion scales as  $\phi^{\gamma(n-m)}$ ,  $\phi \in (1, 2]$ , we have

$$\log \hat{Z}_n = \phi^{\gamma(n-n_0)} \log \hat{Z}_k,\tag{69}$$

where  $k = n_0 + (1 - \gamma)(n - m)$ . During consecutive runs of **a** the value of  $\hat{Z}_i$  does not change with respect to  $\hat{Z}_{i-1}$  when  $S_i = \star$  and it increases as  $\hat{Z}_i = \hat{Z}_{i-1} + \hat{Z}_{i-m} - \hat{Z}_{n-1}\hat{Z}_{i-m}$ when  $S_i = -$ . By construction of  $\{S_{n_0+1}, S_{n_0+2}, \dots, S_n\}$  each state - is preced by m - 1 occurances of  $\star$  therefore if  $S_i = -$  we have  $(S_{i-1}, S_{i-2}, \dots, S_{i-(m-1)}) = (\star, \star, \dots, \star)$ indicating  $\hat{Z}_{i-1} = \hat{Z}_{i-2} = \dots = \hat{Z}_{i-(m-1)}$ . Therefore during each occurance of state - in **a** we see the recursion  $\hat{Z}_{i-1} + \hat{Z}_{i-m} - \hat{Z}_{i-1}\hat{Z}_{i-m} = 2\hat{Z}_{i-1} - \hat{Z}_i^{(i)}$  or equvalently  $1 - \hat{Z}_i = (1 - \hat{Z}_i^{(i)})^2$ . This recursion occurs  $(1 - \gamma)(n - n_0)$ times resulting in  $1 - \hat{Z}_k = (1 - \hat{Z}_{n_0})^{2(1-\gamma)(n-n_0)}$  and  $\hat{Z}_k = 1 - (1 - \hat{Z}_{n_0})^{2(1-\gamma)(n-n_0)}$ . Next, employ the inequality  $\log x \leqslant x - 1, x \in [0, 1]$ , by letting  $x = \hat{Z}_k$  to obtain

$$\log \hat{Z}_k \leqslant -(1 - \hat{Z}_{n_0})^{2(1 - \gamma)(n - n_0)}.$$
(70)

Using (70) in (69) gives

$$\log \hat{Z}_n = -\phi^{\gamma(n-n_0)} (1 - Z_{n_0})^{2(1-\gamma)(n-n_0)},$$
  

$$\leqslant -\phi^{\gamma(n-n_0)} (1 - Z_{n_0})^{2(n-n_0)}$$
  

$$= -\phi^{(\gamma-\epsilon)(n-n_0)} \left( (1 - Z_{n_0})^2 \phi^\epsilon \right)^{(n-n_0)}.$$

Choose  $\zeta \in (0,1)$  so that  $\zeta \leq 1 - \phi^{\frac{-\epsilon}{2}}$  holds. Conditioned on  $C_{n_0}(\zeta) = \{Z_{n_0} \leq \zeta\}$  we have  $(1 - Z_{n_0})^2 \phi^{\epsilon} \geq 1$ , resulting in

$$\log_2 \hat{Z}_n \leqslant -\phi^{(\gamma-\epsilon)(n-m)}, \quad C_{n_0}(\zeta) \cap D_{n_0}^n(\gamma),$$

which proves the lemma.