# The Steiner triple systems of order 21 with a transversal subdesign $TD(3,6)^*$

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#### Abstract

We prove several structural properties of Steiner triple systems (STS) of order 3w+3 that include one or more transversal subdesigns TD(3, w). Using an exhaustive search, we find that there are 2004720 isomorphism classes of STS(21) including a subdesign TD(3,6), or, equivalently, a 6-by-6 latin square.

Keywords: Steiner triple system, subdesign, latin square, transversal design.

MSC: 51E10

#### 1. Introduction

A Steiner triple system of order v, or STS(v), is a pair  $(S, \mathcal{B})$  from a finite set S (called the support, or the point set, of the STS) of cardinality v and a collection  $\mathcal{B}$  of 3-subsets of S, called blocks, such that every two distinct elements of S meet in exactly one block. A transversal design TD(k, w) (in this paper, we only consider the case k = 3) is a triple  $(S, \mathcal{G}, \mathcal{B})$  that consists of a point set S of cardinality kw, a partition  $\mathcal{G}$  of S into k subsets, groups,

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of cardinality w, and a collection  $\mathcal{B}$  of k-subsets of S, blocks, such that every block intersects every group in exactly one point and every two points in different groups meet in exactly one block. As the support and (in the case of TD) the groups are uniquely determined by the block set, it is convenient to identify the system, STS or TD, with its block set. With this agreement, it is correct to say that an STS  $\mathcal{B}$  can include, as a subset, some STS or TD  $\mathcal{C}$ , in which case  $\mathcal{C}$  is called a sub-STS or sub-TD of  $\mathcal{B}$ , respectively. Two systems, STS or TD, are called isomorphic if there is a bijection between their supports, an isomorphism, that sends the blocks of one system to the blocks of the other. An isomorphism of a system  $\mathcal{B}$  to itself is called an automorphism; the set of all automorphisms of  $\mathcal{B}$  is denoted by  $Aut(\mathcal{B})$ .

Transversal designs TD(3, w) are equivalent to latin  $w \times w$  squares and known to exist for every natural w. The isomorphism classes of TD(3, w) correspond to the so-called main classes of latin squares; their number is known for w up to 11, see [6]. Steiner triple systems STS(v) exist if and only if  $v \equiv 1, 3 \mod 6$ , see, e.g., [3], the necessary condition being given by simple counting arguments.

The number of isomorphism classes of Steiner triple system is known for order up to 19 [9]. The classification of STS of higher orders is possible only with additional restrictions on the structure of STS. Among such restrictions, the most popular are restrictions on the automorphisms, see e.g. [1], [4], [5], [13], [16], [18], [8], restrictions on the maximal rank of the system [15], [7], requirement for the system to include a subsystem with certain fixed parameters.

Stinson and Seah [17] found that there are 284457 STS(19) with sub-STS(9). Kaski, Östergård, Topalova, and Zlatarski [11] classified STS(19) with sub-STS(7) and STS(21) that include three sub-STS(7) with disjoint supports (the last class coincides with the class of STS(21) of 3-rank at most 19). Recently, Kaski, Östergård, and Popa [10] counted all STS(21) with sub-STS(9) (and also, the STS(27) with sub-STS(13)). The number 12661527336 (respectively, 1356574942538935943268083236) of isomorphism classes of such systems is too large to admit any constructive enumeration; in particular, one cannot computationally check any required property for all these classes.

In the current paper, we classify the STS(21) with subdesigns TD(3,6), or saying in a different way, the STS(21) that include a latin  $6 \times 6$  square. We establish that there are 2004720 isomorphism classes of Steiner triple systems of order 21 with transversal subdesigns on 3 groups of size 6, including 599

systems with exactly three sub-TD(3,6) and 12 systems with exactly seven sub-TD(3,6). Considered class contains 393 non-isomorphic resolvable STS; none of them is doubly-resolvable.

In the next section, we prove some facts about the Steiner triple systems of order 3w + 3 that have a transversal subdesign on three groups of size w, mainly focused on the case w = 6. In Section 3, we present the results of computer-aided classification of STS(21) with a subdesign TD(3,6), including Table 1, which contains the number of found isomorphism classes classified by the number of subdesigns TD(3,6), STS(9), and the number of automorphisms. Section 4 contains a double-counting argument that validates the results of computing. In Section 5, we discuss the resolvability of the found STS and show that STS(21) with sub-TD(3,6) and only one sub-STS(9) cannot be resolvable.

## 2. Steiner triple systems with transversal subdesigns

We start with some theoretical considerations. If an STS(v) has a sub-TD(3, w), then v = 3w + u, where  $u \equiv 1, 3$  if w is even and  $u \equiv 0, 4$  if w is odd. The case u = 0 corresponds to the Wilson-type STS(3w) [19]; readily, such a system is the union of three STS(w) with mutually disjoint supports and a transversal design TD(3, w). The case u = 1 corresponds to the Wilson-type STS(3w + 1) [19]; again, it is easy to see that such a system is a union of three STS(w + 1) whose supports have one point in common and a transversal design TD(3, w).

The next case is u=3. We introduce a related concept. A subset  $\mathcal{C}$  of an STS  $\mathcal{B}$  is called an almost-sub-STS if  $\mathcal{C}=\mathcal{C}'\setminus\{T\}$  for some STS  $\mathcal{C}'$  and a triple T of  $\mathcal{C}'$  (note that T is not required to be a block of  $\mathcal{B}$ ); this triple is called missing for the almost-sub-STS  $\mathcal{C}$ .

**Lemma 1.** Let  $(A \cup B \cup C \cup D, \mathcal{B})$  be an STS(3w + 3) with a sub-TD  $(A \cup B \cup C, \{A, B, C\}, \mathcal{T})$ , where |A| = |B| = |C| = w and |D| = 3. Then  $\mathcal{B} = \mathcal{T} \cup \mathcal{B}_A \cup \mathcal{B}_B \cup \mathcal{B}_C$ , where the supports of  $\mathcal{B}_A$ ,  $\mathcal{B}_B$ ,  $\mathcal{B}_C$  are  $A \cup D$ ,  $B \cup D$ ,  $C \cup D$  respectively, and two of them are almost-sub-STS with the missing triple D, the remaining one being a sub-STS (in particular,  $w + 3 \equiv 1, 3 \mod 6$ ).

*Proof.* Assume first that D is one of the blocks of  $\mathcal{B}$ . In this case, it is easy to see that  $\mathcal{B}$  has a sub-STS(w+3) with support  $A \cup D$ . Take it as  $\mathcal{B}_A$ .

Similarly,  $\mathcal{B}$  has a sub-STS(w+3) with support  $B \cup D$ . Removing the block D, we obtain an almost-sub-STS  $\mathcal{B}_B$ . Similarly, we find an almost-sub-STS  $\mathcal{B}_C$ .

It remains to consider the case  $D \notin \mathcal{B}$ . In this case, we divide  $\mathcal{B} \setminus \mathcal{T}$  into three subsets:  $\mathcal{B}_A$  consists of the blocks that are subsets of  $A \cup D$ ,  $\mathcal{B}_B$  of the blocks that are subsets of  $B \cup D$ , and  $\mathcal{B}_C$  of subsets of  $C \cup D$  (note that any other block has points in at least two of A, B, C, and hence necessarily belongs to  $\mathcal{T}$ ). The blocks from  $\mathcal{B}_A$ ,  $\mathcal{B}_B$ , and  $\mathcal{B}_C$  cover

$$\frac{1}{2}|A| \cdot (|A|-1) + \frac{1}{2}|B| \cdot (|B|-1) + \frac{1}{2}|C| \cdot (|C|-1) + 3|A| + 3|B| + 3|C| + 3$$

$$= \frac{3}{2}w^2 + \frac{15}{2}w + 3$$

pairs of points, while the blocks from  $\mathcal{B}_A$  (similarly, from  $\mathcal{B}_B$  or from  $\mathcal{B}_C$ ) cover at least

$$\frac{1}{2}|A|\cdot(|A|-1)+3|A|=\frac{1}{2}w^2+\frac{5}{2}w$$

and at most

$$\frac{1}{2}|A| \cdot (|A| - 1) + 3|A| + 3 = \frac{1}{2}w^2 + \frac{5}{2}w + 3$$

of them. Since the number of the pairs covered by  $\mathcal{B}_A$  must be divisible by 3, we see that it is

$$\frac{1}{2}w^2 + \frac{5}{2}w \quad \text{or } \frac{1}{2}w^2 + \frac{5}{2}w + 3 \quad \text{if } w \equiv 0, 1 \bmod 3,$$

and it is

$$\frac{1}{2}w^2 + \frac{5}{2}w + 2 \quad \text{if } w \equiv 2 \bmod 3.$$

The last case is impossible because  $3(\frac{1}{2}w^2 + \frac{5}{2}w + 2) > \frac{3}{2}w^2 + \frac{15}{2}w + 3$ . We conclude that one of  $\mathcal{B}_A$ ,  $\mathcal{B}_B$ ,  $\mathcal{B}_C$ , say  $\mathcal{B}_A$ , has

$$\frac{1}{3} \cdot \left(\frac{1}{2}w^2 + \frac{5}{2}w + 3\right) = \frac{(w+3)(w+2)}{6}$$

blocks, while each of the other has one less. This means that  $(A \cup D, \mathcal{B}_A)$  is an STS(w+3) and  $(A \cup D, \mathcal{B}_B \cup \{D\})$ ,  $(A \cup D, \mathcal{B}_B \cup \{D\})$  are also STS(w+3), which proves the statement.

So, we see that if an STS(3w + 3) has a sub-TD(3, w), then it is split into this sub-TD(3, w), one sub-STS(w+3), and two almost-sub-STS(w+3). In general, there can be more than one sub-TD(3, w) and hence more than one such splittings. So, it is important to understand how these subsystems can intersect. The following lemma on the intersection of two almost-sub-STS generalizes the well-known and obvious fact that the intersection of two sub-STS is always a sub-STS.

**Lemma 2.** Assume that an STS  $\mathcal{B}$  has two almost-sub-STS  $\mathcal{B}'$  and  $\mathcal{B}''$  with the supports S' and S'', respectively. Then

- either  $|S' \cap S''| = 2$  and  $\mathcal{B}' \cap \mathcal{B}''$  is empty
- or  $|S' \cap S''| \equiv 1, 3 \mod 6$  and  $\mathcal{B}' \cap \mathcal{B}''$  can be completed to  $STS(|S' \cap S''|)$  by adding 0, 1, or 2 blocks.

*Proof.* Denote  $D := S' \cap S''$  and  $\mathcal{D} := \mathcal{B}' \cap \mathcal{B}''$ . Assume that  $\mathcal{B}' \cup \{a', b', c'\}$  is an STS and  $\mathcal{B}' \cup \{a'', b'', c''\}$  is an STS. So, any pair of points from S' except  $\{a', b'\}$ ,  $\{a', c'\}$ ,  $\{b', c'\}$  is included in one block of  $\mathcal{B}'$ . Similarly, with S'' and  $\mathcal{B}''$ . Therefore,

(\*) Any pair of points from D different from  $\{a',b'\}$ ,  $\{a',c'\}$ ,  $\{b',c'\}$ ,  $\{a'',b''\}$ ,  $\{a'',c''\}$ ,  $\{b'',c''\}$  is included in one block of D.

Further,

(\*\*) For every point t from D, the number l(t) of pairs of points from D containing t and different from  $\{a',b'\}$ ,  $\{a',c'\}$ ,  $\{b',c'\}$ ,  $\{a'',b''\}$ ,  $\{a'',c''\}$ ,  $\{b'',c''\}$  is even. Indeed, this number is twice the number of blocks of  $\mathcal{D}$  containing t.

By easy check of all cases for the intersections of the sets D,  $\{a', b', c'\}$ , and  $\{a'', b'', c''\}$ , one can find that if both (\*) and (\*\*) are satisfied, then one of the following assertions holds.

- (i) D contains at most one point from a', b', c' and at most one point from a'', b'', c''. In this case,  $\mathcal{D}$  is an STS.
- (ii) D consists of two points from a', b', c' or from a'', b'', c''. So, |D| = 2.

- (iii) D contains a', b', c' and at most one of a'', b'', c''; or D contains a', b', c' and two or three of a'', b'', c'' that coincide with some of a', b', c'. Or, analogously, D contains a'', b'', c'' and at most one of a', b', c'; or D contains a'', b'', c'' and two or three of a', b', c' that coincide with some of a'', b'', c''. In this case,  $\mathcal{D}$  can be completed to an STS by adding the triple  $\{a', b', c'\}$  or  $\{a'', b'', c''\}$ , respectively.
- (iv) D includes both  $\{a', b', c'\}$  and  $\{a'', b'', c''\}$  and these two sets intersect in at most one point. In this case,  $\mathcal{D}$  can be completed to an STS by adding the two triples  $\{a', b', c'\}$  and  $\{a'', b'', c''\}$ .
- (v) |D| = 4 and two points e', f' from D are in  $\{a', b', c'\}$ , while the other two e'', f'' are in  $\{a'', b'', c''\}$ . It is easy to see that the four pairs  $\{e', e''\}$ ,  $\{e', f''\}$ ,  $\{f', e''\}$ ,  $\{f', f''\}$  cannot be covered by blocks of  $\mathcal{D}$  in a proper way.

We have been convinced that the statement of the lemma holds in any non-contradictory case.  $\Box$ 

Remark 1. In contrast to the case of sub-STS's, the supports of two almost-sub-STS's can intersect in exactly two points. One can easily construct such example using known embedding theorems: any set of 3-sets such that no pair of points meets in more than one set can always be embedded as a subset in a Steiner triple system, whose support can in general be larger that the support of the original triple set [12], [2].

Corollary 1. (i) Different supports of two almost-sub-STS(9) of the same STS intersect in at most 3 points. (ii) Different supports of two almost-sub-STS(9) of the same STS(21) intersect in 3 points. These 3 points form either a block of each of the two almost-sub-STS(9), or the missing triple of one or both of the almost-sub-STS(9).

*Proof.* To prove (i), by Lemma 2, it remains to verify that the supports cannot intersect in 7 points. Indeed, if such situation happens, then by Lemma 2, there are STS(7) and STS(9) that have at least 5 blocks in common. It is straightforward to check that this is not possible.

(ii) It remains to prove that the supports, say S' and S'', of the subsystems, say  $\mathcal{B}'$  and  $\mathcal{B}''$ , cannot intersect in 2, 1, or 0 points. Assume first that  $|S' \cap S''| = 2$ . The block including  $S' \cap S''$  does not belong to at least one of  $\mathcal{B}'$  and  $\mathcal{B}''$ . W.l.o.g., assume that it is not in  $\mathcal{B}''$ ; so, the pair  $S' \cap S''$  lies

in the missing triple of the almost-sub-STS  $\mathcal{B}''$ . Hence, 6 points  $a_1, \ldots, a_6$  from  $S'' \setminus S''$  do not belong to the missing triple. Take also one point b from  $S' \setminus S''$  that do not belong to the missing triple of  $\mathcal{B}'$  and consider the block containing b and  $a_i$ ,  $i \in \{1, 2, 3, 4, 5, 6\}$ . This block is not from  $\mathcal{B}'$  or  $\mathcal{B}''$ ; hence, it intersects with each of S' and S'' in only one point. So, the third point  $c_i$  of this block does not belong to  $S' \cup S''$ . But there are only 5 points not in  $S' \cup S''$ , which immediately leads to a contradiction with the definition of STS. Similar contradictions can be found in the cases  $|S' \cap S''| = 1$  and  $|S' \cap S''| = 0$ .

Next, we focus on the order 21. Assume we are given an STS(21)  $(S, \mathcal{B})$ . A partition of S into four sets A, B, C, D of size 6, 6, 6, and and 3 respectively is called a *flower* with *stem* D and *petals* A, B, C if  $\mathcal{B}$  has a sub-STS(9) and two almost-sub-STS(9) with supports  $A \cup D$ ,  $B \cup D$ ,  $C \cup D$ , where the missing triple of each of these almost-sub-STS is D (whenever it belongs to  $\mathcal{B}$  or not). From Lemma 1, we can easily deduce the following.

**Lemma 3.** An STS(21) has a flower  $\{A, B, C, D\}$  with the stem D if and only if it has a sub-TD(3,6) with groups A, B, C.

If there is only one flower  $\{A, B, C, D\}$  (and only one sub  $\mathrm{TD}(3,6)$ ), then we have two subcases, depending on whether the stem D is a block or not. In the first subcase, the  $\mathrm{STS}(21)$  has three sub- $\mathrm{STS}(9)$  with supports  $A \cup D$ ,  $B \cup D$ ,  $C \cup D$ . In the second subcase, the  $\mathrm{STS}(21)$  has only one sub- $\mathrm{STS}(9)$ . Our next goal is to characterize the situation when  $\mathrm{STS}(21)$  has more than one sub- $\mathrm{TD}(3,6)$ .

**Lemma 4.** Assume that an STS(21) has two different flowers  $\{A, B, C, D\}$  and  $\{A', B', C', D'\}$  with stems D, D'. Then

- (i) D and D' are disjoint;
- (ii)  $D \cup E = D' \cup E'$ , for some  $E \in \{A, B, C\}$  and  $E' \in \{A', B', C'\}$ ;
- (iii) the STS(21) has a sub-STS(9) with support  $D \cup E$ , where E is from p.(ii).

*Proof.* Consider a point d from D and assume without loss of generality that it lies in  $D' \cup A'$ . Every point of  $D' \cup A'$  lies in one of  $D \cup A$ ,  $D \cup B$ ,  $D \cup C$  and at least one point lies in all. Hence,  $D' \cup A'$  intersects in more than 3 points with these three sets in average. By Corollary 1, it coincides with one

of them, say  $D \cup A$ ; so, (ii) is proved. If D and D' intersect, then the same can be said about  $D' \cup B'$  and  $D' \cup C'$ , and the flowers coincide. So, (i) holds. The last claim is also easy as the union of two different almost-sub-STS with the same support  $D \cup A$  is necessarily a sub-STS.

**Lemma 5.** Assume that an STS(21)  $\mathcal{B}$  has two different flowers  $\{A, B, C, D\}$  and  $\{A', B', C', D'\}$  with stems D, D'. Assume without loss of generality that  $D \cup A = D' \cup A'$ . Denote

$$A_{001} := D,$$
  $A_{011} := A \setminus D',$   $A_{010} := D',$  
$$A_{101} := B \cap C',$$
  $A_{111} := C \cap C',$   $A_{110} := C \cap B',$  
$$A_{100} := B \cap B'.$$

The following assertions hold.

(i) If both D and D' are blocks of  $\mathcal{B}$ , then  $\mathcal{B}$  includes exactly 7 sub-TD(3, 6) with flowers

$$\{A_{001}, A_{010} \cap A_{011}, A_{100} \cap A_{101}, A_{110} \cap A_{111}\},$$
 (1)

$$\{A_{010}, A_{001} \cap A_{011}, A_{100} \cap A_{110}, A_{101} \cap A_{111}\},$$
 (2)

$$\{A_{011}, A_{001} \cap A_{010}, A_{100} \cap A_{111}, A_{101} \cap A_{110}\},$$
 (3)

$$\{A_{100}, A_{001} \cap A_{101}, A_{010} \cap A_{110}, A_{011} \cap A_{111}\},$$
 (4)

$$\{A_{101}, A_{001} \cap A_{100}, A_{010} \cap A_{111}, A_{011} \cap A_{110}\},$$
 (5)

$$\{A_{110}, A_{001} \cap A_{111}, A_{010} \cap A_{100}, A_{011} \cap A_{101}\},$$
 (6)

$$\{A_{111}, A_{001} \cap A_{110}, A_{010} \cap A_{101}, A_{011} \cap A_{100}\},$$
 (7)

and exactly 7 sub-STS(9).

(ii) If at most one of D, D' is a block of  $\mathcal{B}$ , then  $\mathcal{B}$  includes exactly 3 sub-TD(3,6) with flowers (1)–(3). In this case, if D or D' is a block, then  $\mathcal{B}$  has exactly 3 sub-STS(9); otherwise exactly 1.

*Proof.* We first note that by Corollary 1(ii),  $A_{101}$ ,  $A_{111}$ ,  $A_{110}$ , and  $A_{100}$  are blocks of  $\mathcal{B}$ . Next, we state that

(\*) there is an almost-sub-STS with the support  $A_{011} \cap A_{100} \cap A_{111}$  and the missing triple  $A_{011}$ . Indeed, consider a block containing a point a from  $A_{100}$  and a point b from  $A_{111}$ . The third point c of this block can only belong

to  $A_{011}$  (for example, if  $c \in A_{001}$ , then the pair  $\{a, c\}$  is already covered by a block from the almost-STS on  $A_{001} \cap A_{101} \cap A_{100}$ ; the other cases lead to similar contradictions). So, the 9 such blocks form a TD(3,3); completing by the blocks  $A_{100}$  and  $A_{111}$ , we get an almost-sub-STS(9). Similarly,

(\*\*) there is an almost-sub-STS with the support  $A_{011} \cap A_{101} \cap A_{110}$  and the missing triple  $A_{011}$ .

So, we have a collection from a sub-STS(9) and six almost-sub-STS(9) with different supports, corresponding to the flowers (1)–(3). It is easy to find that

(\*\*\*) there is no sub-STS(9) or almost-sub-STS(9) with any other support. Indeed, if B is the support of a sub-STS(9), then it intersects in at least four points in total with some two sets  $A_{...}$ ; the union of these two sets is included in the support of some of the seven sub-STS(9), and by Corollary 1 B coincides with this support.

Now consider subcases.

If both D and D' are in  $\mathcal{B}$ , then we also have  $A_{011} \in \mathcal{B}$ , and all those six almost-sub-STS are completed to a sub-STS(9), forming seven different flowers in total. From Lemma 5 and Corollary 1, we conclude that there are no more flowers. By (\*\*\*), there are only seven sub-STS(9).

If  $D \notin \mathcal{B}$  or  $D' \notin \mathcal{B}$ , then (4)–(7) are not flowers. Arguments similar as above show that there are only three flowers (1)–(3).

If  $D \in \mathcal{B}$  and  $D' \notin \mathcal{B}$ , then we also have  $A_{011} \notin \mathcal{B}$  (in any STS(9), the complement of two disjoint blocks is necessarily a block too). In this case, we have only three sub-STS(9) with supports  $A_{001} \cup A_{011} \cup A_{010}$ ,  $A_{001} \cup A_{101} \cup A_{100}$ ,  $A_{001} \cup A_{111} \cup A_{110}$ . The subcase  $D \notin \mathcal{B}$  and  $D' \in \mathcal{B}$  is similar.

If  $D \notin \mathcal{B}$  and  $D' \notin \mathcal{B}$ , then the missing triple of any of the six almost-sub-STS is not in  $\mathcal{B}$ , and  $\mathcal{B}$  has only one sub-STS(9), with the support  $A_{001} \cup A_{011} \cup A_{010}$ .

**Remark 2.** One can observe that each of the seven supports of (almost)-sub-STS(9) considered in the lemma above is the union of three of  $A_{001}$ ,  $A_{010}$ ,  $A_{011}$ ,  $A_{100}$ ,  $A_{101}$ ,  $A_{101}$ ,  $A_{111}$ . The corresponding seven triples form an STS(7) on the point set  $\{A_{001}, A_{010}, A_{011}, A_{100}, A_{101}, A_{110}, A_{111}\}$  (the STS(7) is unique up to isomorphism and known as the *Fano plane*).

The next two well-known and straightforward facts will be utilized in our further discussion.

**Proposition 1.** If a TD(3,6)  $(S, \{A, B, C\}, \mathcal{T})$  has a sub-TD(3,3) with the groups  $A_0 \subset A$ ,  $B_0 \subset B$ ,  $C_0 \subset C$ , then  $\mathcal{T}$  has exactly three other sub-TD(3,3), with groups  $A_0$ ,  $B_1$ ,  $C_1$ , with groups  $A_1$ ,  $B_0$ ,  $C_1$ , and with groups  $A_1$ ,  $B_1$ ,  $C_0$ , where  $A_1 := A \setminus A_0$ ,  $B_1 := B \setminus B_0$ ,  $C_1 := C \setminus C_0$ .

**Proposition 2.** If D is a block of an STS(9)  $(S, \mathcal{B})$ , then  $\mathcal{B}$  has exactly two blocks disjoint with D. Moreover, these two blocks are disjoint with each other, and the remaining 9 blocks form a sub-TD(3, 3).

**Lemma 6.** Assume that an STS(21)  $(S, \mathcal{B})$  has a flower  $\{A, B, C, D\}$ , and  $\mathcal{T}$  is a transversal subdesign of  $\mathcal{B}$  on the petals A, B, C, as the groups. Let D' be a 3-subset of A. The system  $\mathcal{B}$  has a second sub-TD(3,6)  $\mathcal{T}'$  with the support  $S \setminus D'$  if and only if it has disjoint blocks  $B_0, B_1 \subset B$  and disjoint blocks  $C_0, C_1 \subset C$  such that  $\mathcal{T}$  is partitioned into four sub-TD(3,3) with groups from D',  $A \setminus D'$ ,  $B_0$ ,  $B_1$ ,  $C_0$ ,  $C_1$ .

Proof. Assume that there is such subdesign  $\mathcal{T}'$ . In this case, there is a flower  $\{D', A', B', C'\}$ , where  $D' \cup A' = D \cup A$ . Denote  $B_0 := B \cap B'$ ,  $B_1 := B \cap C'$ ,  $C_0 := C \cap B'$ ,  $C_1 := C \cap C'$ . By Corollary 1,  $B_0$ ,  $B_1$ ,  $C_0$ ,  $C_1$  are blocks of  $\mathcal{B}$ . Since  $\mathcal{B}$  has a sub-STS with the support  $D' \cup A'$ , by the definition of a flower it has two almost-sub-STS with the supports  $D' \cup B'$  and  $D' \cup C'$ . Removing the blocks  $B_0$ ,  $B_1$ ,  $C_0$ ,  $C_1$  from these almost-sub-STS's, we obtain two TD(3, 3). The remaining two sub-TD(3, 3) of  $\mathcal{T}$  are guaranteed by Proposition 1.

The "if" part of the statement is also straightforward, taking into account Lemma 3. If B is partitioned into two blocks  $B_0$ ,  $B_1$ , then, by the definition of a flower and Proposition 2, we see that there is an almost-sub-STS with the support  $B \cup D$  and the missing triple D (which can be a block or not a block of  $\mathcal{B}$ ). The same can be said about the support  $C \cup D$ . Then the definition of a flower implies that  $\mathcal{B}$  has a complete sub-STS with the support  $A \cup D$ . It remains to find two more petals, to form a flower with the stem D'. By the hypothesis, we have a sub-TD with groups D',  $B_0$  and  $C_0$ , for some block  $C_0 \subset C$ . Completing it by the blocks  $B_0$  and  $C_0$ , we get an almost-sub-STS. Similarly, we find an almost-sub-STS with the support  $D' \cup B_1 \cup C_1$ ,  $C_1 := C \setminus C_0$ . So,  $\{D', D \cup A \setminus D', B_0 \cup C_0, B_1 \cup C_1\}$  is a flower, and by Lemma 3 there is a required sub-TD.

### 3. Classification of STS(21) with sub-TD(3, 6).

Now, based on the lemmas above, we are ready to present the way of the computer-aided classification and its results. We start with describing how to count the number of isomorphism classes of STS(21) with a unique sub-TD(3,6).

We first fix a flower  $\{A_{001}, A_{010} \cap A_{011}, A_{100} \cap A_{101}, A_{110} \cap A_{111}\}$ , where all sets  $A_{...}$  are of size 3. Let  $\mathbf{A}$  be the set of all 840 STS(9) on  $A_{001} \cap A_{010} \cap A_{011}$ . Denote by  $\mathbf{A}'$  the subset of  $\mathbf{A}$  that consists of 120 STS(9) with block  $A_{001}$ ; by deleting this block in all these STS we obtain the set  $\mathbf{A}^*$  of 120 almost-STS(9) with missing  $A_{001}$ . Denote by  $\mathbf{A}''$  the subset of  $\mathbf{A}'$  that consists of 12 STS(9) with blocks  $A_{001}$ ,  $A_{010}$ ,  $A_{011}$ . Similarly, we define the collections  $\mathbf{B}$ ,  $\mathbf{B}'$ ,  $\mathbf{B}''$ ,  $\mathbf{B}^*$  of triple systems on  $A_{001} \cap A_{100} \cap A_{101}$  and the collections  $\mathbf{C}$ ,  $\mathbf{C}''$ ,  $\mathbf{C}''$  of triple systems on  $A_{001} \cap A_{110} \cap A_{111}$ .

Next, we choose a representative  $\mathcal{T}$  of one of 12 (see [21]) isomorphism classes of TD(3,6) with groups  $A_{010} \cap A_{011}$ ,  $A_{100} \cap A_{101}$ ,  $A_{110} \cap A_{111}$ . Moreover, we require that if the representative is divided into sub-TD(3,3)'s, then these sub-TD's have the group sets  $\{A_{010}, A_{100}, A_{110}\}$ ,  $\{A_{010}, A_{101}, A_{111}\}$ ,  $\{A_{011}, A_{100}, A_{111}\}$ ,  $\{A_{011}, A_{101}, A_{101}, A_{110}\}$  (see Proposition 1).

Now, by Lemma 3, every STS(21) with sub-TD  $\mathcal{T}$  is divided into  $\mathcal{T}$ ,  $\mathcal{A}$ ,  $\mathcal{B}$ , and  $\mathcal{C}$ , where

- either  $A \in A'$ ,  $B \in B^*$ ,  $C \in C^*$ ,
- $\bullet \ \, {\rm or} \,\, \mathcal{A} \in \mathbf{A} \backslash \mathbf{A}', \qquad \mathcal{B} \in \mathbf{B}^*, \qquad \mathcal{C} \in \mathbf{C}^*,$
- $\bullet \ \, {\rm or} \,\, \mathcal{A} \in \mathbf{A}^*, \qquad \mathcal{B} \in \mathbf{B} \backslash \mathbf{B}', \qquad \mathcal{C} \in \mathbf{C}^*,$
- $\bullet \ \, {\rm or} \,\, \mathcal{A} \in \mathbf{A}^*, \qquad \mathcal{B} \in \mathbf{B}^*, \qquad \mathcal{C} \in \mathbf{C} \backslash \mathbf{C}'.$

Moreover, by Lemmas 5 and 6, such STS(21) has exactly 7 sub-TD(3,6) if and only if  $\mathcal{T}$  is divided into sub-TD(3,3) and

$$\mathcal{A} \cup A_{001} \in \mathbf{A}'', \qquad \mathcal{B} \cup A_{001} \in \mathbf{B}'', \qquad \mathcal{C} \cup A_{001} \in \mathbf{C}'';$$
 (8)

and it has exactly 3 sub-TD(3,6) if and only if  $\mathcal{T}$  is divides into sub-TD(3,3) and exactly two of (8) are satisfied. We exclude these cases and finally have at most  $120^3 + 3 \cdot 720 \cdot 120^2$  STS(21) with only one sub-TD(3,6), equal to  $\mathcal{T}$ . Using the graph-isomorphism software [14], we can check all of them on isomorphism and keep the representatives. Trivially, any STS(21) that has

a sub-TD isomorphic to  $\mathcal{T}$  is isomorphic to some STS(21) that includes  $\mathcal{T}$ . Repeating the steps above for each of 12 nonisomorphic choices of  $\mathcal{T}$ , we find all equivalence classes of STS(21) with only one sub-STS(9).

Similarly, we can classify the STS(21) with 3 or 7 sub-STS(9). The only difference is that we also need to check for isomorphism between the representatives obtained from different  $\mathcal{T}$ .

The results of the calculation are reflected in Table 1. The last column of the table was calculated by comparing the data in the other columns with the results of [10]. All calculations took few core-hours on a modern PC. We summarize the results in the following theorem.

**Theorem 1.** There are 2004720 Steiner triple systems of order 21 with transversal subdesigns on 3 groups of size 6. 2004109 of them have exactly one such sub-TD(3,6), 599 have exactly three sub-TD(3,6), and 12 have seven sub-TD(3,6) (by Lemma 5, the last group coincides with the 12 STS(21) having 7 sub-STS(7), found in [10]).

#### 4. Validity of the results

In this section, we consider a double-counting argument that validates the results of computing.

**Proposition 3.** Given a point set S of size 21, there are exactly

$$\frac{21!}{3!^2 \cdot 6!^3} \cdot (120^3 + 3 \cdot 720 \cdot 120^2) \cdot 812851200 = 101473423278637842432000000$$

pairs  $(\mathcal{B}, \mathcal{T})$ , where  $(S, \mathcal{B})$  is an STS(21) and  $\mathcal{T}$  is a sub-TD(3,6) of  $\mathcal{B}$ .

*Proof.* We first remind that there are 840 different STS(9) with given support, see, e.g., [22]; a given triple of points belongs to exactly 120 of them.

A set of cardinality 21 can partitioned into a flower  $\{A, B, C, D\}$  in  $21! \cdot 3!^{-2} \cdot 6!^{-3}$  ways. Assuming that D is a block, we can choose an almost-sub-STS with each of the supports  $A \cup D$ ,  $B \cup D$ ,  $C \cup D$  in 120 ways. Assuming that the D is not a block, we can choose which of  $A \cup D$ ,  $B \cup D$ ,  $C \cup D$  is the support of a sub-STS in 3 ways, then choose that sub-STS in 840 - 120 = 720 ways, then choose each of the remaining two almost-sub-STS's in 120 ways. Finally, we choose a transversal design with the groups A, B, C in 812851200 ways (the total number of different  $6 \times 6$  latin squares [20]).

Aut	$\begin{array}{c} \tau_6 = 7 \\ \sigma_9 = 7 \end{array}$	$ \tau_6 = 3 \\ \sigma_9 = 3 $	$ \tau_6 = 3 \\ \sigma_9 = 1 $	$ \tau_6 = 1 \\ \sigma_9 = 3 $	$ \tau_6 = 1 \\ \sigma_9 = 1 $	$ \tau_6 = 0 \\ \sigma_9 = 1 $
1		98 (0)	171 (0)	101621 (355)	1865036 (0)	12656035473
$\frac{2}{3}$		45(0)	36(0)	5271 (14)	30771(0)	3461498
3		37(0)	66(0)	103 (8)	52 (0)	14932
4		18 (0)	14(0)	321 (1)	786 (0)	10328
6	1 (1)	$31\ (0)$	45(0)	24 (1)	8 (0)	157
8	1 (0)	7(0)	1(0)	60 (5)	23(0)	130
9	1 (1)		9(0)			12
12	1 (1)	6(0)	8 (0)	5(0)	5(0)	60
14	1 (0)					
16	1(0)	2(0)		9 (1)		
18	2(1)		3(0)	1 (0)		6
24				7(3)	1 (0)	11
27						3
36			1(0)	1 (0)		3
48				2(0)		
54	1 (0)					
72			1(0)	1 (0)		3
108	1 (0)					
144				1 (0)		
504	1(0)					
1008	1 (1)					
any	12 (5)	244(0)	355(0)	107427 (388)	1896682 (0)	12659522616

Table 1: The number of the isomorphism classes of STS(21) with sub-TD(3,6), sorted by the number  $\tau_6$  of sub-TD(3,6), the number  $\sigma_9$  of sub-STS(9), and the number of automorphisms (the number or nonisomorphic resolvable systems, if known, is given in parenthesis).

On the other hand, we can calculate the same number based on the given representatives of the isomorphism classes.

**Proposition 4.** Let S be a set of 21 points and let S be a set of representatives of all isomorphism classes of STS(21) on S. The number of pairs  $(\mathcal{B}, \mathcal{T})$  where  $(S, \mathcal{B})$  is an STS(21) and  $\mathcal{T}$  is a sub-TD(3, 6) of  $\mathcal{B}$  is calculated by the formula

$$\sum_{\mathcal{B} \in \mathbf{S}} N(\mathcal{B}) \cdot \frac{21!}{|\operatorname{Aut}(\mathcal{B})|},\tag{9}$$

where  $N(\mathcal{B})$  is the number of sub-TD(3,6) in  $\mathcal{B}$ .

Using the data in Table 1, we can compute the nonzero (with  $N(\mathcal{B}) > 0$ ) terms in the sum (9), which happens to coincide with the value in Proposition 3. This approves the results of our computing.

#### 5. Resolvability

A Steiner triple system  $(S, \mathcal{B})$  is called *resolvable* if  $\mathcal{B}$  can be partitioned into parallel classes, where a *parallel class* is a partition of S into blocks. We check all found systems on resolvability and found 393 isomorphism classes of resolvable STS of considered type. As we see from Table 1, there is no resolvable STS(21) with sub-TD(3,6) and only one sub-STS(9). We can prove this fact theoretically.

**Proposition 5.** If an STS(21) has a sub-TD(3,6) and only one sub-STS(9) then  $\mathcal{D}$  is not resolvable.

*Proof.* Let  $(S, \mathcal{D})$  be an STS, let  $(A \cup B \cup C, \{A, B, C\}, \mathcal{T})$  be a sub-TD(3, 6), corresponding to the flower  $\{A, B, C, D\}$  and let  $\mathcal{A}$  be the unique sub-STS. Without loss of generality we assume that the support of  $\mathcal{A}$  is  $A \cup D$ . So, there are two almost-sub-STS with the supports  $B \cup D$  and  $C \cup D$  respectively, and the missing triple D. By the hypothesis,  $D \notin \mathcal{D}$ .

Let  $D = \{a, b, c\}$ . Seeking a contradiction, assume that there is a resolution. Consider the block U containing a and b and consider the parallel class  $\mathcal{P}$  containing this block. Denote  $t := |\mathcal{P} \cap \mathcal{T}|$ . We state that

(\*) the block V from  $\mathcal{P}$  containing c belongs to  $\mathcal{A}$ . Indeed, if it is in  $\mathcal{B}$ , then  $|B\backslash V|=4$ , and t of these 4 points are covered by blocks from  $\mathcal{T}\cap\mathcal{P}$ , the other 4-t being covered by blocks from  $\mathcal{B}\cap\mathcal{P}$ . Hence,  $4-t\equiv 0$  mod 3.

On the other hand, t of the 6 points of C are covered by blocks from  $\mathcal{T} \cap \mathcal{P}$ , the other 6-t being covered by blocks from  $\mathcal{C} \cap \mathcal{P}$ . So,  $6-t \equiv 0 \mod 3$ , a contradiction. Similarly,  $V \not\in \mathcal{C}$ , and (\*) holds.

Since  $|A \setminus U \setminus V| = 3$ , we have  $t \leq 3$ . Therefore,  $\mathcal{P}$  contains at least one block from  $\mathcal{B}$  and at least one block from  $\mathcal{C}$ , and these blocks have no points in D. The same can be said about the parallel class that contains the block with a and c. And similarly, for the parallel class that contains the block with b and c. We conclude that  $\mathcal{B}$  has at least three blocks disjoint with D. This contradicts to Proposition 2.

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#### References

- [1] S. Bays. Recherche des systèmes cycliques de triples de steiner différents pour n premier (ou puissance de nombre premier). J. Math. Pures Appl. (9), 2:73–98, 1923.
- [2] D. Bryant and D. Horsley. A proof of Lindner's conjecture on embeddings of partial Steiner triple systems. J. Comb. Des., 17(1):63–89, 2008. DOI: 10.1002/jcd.20189.
- [3] C. J. Colbourn and R. Mathon. Steiner systems. In C. J. Colbourn and J. H. Dinitz, editors, *Handbook of Combinatorial Designs*, Discrete Mathematics and Its Applications, pages 102–108. Chapman & Hall/CRC, Boca Raton, London, New York, second edition, 2006.
- [4] M. J. Colbourn. An analysis technique for steiner triple systems. In *Proc. Tenth S.-E. Conf. Combinatorics, Graph Theory and Computing, Vol. I*, pages 289–303. Boca Raton, 1979.
- [5] R. H. F. Denniston. Non-isomorphic reverse Steiner triple systems of order 19. *Ann. Discrete Math.*, 7:255–264, 1980. DOI: 10.1016/S0167-5060(08)70183-3.

- [6] A. Hulpke, P. Kaski, and P. R. J. Östergård. The number of Latin squares of order 11. Math. Comput., 80:1197–1219, 2011. DOI: 10.1090/S0025-5718-2010-02420-2.
- [7] D. Jungnickel, S. S. Magliveras, V. D. Tonchev, and A. Wassermann. The classification of Steiner triple systems on 27 points with 3-rank 24. Des. Codes Cryptography, 87(4):831–839, 2019. DOI: 10.1007/s10623-018-0502-5.
- [8] S. N. Kapralov and S. Topalova. On the Steiner triple systems of order 21 with automorphisms of order 3. In *Proc. the Third Int. Workshop on Algebraic and Combinatorial Coding Theory, Voneshta Voda, Bulgaria, 22–28 June 1992*, pages 105–108, 1992.
- [9] P. Kaski and P. R. J. Östergård. The Steiner triple systems of order 19. J. Comb. Theory, Ser. A, 73(248):2075–2092, 2004. DOI: 10.1090/S0025-5718-04-01626-6.
- [10] P. Kaski, P. R. J. Östergård, and A. Popa. Enumeration of steiner triple systems with subsystems. *Math. Comput.*, 84(296):3051–3067, Nov. 2015. DOI: 10.1090/mcom/2945.
- [11] P. Kaski, P. R. J. Östergård, S. Topalova, and R. Zlatarski. Steiner triple systems of order 19 and 21 with subsystems of order 7. *Discrete Math.*, 308(13):2732–2741, July 2008. DOI: 10.1016/j.disc.2006.06.038.
- [12] C. C. Lindner. A survey of embedding theorems for Steiner systems. In C. C. Lindner and A. Rosa, editors, *Topics on Steiner Systems*, volume 7 of *Ann. Discrete Math.*, pages 175–202. North-Holland, 1980.
- [13] R. A. Mathon, K. T. K.T. Phelps, and A. Rosa. A class of Steiner triple systems of order 21 and associated Kirkman systems. *Math. Comput.*, 37(155):209–222, 1981. and 64 (1995) 1355–1356.
- [14] B. D. McKay and A. Piperno. Practical graph isomorphism, II. *J. Symb. Comput.*, 60:94–112, 2014. DOI: 10.1016/j.jsc.2013.09.003.
- [15] O. P. Osuna. There are 1239 Steiner triple systems STS(31) of 2-rank 27. Des. Codes Cryptography, 40(2):187–190, Aug. 2006. DOI: 10.1007/s10623-006-0006-6.

- [16] K. T. Phelps and A. A. Rosa. Rotational Steiner triple systems. *Discrete Math.*, 33(1):57–66, 1981. DOI: 10.1016/0012-365X(81)90258-2.
- [17] D. R. Stinson and E. Seah. 284457 Steiner triple systems of order 19 contain a subsystem of order 9. Math. Comput., 46(174):717-729, 1986.
   DOI: 10.1090/S0025-5718-1986-0829642-7.
- [18] V. D. Tonchev. Steiner triple systems of order 21 with automorphisms of order 7. Ars Combin., 23:93–96, 1987. and 39 (1995) 3–3.
- [19] R. M. Wilson. Nonisomorphic Steiner triple systems.  $Mathematische\ Zeitschrift, 135(4):303-313, 1974.$  DOI: 10.1007/BF01215371.
- [20] Sequence A002860 in The On-Line Encyclopedia of Integer Sequences. Published electronically at http://oeis.org, 2004. Number of Latin squares of order n; or labeled quasigroups.
- [21] Sequence A003090 in The On-Line Encyclopedia of Integer Sequences. Published electronically at http://oeis.org, 2004. Number of species (or "main classes" or "paratopy classes") of Latin squares of order n.
- [22] Sequence A030128 in The On-Line Encyclopedia of Integer Sequences. Published electronically at http://oeis.org, 2004. Steiner triple systems (STS's) on n elements.